An Oriented Distance Function Application to Gap Functions for Vector Variational Inequalities

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Abstract This paper aims to extend some results dealing with gap functions for vector variational inequalities from the literature by using the so-called oriented distance function.

Key words Vector variational inequalities • Gap function • Oriented distance function.

1 Introduction

The so-called gap function approach allows to reduce the investigation of variational inequalities into the study of optimization problems. Let us mention several papers which are devoted to the study of set-valued gap functions for vector variational inequalities. Specially, the generalizations of Auslender's and Giannessi's gap functions for vector variational inequalities have been introduced in [5]. More recently, a conjugate duality approach to the construction of a gap function has been applied to vector variational inequalities (see [2]).

On the other hand, scalarization techniques in vector optimization have been applied to the construction of a gap function for vector variational inequalities.

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For instance, we refer to [3, 11, 13, 17] for vector variational inequalities, to [12] for generalized vector variational inequalities and to [15] for set-valued vector variational-like inequalities.

This paper concentrates on scalar-valued gap functions for vector variational inequalities on the basis of the oriented distance function and the approach presented in [14]. For some investigations dealing with the oriented distance function we refer to [6–10, 16] and [18]. The oriented distance function allows us to extend some results dealing with gap functions for vector variational inequalities from the literature (cf. [11–13, 15] and [17]).

The paper is organized as follows. In section 2 we recall some preliminary results dealing with the oriented distance function. The section 3 is devoted to introduce gap functions for vector variational inequalities. Moreover, we suggest another type of gap functions, which are based on dual problems. For this purpose, we use the powerful approach of the perturbation theory of the conjugate duality. We conclude our paper with the extension to some set-valued problems in section 4.

2 Mathematical preliminaries

Let X be a Hausdorff locally convex space. The dual space of X is denoted by X^*. For $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle := x^*(x)$ be the value of the linear continuous functional x^* at x. For a subset $A \subseteq X$ we define the indicator function $\delta_A : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ by

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a given function $h : X \to \overline{\mathbb{R}}$ the effective domain is

$$\text{dom } h := \{x \in X : \ h(x) < +\infty\}.$$

The function h is called proper if dom $h \neq \emptyset$ and $h(x) > -\infty$ for all $x \in X$. For a nonempty subset $E \subseteq X$ we define the conjugate function of h by

$$h_E^* : X^* \to \overline{\mathbb{R}},\ h_E^*(p^*) = (h + \delta_E)^*(p^*) = \sup_{x \in E}\{\langle x^*, x\rangle - h(x)\}.$$

When $E = X$, one can see that h_E^* turns into the classical Fenchel-Moreau conjugate function of h denoted by h^*.

Let Y be a real Banach space partially ordered by a closed convex pointed cone C with nonempty interior, i.e. $\text{int}\, C \neq \emptyset$. A weak ordering in Y is defined by

$$y \prec x \Leftrightarrow x - y \in \text{int}\, C \ \text{ and } \ y \not\prec x \Leftrightarrow x - y \notin \text{int}\, C,\ x, y \in Y.$$

Let $C^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \ \forall y \in C\}$ be the dual cone of C. The space of the linear continuous mappings from X to Y is denoted by $\mathcal{L}(X,Y)$. Moreover, let $S_Y := \{y \in Y : \|y\| = 1\}$ and $S(M) := \{y \in M : \|y\| = 1\}$, $M \subseteq Y$.
Further, we consider a Hausdorff locally convex vector space Z and a nonempty convex cone $D \subseteq Z$, which induces on Z a partial ordering $\leqq_D$, i.e. for $x, y \in Z$ it holds $x \leqq_D y \Leftrightarrow y - x \in D$. We attach to Z a greatest and a smallest element with respect to "$\leqq_D$", denoted by $+\infty_D$ and $-\infty_D$, respectively, which do not belong to Z and denote $\overline{Z} = Z \cup \{\pm\infty_D\}$. Besides, we define $x \leq_D y$ if and only if $x \leqq_D y$ and $x \neq y$. For all $x \in \overline{Z}$ it holds $-\infty_D \leqq_D x \leqq_D +\infty_D$ and for all $x \in Z$ it holds $-\infty_D \leq_D x \leq_D +\infty_D$.

In this paper, we consider on $\overline{Z}$ the following operations and conventions (cf. [4]): $x + (+\infty_D) = (+\infty_D) + x := +\infty_D \ \forall x \in Z \cup \{+\infty_D\}$, $x + (-\infty_D) = (-\infty_D) + x := -\infty_D \ \forall x \in Z \cup \{-\infty_D\}$, $\lambda \cdot (+\infty_D) := +\infty_D \ \forall \lambda \in (0, +\infty]$, $\lambda \cdot (+\infty_D) := -\infty_D \ \forall \lambda \in [-\infty, 0)$, $\lambda \cdot (-\infty_D) := -\infty_D \ \forall \lambda \in (0, +\infty]$, $\lambda \cdot (-\infty_D) := +\infty_D \ \forall \lambda \in [-\infty, 0)$, $(+\infty_D) + (-\infty_D) = (-\infty_D) + (+\infty_D) := +\infty_D$, $0(+\infty_D) := +\infty_D$ and $0(-\infty_D) := 0$. Further, define $\langle z^*, +\infty_D \rangle := +\infty_D$ for $z^* \in D^*$.
For a vector function $g : X \to \overline{Z}$ the domain is the set $\operatorname{dom} g := \{x \in X : g(x) \neq +\infty_D\}$. If $g(x) \neq -\infty_D$ for all $x \in X$ and $\operatorname{dom} g \neq \emptyset$, then the vector function g is called proper.

When $g(\lambda x + (1-\lambda)y) \leqq_D \lambda g(x) + (1-\lambda)g(y)$ holds for all $x, y \in X$ and all $\lambda \in [0,1]$ the vector function g is said to be D-convex.

Definition 2.1. Let $M \subseteq Y$. Then the function $\Delta_M : Y \to \overline{\mathbb{R}}$ defined by

$$\Delta_M(y) := d_M(y) - d_{M^c}(y), \ y \in Y,$$

is called the oriented distance function, where $d_M(y) = \inf_{z \in M} \|y - z\|$ is the distance function from the point $y \in Y$ to the set M and $M^c := Y \backslash M$.

The oriented distance function was introduced by Hiriart-Urruty ([9], [10]) in order to investigate optimality conditions in nonsmooth optimization. The main properties of Δ_M can be summarized as follows.

Proposition 2.1. *([18]) Let $M \subseteq Y$ be an nontrivial subset of Y, i.e., $M \neq \emptyset$ and $M \neq Y$. Then*

(i) Δ_M *is real-valued.*
(ii) Δ_M *is Lipschitz function with constant 1.*
(iii) $\Delta_M(y) = \begin{cases} < 0, \ \forall y \in \mathit{int} M, \\ = 0, \ \forall y \in \partial M, \\ > 0, \ \forall y \in \mathit{int} M^c. \end{cases}$
(iv) if M is closed, then it holds that $M := \{y : \Delta_M(y) \leq 0\}$.
(v) if M is convex, then Δ_M is convex.
(vi) if M is a cone, then Δ_M is positively homogeneous.

(vii) if M is a closed convex cone, then Δ_M is nonincreasing with respect to the ordering relation induced on Y, i.e., if $y_1, y_2 \in Y$, then

$$y_1 - y_2 \in M \;\Rightarrow\; \Delta_M(y_1) \leq \Delta_M(y_2);$$

if $\operatorname{int} M \neq \emptyset$, then

$$y_1 - y_2 \in \operatorname{int} M \;\Rightarrow\; \Delta_M(y_1) < \Delta_M(y_2).$$

Proposition 2.2. *([14]) Let $M \subseteq Y$ be convex and $ri(M) \neq \emptyset$. Then Δ_M can be represented as*

$$\Delta_M(y) = \sup_{x^* \in S_{Y^*}} \inf_{x \in M} \langle x^*, y - x \rangle, \; \forall y \in Y,$$

where $ri\,(M) := \begin{cases} rint\, M, & \text{if } aff(M) \text{ is closed}, \\ \emptyset, & \text{otherwise} \end{cases}$ and by $rint\, M$ we denote the interior of M with respect to affine hull $aff(M)$.

Remark. The above-mentioned canonical representation of a convex set has been investigated also in [6], [7] and [8].

Corollary 2.1. *([14]) For a convex cone C with $\operatorname{int} C \neq \emptyset$, we have that*

$$\Delta_C(y) = \sup_{x^* \in S(C^*)} \langle -x^*, y \rangle, \; \forall y \in Y.$$

Let $M \subseteq Y$ be a given set. Then one can introduce the function ξ defined by

$$\xi_{M,C}(y) := -\inf_{z \in M} \Delta_C(y - z), \; \forall y \in Y.$$

Proposition 2.3. *(cf. [14]) The following assertions are true.*

(i) $\xi_{M,C}(y) = \sup_{x \in M} \inf_{x^* \in S(C^*)} \langle x^*, y - x \rangle, \; \forall y \in Y.$

(ii) $\xi_{M,C}(y) \geq 0, \; \forall y \in M.$

Proof:

(i) This formula follows directly from Corollary 2.1 and the Definition of $\xi_{M,C}$.

(ii) For $y \in M$ there is

$$\xi_{M,C}(y) = \sup_{x \in M} \inf_{x^* \in S(C^*)} \langle x^*, y - x \rangle \geq \inf_{x^* \in S(C^*)} \langle x^*, y - y \rangle = 0.$$

$\square$

3 Gap functions for vector variational inequalities

Let X be a Hausdorff locally convex space and Y be a real Banach space, $K \subseteq X$ be a nonempty set and $F : K \to \mathcal{L}(X,Y)$ be a given mapping. We consider the weak vector variational inequality which consists in finding $\bar{x} \in K$ such that

$$(WVVI) \qquad \langle F(\bar{x}), y-\bar{x}\rangle \not< 0,\ \forall y \in K,$$

where $\langle F(\bar{x}), y-\bar{x}\rangle \in Y$ denotes the image of $y-\bar{x} \in X$ under the linear continuous mapping $F(\bar{x}) \in \mathcal{L}(X,Y)$, we use this notation synonymously with $F(\bar{x})(y-\bar{x})$.

In this section we concentrate on the investigation of scalar-valued gap functions for the problem $(WVVI)$ on the basis of the oriented distance function. Recently a similar approach was applied to vector optimization in [14]. Let us recall the definition of a gap function for $(WVVI)$.

Definition 3.1. A function $\gamma : K \to \overline{\mathbb{R}}$ is said to be a gap function for the problem $(WVVI)$ if it satisfies the following properties:

(i) $\gamma(x) \geq 0,\ \forall x \in K$;
(ii) $\gamma(\bar{x}) = 0$ if and only if $\bar{x}$ solves the problem $(WVVI)$.

Additionally, we want to consider another type of gap functions which have weaker properties as the gap functions defined above. These functions are called weak gap functions.

Definition 3.2. A function $\gamma : K \to \overline{\mathbb{R}}$ is said to be a weak gap function for the problem $(WVVI)$ if it satisfies the following properties:

(i) $\gamma(x) \geq 0,\ \forall x \in K$;
(ii) $\gamma(\bar{x}) = 0 \Rightarrow \bar{x}$ solves the problem $(WVVI)$.

Let us introduce with

$$F(x)K := \{z \in Y : \exists w \in K \text{ such that } z = \langle F(x), w\rangle\}$$

(notice again that $\langle F(x), w\rangle$ stands for $F(x)w$ and $F(x) \in \mathcal{L}(X,Y)$) the function (setting $M = F(x)K$ for any $x \in K$)

$$\begin{aligned}\gamma^F_{\Delta,x}(y) :&= \xi_{F(x)K,C}(y) = -\inf_{z\in F(x)K} \Delta_C(y-z)\\ &= -\inf_{w\in K} \Delta_C(y - \langle F(x), w\rangle),\ \forall y \in Y.\end{aligned}$$

Then setting $y = \langle F(x), x\rangle$ in $\gamma^F_{\Delta,x}(y)$ we define for $x \in K$

$$\begin{aligned}\gamma^F_{\Delta}(x) &:= \gamma^F_{\Delta,x}(\langle F(x), x\rangle) = \xi_{F(x)K,C}(\langle F(x), x\rangle)\\ &= -\inf_{w\in K} \Delta_C(\langle F(x), x-w\rangle)\end{aligned}$$

$$= -\inf_{w \in K} \sup_{y^* \in S(C^*)} \langle -y^*, \langle F(x), x-w \rangle \rangle$$

$$= \sup_{w \in K} \inf_{y^* \in S(C^*)} \langle y^*, \langle F(x), x-w \rangle \rangle$$

because of Corollary 2.1.

Theorem 3.1. *γ_Δ^F is a gap function for $(WVVI)$.*

Proof. (i) By Proposition 2.3(ii) it holds $\gamma_\Delta^F(x) = \xi_{F(x)K,C}(\langle F(x), x \rangle) \geq 0$, since $\langle F(x), x \rangle \in F(x)K$ for any $x \in K$.

(ii) Let $x \in K$ be fixed. Then $\gamma_\Delta^F(x) > 0$ if and only if $\exists \widetilde{x} \in K$ such that

$$\Delta_C(\langle F(x), x - \widetilde{x} \rangle) < 0 \Leftrightarrow \langle F(x), x - \widetilde{x} \rangle \in \operatorname{int} C.$$

This equivalently means that $\langle F(x), \widetilde{x} - x \rangle \prec 0$, i.e., x is not a solution to $(WVVI)$. Consequently, taking (i) into account, for some $x \in K$ it holds $\gamma_\Delta^F(x) = 0$ if and only if x is a solution to $(WVVI)$. $\square$

In order to suggest some other gap functions, let us consider optimization problems having the composition with a linear continuous mapping in the objective function and formulate some duality results. As mentioned in the introduction we use for our investigations the perturbation theory (cf. [4]), where to a general primal problem

$$(P) \qquad \inf_{x \in X} \Phi(x, 0),$$

with the perturbation function $\Phi : X \times Y \to \overline{\mathbb{R}}$, the conjugate dual problem is defined by:

$$(D) \qquad \sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\}.$$

Assume that $f : Y \to \overline{\mathbb{R}}$ and $g : X \to \overline{Z}$ are proper functions, $S \subseteq X$ a nonempty set and $A \in \mathcal{L}(X,Y)$ fullfilling $(y - A^{-1}(\operatorname{dom} f)) \cap g^{-1}(-D) \cap S \neq \emptyset$. Consider the following primal optimization problem

$$(P^C) \qquad \inf_{x \in K} f(A(y-x))$$
$$K = \{x \in S : g(x) \in -D\},$$

where $y \in K$ is fixed and $S \subseteq X$.

The first dual problem of interest is the well-known Lagrange-dual problem:

$$(D^{C_L}) \qquad \sup_{z^* \in D^*} \inf_{x \in S} \{f(A(y-x)) + \langle z^*, g(x) \rangle\}.$$

To construct another dual problem we introduce the following perturbation function $\Phi^{C_F} : X \times Y \to \overline{\mathbb{R}}$,

$$\Phi^{C_F}(x,p) := \begin{cases} f(A(y-x)+p), \text{ if } x \in S, g(x) \in -D, \\ +\infty, \text{ otherwise,} \end{cases}$$

where $p \in Y$ is the perturbation variable. The perturbation function can be written as

$$\Phi^{C_F}(x,p) = f(A(y-x)+p) + \delta_K(x).$$

For the formula of the conjugate function $(\Phi^{C_F})^* : X^* \times Y^* \to \overline{\mathbb{R}}$ of Φ^{C_F} we get for all $(x^*, p^*) \in X^* \times Y^*$:

$$\begin{aligned}
(\Phi^{C_F})^*(x^*,p^*) &= \sup_{x \in X, p \in Y} \{\langle x^*, x\rangle + \langle p^*, p\rangle - \Phi^{C_F}(x,p)\} \\
&= \sup_{x \in X, p \in Y} \{\langle x^*, x\rangle + \langle p^*, p\rangle - f(A(y-x)+p) - \delta_K(x)\} \\
&= \sup_{x \in X, r \in Y} \{\langle x^*, x\rangle + \langle p^*, r - A(y-x)\rangle - f(r) - \delta_K(x)\} \\
&= \sup_{x \in X, r \in Y} \{\langle x^*, x\rangle + \langle p^*, r\rangle - \langle p^*, Ay\rangle + \langle p^*, Ax\rangle - f(r) - \\
&\quad \delta_K(x)\} \\
&= \sup_{x \in X}\{\langle x^* + A^* p^*, x\rangle - \delta_K(x)\} + \sup_{r \in Y}\{\langle p^*, r\rangle - f(r)\} - \\
&\quad \langle A^* p^*, y\rangle \\
&= \delta_K^*(x^* + A^* p^*) + f^*(p^*) - \langle A^* p^*, y\rangle.
\end{aligned}$$

This leads to the following dual problem to (P^C):

$$(D^{C_F}) \qquad \sup_{p^* \in Y^*} \{\langle A^* p^*, y\rangle - \delta_K^*(A^* p^*) - f^*(p^*)\},$$

which can be interpreted as a Fenchel dual problem. As the above construction shows it applies also if K is any nonempty set not necessarily given in the form as in (P^C).

Remark. From the calculations we made above for the Fenchel dual problem, we can easily conclude that to the primal problem

$$(\overline{P}) \qquad \inf_{x \in X}\{f(A(y-x)) + g(x)\},$$

where $A \in \mathcal{L}(X,Y)$ and $f : Y \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ are proper functions fullfilling $(y - A^{-1}(\operatorname{dom} f)) \cap \operatorname{dom} g \neq \emptyset$, the Fenchel dual problem looks like

$$(\overline{D}) \qquad \sup_{p^* \in Y^*} \{\langle A^* p^*, y\rangle - g^*(A^* p^*) - f^*(p^*)\}.$$

The last perturbation function we consider leads to the Fenchel-Lagrange dual problem and is defined by $\Phi^{C_{FL}} : X \times Y \times Z \to \overline{\mathbb{R}}$,

$$\Phi^{C_{FL}}(x,p,z) := \begin{cases} f(A(y-x)+p), \text{ if } x \in S, g(x) \in z - D, \\ +\infty, \text{ otherwise}, \end{cases}$$

where $(p,z) \in Y \times Z$ are the perturbation variables. We define $(z^*g)(x) := \langle z^*, g(x)\rangle$ and obtain for the conjugate of $\Phi^{C_{FL}}$, $(\Phi^{C_{FL}})^* : X^* \times Y^* \times Z^* \to \overline{\mathbb{R}}$, for all $(x^*,p^*,z^*) \in X^* \times Y^* \times Z^*$:

$$\begin{aligned}
(\Phi^{C_{FL}})^*(x^*,p^*,z^*) &= \sup_{\substack{x\in X, p\in Y \\ z \in Z}} \{\langle x^*,x\rangle + \langle p^*,p\rangle + \langle z^*,z\rangle - \Phi^{C_{FL}}(x,p,z)\} \\
&= \sup_{\substack{x\in S, (p,z)\in Y\times Z \\ g(x)\in z-D}} \{\langle x^*,x\rangle + \langle p^*,p\rangle + \langle z^*,z\rangle - \\
&\quad f(A(y-x)+p)\} \\
&= \sup_{\substack{x\in S, r\in Y \\ s\in -D}} \{\langle x^*,x\rangle + \langle p^*, r-A(y-x)\rangle + \\
&\quad \langle z^*, g(x)-s\rangle - f(r)\} \\
&= \sup_{\substack{x\in S, r\in Y \\ s\in -D}} \{\langle x^*,x\rangle + \langle p^*,r\rangle - \langle p^*,Ay\rangle + \\
&\quad \langle p^*,Ax\rangle + \langle z^*,g(x)\rangle + \langle -z^*,s\rangle - f(r)\} \\
&= \sup_{s\in -D}\{\langle -z^*,s\rangle\} + \sup_{r\in Y}\{\langle p^*,r\rangle - f(r)\} + \\
&\quad \sup_{x\in S}\{\langle x^* + A^*p^*, x\rangle - \langle -z^*, g(x)\rangle\} - \langle A^*p^*, y\rangle \\
&= \delta_{-D^*}(z^*) + f^*(p^*) + (-z^*g)^*_S(x^* + A^*p^*) - \langle A^*p^*, y\rangle.
\end{aligned}$$

As a consequence, the Fenchel-Lagrange dual problem is actually

$$(D^{C_{FL}}) \qquad \sup_{(p^*,z^*)\in Y^*\times D^*} \{\langle A^*p^*,y\rangle - f^*(p^*) - (z^*g)^*_S(A^*p^*)\}.$$

According to the general theory (cf. [4]) the weak duality is always fullfilled, i.e. $v(D^{C_L}) \le v(P^C)$, $v(D^{C_F}) \le v(P^C)$ and $v(D^{C_{FL}}) \le v(P^C)$, where $v(P^C), v(D^{C_L}), v(D^{C_F})$ and $v(D^{C_{FL}})$ are the optimal objective values of $(P^C), (D^{C_L})$, (D^{C_F}) and $(D^{C_{FL}})$, respectively.
We consider now for any $x \in K$ the following optimization problem

$$(P_x^F) \qquad \inf_{y\in K} \Delta_C(\langle F(x), x-y\rangle),$$

where the ground set K is defined by

$$K = \{y \in S : g(y) \in -D\}.$$

It is easy to see that the optimal objective value $v(P_x^F) = -\gamma_\Delta^F(x) \leq 0$, $\forall x \in K$. By using the calculations we made above we get for the dual problems of (P_x^F):

$$(D_x^{F_L}) \qquad \sup_{z^* \in D^*} \inf_{y \in S} \{\Delta_C(\langle F(x), x-y\rangle) + \langle z^*, g(y)\rangle\},$$

$$(D_x^{F_F}) \qquad \sup_{p^* \in Y^*} \{\langle F(x)^* p^*, x\rangle - \Delta_C^*(p^*) - \delta_K^*(F(x)^* p^*)\}$$

(here, K may be any nonempty set) and

$$(D_x^{F_{FL}}) \qquad \sup_{p^* \in Y^*, z^* \in D^*} \{\langle F(x)^* p^*, x\rangle - (z^* g)_S^*(F(x)^* p^*) - \Delta_C^*(p^*)\}.$$

It is well known that for any set $A \subseteq X$ it holds that $\sigma_A(x^*) = \sup_{x \in A} \langle x^*, x\rangle = \sigma_{\operatorname{clco} A}(x^*)$, whereas $\operatorname{clco} A$ is the closed convex hull of the set A. Hence, for a convex cone C with $\operatorname{int} C \neq \emptyset$, by Corollary 2.1 it follows that $\Delta_C(y) = \sup_{y^* \in S(C^*)} \langle -y^*, y\rangle = \sup_{y^* \in S(-C^*)} \langle y^*, y\rangle = \sigma_{S(-C^*)}(y) = \sigma_{\operatorname{clco} S(-C^*)}(y)$, i.e. $\Delta_C(y) = \sigma_{\operatorname{clco} S(-C^*)}(y) = \delta^*_{\operatorname{clco} S(-C^*)}(y)$. Further, since $\operatorname{clco} S(-C^*)$ is a closed convex set we have by the Fenchel-Moreau Theorem (cf. [4, Theorem 2.3.6]) for the conjugate of the oriented distance function $\Delta_C^*(y^*) = \delta^{**}_{\operatorname{clco} S(-C^*)}(y^*) = \delta_{\operatorname{clco} S(-C^*)}(y^*)$. As a result, the Fenchel dual problem and the Fenchel-Lagrange dual problem can be written as

$$(D_x^{F_F}) \qquad \sup_{p^* \in \operatorname{clco} S(-C^*)} \{\langle F(x)^* p^* x\rangle - \delta_K^*(F(x)^* p^*)\}$$

and

$$(D_x^{F_{FL}}) \qquad \sup_{\substack{p^* \in \operatorname{clco} S(-C^*) \\ z^* \in D^*}} \{\langle F(x)^* p^*, x\rangle - (z^* g)_S^*(F(x)^* p^*)\}.$$

Example. Let $X = Y = \mathbb{R}^2$ be equipped with the Euclidean topology and $C = \mathbb{R}^2_+$, then we have $X^* = Y^* = \mathbb{R}^2$ also equipped with the Euclidean topology and $C^* = \mathbb{R}^2_+$. Let the ground set $K \subseteq X$ be a nonempty set and $F : K \to \mathcal{L}(X, Y)$ be a given mapping. For the set $\operatorname{clco} S(-C^*)$ we get

$$\begin{aligned} \operatorname{clco} S(-C^*) &= \{p^* = (p_1^*, p_2^*)^T \in -\mathbb{R}^2_+ \cap B(0,1) : p_1^* + p_2^* \leq -1\} \\ &= \{p^* \in \mathbb{R}^2 : \|p^*\| \leq 1, p_1^* + p_2^* \leq -1\}, \end{aligned}$$

where $B(0,1) = \{p^* \in \mathbb{R}^2 : \|p^*\| \leq 1\}$. Therefore, the corresponding Fenchel dual problem looks like

$$(\widetilde{D}_x^F) \qquad \sup_{\substack{\|p^*\| \leq 1 \\ p_1^* + p_2^* \leq -1}} \{\langle F(x)^* p^*, x\rangle - \delta_K^*(F(x)^* p^*)\}.$$

Remark. If the convex set C is not a cone with $\operatorname{int} C \neq \emptyset$ we refer to [6] for the conjugate of the oriented distance function.

By using the duals $(D_x^{F_L}), (D_x^{F_F})$ and $(D_x^{F_{FL}})$ of the optimization problem (P_x^F), we introduce the following functions for $x \in K$:

$$\begin{aligned}
\gamma_\Delta^{F_L}(x) :&= -\sup_{z^* \in D^*} \inf_{y \in S} \{\Delta_C(\langle F(x), x-y\rangle) + \langle z^*, g(y)\rangle\} \\
&= \inf_{z^* \in D^*} \sup_{y \in S} \{-\Delta_C(\langle F(x), x-y\rangle) - \langle z^*, g(y)\rangle\}, \\
\gamma_\Delta^{F_F}(x) :&= -\sup_{p^* \in \operatorname{clco} S(-C^*)} \{\langle F(x)^* p^*, x\rangle - \delta_K^*(F(x)^* p^*)\} \\
&= \inf_{p^* \in \operatorname{clco} S(-C^*)} \{\delta_K^*(F(x)^* p^*) - \langle F(x)^* p^*, x\rangle\}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_\Delta^{F_{FL}}(x) :&= -\sup_{\substack{p^* \in \operatorname{clco} S(-C^*) \\ z^* \in D^*}} \{\langle F(x)^* p^*, x\rangle - (z^* g)_S^*(F(x)^* p^*)\} \\
&= \inf_{\substack{p^* \in \operatorname{clco} S(-C^*) \\ z^* \in D^*}} \{(z^* g)_S^*(F(x)^* p^*) - \langle F(x)^* p^*, x\rangle\}.
\end{aligned}$$

Remark. A similar approach was introduced in [1] in order to construct a gap function for scalar variational inequalities.

Proposition 3.1. *It holds that*

$$\gamma_\Delta^{F_{FL}}(x) \geq \gamma_\Delta^{F_F}(x), \ \forall x \in K.$$

Proof. We fix $x \in K$ and $p^* \in Y^*$ and consider the following primal problem

$$\begin{aligned}
(P^0) \qquad & \inf_{y \in K} \langle -F(x)^* p^*, y\rangle, \\
& K = \{y \in S : g(y) \in -D\}.
\end{aligned}$$

The corresponding Lagrange dual problem is

$$\begin{aligned}
(D^0) \qquad & \sup_{z^* \in D^*} \{\inf_{y \in S} \langle -F(x)^* p^*, y\rangle + \langle z^*, g(y)\rangle\} \\
&= \sup_{z^* \in D^*} \inf_{y \in S} \{-[\langle F(x)^* p^*, y\rangle - \langle z^*, g(y)\rangle]\} \\
&= \sup_{z^* \in D^*} -\sup_{y \in S} \{\langle F(x)^* p^*, y\rangle - \langle z^*, g(y)\rangle\} \\
&= \sup_{z^* \in D^*} \{-(z^* g)_S^*(F(x)^* p^*)\}.
\end{aligned}$$

By the weak duality it follows that

$$\sup_{z^* \in D^*} \{-(z^* g)^*_S(F(x)^* p^*)\} \leq \inf_{y \in K} \{\langle -F(x)^* p^*, y\rangle\}$$

or, equivalently,

$$- \sup_{z^* \in D^*} \{-(z^* g)^*_S(F(x)^* p^*)\} + \delta_{\operatorname{clco} S(-C^*)}(p^*) - \langle F(x)^* p^*, x\rangle \geq$$

$$- \inf_{y \in K} \{\langle -F(x)^* p^*, y\rangle\} + \delta_{\operatorname{clco} S(-C^*)}(p^*) - \langle F(x)^* p^*, x\rangle.$$

Now we take the infimum over $p^* \in Y^*$ in both sides and get

$$\begin{aligned}\gamma_\Delta^{F_{FL}}(x) &= \inf_{\substack{p^* \in \operatorname{clco} S(-C^*) \\ z^* \in D^*}} \{(z^* g)^*_S(F(x)^* p^*) - \langle F(x)^* p^*, x\rangle\} \\ &\geq \inf_{p^* \in \operatorname{clco} S(-C^*)} \{\delta^*_K(F(x)^* p^*) - \langle F(x)^* p^*, x\rangle\} = \gamma_\Delta^{F_F}(x).\end{aligned}$$

□

Proposition 3.2. *It holds that*

$$\gamma_\Delta^{F_{FL}}(x) \geq \gamma_\Delta^{F_L}(x), \ \forall x \in K.$$

Proof. Let $z^* \in D^*$ be fixed. Since

$$\sup_{p^* \in Y^*} \{-\delta_{\operatorname{clco} S(-C^*)}(p^*) - (z^* g)^*_S(F(x)^* p^*) + \langle F(x)^* p^*, x\rangle\}$$

is the Fenchel dual problem of the primal problem (cf. Remark for the Fenchel dual problem)

$$\inf_{y \in X} \{\Delta_C(\langle F(x), x - y\rangle) + ((z^* g) + \delta_S)(y)\} =$$

$$\inf_{y \in S} \{\Delta_C(\langle F(x), x - y\rangle) + \langle z^*, g(y)\rangle\},$$

we get by the weak duality

$$\sup_{p^* \in Y^*} \{-\delta_{\operatorname{clco} S(-C^*)}(p^*) - (z^* g)^*_S(F(x)^* p^*) + \langle F(x)^* p^*, x\rangle\rangle\} \leq$$

$$\inf_{y \in S} \{\Delta_C(\langle F(x), x - y\rangle) + \langle z^*, g(y)\rangle\}$$

or

$$\inf_{p^* \in Y^*} \{\delta_{\operatorname{clco} S(-C^*)}(p^*) + (z^* g)^*_S(F(x)^* p^*) - \langle F(x)^* p^*, x\rangle\rangle\} \geq$$

$$\sup_{y \in S} \{-\Delta_C(\langle F(x), x - y\rangle) - \langle z^*, g(y)\rangle\}.$$

Taking the infimum over $z^* \in D^*$ in both sides yields the desired result

$$\gamma_\Delta^{F_{FL}}(x) = \inf_{\substack{p^* \in \operatorname{clco} S(-C^*) \\ z^* \in D^*}} \{(z^* g)_S^*(F(x)^* p^*) - \langle F(x)^* p^*, x\rangle\} \geq$$

$$\inf_{z^* \in D^*} \sup_{y \in S} \{-\Delta_C(\langle F(x), x-y\rangle) - \langle z^*, g(y)\rangle\} = \gamma_\Delta^{F_L}(x).$$

□

Proposition 3.3. *It holds for all $x \in K$ that*

$$\gamma_\Delta^F(x) \leq \gamma_\Delta^{F_L}(x), \gamma_\Delta^F(x) \leq \gamma_\Delta^{F_F}(x) \text{ and } \gamma_\Delta^F(x) \leq \gamma_\Delta^{F_L}(x).$$

Proof. Let $x \in K$. Since $\gamma_\Delta^F(x) = -v(P_x^F), \gamma_\Delta^{F_L}(x) = -v(D_x^{F_L}), \gamma_\Delta^{F_F}(x) = -v(D_x^{F_F})$ and $\gamma_\Delta^{F_{FL}}(x) = -v(D_x^{F_{FL}})$ the assertions follow from the weak duality between (P_x^F) and the corresponding different dual problems. □

Remark. By the last three propositions we obtain the following relations between the introduced functions

$$\gamma_\Delta^{F_{FL}}(x) \geq \begin{array}{c} \gamma_\Delta^{F_L}(x) \\ \gamma_\Delta^{F_F}(x) \end{array} \geq \gamma_\Delta^F(x) \; \forall x \in K,$$

which is equivalent to

$$v(P_x^F) \geq \begin{array}{c} v(D_x^{F_L}) \\ v(D_x^{F_F}) \end{array} \geq v(D_x^{F_{FL}}) \; \forall x \in K.$$

Remark. The relations in the Remark above show that if strong duality for the pair $(P_x^F) - (D_x^{F_{FL}})$ holds, then strong duality holds also for the pairs $(P_x^F) - (D_x^{F_L})$ and $(P_x^F) - (D_x^{F_F})$.

Proposition 3.4. *$\gamma_\Delta^{F_L}, \gamma_\Delta^{F_F}$ and $\gamma_\Delta^{F_{FL}}$ are weak gap functions for the problem $(WVVI)$ where $K = \{y \in S : g(y) \in -D\} \neq \emptyset$. Concerning $\gamma_\Delta^{F_F}$, K may be any nonempty set.*

Proof. (i) By Theorem 3.1 and Propositions 3.1, 3.2 and 3.3 it holds that

$$\gamma_\Delta^{F_{FL}}(x) \geq \begin{array}{c} \gamma_\Delta^{F_L}(x) \\ \gamma_\Delta^{F_F}(x) \end{array} \geq \gamma_\Delta^F(x) \geq 0 \; \forall x \in K.$$

(ii) Let $\gamma_\Delta^{F_L}(\bar{x}) = 0$ for some $\bar{x} \in K$. Then we obtain by (i) that $\gamma_\Delta^F(\bar{x}) = 0$. From Theorem 3.1 we have that $\bar{x}$ solves the problem $(WVVI)$.
For $\gamma_\Delta^{F_F}$ and $\gamma_\Delta^{F_{FL}}$ it follows analogously. □

In order to guarantee the strong duality between the primal problem (P^C) and the corresponding dual problems (D^{C_L}), (D^{C_F}) and $(D^{C_{FL}})$ we assume for the rest of this chapter that S is a convex set, f is a convex function and g is a D-convex function.

First, we state a strong duality proposition for the primal-dual pair $(P^C) - (D^{C_L})$, which is a direct conclusion of [4, Theorem 3.2.1].

Proposition 3.5. *If there exists $x' \in (y - A^{-1}(\mathrm{dom}\, f)) \cap S$ such that $g(x') \in -\mathrm{int} D$, then $v(P^C) = v(D^{C_L})$ and (D^{C_L}) has an optimal solution.*

In the case where $f = \Delta_C$ and $A = F(x)$, we have (notice that x and y are changed in (P_x^F) compared with (P^C))

$$\begin{aligned} & y' \in (x - F(x)^{-1}(\mathrm{dom}\, \Delta_C)) \cap S \\ \Leftrightarrow\ & y' \in (x - F(x)^{-1}(Y)) \cap S \\ \Leftrightarrow\ & y' \in (x - X) \cap S \\ \Leftrightarrow\ & y' \in S. \end{aligned}$$

Therefore we have for the pair $(P_x^F) - (D_x^{F_L})$, $x \in K$, the following strong duality proposition.

Proposition 3.6. *If there exists $y' \in S$ such that $g(y') \in -\mathrm{int} D$, then $v(P_x^F) = v(D_x^{F_L})$ and $(D_x^{F_L})$ has an optimal solution.*

Next, we give for any convex set $K \neq \emptyset$ a strong duality proposition for the primal-dual problems $(P^C) - (D^{C_F})$ by using [4, Theorem 3.2.1] again.

Proposition 3.7. *If there exists $x' \in (y - A^{-1}(\mathrm{dom}\, f)) \cap K$ such that f is continuous at $A(y - x')$, then $v(P^C) = v(D^{C_F})$ and (D^{C_F}) has an optimal solution.*

Since Δ_C is a Lipschitz function, i.e. Δ_C is also continuous everywhere on Y, the Proposition 3.7 can be rewritten for the pairs $(P_x^F) - (D_x^{F_F})$, $x \in K$, as follows.

Proposition 3.8. *If $K \neq \emptyset$ is any convex set, then $v(P_x^F) = v(D_x^{F_F})$ and $(D_x^{F_F})$ has an optimal solution.*

Remark. Note that in this case there is no regularity condition needed.

Finally, we state a strong duality proposition for the primal-dual pair $(P^C) - (D^{C_{FL}})$ which also follows as a simple conclusion of [4, Theorem 3.2.1].

Proposition 3.9. *If there exists $x' \in (y - A^{-1}(\mathrm{dom}\, f)) \cap S$ such that f is continuous at $A(y - x')$ and $g(x') \in -\mathrm{int} D$, then $v(P^C) = v(D^{C_{FL}})$ and $(D^{C_{FL}})$ has an optimal solution.*

As application we can establish strong duality for (P_x^F) and $(D_x^{F_{FL}})$, $x \in K$.

Proposition 3.10. *If there exists $y' \in S$ such that $g(y') \in -\mathrm{int} D$, then $v(P_x^F) = v(D_x^{F_{FL}})$ and $(D_x^{F_{FL}})$ has an optimal solution.*

Theorem 3.2. *(i)* $\gamma_{\Delta}^{F_F}$ *is a gap function for* $(WVVI)$ *for any convex set* $K \neq \emptyset$.
(ii) If there exists $y' \in S$ *such that* $g(y') \in -\operatorname{int} D$ *then* $\gamma_{\Delta}^{F_L}$ *and* $\gamma_{\Delta}^{F_{FL}}$ *are gap functions for* $(WVVI)$.

Proof. (i) By Proposition 3.4, it follows that $\gamma_{\Delta}^{F_F}$ is a weak gap function. For that reason, we need only to prove that if $\bar{x} \in K$ solves $(WVVI)$, then it holds that $\gamma_{\Delta}^{F_F}(\bar{x}) = 0$. According to Theorem 3.1, for some $\bar{x} \in K$ it holds that $\gamma_{\Delta}^{F}(\bar{x}) = 0$ if and only if $\bar{x}$ is a solution to $(WVVI)$. That means $v(P_{\bar{x}}^{F}) = -\gamma_{\Delta}^{F}(\bar{x}) = 0$. On the other hand, by Proposition 3.8 strong duality holds, i.e. if $\bar{x} \in K$ solves $(WVVI)$, then $\gamma_{\Delta}^{F_F}(\bar{x}) = -v(D_{\bar{x}}^{F_F}) = -v(P_{\bar{x}}^{F}) = 0$.
(ii) This can be proved in a similar way taking into account Proposition 3.6 and Proposition 3.10 instead of Proposition 3.8 as in the proof of i). □

4 Extension to set-valued problems

In this section we discuss how the presented approach can be extended to some variational inequalities with set-valued mappings investigated in the literature (see [11], [12], [13], [15] and [17]). Let us notice that in all mentioned works the space Y was supposed to be Euclidean one. Under compactness assumptions we will extend above results in Banach spaces.

4.1 Vector variational inequalities with set-valued mappings

Let X, Y be real Banach spaces, Y be partially ordered by a closed convex pointed cone C with $\operatorname{int} C \neq \emptyset$ and $\emptyset \neq K \subseteq X$ be a compact set. Further let $T : K \rightrightarrows \mathcal{L}(X,Y)$ be a set-valued mapping, where $\mathcal{L}(X,Y)$ is equipped with the usual operator norm, i.e. $(\mathcal{L}(X,Y), \|\cdot\|)$ is a Banach space. We consider the vector variational inequality with set-valued mapping which consists in finding $\bar{x} \in K$ such that

$$(SVVI) \qquad \exists \bar{t} \in T(\bar{x}) : \quad \langle \bar{t}, y - \bar{x} \rangle \not< 0, \ \forall y \in K.$$

Let us introduce the function

$$\begin{aligned} \gamma_S^T(x) &= - \sup_{t \in T(x)} \inf_{y \in K} \Delta_C(\langle t, x - y \rangle) \\ &= \inf_{t \in T(x)} \sup_{y \in K} \{-\Delta_C(\langle t, x - y \rangle)\}, \ x \in K. \end{aligned}$$

Theorem 4.1. *Assume that for each* $x \in K$, $T(x)$ *is nonempty and compact. Then* γ_S^T *is a gap function for* $(SVVI)$.

Proof: Let $x \in K$ and $t \in T(x)$. Then, from Proposition 2.3(ii) follows

$$\begin{aligned}\xi_{tK,C}(\langle t,x\rangle) := & -\inf_{z\in tK} \Delta_C(\langle t,x\rangle - z) \text{ (set } z := \langle t,y\rangle,\ y \in K)\\ = & -\inf_{y\in K} \Delta_C(\langle t,x\rangle - \langle t,y\rangle)\\ = & -\inf_{y\in K} \Delta_C(\langle t,x-y\rangle) \geq 0.\end{aligned}$$

Consequently, we have

$$\gamma_S^T(x) = \inf_{t\in T(x)} \xi_{tK,C}(\langle t,x\rangle) \geq 0.$$

Since K and $T(x)$ are compact and Δ_C is continuous, then by standard arguments (uniform continuity), we obtain that the function $\sup_{y\in K}\{-\Delta_C(\langle t,x-y\rangle)\}$ is continuous with respect to $t \in T(x)$. Moreover, the function γ_S^T is well defined and can be written as

$$\gamma_S^T(x) = \min_{t\in T(x)} \sup_{y\in K}\{-\Delta_C(\langle t,x-y\rangle)\}$$

(the infimum is attained). For some $\bar{x} \in K$ it holds $\gamma_S^T(\bar{x}) = 0$ if and only if $\exists \bar{t} \in T(\bar{x})$ such that

$$\sup_{y\in K}\{-\Delta_C(\langle \bar{t},\bar{x}-y\rangle)\} = 0$$

or

$$\Delta_C(\langle \bar{t},\bar{x}-y\rangle) \geq 0,\ \forall y \in K.$$

This equivalently means (cf. Proposition 2.1) that

$$\langle \bar{t},\bar{x}-y\rangle \notin \operatorname{int} C \ \Leftrightarrow \langle \bar{t},y-\bar{x}\rangle \not< 0,\ \forall y \in K,$$

i.e., $\bar{x}$ is a solution to $(SVVI)$. $\square$

Example. If $Y = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, then $Y^* = Y$, $C^* = \mathbb{R}^m_+$. Let $x,\ y \in K$. Then $T(x) = \prod_{i=1}^m T_i(x),\ T_i : K \rightrightarrows X^*$. For any $t \in T$ it holds $t = (t_1,\ldots,t_m)$ and

$$\langle t,x-y\rangle = (\langle t_1,x-y\rangle,\ldots,\langle t_m,x-y\rangle).$$

According to Corollary 2.1 and Proposition 3(iv) in [14], we have

$$\Delta_C(\langle t,x-y\rangle) = \sup_{\substack{z\in\mathbb{R}^m_+\\ \|z\|=1}} \langle -z,\langle t,x-y\rangle\rangle = \max_{1\leq i\leq m} \langle t_i,y-x\rangle.$$

Consequently, we get

$$\gamma_S^T(x) = \inf_{t\in T(x)} \sup_{y\in K} \min_{1\le i\le m} \langle t_i, x-y\rangle$$

which is nothing else than the gap function for $(SVVI)$ investigated in [13] and [17].

4.2 Vector variational-like inequalities with set-valued mappings

Under the general assumptions of section 4.1 let $\eta : K\times K \to X$ be a vector-valued mapping such that $\eta(x,x) = 0,\ \forall x\in K$, which is continuous with respect to the first variable for any fixed second variable in K. Then the vector variational-like inequality with set-valued mapping consists in finding $\bar{x}\in K$ such that

$$(SVVLI) \qquad \exists \bar{t}\in T(\bar{x}): \quad \langle \bar{t}, \eta(y,\bar{x})\rangle \not\leq 0,\ \forall y\in K.$$

Let us introduce the function

$$\begin{aligned}\gamma_S^L(x) &= -\sup_{t\in T(x)} \inf_{y\in K} \Delta_C(-\langle t, \eta(y,x)\rangle)\\ &= \inf_{t\in T(x)} \sup_{y\in K}\{-\Delta_C(-\langle t,\eta(y,x)\rangle)\},\ x\in K,\end{aligned}$$

and verify the following assertion.

Theorem 4.2. *Assume that for each $x\in K,\ T(x)$ is nonempty and compact. Then γ_S^L is a gap function for $(SVVLI)$.*

Proof. First we prove that $\gamma_S^L(x)\ge 0\ \forall x\in K$. It holds $\eta(x,x) = 0\ \forall x\in K$ and hence $\langle t,\eta(x,x)\rangle = 0\ \forall x\in K, t\in T(x)$. Further we have by Corollary 2.1 that $\sup_{x^*\in S(C^*)}\langle -x^*, -\langle t,\eta(x,x)\rangle\rangle = 0\ \ \forall x\in K, t\in T(x)$, i.e. $\Delta_C(-\langle t,\eta(x,x)\rangle) = \Delta_C(0) = 0\ \ \forall x\in K, t\in T(x)$. By taking the infimum over $y\in K$ we get $\inf_{y\in K}\Delta_C(-\langle t,\eta(y,x)\rangle)\le 0\ \forall x\in K, t\in T(x)$, which is equivalent to $\sup_{y\in K}\{-\Delta_C (-\langle t,\eta(y,x)\rangle)\}\ge 0\ \forall x\in K, t\in T(x)$. Finally, it follows that

$$\gamma_S^L(x) = \inf_{t\in T(x)} \sup_{y\in K}\{-\Delta_C(-\langle t,\eta(y,x)\rangle)\}\ge 0\ \forall x\in K.$$

Next, we show that $\gamma_S^L(\bar{x}) = 0$ if and only if $\bar{x}$ solves $(SVVLI)$. As in the proof of Theorem 4.1 it follows that $\sup_{y\in K}\{-\Delta_C(-\langle t,\eta(y,x)\rangle)\}$ is a continuous function with respect to $t\in T(x)$. Moreover, from the assumption for $T(x)$, the function γ_S^L is well defined and can be formulated as

$$\gamma_S^L(x) = \min_{t\in T(x)} \sup_{y\in K}\{-\Delta_C(-\langle t,\eta(y,x)\rangle)\}.$$

Further, let $\bar{x} \in K$, then $\gamma_S^L(\bar{x}) = 0$ if and only if $\exists \bar{t} \in T(\bar{x})$ such that

$$\sup_{y \in K}\{-\Delta_C(-\langle \bar{t}, \eta(y,\bar{x})\rangle)\} = 0$$

and hence follows

$$\Delta_C(-\langle \bar{t}, \eta(y,\bar{x})\rangle) \geq 0 \quad \forall y \in K.$$

This implies

$$-\langle \bar{t}, \eta(y,\bar{x})\rangle \notin \operatorname{int} C \Leftrightarrow \langle \bar{t}, \eta(y,\bar{x})\rangle \not< 0, \quad \forall y \in K,$$

which means that $\bar{x}$ is a solution to $(SVVLI)$. □

Remark. As mentioned before, if $Y = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, then it can be shown that γ_S^L reduces to the gap function investigated in [15].

4.3 Generalized vector variational-like inequalities with set-valued mappings

Under the general suppositions as given in section 4.1 let $\eta : K \times K \to X$ and $h : K \times K \to Y$ be two vector-valued mappings satisfying $\eta(x,x) = 0$ and $h(x,x) = 0$, $\forall x \in K$, which are continuous with respect to the first variable for any fixed second variable in K. Let us consider the generalized vector variational-like inequality with set-valued mapping which consists in finding $\bar{x} \in K$ such that

$$(SGVVLI) \qquad \exists \bar{t} \in T(\bar{x}) : \quad \langle \bar{t}, \eta(y,\bar{x})\rangle + h(y,\bar{x}) \not< 0, \ \forall y \in K$$

and introduce the function

$$\begin{aligned} \gamma_S^{GL}(x) &= -\sup_{t \in T(x)} \inf_{y \in K} \Delta_C(-\langle t, \eta(y,x)\rangle - h(y,x)) \\ &= \inf_{t \in T(x)} \sup_{y \in K}\{-\Delta_C(-\langle t, \eta(y,x)\rangle - h(y,x))\}, \ x \in K. \end{aligned}$$

Analogously, we can verify the following assertion.

Theorem 4.3. *Assume that for each $x \in K$, $T(x)$ is nonempty and compact. Then γ_S^{GL} is a gap function for $(SGVVLI)$.*

Proof. The proof is similary to the proof of Theorem 4.2. □

Remark. If $Y = \mathbb{R}^m$, $C = \mathbb{R}^m_+$, then it is easy to verify that γ_S^{GL} can be reduced to the gap function investigated in [12].

Remark. For the readers who are interested in the existence of solutions to vector variational inequalities, we refer to [12] and [17].

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