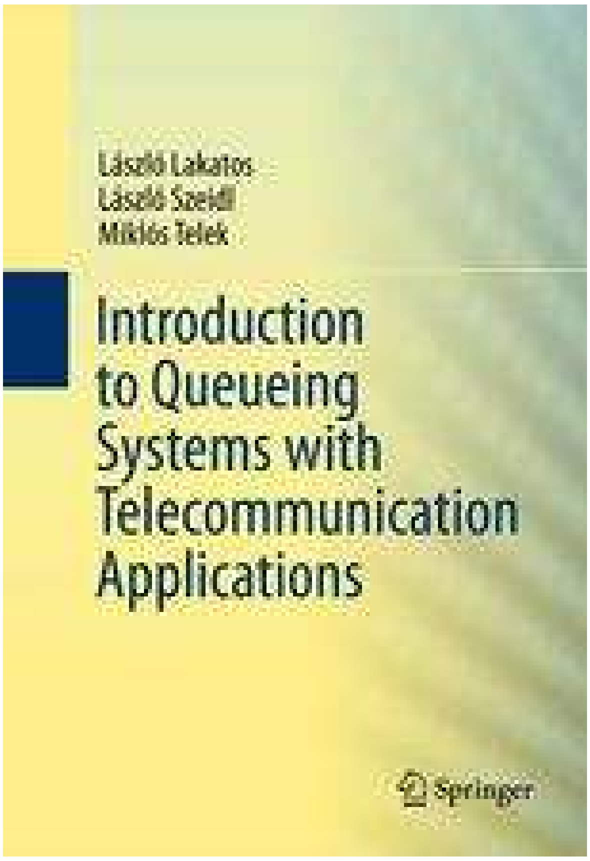
Introduction to queueing systems with telecommunication applications

Solution manual



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Part I

Introduction to probability theory and stochastic processes

Chapter 1

Introduction to probability theory

Exercise 1.1. *Let X be a nonnegative r.v. with c.d.f. F_X. Suppose that given $0 \le t \le X$ ($\mathbf{P}(X > t) \neq 0$), find the c.d.f of residual life time X.*

Solution 1.1. *Suppose $0 \le t \le z$, then*

$$\mathbf{P}(X \le z |\ X > t) = \frac{\mathbf{P}(X \le z, X > t)}{\mathbf{P}(X > t)} = \frac{\mathbf{P}(t < X \le z)}{\mathbf{P}(X > t)} = \\ = \frac{\mathbf{P}(X \le z) - \mathbf{P}(X \le t)}{1 - \mathbf{P}(X \le t)} = \frac{F_X(z) - F_X(t)}{1 - F_X(t)}.$$

Exercise 1.2. *Let X and Y be independent r.v.s with Poisson distribution of parameters λ and μ, respectively. Verify that*
a) the sum $X + Y$ has Poisson distribution with parameter $\lambda + \mu$,
b) for any nonnegative integers $m \le n$ the conditional distribution $\mathbf{P}(X = m \mid X + Y = n)$ is binomial with parameter $(n, \frac{\lambda}{\lambda+\mu})$, i.e.

$$\mathbf{P}(X = m \mid X + Y = n) = \binom{m}{n} \left(\frac{\lambda}{\lambda + \mu}\right)^m \left(1 - \frac{\lambda}{\lambda + \mu}\right)^{n-m}.$$

Solution 1.2. *a) Since the r.v.s X and Y are independent, therefore the generating function of the r.v. $X + Y$ has the form*

$$G_{X+Y}(z) = G_X(z)G_Y(z) = e^{\lambda(z-1)}e^{\mu(z-1)} = e^{(\lambda+\mu)(z-1)},$$

which justifies that $X + Y$ has Poisson distribution with parameter $\lambda + \mu$ Poisson.
b) With a simple calculation we have

$$\mathbf{P}(X = m\ |X + Y = n) = \frac{\mathbf{P}(X = m, X + Y = n)}{\mathbf{P}(X + Y = n)} = \frac{\mathbf{P}(X = m)\mathbf{P}(Y = n - m)}{\mathbf{P}(X + Y = n)} = \\ = \frac{\frac{\lambda^m}{m!}e^{-\lambda}\frac{\mu^{n-m}}{(n-m)!}e^{-\mu}}{\frac{(\lambda+\mu)^n e^{-(\lambda+\mu)}}{n!}} = \binom{n}{m}\frac{\lambda^m \mu^{n-m}}{(\lambda + \mu)^n} = \binom{m}{n}\left(\frac{\lambda}{\lambda + \mu}\right)^m \left(\frac{\mu}{\lambda + \mu}\right)^{n-m}.$$

Exercise 1.3. *Let X and Y be independent r.v.s having uniform distribution on interval $(0, 1)$, and exponential distribution with parameter 1, respectively. Find the probability (concrete number) that $X < Y$.*

Solution 1.3. *It is clear that the density function of* X *is* $f_X(x) = \mathcal{I}_{\{0<x<1\}}$*, and* Y *has density function* $f_Y(y) = e^{-y}\mathcal{I}_{\{y>0\}}$*, therefore*

$$\mathbf{P}(X<Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{I}_{\{x<y\}} f_X(x) f_Y(y)dxdy = \int_0^{\infty}\int_0^1 \mathcal{I}_{\{x<y\}} e^{-y}dxdy =$$

$$= \int_0^{\infty}\int_0^{\min(1,y)} e^{-y}dxdy = \int_0^{\infty} \min(1,y)e^{-y}dy = \int_0^1 ye^{-y}dy + \int_1^{\infty} e^{-y}dy =$$

$$= \left[-ye^{-y}\right]_0^1 + \int_0^1 e^{-y}dy + \left[-e^{-y}\right]_1^{\infty} = 1 - e^{-1} = 0.63.$$

Exercise 1.4. *Divide the interval* $(0,1)$ *into three pieces with two independently and randomly chosen points* U_1 *and* U_2 *of the interval* $(0,1)$*. Find the probability of the event* A *that the three pieces can determine a triangle.*

Solution 1.4. *The r.v.s* U_1 *and* U_2 *are independent and uniformly distributed on the interval* $(0,1)$*. The lenght of the three pieces are:*

$$\begin{array}{ll} U_1, U_2 - U_1, 1 - U_2, & \textit{if} U_1 \le U_2, \\ U_2, U_1 - U_2, 1 - U_1, & \textit{if } U_1 > U_2. \end{array}$$

The three pieces determine a triangle if and only if the triangle inequality is satisfied, then using the formula of the total probability

$$\begin{aligned} \mathbf{P}(A) &= \mathbf{P}(A\ |U_1 \le U_2)\mathbf{P}(U_1 \le U_2) + \mathbf{P}(A \mid U_1 > U_2)\mathbf{P}(U_1 > U_2) = \\ &= 2\mathbf{P}(A \mid U_1 \le U_2)\mathbf{P}(U_1 \le U_2) = \\ &= \int_0^1\int_0^1 I(x \le (y-x) + (1-y), y - x \le x + (1-y), (1-y) \le x + (y-x))dxdy = \\ &= \int_0^1\int_0^1 I(x \le \frac{1}{2}, y \le x + \frac{1}{2}, \frac{1}{2} \le y)dxdy = \int_0^{1/2}\int_{1/2}^{1/2+x} dydx = \int_0^{1/2} xdx = \frac{1}{4}. \end{aligned}$$

Exercise 1.5. *Show that for a nonnegative r.v.* X *with finite n-th* $(n \ge 1)$ *moment it is true* $\mathbf{E}(X^n) = \int_0^{\infty} \mathbf{P}(x < X)nx^{n-1}dx$.

Solution 1.5. *Denote by* $F_X(x)$ *the c.d.f of the r.v.* X*. Since* $\mathbf{E}(X^n) < \infty$*, then for* $a \to \infty$, $\mathbf{E}\big(X^n\mathcal{I}_{\{X>a\}}\big) = \int_a^{\infty} x^n dF_X(x) = -\int_a^{\infty} x^n d(1-F_X(x)) \to 0$ *and consequently* $-\int_a^{\infty} x^n d(1 - F_X(x)) \ge a^n(1-F(a)) \to 0$*. Integrating by part, we have*

$$\mathbf{E}\big(X^n\mathcal{I}_{\{X\le a\}}\big) = -\int_0^a x^n d(1-F_X(x)) = -\left[x^n(1-F_X(x))\right]_0^a + \int_0^a (1-F_X(x))d(x^n) =$$

$$= -a^n(1-F(a)) + \int_0^a (1-F_X(x))nx^{n-1}dx,$$

from this it follows

$$\begin{aligned}\mathbf{E}(X^n) &= \lim_{a\to\infty}\left[-a^n(1-F(a)) + \int\limits_0^a (1-F_X(x))nx^{n-1}dx\right] = \\ &= \int\limits_0^\infty (1-F_X(x))nx^{n-1}dx = \int\limits_0^\infty \mathbf{P}(x<X)nx^{n-1}dx.\end{aligned}$$

Exercise 1.6. *Let X and Y be independent r.v.s with uniform distribution on the interval $(0,1)$. Find the quantities*

a) $\mathbf{E}(|X-Y|)$, $\mathbf{D}^2\left(|X-Y|\right)$,

b) $\mathbf{P}(|X-Y|) > \frac{1}{2}$.

Solution 1.6. *a) Since $f_X(u) = f_Y(u) \equiv \mathcal{I}_{\{0<u<1\}}$, then*

$$\begin{aligned}\mathbf{E}(|X-Y|) &= \int\limits_0^1\int\limits_0^1 |x-y|\, f_X(x)f_Y(y)dxdy = \int\limits_0^1\int\limits_0^1 |x-y|\, dxdy = \\ &= 2\int\limits_0^1\int\limits_0^1 |x-y|\, I(x\le y)dxdy = \\ &= 2\int\limits_0^1\int\limits_0^y (y-x)dxdy = 2\int\limits_0^1\left[y^2-\frac{1}{2}y^2\right]dy = \frac{1}{3},\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}^2\left(|X-Y|\right) &= \mathbf{E}\big(|X-Y|^2\big) - (\mathbf{E}(|X-Y|))^2 = \mathbf{E}(X^2-2EXY+EY^2) - \frac{1}{9} = \\ &= 2\int\limits_0^1 x^2dx - 2\left(\int\limits_0^1 xdx\right)^2 - \frac{1}{9} = 2\cdot\frac{1}{3} - 2\frac{1}{2^2} - \frac{1}{9} = \frac{1}{18}.\end{aligned}$$

b) It is easy to see

$$\begin{aligned}\mathbf{P}\big(|X-Y|>\tfrac{1}{2}\big) &= \int\limits_0^1\int\limits_0^1 \mathcal{I}_{\left\{|x-y|>\frac{1}{2}\right\}}dxdy = 2\int\limits_0^1\int\limits_0^1 \mathcal{I}_{\left\{|x-y|>\frac{1}{2},x\ge y\right\}}dxdy = \\ &= 2\int\limits_0^{1/2} dxdy = 2\int\limits_0^{1/2}\left(\frac{1}{2}-y\right)dy = 2\int\limits_0^{1/2} ydy = 2\left[\frac{1}{2}\frac{1}{2^2}\right] = \frac{1}{4}.\end{aligned}$$

Exercise 1.7. *Let X and Y be independent r.v.s having exponential distribution with parameters λ and μ, respectively.*

a) Determine the density function of r.v. $Z = X+Y$.

b) Find the density function of r.v. $W = \min(X, Y)$.

Solution 1.7. *(a) Applying the convolution formula for the sum of independent r.v.s, we have*

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \mu e^{-\mu(z-x)} I(x>0, z-x>0)dx = \\ &= \lambda\mu e^{-\mu z} \int_{-\infty}^{z} e^{-(\lambda-\mu)x}dx = \begin{cases} \lambda^2 z e^{-\lambda z}, & \text{if } \lambda = \mu \\ \frac{\lambda\mu}{\lambda-\mu}\left[e^{-\mu z} - e^{-\lambda z}\right], & \text{if } \lambda \neq m \end{cases}. \end{aligned}$$

As a result we get gamma distribution with parameter $(\lambda, 2)$ if $\lambda = \mu$ and Erlang distribution of first degree, if $\lambda \neq \mu$.
(b) It is clear that

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \le z) = \mathbf{P}(\min(X,Y) \le z) = 1 - \mathbf{P}(\min(X,Y) > z) = \\ &= 1 - \mathbf{P}(X > z)\mathbf{P}(Y > z) = 1 - (1 - F_X(z)(1 - F_Y(z) = \\ &= F_X(z) + F_Y(z) - F_X(z)F_Y(z), \end{aligned}$$

from which

$$f_Z(z) = f_X(z) + f_Y(z) - f_X(z)F_Y(z) + f_Y(z)(1 - F_X(z).$$

Using the exponential distributions with parameters λ and μ, we have

$$\begin{aligned} f_Z(z) &= \lambda e^{-\lambda z}\left(1 - \left[1 - e^{-\mu z}\right]\right) + \mu e^{-\mu z}\left(1 - \left[1 - e^{-\lambda z}\right]\right) = \\ &= \lambda e^{-\lambda z - \mu z} + \mu e^{-\lambda z - \mu z} = (\lambda + \mu)e^{-(\lambda+\mu)z}. \end{aligned}$$

Exercise 1.8. *Let $X_1, ..., X_n$ be independent random variables having exponential distribution with parameter λ.*
Find the expected values of the r.v.s $V_n = \max(X_1, ..., X_n)$, and $W_n = \min(X_1, ..., X_n)$.

Solution 1.8. *Clearly*

$$\begin{aligned} F_{V_n}(x) &= \mathbf{P}(V_n \le x) = \mathbf{P}(X_1 \le x, ..., X_n \le x) = (\mathbf{P}(X_1 \le x))^n = (1 - e^{-\lambda x})^n, \quad x \ge 0, \\ F_{W_n}(x) &= \mathbf{P}(W_n \le x) = 1 - \mathbf{P}(W_n > x) = 1 - \mathbf{P}(\min(X_1, ..., X_n) > x) = \\ &= 1 - \mathbf{P}(X_1 > x, ..., X_n > x) = 1 - (\mathbf{P}(X_1 > x))^n = 1 - \left(e^{-\lambda x}\right)^n = 1 - e^{-\lambda n x}. \end{aligned}$$

Using $\int_{-\infty}^{\infty} x dF(x) = \int_{-\infty}^{\infty} (1 - F(x))dx$ (see Exercise 1.5) and introducing in the integral a new variable $y = 1 - e^{-\lambda x}$, we get the expected value of V_n as follows

$$\begin{aligned} \mathbf{E}(V_n) &= \int_0^{\infty} (1 - F_V(x))dx = \int_0^{\infty} (1 - (1 - e^{-\lambda x})^n)dx = \int_0^{\infty} (1 - (1 - e^{-\lambda x})^n)dx = \\ &= \frac{1}{\lambda}\int_0^1 (1 - y^n)\frac{1}{1-y}dy = \frac{1}{\lambda}\int_0^1 (1 + y + ... + y^{n-1})dy = \frac{1}{\lambda}\sum_{i=0}^{n-1}\frac{1}{i}. \end{aligned}$$

From the formula $F_{W_n}(x) = 1 - e^{-\lambda n x}$ it can be seen that W_n has exponential distribution with parameter λn, therefore $\mathbf{E}(W_n) = \frac{1}{\lambda n}$.
Note that the sequence of r.v.s W_n, $n = 1, 2, ...$ has exponential limit distribution with parameter λ_0 if the limit $\lim\limits_{n\to\infty} \lambda n \to \lambda_0$ is satisfied.

Exercise 1.9. *Let X and Y be independent r.v.s with density functions $f_X(x)$ and $f_Y(x)$, respectively. Determine the conditional expected value $E(X \mid X < Y)$.*

Solution 1.9. *By the definition*

$$\mathbf{E}(X|X<Y) = \frac{\mathbf{E}\big(X\mathcal{I}_{\{X<Y\}}\big)}{\mathbf{P}(X<Y)},$$

where

$$\mathbf{P}(X<Y) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mathcal{I}_{\{x<y\}} f_X(x) f_Y(y) dx dy = \int\limits_{-\infty}^{\infty} f_X(x) \left(\int\limits_{x}^{\infty} f_Y(y) dy\right) dx =$$

$$= \int\limits_{-\infty}^{\infty} f_X(x)\left(1 - F_Y(x)\right) dx = \mathbf{E}((1 - F_Y(X)))$$

and

$$\mathbf{E}\big(X\mathcal{I}_{\{X<Y\}}\big) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x\mathcal{I}_{\{x<y\}} f_X(x) f_Y(y) dx dy = \int\limits_{-\infty}^{\infty} x f_X(x) \left(\int\limits_{x}^{\infty} f_Y(y) dy\right) dx =$$

$$= \int\limits_{-\infty}^{\infty} x f_X(x)\left(1 - F_Y(x)\right) dx = \mathbf{E}(X(1 - F_Y(X))).$$

Consequently,

$$\mathbf{E}(X \mid X<Y) = \frac{\int_{-\infty}^{\infty} x f_X(x)\left(1 - F_Y(x)\right) dx}{\int_{-\infty}^{\infty} f_X(x)\left(1 - F_Y(x)\right) dx} = \frac{\mathbf{E}(X(1 - F_Y(X)))}{\mathbf{E}(1 - F_Y(X))}.$$

Exercise 1.10. *Determine the conditional expectation $\mathbf{E}(X \mid Y = y)$ and $\mathbf{E}(X \mid Y)$, if the joint probability density function of r.v.s X and Y has the form*

(a) $f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } 0 < x, y \text{ and } x + y < 1 \\ 0, & \text{otherwise} \end{cases}$,

(b) $f_{X,Y}(x,y) = \begin{cases} 3(x+y), & \text{if } 0 < x, y \text{ and } x + y < 1 \\ 0, & \text{otherwise} \end{cases}$.

Solution 1.10. *(a) Since* $f_Y(y) = \int_0^1 f_{X,Y}(x,y)dx = \int_0^{1-y} 2dx = 2(1-y)$, $0 < y < 1$, *thus the conditional density function is* $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}$, *if* $0 < x, y$ *and* $x + y < 1$, *so*

$$\mathbf{E}(X \mid Y = y) = \int\limits_{0}^{1-y} x f_{X|Y}(x|y) dx = \int\limits_{0}^{1-y} x \frac{1}{1-y} dx = \frac{1}{2}\frac{(1-y)^2}{1-y} = \frac{1}{2}(1-y)$$

and

$$\mathbf{E}(X \mid Y) = \frac{1}{2}(1-Y).$$

(b) Analogously we get $f_Y(y) = \int\limits_0^1 f_{X,Y}(x,y)dx = \int\limits_0^{1-y} 3(x+y)dx = 3\left[\frac{1}{2}(1-y)^2 + y(1-y)\right] = \frac{3}{2}(1-y^2)$ *and* $f_{X|Y}(x|y) = \frac{3(x+y)}{\frac{3}{2}(1-y^2)} = 4\frac{x+y}{1-y^2}$. *From this it follows*

$$\mathbf{E}(X \mid Y = y) = \int\limits_0^{1-y} x f_{X|Y}(x|y)dx = \int\limits_0^{1-y} x\frac{2(x+y)}{1-y^2}dx = 2\frac{\frac{1}{3}(1-y)^3 + y(1-y)}{1-y^2} = \frac{2}{3}\frac{1+y+y^2}{1+y},$$

$$E(X \mid Y) = \frac{2}{3}\frac{1+Y+Y^2}{1+Y}.$$

Exercise 1.11. *Let $X_1, X_2, \ldots$ be independent r.v.s with exponential distribution of parameter λ. Let N be geometrically distributed r.v. with parameter p ($p_k = \mathbf{P}(N = k) = p(1-p)^k,\ k = 1, 2, \ldots$), which does not depend on r.v.s $(X_1, X_2, \ldots)$. Prove that the sum $Y = X_1 + \ldots + X_N$ has exponential distribution with parameter $p\lambda$.*

Solution 1.11. *Since the sum of r.v.s $Y_n = X_1 + \ldots + X_n$ has gamma distribution with parameter (n, λ) thus*

$$f_{Y_n}(x) = \frac{\lambda^n}{(n-1)!}x^{n-1}e^{-\lambda x},\ x > 0$$

and

$$\begin{aligned} F_Y(y) &= \mathbf{P}(X_1 + \ldots + X_N \le y) = \sum_{n=1}^{\infty}\mathbf{P}(X_1 + \ldots + X_N \le y \mid N = n)\mathbf{P}(N = n) = \\ &= \sum_{n=1}^{\infty}\mathbf{P}(Y_n \le y)\mathbf{P}(N = n) = \sum_{n=1}^{\infty}\int\limits_{-\infty}^{y}\frac{\lambda^n}{(n-1)!}x^{n-1}e^{-\lambda x}\left[p(1-p)^{n-1}\right]dx = \\ &= p\lambda\int\limits_{-\infty}^{y}\left(\sum_{n=0}^{\infty}\frac{[(1-p)\lambda x]^n}{n!}\right)e^{-\lambda x}dx = p\lambda\int\limits_{-\infty}^{y}\left(e^{(1-p)\lambda x}e^{-\lambda x}\right)dx = p\lambda\int\limits_{-\infty}^{y}e^{-p\lambda x}dx. \end{aligned}$$

From this

$$f_Y(y) = p\lambda e^{-p\lambda x},\ y > 0,$$

therefore the r.v. Y really has exponential distribution with parameter $p\lambda$.

Exercise 1.12. *Consider the distribution function of the sum Y_{40} of independent r.v.s $X_1, \ldots, X_{40}$ having exponential distribution with parameter 1. Give an estimate for the probability $p = \mathbf{P}\left(\frac{|Y_{40} - \mathbf{E}(Y_{40})|}{\mathbf{D}(Y_{40})} > 0.05\right)$ calculated with the help of the central limit theorem. We can numerically calculate this probability, because the r.v. Y_{40} has gamma distribution with parameter $(40, 1)$. Using this fact, what result can we obtain for the considered probability? (According to the numerical calculation of the gamma distribution see, for example, NIST: National Institute of Standards and Technology. Digital library of mathematical functions. http://dlmf.nist.gov, or A. Lewandowski. Statistical tables. http://www.alewand.de. .)*

Solution 1.12. *By the use of the central limit theorem the r.v. $(Y_{40} - \mathbf{E}(Y_{40}))/DY_{40}$ has approximately $N(0, 1)$ normal distribution, thus*

$$\mathbf{P}\left(\frac{|Y_{40} - \mathbf{E}(Y_{40})|}{\mathbf{D}(Y_{40})} > 0.05\right) = \mathbf{P}\left(\frac{|Y_{40} - 40|}{\sqrt{40}} > 0.05\right) \approx 1 - (\Phi(0.05) - \Phi(-0.05)) = 0.0612,$$

where $\Phi(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{x} e^{-u^2/2} du$ *denotes the standard distribution normal function. Compute numerically the probability* p *using the software from http://www.alewand.de. Then*

$$\mathbf{P}\left(\frac{|Y_{40}-40|}{\sqrt{40}} > 0.05\right) = \mathbf{P}\left(\frac{|Y_{40}-40|}{\sqrt{40}} > 0.05\right) =$$
$$= 1 - \mathbf{P}(39.6837 \leq Y_{40} \leq 40.3163) = 1 - (0.5409 - 0.5010) = 0.0601.$$

It can be seen that the difference between the estimated and numerically computed values is only 0.0011.

Chapter 2

Introduction to stochastic processes

Exercise 2.1. *Let $X_1, X_2, ...$ be independent identically distributed r.v.s with finite absolute moment $\mathbf{E}(|X_1|) < \infty$. Let N be a r.v. taking positive integer numbers and independent of r. v. $(X_i, i = 1, 2, ...)$. Prove that*

a) $\mathbf{E}(X_1 + ... + X_N) = \mathbf{E}(X_1)\mathbf{E}(N)$,

b) $\mathbf{D}^2(X_1 + ... + X_N) = \mathbf{D}^2(X_1) + (\mathbf{E}(X_1))^2(\mathbf{E}(N))^2$

(Wald identities or Wald lemma).

Solution 2.1. *a) Using the formula of the total expected value we have*

$$\begin{aligned}
\mathbf{E}(X_1 + ... + X_N) &= \sum_{n=1}^{\infty} \mathbf{E}(X_1 + ... + X_N)|\ N = n\}\mathbf{P}(N = n) = \\
&= \sum_{n=1}^{\infty} \mathbf{E}(X_1 + ... + X_n)\mathbf{P}(N = n) = \\
&= \sum_{n=1}^{\infty} n\mathbf{E}(X_1)\mathbf{P}(N = n) = \mathbf{E}(X_1)\sum_{n=1}^{\infty} n\mathbf{P}(N = n) = \mathbf{E}(X_1)\mathbf{E}(N).
\end{aligned}$$

b) It is easy to see that

$$\mathbf{D}^2(X_1 + ... + X_N) = \mathbf{E}(X_1 + ... + X_N\mathbf{E}(X_1)\mathbf{E}(N))^2 = \mathbf{E}(X_1 + ... + X_N)^2 - (\mathbf{E}(X_1)\mathbf{E}(N))^2$$

and

$$\begin{aligned}
\mathbf{E}(X_1 + ... + X_N)^2 &= \sum_{n=1}^{\infty} \mathbf{E}((X_1 + ... + X_N)^2 \mid N = n)\mathbf{P}(N = n) = \\
&= \sum_{n=1}^{\infty} \mathbf{E}(X_1 + ... + X_n)^2\mathbf{P}(N = n) = \\
&= \sum_{n=1}^{\infty} \mathbf{E}((X_1 - EX_1) + ... + (X_n - EX_n)) + nEX_n\}^2\mathbf{P}(N = n) = \\
&= \sum_{n=1}^{\infty} \left[n\mathbf{D}^2(X_1) + n^2(\mathbf{E}(X_1))^2\right]\mathbf{P}(N = n) = \\
&= \mathbf{E}(N)\ \mathbf{D}^2(X_1) + \mathbf{E}(N^2)(\mathbf{E}(X_1))^2.
\end{aligned}$$

Note that the identities remain valid if the r.v. N is a stopping time with respect to the sequence of r.v.s $(X_i, i = 1, 2, ...)$, which means that the event $\{N = n\}$ depends only on $(X_1, ..., X_n)$ for all $n = 1, 2, ...$

Exercise 2.2. *Let $X_0, X_1, ...$ be independent r.v.s with joint distribution $\mathbf{P}(X_i = 1) = \mathbf{P}(X_i = -1) = \frac{1}{2}$.*
Define $Z_0 = 0,\ Z_k = Z_{k-1} + X_k,\ k = 0, 1, ...$ Determine the expectation and covariance function of the process $(Z_k, k = 1, 2, ...)$ (random walk on the integer numbers).
Let a and b be real numbers, $|b| < 1$. Denote $W_0 = aX_0,\ W_k = bW_{k-1} + X_k,\ k = 1, 2, ...$ (here the process $(W_k, k = 0, 1, ...)$ constitutes a first degree autoregressive process with the initial value aX_0, and with the innovation process $(X_k, k = 1, 2, ...)$). If we fix the value b, for which value of a will the process W_k be stationary in weak sense?

Solution 2.2. *a) It is clear that $\mathbf{E}(X_k) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0$, $\sigma_X^2 = D^2\{X_k\} = \mathbf{E}(X_k^2) - (\mathbf{E}(X_k))^2 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1)^2 = 1$, moreover $\operatorname{cov}(X_i, X_i) = \sigma_X^2$ and by the independence of the r.v.s X_i, $\operatorname{cov}(X_i, X_j) = 0$, if $i \neq j$. Since $Z_k = Z_{k-1} + X_k = ... = X_k + ... + X_1,\ k = 1, 2, ...,$ then*

$$\begin{aligned} \mathbf{E}(Z_k) &= \mathbf{E}(X_k) + ... + \mathbf{E}(Z_1) = 0, \\ \mathbf{D}^2(Z_k) &= \mathbf{D}^2(X_k) + ... + \mathbf{D}^2(X_1) = k\sigma_X^2. \end{aligned}$$

b) Determine the expectation and covariance function of the process W_k. With a simple calculation we get

$$W_k = bW_{k-1} + X_k = X_k + bX_{k-1} + b^2 W_{k-2} = X_k + bX_{k-1} + ... + b^{k-1}X_1 + b^k aX_0,\ k = 1, 2, ...,$$

then the expectation function $\mathbf{E}(W_k) = E\{X_k + bX_{k-1} + ... + b^{k-1}X_1 + b^k aX_0\} = 0$, the deviation of W_k is the following

$$\begin{aligned} \mathbf{D}^2(W_k) &= \mathbf{D}^2\left(X_k + bX_{k-1} + ... + b^{k-1}X_1 + b^k aX_0\right) = \\ &= \mathbf{D}^2(X_k) + b^2\mathbf{D}^2(X_{k-1}) + ... + b^{2(k-1)}\mathbf{D}^2(X_1) + b^{2k}a\mathbf{D}^2(X_0) = \\ &= \sigma_X^2(1 + b^2 + ... + b^{2(k-1)}) + b^{2k}a\sigma_X^2 = \frac{1 - b^{2k}}{1 - b^2}\sigma_X^2 + b^{2k}a\sigma_X^2. \end{aligned}$$

Since the r.v.s X_k are independent with 0 expectation,then the covariance function is

$$\begin{aligned} R_W(k+m, k) &= \operatorname{cov}(W_{k+m}, W_k) = E(W_{k+m}W_k) = \\ &= \mathbf{E}([X_{k+m} + bX_{k+m-1} + ... + b^{m-1}X_k + b^m W_k]W_k) = \\ &= b^m\mathbf{E}(W_k^2) = b^m\mathbf{D}^2(W_k) = b^m\left(\frac{1 - b^{2k}}{1 - b^2} + b^{2k}a\right)\sigma_X^2. \end{aligned}$$

From this it can be seen that if we choose $a = \frac{1}{1-b^2}$, then the process $(W_k, k = 0, 1, ...)$ will be stationary with expectation function $\mathbf{E}(W_k) = 0,\ k = 0, 1, ...$ and covariance function $R_W(k+m, k) = R_W(k, k+m) = \frac{b^m}{1-b^2}\sigma_X^2,\ k, m = 0, 1, ...$

Exercise 2.3. *Let a and b real numbers and let U be a r.v. uniformly distributed on the interval $(0, 2\pi)$. Denote by $X_t = a\cos(bt + U),\ -\infty < t < \infty$. Prove that the the random cosine process $(X_t,\ -\infty < t < \infty)$ is stationary.*

Solution 2.3. *Since the expectation function of the process* $(X_t, -\infty < t < \infty)$ *takes the form* $\mu_t = \mathbf{E}(X_t) = \int\limits_0^{2\pi} a\cos(bt+x)\frac{1}{2\pi}dx = \frac{a}{2\pi}\int\limits_0^{bt+2\pi} \cos x dx = 0$, *therefore it does not depend on the parameter* t. *The covariance function is*

$$\begin{aligned}
R_X(t,s) &= \operatorname{cov}(X_t, X_s) = \mathbf{E}(X_t X_s) = \int\limits_0^{2\pi} a^2 \cos(bt+x)\cos(bs+x)\frac{1}{2\pi}dx = \\
&= \frac{a^2}{2\pi}\int\limits_0^{2\pi} \frac{1}{2}[\cos(b(t+s)+2x) + \cos(b(t-s)]dx = \frac{a^2}{4\pi}\cos(b(t-s)) = \\
&= \frac{a^2}{4\pi}\cos(b\,|t-s|),
\end{aligned}$$

which means that $(X_t, -\infty < t < \infty)$ *is stationary process.*

Exercise 2.4. *Let* $N(t), T \geq 0$ *be a homogeneous Poisson process with intensity* λ.
a) Determine the covariance and correlation function of $N(t)$.
b) Determine the conditional expectation $\mathbf{E}(N(t+s) \mid N(t))$.

Solution 2.4. *a) Since* $N(t)$ *has Poisson distribution with parameter* λt, *then* $\mathbf{E}(N(t)) = \lambda t$ *and* $D^2\{N(t)\} = \lambda t,\ t \geq 0$. *By the use of the property of independent increments we can get the covariance and correlation functions*

$$\operatorname{cov}(N(t+s), N(t)) = \operatorname{cov}([N(t+s) - N(t)] + N(t), N(t)) = \operatorname{cov}(N(t), N(t)) = \lambda t,\ \ t, s \geq 0,$$

$$\operatorname{corr}(N(t+s), N(t)) = \frac{\operatorname{cov}(N(t), N(t))}{\mathbf{D}(N(t+s))\mathbf{D}(N(t))} = \frac{\lambda t}{\sqrt{\lambda(t+s)}\sqrt{\lambda t}} = \frac{1}{\sqrt{1+\frac{s}{t}}}.$$

b) Repeating the use of the property of independent increments, we have

$$\begin{aligned}
\mathbf{E}(N(t+s) \mid N(t)) &= \mathbf{E}([N(t+s) - N(t)] + N(t) \mid N(t)) = \\
&= \mathbf{E}([N(t+s) - N(t)] \mid N(t)) + \mathbf{E}(N(t) \mid N(t)) = \\
&= \mathbf{E}(N(t+s) - N(t)) + N(t) = \lambda s + N(t).
\end{aligned}$$

Chapter 3

Markov chains

Exercise 3.1. *Compute the probability that the CTMC with generator matrix* $\begin{pmatrix} -1 & 0.5 & 0.5 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ *stays in state* 1 *after the second state transition, if the initial distribution is* $(0.5, 0.5, 0)$.

Solution 3.1. *Using the formulae (3.15) and (3.16), let us compute the transition probability of the embedding Markov chain. We get*

$$q_0 = 1,\ q_1 = 2,\ q_2 = 1$$

and the transition probability matrix is

$$\mathbf{\Pi} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since the initial distribution is $p = (1/2, 1/2, 0)$, *then the distribution of the embedded Markov chain after the second state transition is*

$$p_2 = p\mathbf{\Pi}^2 = (5/8, 1/8, 2/8).$$

From this, we have the resulting probability for the state 1 *as* $5/8$.

Exercise 3.2. *Compute the stationary distribution of the CTMC with generator matrix* $\begin{pmatrix} -3 & 3 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *if the initial distribution is* $(0.5, 0, 0.5)$.

Solution 3.2. *The Markov chain is composed by two irreducible sets of states,* $\{1, 2\}$ *and* $\{3\}$. *The probability of being in these irreducible sets are determined by the initial probability vector. The process starts in both sets with probability* 0.5.

The stationary solution of the Markov chain on set $\{1, 2\}$ *assuming that the process starts in that set is* $(0.5, 0.5)$. *The overall stationary distribution is* $(0.5, 0.5, 0)0.5 + (0, 0, 1)0.5 = (0.25, 0.25, 0.5)$.

Exercise 3.3. Z_n *and* Y_n, $n = 1, 2, \ldots$, *are discrete independent random variables.* $\mathbf{P}(Z_n = 0) = 1-p, \mathbf{P}(Z_n = 1) = p$ *and* $\mathbf{P}(Y_n = 0) = 1-q$, $\mathbf{P}(Y_n = 1) = q$. *Define the transition probability matrix of the DTMC* X_n *if*

$$X_{n+1} = (X_n - Y_n)^+ + Z_n,$$

where $(x)^+ = \max(x, 0)$*. This equation is commonly referred to as the evolution equation of the DTMC.*

Solution 3.3. *It can be seen that the state space of the Markov chain* $\{X_n,\ n = 1, 2, ...\}$ *is the set* $\{0, 1, ...\}$*.Using the evolution equation of the DMTC process we get the transition probability matrix as follows*

$$\mathbf{P}(X_{n+1} = j \mid X_n = i) = \mathbf{P}((X_n - Y_n)^+ + Z_n = j \mid X_n = i) = \mathbf{P}((i - Y_n)^+ + Z_n = j) =$$

$$= \mathbf{P}((i - Y_n)^+ + Z_n = j \mid Y_n = k, Z_n = m)\mathbf{P}(Y_n = k, Z_n = m) =$$

$$= \mathbf{P}((i - k)^+ + m = j)\mathbf{P}(Y_n = k, Z_n = m) = \mathbf{P}(Y_n = k)\mathbf{P}(Z_n = m), \quad i, j = 0, 1, ...$$

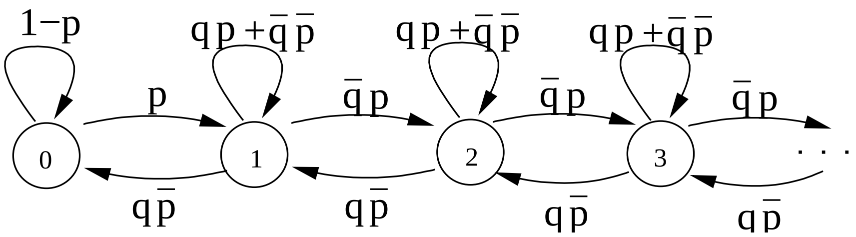
Since

$$\mathbf{P}(Y_n = k, Z_n = m) = \mathbf{P}(Y_n = k)\mathbf{P}(Z_n = m) = \begin{cases} (1-q)(1-p), & \textit{if} \quad k = m = 0, \\ (1-q)p, & \textit{if} \quad k = 0,\ m = 1, \\ q(1-p), & \textit{if} \quad k = 1,\ m = 0, \\ pq, & \textit{if} \quad k = m = 1, \end{cases}$$

the transition probability matrix has the form $(i, j = 0, 1, ...)$

$$p_{ij} = \begin{cases} (1-q)(1-p), & \textit{if} \quad j = i, \\ (1-q)p, & \textit{if} \quad j = i+1, \\ q(1-p), & \textit{if} \quad j = (i-1)^+, \\ pq, & \textit{if} \quad j = (i-1)^+ + 1, \\ 0, & \textit{in other cases.} \end{cases}$$

The transition probability graph with $\bar{p} = 1 - p$ *and* $\bar{q} = 1 - q$ *is*



Exercise 3.4. X_n*,* $n = 1, 2, \ldots,$ *is a DTMC with transition probability matrix* $\boldsymbol{P} = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1/3 & 2/3 \end{pmatrix}$. *Compute* $\mathbf{E}(X_0X_1)$ *and* $corr(X_0, X_1)$ *if the initial distribution is* $(0.5, 0, 0.5)$ *and the state space is* $S = \{0, 1, 2\}$*.*

Solution 3.4. *Let us denote the transition probability matrix*

$$\boldsymbol{P} = (p_{ij}) = \begin{pmatrix} 3/6 & 1/6 & 2/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1/3 & 2/3 \end{pmatrix},$$

and the initial distribution

$$p = (p_0, p_1, p_2) = (1/2, 0, 1/2).$$

Using the Markov property we have

$$\mathbf{E}(X_0X_1) = \sum_{i=0}^{2}\sum_{j=0}^{2} ij\mathbf{P}(X_0 = i, X_1 = j) = \sum_{i=1}^{2}\sum_{j=1}^{2} ij\mathbf{P}(X_1 = j \mid X_0 = i)\mathbf{P}(X_0 = i) =$$

$$= \sum_{j=1}^{2} 2jp_{2j}p_2 = 2\cdot\frac{1}{3}\cdot\frac{1}{2} + 2\cdot 2\cdot\frac{2}{3}\cdot\frac{1}{2} = \frac{5}{3}.$$

It is clear that $corr(X_0, X_1) = \frac{\mathbf{E}(X_0X_1) - \mathbf{E}(X_0)\mathbf{E}(X_1)}{\mathbf{D}(X_0)\mathbf{D}(X_1)}$. *Since the distribution of RVs* X_1 *is*

$$q = (q_0, q_1, q_2) = p^T P = (1/4, 1/4, 1/2),$$

then by simple calculations we have

$$\mathbf{E}(X_0) = \sum_{i=0}^{2} ip_i = 1,\quad \mathbf{E}(X_0^2) = \sum_{i=0}^{2} i^2p_i = 2,\quad \mathbf{D}(X_0) = \sqrt{\mathbf{E}(X_0^2) - (\mathbf{E}(X_0))^2} = 1,$$

$$\mathbf{E}(X_1) = \sum_{i=0}^{2} iq_i = \frac{5}{4},\quad \mathbf{E}(X_1^2) = \sum_{i=0}^{2} i^2q_i = \frac{9}{4},\quad \mathbf{D}(X_1) = \sqrt{\mathbf{E}(X_1^2) - (\mathbf{E}(X_1))^2} = \frac{3}{4}$$

and

$$corr(X_0, X_1) = \frac{5/3 - 1\cdot 5/4}{1\cdot 3/4} = \frac{5}{9}.$$

Exercise 3.5. *The generator of a CTMC is defined by*

$$q_{0j} = \begin{cases} \frac{1}{3} & \text{if } j = 1, \\ \frac{1}{3} & \text{if } j = 2, \\ -\frac{2}{3} & \text{if } j = 0, \\ 0 & \text{otherwise}; \end{cases} \qquad q_{ij} = \begin{cases} \frac{1}{3i} & \text{if } j = i+1, \\ \frac{1}{3i} & \text{if } j = i+2, \\ -\frac{2}{3i} - \mu & \text{if } j = i, \\ \mu & \text{if } j = i-1, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } i = 1, 2, \ldots.$$

Evaluate the properties of this Markov chain using e.g., the Foster theorem.

Solution 3.5. *Firstly, let us compute the transition probabilities of the embedding Markov chain. Using the formulae (3.15) and (3.16) we have* $q_0 = 2/3$, $q_i = \frac{2+3i\mu}{3i}$, $i = 1, 2, \ldots$, *thus*

$$\pi_{0j} = \begin{cases} 1/2, & j = 1 \\ 1/2, & j = 2 \\ 0, & j = 0 \\ 0, & \text{otherwise} \end{cases} \qquad \pi_{ij} = \begin{cases} \frac{1}{2+3i\mu}, & j = i+1 \\ \frac{1}{2+3i\mu}, & j = i+2 \\ \frac{3i\mu}{2+3i\mu}, & j = i-1 \\ 0, & j = 1 \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, 2, \ldots$$

Denote by $X = (X_0, X_1, \ldots)$ *the embedding Markov chain of the CTMC, then the Foster theorem (Th. 3.42) says that the Markov chain* X *is ergodic if there exist constants* $a, b > 0$ *and* $\ell \geq 0$ *such that the innequalities*

$$\mathbf{E}(X_{n+1} \mid X_n = i) \leq a,\ i \leq \ell,$$

$$\mathbf{E}(X_{n+1} \mid X_n = i) \le i - b, \;\; i > \ell$$

hold. Since

$$\mathbf{E}(X_{n+1} \mid X_n = i) = (i+1)\frac{1}{2+3i\mu} + (i+2)\frac{1}{2+3i\mu} + (i-1)\frac{3i\mu}{2+3i\mu} =$$

$$= i - 1 + \frac{5}{2+3i\mu}, \;\; i = 1, 2, ...,$$

and

$$\mathbf{E}(X_{n+1} \mid X_n = 0) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2},$$

then $\mathbf{E}(X_{n+1} \mid X_n = i) \le i + 3/2, \;\; i = 0, 1, ...$ *Choosing* $\ell = \lceil 8/3\mu \rceil$, $a = \ell + 3/2$ *and* $b = 1/2$ *we have*

$$\mathbf{E}(X_{n+1} \mid X_n = i) \le a, \;\; \text{if } i \le \ell,$$
$$\mathbf{E}(X_{n+1} \mid X_n = i) \le i - b, \; \text{if } i > \ell.$$

This means that the Foster's conditions of ergodicity hold, i.e. the Markov chain X is ergodic.

Exercise 3.6. *Show examples for*

- *reducible,*
- *periodic (and irreducible) and*
- *transient (and irreducible)*

DTMCs. Evaluate $\lim_{n\to\infty} \mathbf{P}(X_n = i)$ for these DTMCs, where i is a state of the Markov chain.

Solution 3.6.

- *Reducible*

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1/2 & 1/6 \\ 0 & 0 & 1 \end{pmatrix}$$

  The stationary probabilities depend on the initial distribution.

  - *If the process starts from state 1 then the stationary distribution is* $(1, 0, 0)$.
  - *If the process starts from state 2 then the stationary distribution is* $(2/3, 0, 1/3)$.
  - *If the process starts from state 3 then the stationary distribution is* $(0, 0, 1)$.

- *Periodic*

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

  The stationary probabilities depend on the initial distribution. If the process starts from state 1 at time $n = 0$ then $\lim_{n\to\infty} \mathbf{P}(X_{2n} = i) = 1/2$ for $i = 1, 3$, $\lim_{n\to\infty} \mathbf{P}(X_{2n} = i) = 0$ for $i = 2, 4$ and $\lim_{n\to\infty} \mathbf{P}(X_{2n+1} = i) = 1/2$ for $i = 2, 4$, $\lim_{n\to\infty} \mathbf{P}(X_{2n+1} = i) = 0$ for $i = 1, 3$. That is, as n tends to infinity the distribution is $(1/2, 0, 1/2, 0)$ in the odd steps and it is $(0, 1/2, 0, 1/2)$ in the even steps.

- *Transient:*

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \ddots & 0 & \ddots \end{pmatrix}$$

$\lim_{n\to\infty} \mathbf{P}(X_n = i) = 0$ *for* $i = 0, 1, \ldots$

Exercise 3.7. *Two players,* A *and* B, *play with dice according to the following rule. They throw the dice and if the number is* 1 *then* A *gets* 2£ *from* B, *if the number is* 2 *or* 3 *then* A *gets* 1£ *from* B, *if the number is greater than* 3 *then* B *gets* 1£ *from* A. *At the beginning of the game both* A *and* B *have* 3£. *The game lasts until someone could not pay. What is the probability that* A *wins?*

Solution 3.7. *Before giving a solution for excercise 3.7 , we consider the problem in a more general setting (see, for example, [Shiryaev, 1994]). Let* $K < L < M < N$ *be integer numbers and let* $X = \{X_n, n = 0, 1, ...\}$ *be a Markov chain with finite state space* $\mathcal{S} = \{K, K+1, ..., N\}$. *Let* $\mathcal{S}_0 = \{K, K+1, ..., L-1\}, \mathcal{S}_1 = \{L, L+1, ..., M\}, \mathcal{S}_2 = \{M+1, ..., N\}$, *then they are disjoint and non-empty subsets of* $\mathcal{S}$, *for which* $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}$. *Denote by* $P = (p_{ij})$ *the transition probability matrix of the Markov chain* X.

The problem is to give a system of recurrent equations which describes the probability of the first hit for some state of the set $\mathcal{S}_2$ *from a state* $i_0 \in \mathcal{S}_1$, *which means the probability that the process starts from a state* $i_0 \in \mathcal{S}_1$ *at the time point* 0 *and it will be first in a state from the set* $\mathcal{S}_2$ *without arriving some state from* $\mathcal{S}_0$.

Let $X_0 = i_0 \in \mathcal{S}_1$ *be an initial state. Let us introduce the set*

$$W_{n+1} = \{(i_0, i_1, ..., i_n) : \ i_k \in \mathcal{S}_1, \ 0 \le k \le n-1, \ i_n \in \mathcal{S}_2$$

and denote

$$r_n(i_0) = \mathbf{P}((X_0, ..., X_n) \in W_{n+1} \mid X_0 = i_0),$$
$$r_n(i) = 1, \ \textit{if } i \in \mathcal{S}_2, \ \textit{and } r_n(i) = 0, \ \textit{if } i \in \mathcal{S}_0$$

and

$$R_n(i_0) = r_1(i_0) + ... + r_n(i_0), \ n = 1, 2, ...,$$
$$R_n(i) = 1, \ \textit{if } i \in \mathcal{S}_2, \ \textit{and } R_n(i) = 0, \ \textit{if } i \in \mathcal{S}_0.$$

Using the Markov property of the chain X *we get the relations*

$$r_1(i_0) = \sum_{i_1 \in \mathcal{S}_2} p_{i_0 i_1}$$

and for $n \ge 2$

$$\begin{aligned} r_n(i_0) &= \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} \mathbf{P}((X_0, X_1, ..., X_n) \in W_{n+1} \mid X_0 = i_0, X_1 = i_1) = \\ &= \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} \mathbf{P}((X_1, ..., X_n) \in W_n \mid X_1 = i_1) = \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} r_{n-1}(i_1). \end{aligned}$$

Analogous equations can be derived for the probabilities $R_n(i_0),\ n \geq 1$ *as follows*

$$R_n(i_0) = r_1(i_0) + \sum_{k=2}^{n} r_k(i_0) = r_1(i_0) + \sum_{k=2}^{n} \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} r_{k-1}(i_1) =$$

$$= r_1(i_0) + \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} \sum_{k=2}^{n} r_{n-1}(i_1) = r_1(i_0) + \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} R_{n-1}(i_1).$$

If $n \to \infty$, *then* $R_n(i_0) \to R(i)$ *and we have the equations*

$$R(i_0) = r_1(i_0) + \sum_{i_1 \in \mathcal{S}_1} p_{i_0 i_1} R(i_1), \quad R(i_0) = 1,\ \textit{if } i_0 \in \mathcal{S}_2,\ \textit{and } R(i_0) = 0,\ \textit{if } i \in \mathcal{S}_0.$$

Since the sequences $R_n(i), n = 1, 2, ...,\ i \in \mathcal{S}_1$ *are monotonically nondecreasing, then the limits* $R(i),\ i \in \mathcal{S}_1$ *exist.*

Remark. *Let* $i_0 \in \mathcal{S}_1$ *and denote*

$$T_n = \min\{k:\ X_k \in \mathcal{S}_1,\ 0 \leq k \leq n-1,\ X_n \in \mathcal{S}_1 \cup \mathcal{S}_2\},$$

$$T_n = n \quad \textit{if } X_k \in \mathcal{S}_1,\ 0 \leq k \leq n$$

and

$$E_n(i_0) = \mathbf{E}(T_n \mid X_0 = i_0).$$

The equations for the expectations $E_n(i),\ n \geq 0$ *are similar to the equations* $R_n(i),\ n \geq 0$ *equations are valid, .*
Denote

$$T(i_0) = \min\{n:\ X_k \in \mathcal{S}_1,\ 0 \leq k \leq n-1, \quad X_n \in \mathcal{S}_1 \cup \mathcal{S}_2 \mid X_0 = i_0\}, \quad i_0 \in \mathcal{S}_1.$$

The RV $T(i_0),\ i_0 \in \mathcal{S}_1$ *is finite with probability* 1, *therefore after very long run we have*

$$\mathbf{P}(B \textit{ wins} \mid X_0 = 3) \approx 1 - \mathbf{P}(A \textit{ wins} \mid X_0 = 3).$$

Now, let us return to Solution 3.7. The possible (generalized) state space of the process X *is* $S = \{w_B, 0, ..., 6, w_A\}$, *where* $w_B = \{-1\}$, *i.e. player* B *wins and* $w_A = \{7, 8\}$ *i.e. player* A *wins. Note that* w_A *and* w_B *are absorbing states. Denote by* $S_0 = \{w_A\},\ S_1 = \{0, 1, ..., 6\}$ *and* $S_2 = \{w_A\}$. *The probability transition matrix of the MC* X *is*

$$\boldsymbol{P} = \begin{array}{c} \\ w_B \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ w_A \end{array} \begin{array}{c} \begin{array}{ccccccccc} w_B & 0 & 1 & 2 & 3 & 4 & 5 & 6 & w_A \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/3 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/3 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/3 & 1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

then we have a system of equations for the probabilities $R(i)$ as follows ($r_1(3) = 0$ because the initial state is $i_0 = 3$)

$$\begin{aligned}
R(i) &= 0,\ i \in S_0,\ R(i) = 1,\ i \in S_2, \qquad (3.1)\\
R(0) &= \frac{1}{3}R(1) + \frac{1}{6}R(2), \quad R(1) = \frac{1}{2}R(0) + \frac{1}{3}R(2) + \frac{1}{6}R(3),\\
R(i) &= \frac{1}{2}R(i-1) + \frac{1}{3}R(i-1) + \frac{1}{6}R(i+1), \quad 2 \le i \le 5,\\
R(6) &= \frac{1}{2}R(5) + \frac{1}{2}.
\end{aligned}$$

Solving this system of linear equations we get

$$\begin{aligned}
R(5) &= 2R(6) - 1,\ R(4) = \frac{10}{3}R(6) - \frac{7}{3},\ R(3) = 5R(6) - 4,\\
R(2) &= \frac{64}{9}R(6) - \frac{55}{9}, \quad R(1) = \frac{88}{9}R(6) - \frac{79}{9},\ R(0) = \frac{355}{27}R(6) - \frac{328}{27}.
\end{aligned}$$

Thus from the equation $R(0) = \frac{1}{3}R(1) + \frac{1}{6}R(2)$ it follows

$$R(6) = \frac{443}{470} = 0.943.$$

If $X_0 = 3$ is the initial state, then the asymptotic probability that player A wins is (i.e. after very long run)

$$R(3) = 5R(6) - 4 = 0.713.$$

Comment. *The system of linear equations for values $\mathbf{P}(A \text{ wins} \mid X_0 = i)$, $0 \le i \le 6$ can be obtained easier based on intuitively considerations. Denote by $D = \{A \text{ wins}\}$ the event that the player A wins if the starting state is $0 \le i \le 6$. Then by the Markov property we have*

$$\mathbf{P}(D \mid X_0 = i) = \sum_{j \in \mathcal{S}_2} p_{ij} + \sum_{j \in \mathcal{S}_1} p_{ij}\mathbf{P}(D \mid X_1 = j) = \sum_{j \in \mathcal{S}_2} p_{ij} + \sum_{j \in \mathcal{S}_1} p_{ij}\mathbf{P}(D \mid X_0 = j). \quad (3.2)$$

Denote $R(i) = \mathbf{P}(D \mid X_0 = i)$, $i \in \mathcal{S}_1$, the system of equations (3.1) follows immediately from (3.2).

Exercise 3.8. *Two players,* A *and* B*, play with dice according to the following rule. They throw the dice and if the number is* 1 *then* A *gets* 2£ *from* B*, if the number is* 2 *or* 3 *then* A *gets* 1£ *from* B*, if the number is greater than* 3 *then* B *gets* 1£ *from* A*. At the beginning of the game both* A *and* B *have* 3£*. If one of them could not pay the required amount he gives all of his money to the other and the game goes on. What is the expected amount of money at* A *after a very long run? What is the probability that* B *cannot pay the required amount in the next step of the game after a very long run?*

Solution 3.8. *If one of the players cannot pay the required amount of money in a step of the game, then he must give all his money to the other player and the game goes on. Denote* $X_n,\ n = 0, 1, ...$ *the amount of money the player A has in* n*th step of game. The state space of the process* $X = (X_n,\ n = 0, 1, ...)$ *is* $\mathcal{X} = \{0, 1, ..., 6\}$. *Let us introduce a sequence* $(Y_1, Y_2, ...)$ *of independent RVs with identically distribution*

$$Y_n = \begin{cases} 1/2, & \textit{if } -1, \\ 1/3, & \textit{if } 1, \\ 1/6, & \textit{if } 2, \end{cases}$$

then the process can be represented with the evolution equation $((x)^+ = \max(x, 0))$

$$X_{n+1} = \begin{cases} (X_n + Y_{n+1})^+, & \textit{if } X_n + Y_{n+1} \le 4, \\ \min(X_n + Y_{n+1}, 6), & \textit{if } X_n + Y_{n+1} \ge 5, \end{cases} \qquad n = 0, 1, ...$$

Contrary to the exercise 3.6., in this case there are not absorbing states and the probability transition matrix is

$$\boldsymbol{P} = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1/2 & 1/3 & 1/6 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 1/6 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/3 & 1/6 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1/6 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{array}.$$

The process X *is homogeneous, irreducible and aperiodic MC with finite state space, therefore it is ergodic (see Th. 3.40.) and his stationary distribution, which does not depend on the initial distribution, satisfies the equations*

$$\pi \boldsymbol{P} = \pi,\ \ \pi = (\pi_0, ..., \pi_6),$$

$$\pi_0 + ... + \pi_6 = 1, \quad \pi_i \ge 0,$$

From this we get the system of linear equations

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1,\ \ \pi_1 = \frac{1}{3}\pi_0 + \frac{1}{2}\pi_2, \quad \pi_2 = \frac{1}{6}\pi_0 + \frac{1}{3}\pi_1 + \frac{1}{2}\pi_3,$$

$$\pi_3 = \frac{1}{6}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_4, \quad \pi_4 = \frac{1}{6}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{2}\pi_5,$$

$$\pi_5 = \frac{1}{6}\pi_3 + \frac{1}{3}\pi_4 + \frac{1}{2}\pi_6, \quad \pi_6 = \frac{1}{6}\pi_4 + \frac{1}{2}\pi_5 + \frac{1}{2}\pi_6,$$

$$\pi_0 + ... + \pi_6 = 1.$$

Expressing the probabilities π_i *one after the other, we get* $\pi_1 = \pi_0$, $\pi_2 = \frac{4}{3}\pi_0$, $\pi_3 = \frac{5}{3}\pi_0$, $\pi_4 = \frac{19}{9}\pi_0$, $\pi_5 = \frac{8}{3}\pi_0$, $\pi_6 = \frac{91}{27}\pi_0$, $1 = \pi_0 + ... + \pi_6 = \frac{27}{355}\pi_0$ *and from this* $\pi_0 = \frac{27}{355} = 0.076.$ *Finally we can compute the stationary distribution of the MC X*

$$\pi = (0.076,\ 0.076,\ 0.101,\ 0.127,\ 0.161,\ 0.203,\ 0.256).$$

Using the stationary distribution, the expected amount of money that A will have after very long run

$$\lim_{n\to\infty} \mathbf{E}(X_n) = \pi(0,1,2,3,4,5,6)^T = \sum_{i=0}^{6} i\pi_i = 3.904.$$

Denote D the event that B will not be able to pay the required amount in the next step of the game after a very long run. Then

$$\begin{aligned}
\mathbf{P}(D) &= \lim_{n\to\infty}\Big(\mathbf{P}(D \mid X_n = 5)\mathbf{P}(X_n = 5) + \mathbf{P}(D \mid X_n = 6)\mathbf{P}(X_n = 6)\Big) = \\
&= \lim_{n\to\infty}\Big(\mathbf{P}(Y_{n+1} = 2)\mathbf{P}(X_n = 5) + \mathbf{P}(Y_{n+1} = 1)\mathbf{P}(X_n = 6)\Big) = \\
&= \frac{1}{6}\pi_5 + \frac{1}{3}\pi_6 = 0.203 \cdot \frac{1}{6} + 0.256 \cdot \frac{1}{3} = 0.119.
\end{aligned}$$

Exercise 3.9. *There are two machines at a production site* A *and* B. *Their failure times are exponentially distributed with parameter λ_A and λ_B, respectively. Their repair times are also exponentially distributed with parameter μ_A and μ_B, respectively. There is a single repair man associated with the two machines, which can work on one machine at a time. Compute the probability that at least one of the machines works.*

Solution 3.9. *The system has five states as follows:*
0 - A and B work at same time;
1 - A in repair and B works;
2 - A works and B in repair,
3 - A is waiting for the repair and B in repair ;
4 - A in repair and B is waiting for the repair.
Denote by $Z = (Z_t,\ t \geq 0)$ the process with state spaces $\{0, 1, ..., 4\}$ which desribes the state of the system at the time point t, and let $W_0 = 0 < W_1 < W_2 < ...$be the consecutive sequence of the transition points of time of the system (i.e. the embedding MC of Z). Denote by P the transition probability matrix of the MC $W = (W_0, W_1, ...)$.

First solution. *Let X, Y, U and V be independent exponentially distributed random variables with parameters $\lambda_A, \lambda_B, \mu_A$ and μ_B, respectively. Firstly, we compute the probabilities p_{01},p_{01} and p_{20}. It is clear that*

$$p_{01} = \mathbf{P}(X < Y) = \int_0^\infty\int_0^\infty \mathcal{I}_{\{x<y\}}\lambda_A\lambda_B e^{-\lambda_A x}e^{-\lambda_B y}dxdy = \int_0^\infty\int_0^y \lambda_A\lambda_B e^{-\lambda_A x}e^{-\lambda_B y}dxdy =$$

$$= \int_0^\infty \lambda_B[1-e^{-\lambda_A y}]e^{-\lambda_B y}dy = 1-\frac{\lambda_B}{\lambda_A+\lambda_B}\int_0^\infty(\lambda_A+\lambda_B)e^{-(\lambda_A+\lambda_B)y}dy = 1-\frac{\lambda_B}{\lambda_A+\lambda_B} = \frac{\lambda_A}{\lambda_A+\lambda_B}.$$

Analogously, with the change of parameters, we have

$$p_{10} = \frac{\mu_A}{\mu_A+\lambda_B}, \quad p_{20} = \frac{\mu_B}{\mu_B+\lambda_A}.$$

Denote by $P_{ij}(t),\ t>0$ the distribution function of the holding time from a state i to an other state j, then $P_{ij}(t)=\mathbf{P}(W_1-W_0<t\mid W_0=i,W_1=j)$ and

$$P_{01}(t)=\mathbf{P}(X\le t,\ X\le Y),\quad P_{02}(t)=\mathbf{P}(Y\le t,\ X>Y),$$

$$P_{10}(t)=\mathbf{P}(U\le t,\ U\le Y),\quad P_{14}(t)=\mathbf{P}(U\le t,\ U>Y),$$
$$P_{20}(t)=\mathbf{P}(V\le t,\ V\le X),\quad P_{23}(t)=\mathbf{P}(X\le t,\ V>X),$$
$$P_{31}(t)=\mathbf{P}(V\le t),\quad P_{42}(t)=\mathbf{P}(U\le t).$$

It is clear that

$$P_{01}(t)=\mathbf{P}(X\le t,\ X<Y)=\int_0^\infty\int_0^\infty \mathcal{I}_{\{u\le t,\ u<v\}}\lambda_A\lambda_B e^{-\lambda_A x}e^{-\lambda_B y}dxdy=$$

$$=\int_0^t\left(\int_u^\infty \lambda_A\lambda_B e^{-\lambda_A u}e^{-\lambda_B y}dv\right)du=\int_0^t \lambda_A e^{-(\lambda_A+\lambda_B)u}du=1-\frac{\lambda_A}{\lambda_A+\lambda_B}e^{-(\lambda_A+\lambda_B)u}$$

then

$$P_{01}^{'}(t)=\lambda_A e^{-(\lambda_A+\lambda_B)u}.$$

With the same computations we have

$$P_{02}^{'}(t)=\lambda_B e^{-(\lambda_A+\lambda_B)u},\quad P_{10}^{'}(t)=\mu_A e^{-(\mu_A+\lambda_B)u},\quad P_{14}^{'}(t)=\mu_A e^{-(\mu_A+\lambda_B)u},$$

$$P_{20}^{'}(t)=\mu_B e^{-(\lambda_A+\mu_B)u},\quad P_{23}^{'}(t)=\lambda_A e^{-(\lambda_A+\mu_B)u},\quad P_{31}^{'}(t)=\mu_B e^{-\mu_B t},\quad P_{42}^{'}(t)=\mu_A e^{-\mu_A t}$$

and in other cases $P_{ij}^{'}(t)=0$. The transition rate matrix of the system

$$\boldsymbol{Q}=(q_{ij})=\begin{pmatrix} -(\lambda_A+\lambda_B) & \lambda_A & \lambda_B & 0 & 0\\ \mu_A & -(\mu_A+\lambda_B) & 0 & 0 & \lambda_B\\ \mu_B & 0 & -(\lambda_A+\mu_B) & \lambda_A & 0\\ 0 & \mu_B & 0 & -\mu_B & 0\\ 0 & 0 & \mu_A & 0 & -\mu_A \end{pmatrix}.$$

By the Kolmogorov forward differential equation

$$\mathbf{\Pi}^{'}(t)=\mathbf{\Pi}(t)\boldsymbol{Q},\quad t\ge 0.$$

This ordinary differential equation is linear and has constant coefficient matrix $\boldsymbol{Q}$ with special structure, therefore (see, for example, Bellmann: Introduction to Matrix Analysis, McGraw-Hill, 1960., Ch. 14, §13) it has a unique solution $\Pi(t), t\ge t$ for all initial value $\mathbf{\Pi}(0)=(p_0,...,p_4),\ p_i\ge 0,\ p_0+...+p_4=1$ and $\mathbf{\Pi}(t)$ determines a distribution for any $t\ge 0$. The stationary distribution of the system can be computed from the linear algebraic equations $\pi\boldsymbol{Q}=0$,where $\pi=(\pi_0,...,\pi_4),\ \pi_i\ge 0,:\pi_0+...+\pi_4=1:$

$$(\lambda_A+\lambda_B)\pi_0=\mu_A\pi_1+\mu_B\pi_2,$$

$$(\mu_A+\lambda_B)\pi_1=\lambda_A\pi_0+\pi_3,$$
$$(\mu_B+\lambda_A)\pi_2=\lambda_B\pi_0+\pi_4,$$

$$\begin{aligned}\pi_3 &= \lambda_A \pi_2,\\ \pi_4 &= \lambda_B \pi_1,\end{aligned}$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1.$$

Solving this system of equation, the quantity $\pi_0 + \pi_1 + \pi_2$ *will be the probability that at least one of the machines works.*

Second solution. *The transition probability matrix of the embedding Markov chain is*

$$\boldsymbol{P} = \begin{pmatrix} 0 & p_{01} & 1-p_{01} & 0 & 0 \\ p_{10} & 0 & 0 & 0 & 1-p_{10} \\ p_{20} & 0 & 0 & 0-p_{20} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

where $p_{01} = \mathbf{P}(X < Y)$, $p_{02} = \mathbf{P}(Y \leq X) = 1 - p_{01}$, $p_{10} = \mathbf{P}(U < Y)$, $p_{14} = \mathbf{P}(Y \leq U) = 1 - p_{10}$, $p_{20} = \mathbf{P}(V < X)$, $p_{23} = \mathbf{P}(X \leq V) = 1 - p_{20}$.

Firstly, we compute the probabilities p_{01},p_{01} *and* p_{20}. *It is clear that*

$$p_{01} = \mathbf{P}(X < Y) = \int_0^\infty \int_0^\infty In\mathbf{D}(x<y)\lambda_A\lambda_B e^{-\lambda_A x}e^{-\lambda_B y}dxdy = \int_0^\infty \int_0^y \lambda_A\lambda_B e^{-\lambda_A x}e^{-\lambda_B y}dxdy =$$

$$= \int_0^\infty \lambda_B[1-e^{-\lambda_A y}]e^{-\lambda_B y}dy = 1 - \frac{\lambda_B}{\lambda_A+\lambda_B}\int_0^\infty (\lambda_A+\lambda_B)e^{-(\lambda_A+\lambda_B)y}dy = 1 - \frac{\lambda_B}{\lambda_A+\lambda_B} = \frac{\lambda_A}{\lambda_A+\lambda_B}.$$

Analogously, with the change of parameters, we have

$$p_{10} = \frac{\mu_A}{\mu_A+\lambda_B}, \quad p_{20} = \frac{\mu_B}{\mu_B+\lambda_A}.$$

The stationary distribution of the embedded Markov chain is the solution of the system of equations

$$r\boldsymbol{P} = r, \quad r = (r_0, ..., r_4), \quad r_i \geq 0 \quad \text{and} \quad \sum_{i=0}^4 r_i = 1.$$

Let us use the method of stationary analysis based on the embedded MC (see p.149.). For this we need to determine the stationary distribution $r = (r_1, ..., r_4)$ *and the mean times* $\hat{\tau}_j$, $j = 0, ..., 4$ *that the system spents in a state* j. *Then, by the proposed method, the stationary distribution of the process* Z *is*

$$\pi_j = \frac{r_j\hat{\tau}_j}{\sum\limits_{j=0}^4 r_j\hat{\tau}_j}, \quad j = 0, ..., 4.$$

The holding time of state 1 *has exponential distributions with parameter* $(\lambda_A + \lambda_B)$, *because*

$$\begin{aligned}\mathbf{P}(\tau_1 \leq t) &= \mathbf{P}(\min(X,Y) < t) = 1 - \mathbf{P}(\min(X,Y) \geq t) = \\ &= 1 - \mathbf{P}(X \geq t, Y \geq t) = 1 - \mathbf{P}(X \geq t)\mathbf{P}(Y \geq t) = 1 - e^{-(\lambda_A+\lambda_B)t},\end{aligned}$$

therefore $\hat{\tau}_0 = \frac{1}{\lambda_A+\lambda_B}$. *With the same way we get*

$$\hat{\tau}_1 = \frac{1}{\mu_A+\lambda_B}, \quad \hat{\tau}_2 = \frac{1}{\lambda_A+\mu_B}, \quad \hat{\tau}_3 = \frac{1}{\mu_b}, \quad \hat{\tau}_4 = \frac{1}{\mu_A}.$$

Exercise 3.10. *Let* $X = (X_0, X_1, ...)$ *be a two-state Markov chain with state space* $\mathcal{X} = \{0,1\}$ *and with the probability transition matrix* $\boldsymbol{P} = \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix}$, *where* $0 < a, b < 1$. *Prove that* $\boldsymbol{P}^n = \frac{1}{2-a-b}\boldsymbol{\Pi} + \frac{(a+b-1)^n}{2-a-b}(\boldsymbol{I} - \boldsymbol{P})$, *where* $\boldsymbol{\Pi} = \begin{bmatrix} 1-b & 1-a \\ 1-b & 1-a \end{bmatrix}$ *and* $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution 3.10. *For* $n = 1$ *it is true*

$$\frac{1}{2-a-b}\boldsymbol{\Pi} + \frac{a+b-1}{2-a-b}(\boldsymbol{I} - \boldsymbol{P}) = -(\boldsymbol{I} - \boldsymbol{P}) + \frac{1}{2-a-b}[\boldsymbol{\Pi} + (\boldsymbol{I} - \boldsymbol{P})] =$$

$$= -\boldsymbol{I} + \boldsymbol{P} + \frac{1}{2-a-b}\begin{bmatrix} (1-b)+1-a & (1-a)+0-(1-a) \\ (1-b)+0-(1-b) & (1-a)+1-b \end{bmatrix} = -\boldsymbol{I} + \boldsymbol{P} + \boldsymbol{I} = \boldsymbol{P}.$$

It is easy to check that $\boldsymbol{\Pi}(\boldsymbol{I} - \boldsymbol{P}) = (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{\Pi} = \mathbf{0}$ *and* $\boldsymbol{P} - \boldsymbol{\Pi} = (a+b-1)\boldsymbol{I}$. *Apply the method of induction. Suppose that the equation*

$$\boldsymbol{P}^n = \frac{1}{2-a-b}\boldsymbol{\Pi} + \frac{(a+b-1)^n}{2-a-b}(\boldsymbol{I} - \boldsymbol{P})$$

is true for $n \geq 2$ *then we prove that it is true for* $(n+1)$. *Thus*

$$\begin{aligned} \boldsymbol{P}^{n+1} &= \boldsymbol{P}\boldsymbol{P}^n = \frac{1}{2-a-b}\boldsymbol{P}\boldsymbol{\Pi} + \frac{(a+b-1)^n}{2-a-b}\boldsymbol{P}(\boldsymbol{I} - \boldsymbol{P}) = \\ &= \frac{1}{2-a-b}\boldsymbol{\Pi} + \frac{(a+b-1)^n}{2-a-b}[(a+b-1)\boldsymbol{I} + \boldsymbol{\Pi}](\boldsymbol{I} - \boldsymbol{P}) = \\ &= \frac{1}{2-a-b}\boldsymbol{\Pi} + \frac{(a+b-1)^{n+1}}{2-a-b}(\boldsymbol{I} - \boldsymbol{P}). \end{aligned}$$

From this it follows

$$\lim_{n\to\infty} \boldsymbol{P}^n = \frac{1}{2-a-b}\begin{bmatrix} 1-b & 1-a \\ 1-b & 1-a \end{bmatrix}.$$

Note that the convergence rate is exponentional because of the inequality $|a+b-1| < 1$.
It can also be seen that for both initial values $X_0 = 0$ *and* $X_0 = 1$ *of the Markov chain there exists the limit matrix of the n-step transition matrix of the chain, which does not depend on the initial value. Thus the Markov chain has limit distribution, for which*

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}(X_n = 1 \mid X_0 = 0) &= \lim_{n\to\infty} \mathbf{P}(X_n = 1 \mid X_0 = 1) = \frac{1-a}{2-a-b}, \\ \lim_{n\to\infty} \mathbf{P}(X_n = 0 \mid X_0 = 0) &= \lim_{n\to\infty} \mathbf{P}(X_n = 0 \mid X_0 = 1) = \frac{1-b}{2-a-b}. \end{aligned}$$

Chapter 4

Renewal and regenerative processes

Exercise 4.1. *Applying the Theorem 4.42 (page 140), find the limit (stationary) distributions of age, residual lifetime and total lifetime ($\delta(t) = t - t_{N(t)}$, $\gamma(t) = t_{N(t)+1} - t$, $\beta(t) = t_{N(t)+1} - t_{N(t)}$), if the interarrival times are independent random variables having joint exponential distribution with parameter λ. Show the expected values for the limit distributions.*

Solution 4.1. *By the use of Theorem 4.42 (page 140) we get the limit distributions in the following forms*

$$G(x) = \lim_{t\to\infty} \mathbf{P}(\delta(t) \le x) = \lim_{t\to\infty} \mathbf{P}(\gamma(t) \le x) = \frac{1}{(1/\lambda)} \int_0^x [1 - F(s)]ds = \lambda \int_0^x e^{-\lambda s} ds = 1 - e^{-\lambda x},$$

$$H(x) = \lim_{t\to\infty} \mathbf{P}(\beta(t) \le x) = \frac{1}{(1/\lambda)} \int_0^x s dF(s) = \frac{1}{(1/\lambda)} \int_0^x \lambda e^{-\lambda s} ds = \int_0^x \lambda^2 e^{-\lambda s} ds.$$

From these we can see that the limit distributions of $\delta(t)$ and $\gamma(t)$ as $t \to \infty$ coincide with the exponential distribution of parameter λ. The limit distribution of $\beta(t)$ as $t \to \infty$ is gamma distribution with parameter $(2, \lambda)$, which coincides with the distribution of the sum of two independent exponentially distributed r.v.s with parameter λ. The expected values for the limit distributions are $1/\lambda,\ 1/\lambda,\ 2/\lambda$.

Exercise 4.2. *(Egodic property of semi-Markov processes) Consider a system with finite state space $\mathcal{X} = \{1, ..., N\}$. The system begins to work at the moment $T_0 = 0$ in a state $X_0 \in \mathcal{X}$ and it changes the states at the random moments $0 < T_1 < T_2 < ...$ Denote by $X_1, X_2, ...$ the sequence of consecutive states of the system and suppose that it constitutes a homogeneous, irreducible and aperiodic Markov chain with initial distribution $(p_i = \mathbf{P}(X_0 = i),\ 1 \le i \le N)$ and probability transition matrix $\Pi = (p_{ij})_{i,j=1}^n$. Define the process $X(t) = X_{n-1},\ T_{n-1} \le t < T_n,\ n = 1, 2, ...$ and assume that the sequence of holding times $Y_k = T_k - T_{k-1},\ k = 1, 2, ...$ depends only conditionally on the states $X_{k-1} = i$ and $X_k = j$ and denote $F_{ij}(x) = \mathbf{P}(Y_k \le x \mid X_{k-1} = i,\ X_k = j)$ if $p_{ij} > 0$, where $\nu_{ij} = \int_0^\infty x dF_{ij}(x) < \infty$.*

Find the limits for
(a) average number of transitions/time,
(b) relative frequencies of the states i in the sequence $X_0, X_1, ...$
(c) limit distribution $\mathbf{P}(X_t = i),\ i \in X$,
(d) average time which is spent in a state $i \in X$.

Solution 4.2. *Since the Markov chain $(X_k,\ k=0,1,...)$ with finite state space is homogeneous, irreducible and aperiodic, then it is ergodic, consequently, the expected values of return times $\mu_i = \sum_{k=1}^{\infty} k f_{ii}(k) < \infty,\ 1 \le i \le N$ are finite and its stationary distribution $\pi = (\pi_1, ..., \pi_N)$ can be given in the form $\pi_i = 1/\mu_i,\ 1 \le i \le N$. From the ergodic property of the Markov chain it also follows that*

$$\frac{1}{n}\sum_{k=1}^{n} \mathcal{I}_{\{X_k=i\}} \to \pi_i \quad as \quad n\to\infty \quad with\ probability\ 1.$$

Let us introduce the notations

$$\begin{aligned} K_i(t) &= \sum_{k=0}^{K(t)} \mathcal{I}_{\{X_k=i\}}, \quad K(t) = \max\{k:\ T_k \le t\}, \quad S_i(t) = \int_0^t \mathcal{I}_{\{X(s)=i\}} ds, \\ n_1^{(i)} &= \min\{k:\ X_k = i,\ k\ge 1\}, \quad n_m^{(i)} = \min\{k:\ \ X_k = i,\ k > n_{m-1}^{(i)}\},\ m=2,3,..., \\ T_m^{(i)} &= T_{n_m^{(i)}} = \sum_{k=1}^{n_m^{(i)}} Y_k,\ m=1,2,...,\ \tau_1^{(i)} = T_1^{(i)},\ \ \tau_m^{(i)} = T_m^{(i)} - T_{m-1}^{(i)},\ m=2,3,... \end{aligned}$$

Note that $S_i(t)$ denotes the amount of time is spent by the process $X(t)$ on the interval $(0,t)$ in a state i.

Let us consider the process $(X(t),\ t\ge 0)$ for a fixed i. Since $(X_0, X_1, ...)$ is a Markov chain and the sequence $(Y_1, Y_2, ...)$ only conditionally depends on $(X_0, X_1, ...)$, then the cycles $(X(t),\ t\in [T_{m-1}^{(i)}, T_m^{(i)})),\ m=2,3,...$ are independent and stochastically equivalent. Consequently, the process $(X(t),\ t\ge 0)$ is regenerative under the condition $X_0 = i$, otherwise it is delayed regenerative. It is also clear that the r.v.s $\tau_m^{(i)},\ m=1,2,...$ are independent and moreover, $\tau_m^{(i)},\ m=2,3,...$ are identically distributed, which means that $(\tau_m^{(i)},\ m=1,2,...)$ forms a renewal or delayed renewal (in the case $X_0 \ne i$) process.
First we prove that $\mathbf{E}\Big(\tau_1^{(i)} \mid X_0 = j\Big),\ 1\le j\le N$ are finite. Note that $\mathbf{E}\Big(\tau_2^{(i)}\Big) = \mathbf{E}\Big(\tau_1^{(i)} \mid X_0 = i\Big)$ and $\mathbf{E}\Big(\tau_1^{(i)}\Big) = \sum_{j=1}^{N} \mathbf{E}\Big(\tau_1^{(i)} \mid X_0 = j\Big)\mathbf{P}(X_0 = j)$. Denote $\nu = \max\{\nu_{ij}:\ 1\le i,j\le N,\ p_{ij}>0\}$ and $A_1^{(i)} = \{X_0 = i, X_1 = i\},\ A_k^{(i)} = \{X_0 = i, X_1 \ne i, ..., X_{k-1} \ne i, X_k = i\},\ k\ge 2$. Since for all $1\le i, i_1, ..., i_{k-1} \le N,\ k=1,2,...$ we have

$$\mathbf{E}\Big(Y_1 + ... + Y_k \mid A_k^{(i)}\Big) = \nu_{i,i_1} + \nu_{i_1,i_2} + ... + \nu_{i_{k-1},i} \le k\nu,$$

then

$$\begin{aligned} a_i &= \mathbf{E}\Big(\tau_2^{(i)}\Big) = \mathbf{E}\Big(\tau_1^{(i)} \mid X_0 = i\Big) = \sum_{k=1}^{\infty} \mathbf{E}\Big((Y_1 + ... + Y_k)\mathcal{I}_{\{A_k^{(i)}\}} \mid X_0 = i\Big) = \\ &= \sum_{k=1}^{\infty} \mathbf{E}\Big((Y_1 + ... + Y_k) \mid A_k^{(i)}\Big)\mathbf{P}\Big(A_k^{(i)}\Big) \le \sum_{k=1}^{\infty} k\nu\mathbf{P}\Big(A_k^{(i)}\Big) = \sum_{k=1}^{\infty} k\nu f_{ii}(k) = \nu\mu_i < \infty. \end{aligned}$$

Denote $m_{ij} = \min\{k:\ p_{ij}(k) > 0\}$, where $p_{ij}(k)$ is the k-step probability transition function. The definition of m_{ij} is correct because the Markov chain $(X_n,\ n\ge 0)$ is irreducible. From *this it follows that there exist* $i_1, ..., i_{m-1} \in \mathcal{X}$, $i_s \neq i,\ 1 \leq s \leq m_{ij}$ *such that* $p = \mathbf{P}\Big(B_{m_{ij}}^{(i,j)}\Big) > 0$, *where* $B_{m_{ij}}^{(i,j)} = \{X_0 = i, X_1 = i_1, ..., X_{m_{ij}-1} = i_{m_{ij}-1}, X_{m_{ij}} = j\}$. *Then*

$$\begin{aligned} a_i &\geq \mathbf{E}\Big(\tau_1^{(i)} \mathcal{I}_{\big\{B_{m_{ij}}^{(i,j)}\big\}} \mid X_0 = i\Big) = \mathbf{E}\Big(\tau_1^{(i)} \mid B_{m_{ij}}^{(i,j)}\Big)\mathbf{P}\Big(B_{m_{ij}}^{(i,j)}\Big) \geq \\ &\geq \mathbf{E}\Big(Y_{m_{ij}} + ... + Y_{n_1^{(i)}} \mid B_{m_{ij}}^{(i,j)}\Big)p = \mathbf{E}\Big(Y_{m_{ij}} + ... + Y_{n_1^{(i)}} \mid X_{m_{ij}} = j\Big)p = \\ &= \mathbf{E}\Big(\tau_1^{(i)} \mid X_0 = j\Big)p, \end{aligned}$$

consequently

$$\mathbf{E}\Big(\tau_1^{(i)} \mid X_0 = j\Big) \leq \frac{1}{p} a_i < \infty.$$

(a) By the Remark 4.29 (page 135) the strong law of the large number is also valid for the delayed renewal process $(\tau_m^{(i)},\ m = 1, 2, ...)$, *therefore with probability* 1

$$\frac{K_i(t)}{t} \to \frac{1}{a_i}$$

and consequently with probability 1

$$\frac{K(t)}{t} = \sum_{i=1}^{N} \frac{K_i(t)}{t} \to a = \sum_{i=1}^{N} \frac{1}{a_i}.$$

(b) Firstly, we prove that the convergence

$$\frac{K_i(t)}{K(t)} = \frac{1}{K(t)} \sum_{i=1}^{K(t)} \mathcal{I}_{\{X_{k-1}=i\}} \to \pi_i, \quad as \ \ t \to \infty$$

is true with probability 1. *Note that from the convergence of* $K(t)/t \overset{a.s.}{\to} a$ *it follows that* $K(t) \overset{a.s.}{\to} \infty$, *as* $t \to \infty$. *Since the Markov chain* $(X_k,\ k = 0, 1, ...)$ *is ergodic with stationary distribution* $(\pi_1, ..., \pi_N)$, *therefore*

$$\frac{1}{m} \sum_{i=1}^{m} \mathcal{I}_{\{X_{k-1}=i\}} \overset{a.s.}{\to} \pi_i,\ i = 1, ..., N \quad as \ \ m \to \infty.$$

On the one hand

$$\frac{K_i(t)}{K(t)} = \frac{[at]}{K(t)} \frac{1}{[at]} \sum_{i=1}^{[at]} \mathcal{I}_{\{X_{k-1}=i\}} + \frac{[at]}{K(t)} \frac{1}{[at]} \left(\sum_{i=1}^{K(t)} \mathcal{I}_{\{X_{k-1}=i\}} - \sum_{i=1}^{[at]} \mathcal{I}_{\{X_{k-1}=i\}} \right)$$

and on the other hand

$$\frac{1}{[at]} \left| \sum_{i=1}^{K(t)} \mathcal{I}_{\{X_{k-1}=i\}} - \sum_{i=1}^{[at]} \mathcal{I}_{\{X_{k-1}=i\}} \right| \leq \frac{1}{[at]} \left| K(t) - [at] \right| = \left| \frac{K(t)}{[at]} - 1 \right| \overset{a.s.}{\to} 0$$

because $\frac{[at]}{[t]} \frac{t}{K(t)} \overset{a.s.}{\to} 1,\ t \to \infty$, *therefore* $\frac{K_i(t)}{K(t)} \overset{a.s.}{\to} \pi_i$ *as* $t \to \infty$.

(c) The process $(X(t),\ t \geq 0)$ is a (delayed) regenerative one with regenerative cycles $\tau_k^{(i)}$ (the distribution of the cycle $\tau_1^{(i)}$ can differ in distribution from the distribution of other cycles), therefore the convergence with probability 1

$$
\begin{aligned}
\frac{S_i(t)}{t} &= \frac{1}{t}\int\limits_0^t \mathcal{I}_{\{X(s)=i\}}ds \xrightarrow{a.s.} \lim_{t\to\infty}\mathbf{P}(X(t)=i \mid X_0=i) = \\
&= \frac{1}{a_i}\mathbf{E}\left(\int\limits_0^{\tau_1^{(i)}} \mathcal{I}_{\{X(s)=i\}}ds \mid X_0=i\right) = \frac{1}{a_i}\mathbf{E}(Y_1 \mid X_0=i) = \\
&= \frac{1}{a_i}\sum_{i=1}^N \mathbf{E}(Y_1 \mid X_0=i, X_1=j)\mathbf{P}(X_1=j \mid X_0=i) = \frac{1}{a_i}\sum_{i=1}^N p_{ij}\nu_{ij} = \frac{\nu_i}{a_i}
\end{aligned}
$$

is true, if $\tau_2^{(i)}$ has non-lattice distribution (this condition is satisfied, for example, if $F_{ij}(x)$ are non-lattice distribution functions.). Note that the convergence $\frac{S_i(t)}{t} \xrightarrow{a.s.} \frac{\nu_i}{a_i}$ can be proved directly with the help of strong law of large numbers, because

$$
\frac{K_i(t)-1}{t}\frac{1}{K_i(t)-1}\sum_{k=1}^{K_i(t)-1} Y_{n_k^{(i)}} \leq \frac{S_i(t)}{t} \leq \frac{K_i(t)}{t}\frac{1}{K_i(t)}\sum_{k=1}^{K_i(t)} Y_{n_k^{(i)}},
$$

where $K_i(t) \xrightarrow{a.s.} \infty,\ t \to \infty$ and the r.v.s $Y_{n_k^{(i)}},\ k = 1, 2, ...$ are independent and identically distributed with $E(Y_{n_k^{(i)}}) = \mathbf{E}(Y_1 \mid X_0 = i)$.

From the relations proved above it follows $\frac{K_i(t)}{K(t)}\frac{S_i(t)/t}{K_i(t)/t} \xrightarrow{a.s.} \pi_i\nu_i$ (if $\tau_1^{(i)} \leq t$ then $K_i(t) \geq 1$) and the sum of average time which is spent in a state i equals to 1*, thus*

$$
1 = \sum_{i=1}^N \frac{S_i(t)}{t} = \frac{K(t)}{t}\sum_{i=1}^N \frac{K_i(t)}{K(t)}\frac{S_i(t)/t}{K_i(t)/t},
$$

where

$$
\sum_{i=1}^N \frac{K_i(t)}{K(t)}\frac{S_i(t)/t}{K_i(t)/t} \xrightarrow{a.s.} \sum_{i=1}^N \pi_i\nu_i.
$$

Then we get

$$
\frac{K(t)}{t} \xrightarrow{a.s.} \left(\sum_{i=1}^N \pi_i\nu_i\right)^{-1}, \quad t \to \infty,
$$

$$
\frac{K_i(t)}{t} = \frac{K_i(t)}{K(t)}\frac{K(t)}{t} \xrightarrow{a.s.} \pi_i\left(\sum_{i=1}^N \pi_i\nu_i\right)^{-1} = \frac{1}{a_i}
$$

and with probability 1

$$
\lim_{t\to\infty}\frac{S_i(t)}{t} = \lim_{t\to\infty}\frac{S_i(t)}{K_i(t)}\frac{K_i(t)}{K(t)}\frac{K(t)}{t} = \frac{\pi_i\nu_i}{\sum\limits_k \pi_k\nu_k}.
$$

As a consequence, can be obtained the expected values of regenerative cycles

$$a_i = \frac{1}{\pi_i} \sum_{j=1}^{N} \pi_j \nu_j, \quad a = \sum_{i=1}^{N} \frac{1}{a_i} = \sum_{i=1}^{N} \frac{1}{\pi_i} \sum_{j=1}^{N} \pi_j \nu_j.$$

Chapter 5

Markov chains with special structures

Exercise 5.1. *X and Y are independent continuous PH distributed r.v. with representations $(\boldsymbol{\alpha}, \boldsymbol{A})$ and $(\boldsymbol{\beta}, \boldsymbol{B})$, respectively. Define the distribution of the following r.v.*

- $Z_1 = c_1 X$,
- Z_2 *equals to* X *with probability* p *and to* Y *with probability* $1-p$,
- $Z_3 = c_1 X + c_2 Y$,
- $Z_4 = Min(X, Y)$,
- $Z_5 = Max(X, Y)$.

Solution 5.1. *$Z_1, \ldots, Z_5$ are continuous PH distributed with the following representations. Denote the size of $(\boldsymbol{\alpha}, \boldsymbol{A})$ by n and the size of $(\boldsymbol{\beta}, \boldsymbol{B})$ by m.*

- *PH representation of* $Z_1 = c_1 X$ *is of size* n

$$(\boldsymbol{\gamma}, \boldsymbol{G}) = (\boldsymbol{\alpha}, \frac{1}{c_1}\boldsymbol{A})$$

- *PH representation of* Z_2 *is of size* $n + m$

$$\boldsymbol{\gamma} = (p\boldsymbol{\alpha}, (1-p)\boldsymbol{\beta}), \boldsymbol{G} = \begin{array}{|c|c|} \hline \boldsymbol{A} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{B} \\ \hline \end{array}\,.$$

- *PH representation of* $Z_3 = c_1 X + c_2 Y$ *is of size* $n + m$

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha}, \boldsymbol{0}), \boldsymbol{G} = \begin{array}{|c|c|} \hline \frac{1}{c_1}\boldsymbol{A} & \frac{1}{c_1}\boldsymbol{a}\boldsymbol{\beta} \\ \hline \boldsymbol{0} & \frac{1}{c_2}\boldsymbol{B} \\ \hline \end{array}\,,$$

  where $\boldsymbol{a} = -\boldsymbol{A}\mathbb{1}$ *is the column vector of transition rates to the absorbing state.*

- *PH representation of* $Z_4 = Min(X, Y)$ *is of size* nm

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \boldsymbol{G} = \boldsymbol{A} \oplus \boldsymbol{B},$$

- *PH representation of $Z_5 = Max(X, Y)$ is of size $nm + n + m$*

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \mathbf{0}, \mathbf{0}), \boldsymbol{G} = \begin{array}{|c|c|c|}\hline \boldsymbol{A} \oplus \boldsymbol{B} & \boldsymbol{I} \otimes \boldsymbol{b} & \boldsymbol{a} \otimes \boldsymbol{I} \\ \hline \mathbf{0} & \boldsymbol{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{B} \\ \hline \end{array},$$

where $\boldsymbol{a} = -\boldsymbol{A}\mathbb{1}$ and $\boldsymbol{b} = -\boldsymbol{B}\mathbb{1}$.

Exercise 5.2. *X and Y are independent discrete time PH distributed r.v. with representations $(\boldsymbol{\alpha}, \boldsymbol{A})$ and $(\boldsymbol{\beta}, \boldsymbol{B})$, respectively. Define the distribution of the following r.v.*

- *$Z_1 = c_1 X$,*
- *Z_2 equals to X with probability p and to Y with probability $1 - p$,*
- *$Z_3 = c_1 X + c_2 Y$,*
- *$Z_4 = Min(X, Y)$,*
- *$Z_5 = Max(X, Y)$.*

Solution 5.2. *$Z_1, \ldots, Z_5$ are discrete PH distributed with the following representations. Denote the size of $(\boldsymbol{\alpha}, \boldsymbol{A})$ by n and the size of $(\boldsymbol{\beta}, \boldsymbol{B})$ by m.*

- *When c_1 is a positive integer the PH representation of $Z_1 = c_1 X$ is of size $c_1 n$*

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha}, \mathbf{0}, \ldots, \mathbf{0}), \boldsymbol{G} = \begin{array}{|c|c|c|c|}\hline \mathbf{0} & \boldsymbol{I} & \cdots & \mathbf{0} \\ \hline \vdots & \ddots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{I} \\ \hline \boldsymbol{A} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \end{array},$$

- *PH representation of Z_2 is of size $n + m$*

$$\boldsymbol{\gamma} = (p\boldsymbol{\alpha}, (1 - p)\boldsymbol{\beta}), \boldsymbol{G} = \begin{array}{|c|c|}\hline \boldsymbol{A} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{B} \\ \hline \end{array}.$$

- *When c_1 and c_2 are positive integers Z_3 is PH distributed with size $c_1 n + c_2 m$. In case of $c_1 = c_2 = 2$ the representation of Z_3 is*

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha}, \mathbf{0}, \mathbf{0}, \mathbf{0}), \boldsymbol{G} = \begin{array}{|c|c|c|c|}\hline \mathbf{0} & \boldsymbol{I} & \mathbf{0} & \mathbf{0} \\ \hline \boldsymbol{A} & \mathbf{0} & \boldsymbol{a}\boldsymbol{\beta} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{I} \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{B} & \mathbf{0} \\ \hline \end{array},$$

where $\boldsymbol{a} = \mathbb{1} - \boldsymbol{A}\mathbb{1}$ is the column vector of transition probabilities to the absorbing state. For $c_1, c_2 > 2$ the representation is obtained similarly.

- *PH representation of $Z_4 = Min(X, Y)$ is of size nm*

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \boldsymbol{G} = \boldsymbol{A} \otimes \boldsymbol{B},$$

- *PH representation of $Z_5 = Max(X, Y)$ is of size $nm + n + m$*

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}, \mathbf{0}, \mathbf{0}), \boldsymbol{G} = \begin{array}{|c|c|c|} \hline \boldsymbol{A} \otimes \boldsymbol{B} & \boldsymbol{A} \otimes \boldsymbol{b} & \boldsymbol{a} \otimes \boldsymbol{B} \\ \hline \mathbf{0} & \boldsymbol{A} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \boldsymbol{B} \\ \hline \end{array},$$

where $\boldsymbol{a} = \mathbb{1} - \boldsymbol{A}\mathbb{1}$ and $\boldsymbol{b} = \mathbb{1} - \boldsymbol{B}\mathbb{1}$.

Exercise 5.3. *There are two machines at a production site* A *and* B. *Their failure times are exponentially distributed with parameter λ_A and λ_B, respectively. Their repair times are also exponentially distributed with parameter μ_A and μ_B, respectively. There is a single repair man associated with the two machines, which can work on one machine at a time. At a given time both machines work. Compute the distribution and the moments of the time to the first complete breakdown, when both machines are failed.*

Solution 5.3. *The time to the complete breakdown is continuous PH distributed with representation*

$$\boldsymbol{\gamma} = (1, 0, 0), \boldsymbol{G} = \begin{array}{|c|c|c|} \hline -\lambda_A - \lambda_B & \lambda_A & \lambda_B \\ \hline \mu_A & -\lambda_B - \mu_A & 0 \\ \hline \mu_B & 0 & -\lambda_A - \mu_B \\ \hline \end{array}.$$

The distribution and the moments of the time to complete breakdown, denoted by T, can be obtained from this PH representation. E.g. its cumulated density function is

$$F_T(t) = \mathbf{P}(T < t) = 1 - \boldsymbol{\gamma}\ e^{\boldsymbol{G}t}\mathbb{1},$$

and its moments are

$$\mathbf{E}(T^n) = \ n!\ \boldsymbol{\gamma}(-\boldsymbol{G})^{-n}\mathbb{1}.$$

Part II

Queueing systems

Chapter 6

Introduction to queueing systems

Exercise 6.1. *Interpret the following Kendall's notations*

- *$M/M/1/\infty/\infty$-FIFO, M/M/1*
- *M/M/2//4,*
- *M/M/1/m-PS,*
- *M/M/m-LIFO.*

Solution 6.1. *The default values are usually eliminated from the Kendall's notations.*

- *$M/M/1/\infty/\infty$-FIFO and M/M/1 refer to the same queueing system with (time homogeneous) memoryless arrival and service processes, single server, infinite buffer, and infinite population. The first version of the notation is the extended version of the second one.*
- *M/M/2//4 refers to the queueing system with (time homogeneous) memoryless arrival and service processes, two server units, infinite buffer, and a finite population of 4 customers.*
- *M/M/1/m-PS refers to the queueing system with (time homogeneous) memoryless arrival and service processes, single server, finite buffer of $m-1$ positions, and processor sharing service discipline. In case of processor sharing the server serves as many customers as many are present in the system at the same time. The service capacity is uniformly distributed among the customers. With this service discipline the size of the buffer does not play role.*
- *M/M/m-LIFO refers to the queueing system with (time homogeneous) memoryless arrival and service processes, m servers, finite buffer and infinite population and last in first our service discipline. It means that an arriving customer always enter a server independent of the number of customers in the system. If all servers are busy upon the arrival of the new customer the new customer moves one of the customers under service to the waiting queue and its server starts the service of the newly arrived customer.*

Exercise 6.2. *In a single server infinite buffer queueing model the arrival rate is λ and the service time is exponentially distributed with parameter μ.*

- *Define the Little's law for the whole queueing system, for the buffer and for the server.*

- *Which one of these expressions define the server utilization?*
- *What is the utilization?*

Solution 6.2. *It is an M/M/1 queuing system.*

- *Let T, W, $\bar{Y}$, be the mean system time, the mean waiting time and the mean service time, and L, L_w, L_s be the mean number of customers in the system, in the buffer and in the server, respectively, and λ be the mean arrival rate. The application of Little's law for the whole queueing system, for the buffer and for the server result*

$$L = \lambda T,$$

$$L_w = \lambda W,$$
$$L_s = \lambda \bar{Y}.$$

- *The Little's law applied for the server is related with the utilization, because the mean number of customers in the server queue (L_s) define it.*
- *The mean number of customers in a single server queue (L_s) is the utilization of the queue, $\rho = L_s$. In case of m servers L_s/m defines the utilization, $\rho = L_s/m$.*

Exercise 6.3. *Which ones of the following queueing systems are lossless:*

- *M/M/1,*
- *M/M/2/5/4,*
- *M/M/1/2-PS,*
- *M/M/m/m,*
- *M/M/m.*

Solution 6.3.

- *M/M/1 – it is lossless, because there is an infinite buffer,*
- *M/M/2/5/4 – it is lossless, because the population is 4 customers and there are 2 server and 3 buffer positions for customers.*
- *M/M/1/2-PS – it is lossless, because the server serves all customers at the same time and buffer is not used.*
- *M/M/m/m – it is lossy queue, because the customers arrive when all servers are busy are lost.*
- *M/M/m – it is lossless, because there is an infinite buffer.*

Exercise 6.4. *Which ones of the following queueing systems provides immediate service for the customers:*

- *M/M/1,*

- *M/M/4/5/3,*
- *M/M/1/2-PS,*
- *M/M/m/m,*
- *M/M/m.*

Solution 6.4.

- *M/M/1 – it is a waiting system, because the customers arrive when the server is busy wait in the buffer.*
- *M/M/2/5/4 – it is a waiting system, because the customers arrive when both servers are busy wait in the buffer.*
- *M/M/1/2-PS – it is an immediate service system, because the server serves all customers at the same time with a portion of the server capacity.*
- *M/M/m/m – it is an immediate service system, because the customers arriving when all servers are busy are lost and the ones which are not lost starts the service immediately at arrival.*
- *M/M/m – it is a waiting system, because the customers arrive when all servers are busy wait in the buffer.*

Chapter 7

Markovian queueing systems

Exercise 7.1. *Compute the mean and the variance of the waiting time in an M/M/1 queue based on the Wald's identity.*

Solution 7.1. *The waiting of a customer in an M/M/1 queue, W, is the sum of the service times of the customers, X_i, which are in the system at its arrival. If the number of customers in the system is N at the arrival of the customer, then its waiting time is*

$$W = \sum_{i=1}^{N} X_i,$$

from which

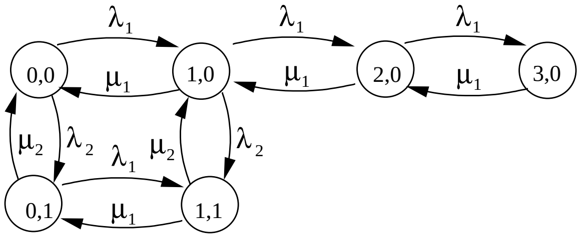
$$\mathbf{E}(W) = \mathbf{E}\left(\sum_{i=1}^{N} X_i\right) = \sum_{n=0}^{\infty} \mathbf{E}\left(\sum_{i=1}^{n} X_i\right) \mathbf{P}(N=n) = \sum_{n=0}^{\infty} n\mathbf{E}(X)\mathbf{P}(N=n) = \mathbf{E}(X)\mathbf{E}(N),$$

and

$$\sigma_W = \sum_{n=0}^{\infty} \sigma_{\sum_{i=1}^{n} X_i} \mathbf{P}(N=n) = \sum_{n=0}^{\infty} n\sigma_X \mathbf{P}(N=n) = \sigma_X \mathbf{E}(N).$$

Exercise 7.2. *Two kinds of customers arrive to a queueing system with 3 servers. Type 1 customers arrive according to a Poisson process with rate λ_1. A type 1 customer occupies one server for an exponentially distributed amount of time with parameter μ_1. Type 2 customers arrive according to a Poisson process with rate λ_2. A type 2 customer occupies two servers for an exponentially distributed amount of time with parameter μ_2. Compute the loss probability of type 2 customers if there is no buffer in the system.*

Solution 7.2. *The stochastic process describing the number of type 1 (first coordinate) and type 2 (second coordinate) customers in the system is a CTMC with the following transition graph.*



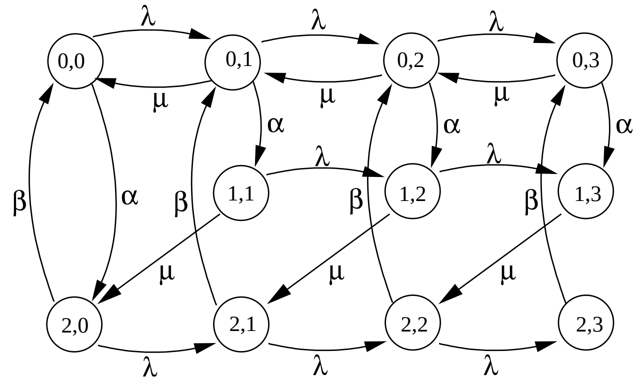
Compute the stationary probability distribution, $p_{i,j}$, of this CTMC and the loss probability is

$$p_{loss} = \frac{\lambda_2}{\lambda_1 + \lambda_2} \left(p_{0,2} + p_{1,0}\right) + p_{1,1} + p_{0,3} \ .$$

In state $p_{0,2}$ and $p_{1,0}$ only the type 2 customers are lost.

Exercise 7.3. *One shop assistant serves the customers in a shop with exponentially distributed service time with parameter μ. The shop assistant wants to smoke after an exponentially distributed time with parameter α. If the shop is idle leaves for smoking immediately. If he is busy when he wants to smoke then he serves the customers while shop is not idle and then he leaves for smoking. The length of the smoke break is exponentially distributed with parameter β. The customers arrive according to a Poisson process with rate λ. Compute the mean shopping time of customers if at most 3 customers can enter the shop. (Compute the same measure if infinitely many customers can enter the shop.)*

Solution 7.3. *The stochastic process describing the state of the shop assistant (first coordinate) and the number of customers in the shop (second coordinate) is a CTMC with the following transition graph. The state of the shop assistant is* 0 *if it works and does not miss a cigarette, is* 1 *if it works and misses a cigarette, is* 2 *if it smokes.*



We compute the mean shopping time of customers with the help of Little's law. For that we need the mean number of customers in the shop, L, and the mean customer arrival rate, $\bar{\lambda}$. Based on the stationary probability distribution, $p_{i,j}$, of the CTMC the mean number of customers in the shop is

$$L = \sum_{j=1}^{3} j \sum_{i=0}^{2} p_{i,j}$$

and mean customer arrival rate is

$$\bar{\lambda} = \lambda p_{loss}$$

where

$$p_{loss} = p_{03} + p_{13} + p_{23}.$$

Finally, the mean shopping time of customers is

$$T = L/\bar{\lambda}.$$

If infinitely many customers can enter the shop then the Markov chain is infinite according *to the second coordinate and we obtain a quasi birth death process of the form*

$$\boldsymbol{Q} = \begin{array}{|c|c|c|c|} \hline \boldsymbol{L'} & \boldsymbol{F'} & \boldsymbol{0} & \cdots \\ \hline \boldsymbol{B'} & \boldsymbol{L} & \boldsymbol{F} & \ddots \\ \hline \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{L} & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \\ \hline \end{array},$$

where

$$\boldsymbol{L} = \begin{array}{|c|c|c|} \hline \bullet & \alpha & 0 \\ \hline 0 & \bullet & 0 \\ \hline \beta & 0 & \bullet \\ \hline \end{array}, \quad \boldsymbol{B} = \begin{array}{|c|c|c|} \hline \mu & 0 & 0 \\ \hline 0 & 0 & \mu \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \quad \boldsymbol{F} = \begin{array}{|c|c|c|} \hline \lambda & 0 & 0 \\ \hline 0 & \lambda & 0 \\ \hline 0 & 0 & \lambda \\ \hline \end{array},$$

$$\boldsymbol{L'} = \begin{array}{|c|c|} \hline \bullet & \alpha \\ \hline \beta & \bullet \\ \hline \end{array}, \quad \boldsymbol{B'} = \begin{array}{|c|c|} \hline \mu & 0 \\ \hline 0 & \mu \\ \hline \end{array}, \quad \boldsymbol{F'} = \begin{array}{|c|c|c|} \hline \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda \\ \hline \end{array}.$$

and $\bullet$ are the negative diagonal elements which are set such that all row sum of $\boldsymbol{Q}$ is zero. Assuming that process is positive recurrent and the stationary solution of this QBD process with irregular level zero is vector p_0, and vectors $p_i = p_1 \boldsymbol{R}^{i-1}$ for $i \geq 1$, we can compute the mean number of customers in the shop as

$$L = \sum_{i=1}^{\infty} i p_1 \boldsymbol{R}^{i-1} \mathbb{1} = p_1 \left(\sum_{i=1}^{\infty} (i-1) \boldsymbol{R}^{i-1} + \sum_{i=1}^{\infty} \boldsymbol{R}^{i} \right) \mathbb{1} =$$
$$p_1 \Big(\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-2} + \boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1} \Big) \mathbb{1} = p_1 \Big(\boldsymbol{R}(\boldsymbol{I}-\boldsymbol{R})^{-1} \left((\boldsymbol{I}-\boldsymbol{R})^{-1} + \boldsymbol{I} \right) \Big) \mathbb{1}.$$

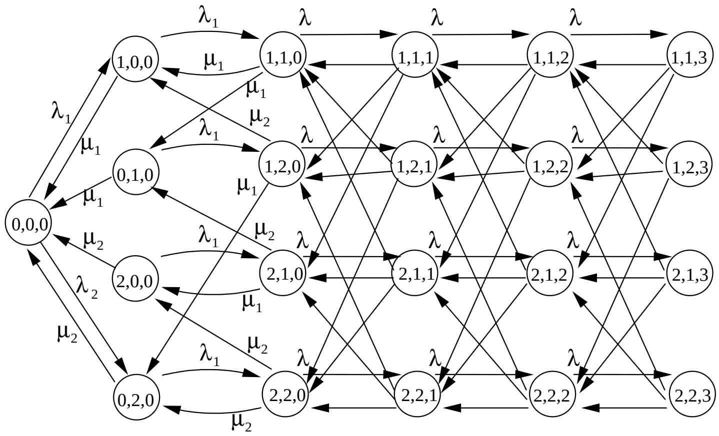
Due to the fact that there is no loss in case of infinite shop capacity ($\bar{\lambda} = \lambda$) the mean shopping time of a customer is

$$T = L/\bar{\lambda} = L/\lambda.$$

Exercise 7.4. *There is a queueing system with two servers and two types of customers. Type i customers arrive according to a Poisson process with rate λ_i and their service time is exponentially distributed with parameter μ_i, $i = 1, 2$. Server i is typically assigned with type i customers. If there is a type i customer in the system when server i is idle then it serves a type i customer. If there is no type i customer in the system when server i is idle then it can serve a customer of the other type. The arrival of a new customer does not interrupt the ongoing service. Compute the loss probability of type i customers if the buffer size is 3.*

Solution 7.4. *A finite CTMC describes the behavior of the queueing system, where the states are identified by the triple (i, j, k). i indicates the state of server 1. $i = 0$ when server 1 is idle, $i = 1$ when server 1 serves a type 1 customer, and $i = 2$ when server 1 serves a type 2 customer. j indicates the state of server 2 in a similar manner, and k indicates the number of customers in the buffer.*

As it is visible from the (i, j, k) state description the type of the customers in the buffer is not identified in the system state. It is a widely applicable trick to reduce the complexity of the Markov chains describing the behavior of queueing systems. When the servers are busy customers arrive to the buffer with rate $\lambda = \lambda_1 + \lambda_2$ and a given customer in the buffer is of type 1 with probability λ_1/λ and of type 2 with probability λ_2/λ.



The transition rates from higher buffer occupancy to lower one are as follows

- *from* $(1,1,i)$ *to* $(1,1,i-1)$ *at rate* $2\mu_1\lambda_1/\lambda$ *because either server 1 completes the service at rate* μ_1 *and the next customer is type 1 with probability* λ_1/λ *or server 2 completes the service at rate* μ_1 *and the next customer is type 1 with probability* λ_1/λ,
- *from* $(1,1,i)$ *to* $(1,2,i-1)$ *at rate* $2\mu_1\lambda_2/\lambda$ *because server 2 completes the service at rate* μ_1 *and the next customer is type 2 with probability* λ_2/λ,
- ...

The loss probability of both customers type are identical and are associated with the states when the buffer is full.

$$p_{loss1} = p_{loss2} = p_{1,1,3} + p_{1,2,3} + p_{2,1,3} + p_{2,2,3}.$$

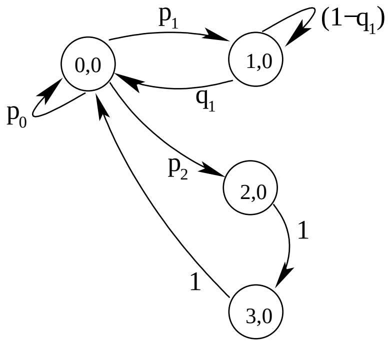
Further type related measures can be computed based on the stationary distribution of this Markov chain. E.g., the probability that a type 1 customer is served by server 2 is

$$p_{type1\text{-}server2} = \frac{\sum_{i=0}^{2}\sum_{k=0}^{3} p_{i,1,k}}{\sum_{i=0}^{2}\sum_{k=0}^{3} p_{i,1,k} + \sum_{j=0}^{2}\sum_{k=0}^{3} p_{1,j,k}}.$$

Exercise 7.5. *Two kinds of customers arrive to a discrete time queueing system. In every time slot a type* i *customer arrives with probability* p_i, $i = 1, 2$, *and no customer arrives with probability* $1 - p_1 - p_2$. *There is a single server. The service time of a type* 1 *customer is geometrically distributed with parameter* q_1. *The service time of a type* 2 *customer is* k *time slot and the buffer size is* b. *Compute the mean system time of type* i *customers for* $i = 1, 2$, *if* $k = 1, 2$ *and* $b = 0, 3, \infty$.

Solution 7.5. *We present the solution for* $k = 2$. *The solution for* $k = 1$ *is straight forward based on that. A finite DTMC describes the behavior of the queueing system, where the states are identified by the couple* (i, j). i *indicates the state of the server.* $i = 0$ *when the server is idle,* $i = 1$ *when the server serves a type 1 customer,* $i = 2$ *when the server serves a type 2 in the first time slot and* $i = 3$ *when the server serves a type 2 in the second time slot.* j

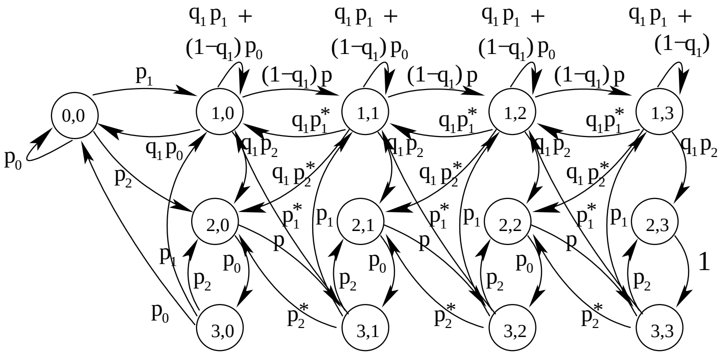
indicates the number of customers in the buffer. The following Markov chain describes the behaviour when there is no buffer ($b = 0$).



The Markov chain also indicate that the customers that receive service, because the system is idle at their arrival, start service immediately after arrival. Consequently, the total system time is the service time for both types of customers and their means are

$$T_1 = \frac{1}{q_1} \ , \qquad T_2 = 2 \ .$$

For the case when there is buffer in the system we follow the same approach as in Exercise 7.4 and the state of the Markov chain does not identify the type of the customers in the buffer only the type of the customer in the server. That is, when the server is busy customer arrive to the buffer with probability $p = p_1 + p_2$ and a given customer in the buffer is of type 1 with probability p_1/p and of type 2 with probability p_1/p. In case of buffer capacity 3 ($b = 3$) the following Markov chain describes the queueing system, where $p_1^* = p_0 p_1/p$ and $p_2^* = p_0 p_2/p$.



When $b = 3$ the computation of the system time requires the stationary solution of the Markov chain ($p_{i,j}$ denote the stationary probabilities) and the application of the Little's law in a similar way as in Exercise 7.3. The loss probability of both customers type are identical and are associated with the states when the buffer is full and the arriving customer is lost.

$$p_{loss} = p_{loss1} = p_{loss2} = \frac{p_{1,3}(1-q_1)p + p_{2,3}p}{p} = p_{1,3}(1-q_1) + p_{2,3},$$

and the mean arrival rate of type 1 (type 2) customers is $\bar{\lambda}_1 = p_1 p_{loss}$ ($\bar{\lambda}_2 = p_2 p_{loss}$). The mean number of type 1 and type 2 customers in the queue are

$$L_1 = \sum_{j=0}^{3} p_{1,j} \left(1 + \frac{jp_1}{p}\right) + \sum_{i=2}^{3} \sum_{j=0}^{3} p_{i,j} \frac{jp_1}{p} \ , \quad L_2 = \sum_{j=0}^{3} p_{1,j} \frac{jp_2}{p} + \sum_{i=2}^{3} \sum_{j=0}^{3} p_{i,j} \left(1 + \frac{jp_2}{p}\right)$$

Finally, from the Little's law we have

$$T_1 = L_1/\bar{\lambda_1} \ , T_2 = L_2/\bar{\lambda_2} \ .$$

The case when the buffer is infinite results in a quasi birth death process with special level zero. Both, the special structure of level zero and the regular structure of the higher levels are readable from the Markov chain of buffer capacity 3. For example, the matrices describing the regular part are

$$\boldsymbol{L} = \begin{array}{|c|c|c|}\hline q_1p_1 + (1-q_2)p_0 & q_1p_2 & 0 \\ \hline 0 & 0 & p_0 \\ \hline p_1 & p_2 & 0 \\ \hline \end{array} \ , \quad \boldsymbol{B} = \begin{array}{|c|c|c|}\hline q_1p_1^* & q_1p_2^* & 0 \\ \hline 0 & 0 & 0 \\ \hline p_1^* & p_2^* & 0 \\ \hline \end{array} \ , \quad \boldsymbol{F} = \begin{array}{|c|c|c|}\hline (1-q_1)p & 0 & 0 \\ \hline 0 & 0 & p \\ \hline 0 & 0 & 0 \\ \hline \end{array} \ .$$

Note that $(\boldsymbol{B}+\boldsymbol{L}+\boldsymbol{F})\mathbb{1} = \mathbb{1}$ holds, that is, the sum of the exit probabilities of each state is one. The stability stability condition of the QBD process can also be obtained by work load consideration. The service of a type 1 customer takes $1/q_1$ time slots in average and the service of a type 2 customer takes 2 time slots. This way $p_1/q_1 + p_2\ 2$ workload arrive to the server in a time slot, which has to be less than 1 in a sable system.

When the buffer is infinite we can compute the system time based on the stationary solution of the QBD type Markov chain. In this case $\bar{\lambda_1} = p_1$ and $\bar{\lambda_2} = p_2$, because there is no loss due to the infinite buffer. The mean number of type 1 and type 2 customers in the queue are

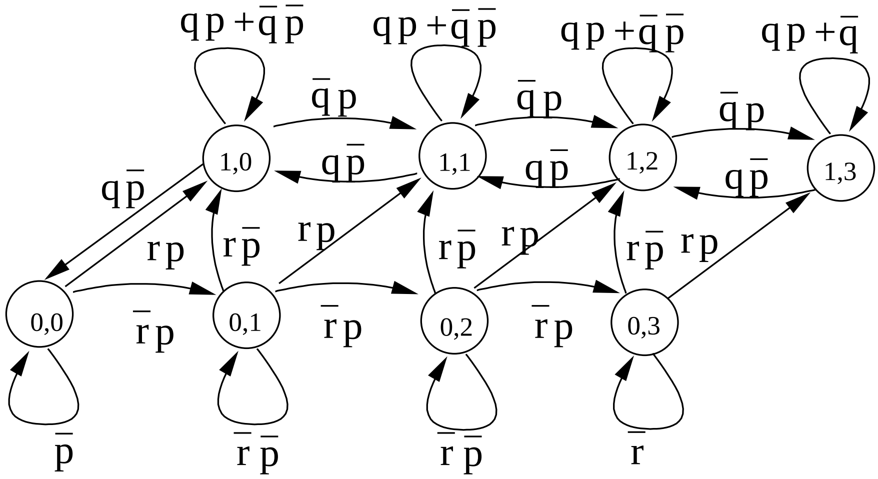
$$L_1 = \sum_{j=0}^{\infty} p_{1,j}\left(1 + \frac{jp_1}{p}\right) + \sum_{i=2}^{3}\sum_{j=0}^{\infty} p_{i,j}\frac{jp_1}{p} \ , \quad L_2 = \sum_{j=0}^{\infty} p_{1,j}\frac{jp_2}{p} + \sum_{i=2}^{3}\sum_{j=0}^{\infty} p_{i,j}\left(1 + \frac{jp_2}{p}\right)$$

and similarly

$$T_1 = L_1/\bar{\lambda_1} \ , T_2 = L_2/\bar{\lambda_2} \ .$$

Exercise 7.6. *To improve the energy efficiency of a discrete time queueing system the server is switched off (goes on vacation) for a geometrically distributed amount of time with parameter r if the system is idle at the end of a time slot. At the end of the vacation period the server starts serving the arrived customers (if any) or goes for an other vacation (if none). In every time slot one customer arrives with probability p and no customer arrives with probability $1-p$. The service time is geometrically distributed with parameter q and the buffer size is b. Compute the mean system time, the mean vacation time and the mean idle time of the server for $b = 3, \infty$.*

Solution 7.6. *When $b = 3$ the following finite DTMC describes the behavior of the queueing system. The states are identified by the couple (i, j). i indicates the state of the server. $i = 0$ when the server is on vacation, $i = 1$ when the server is active and serving a customer. j the number of customers in the buffer. On the figure $\bar{q} = 1-q$, $\bar{p} = 1-p$ and $\bar{r} = 1-r$.*



We compute the mean system time of customers, T, based on the stationary solution of the Markov chain (denoted as $p_{i,j}$) and the Little's law in a similar way as in Exercise 7.3. The loss probability is

$$p_{loss} = \frac{p_{1,3}(1-q)p + p_{0,3}(1-r)p}{p} = p_{1,3}(1-q) + p_{0,3}(1-r),$$

and the mean arrival rate customers is $\bar{\lambda} = pp_{loss}$. The mean number of customers in the queue is

$$L = \sum_{i=0}^{1}\sum_{j=0}^{3}(i+j)p_{i,j} \text{ and } T = L/\bar{\lambda} \ .$$

The idle time and the vacation time of the server is identical because the server immediately starts the vacation when it becomes idle and it finishes the vacation only when there is customer to serve. The vacation time of the server is discrete PH distributed and its representation can be extracted from the Markov chain by interpreting the transition from the lower row of states (server in vacation) to the upper ones (server busy) as transitions to an absorbing state and recognizing that the vacation always starts in state $(0,0)$.

$$\beta = \{1,0,0,0\} \ , \quad \boldsymbol{B} = \begin{array}{|c|c|c|c|}\hline 1-p & (1-r)p & 0 & 0 \\ \hline 0 & (1-p)(1-r) & (1-r)p & 0 \\ \hline 0 & 0 & (1-p)(1-r) & (1-r)p \\ \hline 0 & 0 & 0 & 1-r \\ \hline \end{array} \ .$$

The mean of the vacation time can be computed as $\beta\boldsymbol{B}^{-1}\mathbb{1}$.

In the case when the buffer is infinite the regular structure of the above Markov chain follows for all buffer levels. There are more than one ways to define a QBD process based on this regular Markov chain. It is possible to define a QBD such that level j is composed by states $(0,j)$ and $(1,j)$ in this case the structure is regular for level zero; and it is also possible to define a QBD such that level j is composed by states $(0,j)$ and $(1,j-1)$ in this case the structure is special for level zero. In the second case level zero has different dimension, it is composed by a single state, state $(0,0)$.

The regular matrix blocks of the QBD whose level j is composed by states $(0,j)$ and $(1,j)$ are

$$\boldsymbol{L} = \begin{array}{|c|c|}\hline (1-r)(1-p) & rp \\ \hline 0 & pq+(1-p)(1-q) \\ \hline \end{array} , \quad \boldsymbol{B} = \begin{array}{|c|c|}\hline 0 & r(1-p) \\ \hline 0 & q(1-p) \\ \hline \end{array} , \quad \boldsymbol{F} = \begin{array}{|c|c|}\hline (1-r)p & 0 \\ \hline 0 & (1-q)p \\ \hline \end{array} .$$

Note again that $(\boldsymbol{B}+\boldsymbol{L}+\boldsymbol{F})\mathbb{1}=\mathbb{1}$ *holds.*

When the buffer is infinite and the system is stable we can compute the mean system time of customers based on the stationary solution of the QBD type Markov chain. In this case $\bar{\lambda}=p$*, because there is no loss due to the infinite buffer. The mean number of customers in the queue is*

$$L=\sum_{i=0}^{1}\sum_{j=0}^{\infty}(i+j)p_{i,j} \quad \text{and} \quad T=L/\bar{\lambda}\ .$$

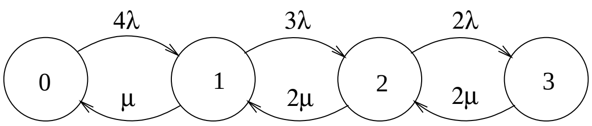
We compute the mean vacation time with infinite buffer based on the stochastic interpretation of the system behavior. The vacation starts in state $(0,0)$ *and there are two conditions to finish the vacation. There is an arrival and at the same time or after the arrival a transition with probability* r *occurs. This stochastic interpretation allow a simpler PH representation of the vacation time*

$$\gamma=\{1,0\}\ ,\quad \boldsymbol{G}=\begin{array}{|c|c|}\hline 1-p & (1-r)p \\ \hline 0 & 1-r \\ \hline\end{array}\ .$$

Note that $(\beta,\boldsymbol{B})$ *and* $(\gamma,\boldsymbol{G})$ *define the same distribution.*

Exercise 7.7. *Compute the stationary number of customers in a* $M/M/2/3/4$ *queue if* $\lambda=1$ *and* $\mu=2$*.*

Solution 7.7. *The Markov chain of* $M/M/2/3/4$ *queue is*



With $\lambda=1$ *and* $\mu=2$ *the stationary probabilities satisfy the following local balance and normalizing equations.*

$$p_1=\frac{4}{2}p_0, p_2=\frac{3}{4}p_1=\frac{3}{2}p_0, p_3=\frac{2}{4}p_2=\frac{3}{4}p_0, \sum_{i=0}^{3}p_i=1\ ,$$

from which

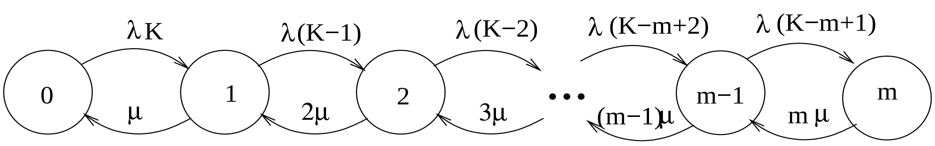
$$p_0=\frac{4}{21}p_1=\frac{8}{21}, p_2=\frac{6}{21}, p_3=\frac{3}{21}\ ,$$

and the mean number of customers is

$$L=\sum_{i=0}^{3}ip_i\ =\frac{29}{21}.$$

Exercise 7.8. *Compute the loss probability of the* $M/M/m/m/K$ *system for* $K>m$*.*

Solution 7.8. *The Markov chain of* $M/M/m/m/K$ *queue is*



The stationary probabilities satisfy the following local balance and normalizing equations.

$$p_i = \frac{(K-i+1)\lambda}{i\mu} p_{i-1}, \quad i = 1, \ldots, m, \qquad \sum_{i=0}^{m} p_i = 1,$$

from which

$$p_i = p_0 \prod_{j=1}^{i} \frac{(K-j+1)\lambda}{j\mu} = p_0 \frac{K!\ \lambda^i}{K-i!\ i!\ \mu^i} = p_0 \binom{K}{i} \left(\frac{\lambda}{\mu}\right)^i, \quad i = 0, \ldots, m,$$

$$p_i = \frac{\binom{K}{i} \left(\frac{\lambda}{\mu}\right)^i}{\sum_{j=0}^{m} \binom{K}{j} \left(\frac{\lambda}{\mu}\right)^j}, \quad i = 0, \ldots, m.$$

The loss probability of the $M/M/m/m/K$ queue is $p_{loss} = p_m$.

Exercise 7.9. *Compare the probability of waiting in an $M/M/m$ queue with the loss probability in an $M/M/m/m$ queue for $m = 1, 2, 3$, where the arrival and service intensities are identical. Interpret the relation of the results.*

Solution 7.9. *In an $M/M/1$ queue the probability of waiting is $p_{wait} = \frac{\lambda}{\mu}$ and in an $M/M/1/1$ queue (Markov chain with 2 states) the loss probability is $p_{loss} = \frac{\frac{\lambda}{\mu}}{1+\frac{\lambda}{\mu}}$.*

Next we compute the probability of waiting and the loss probability for $m = 3$ only. The stationary state probabilities of the $M/M/3$ queue satisfy

$$p_i = \frac{\lambda}{i\mu} p_{i-1}, \quad i = 1, 2, \quad p_i = \frac{\lambda}{3\mu} p_{i-1}, \quad i = 3, 4, \ldots, \qquad \sum_{i=0}^{\infty} p_i = 1,$$

$$p_i = p_0 \frac{\lambda^i}{i!\mu^i}, \quad i = 1, 2, \quad p_i = p_2 \left(\frac{\lambda}{3\mu}\right)^{i-2}, \quad i = 2, \ldots, m, \quad \text{and} \quad \sum_{i=2}^{\infty} p_i = p_2 \frac{3\mu}{3\mu - \lambda}$$

from which

$$p_{wait} = \frac{\sum_{i=3}^{\infty} p_i}{p_0 + p_1 + \sum_{i=2}^{\infty} p_i} = \frac{\frac{\lambda^3}{3!\mu^3} \frac{3\mu}{3\mu-\lambda}}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} \frac{3\mu}{3\mu-\lambda}}.$$

The stationary state probabilities of the $M/M/3/3$ queue (Markov chain with 4 states) satisfy

$$p_i = \frac{\lambda}{i\mu} p_{i-1}, \quad i = 1, 2, 3, \qquad \sum_{i=0}^{3} p_i = 1,$$

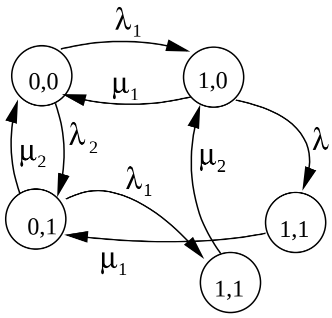
from which

$$p_{loss} = \frac{p_3}{p_0 + p_1 + p_2 + p_3} = \frac{\frac{\lambda^3}{3!\mu^3}}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{3!\mu^3}}.$$

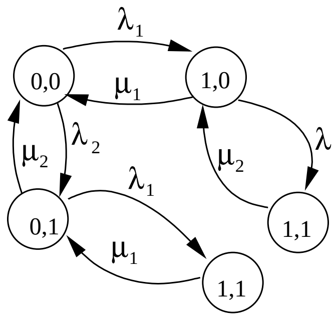
In both cases the waiting probability is larger than the loss probability because the Markov chains spend the same amount of time in the $0, \ldots, m-1$ part of the state space, but in case of $M/M/m$ queue the Markov chain spends more time in the other part of the state space, which is composed by infinitely many states, than in the case of $M/M/m/m$ queue, when the other part of the state space is composed by a single state.

Exercise 7.10. *A complex system is composed by two main units. The failure and the repair time of unit i, $i = 1, 2$, are exponentially distributed with parameter λ_i and μ_i, respectively. The units are maintained by a single repairman. Define the Markov chain of the system behavior if the service discipline of the repairman is FIFO, preemptive LIFO, processor sharing, if the repair of unit 1 has a preemptive priority over the one of unit 2, if the repair of unit 1 has a non-preemptive priority over the one of unit 2.*

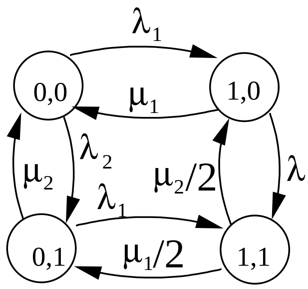
Solution 7.10.



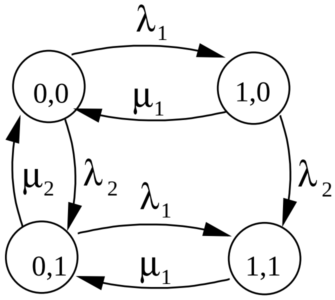
FIFO service



Preemptive LIFO



Processor sharing

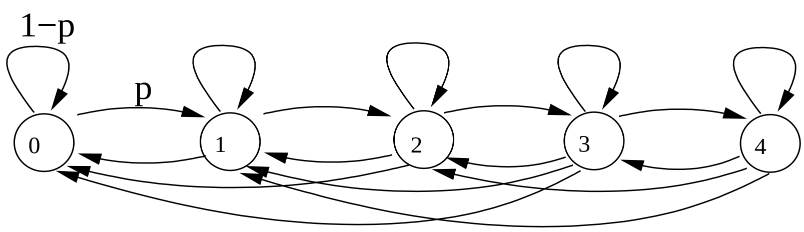


Preemptive priority of unit 1

The non-preemptive priority of unit 1 is identical with the FIFO case, because the repairman does not interrupt the ongoing service process and at the time when the ongoing service process is completed there is only only one failed unit to repair.

Exercise 7.11. *Customers of a discrete time queueing system (under service and waiting) can be lost. Each customer is lost with probability r in each time slot. One customer arrives with probability p (and with $1 - p$ no customer arrives) in each time slot and the service time is geometrically distributed with parameter q. Compute the probability of successful service completion if the buffer size is 3.*

Solution 7.11. *The following DTMC describe the system behavior,*



with transition probabilities

$$\begin{aligned} p_{i,i+1} &= (1-q)pr(i,0), \\ p_{i,i} &= qpr(i,0) + (1-q)(1-p)r(i,0) + (1-q)pr(i,1), \\ p_{i,i-1} &= qpr(i,1) + q(1-p)r(i,0) + (1-q)(1-p)r(i,1) + \underbrace{(1-q)pr(i,2)}_{if\ i\geq 2}, \\ \dots & \qquad \dots, \end{aligned}$$

where $r(i,j)$ *denotes the probability that* j *customers are lost in a time slot, when there are* i *customers in the system at the beginning of the time slot. The number of lost customers is binomially distributed with parameters* i, r*, that is,* $r(i,j) = \binom{i}{j} r^j (1-r)^{i-j}$.

The transition probabilities are determined by the distribution of the number of served customers, Y*, the number of arrived customers,* V*, and the number of lost customers,* Z*. Which are Beroulli distributed with parameters* q*, Beroulli distributed with parameters* p *and in case of* i *customers binomially distributed with parameters* i, r*, respectively. When there are* i *customers* $2 \times 2 \times (i+1)$ *cases needs to be evaluated to obtain the transition probabilities. The evolution equation presents a rather compact way to describe the same DTMC. Let* X_n *be the number of customers in the system in time slot* n *and* Y_n, Z_n, V_n *the number of served, lost and arrived customers in time slot* n*, than*

$$X_n = \min(\max(X_{n-1} - Y_n - Z_n, 0) + V_n, 3) \ .$$

Chapter 8

Non-Markovian queueing systems

Exercise 8.1. *There is an M/G/1 queue. The arrival intensity is λ and the service time is exponentially distributed with parameter μ_2 with probability $1-p$ and it is the sum of two independent exponentially distributed random variable with parameters μ_1 and μ_2 with probability p.*

- *Compute the utilization of the server.*
- *Compute the coefficient of variation of the service time.*
- *Compute the mean system time of customers.*
- *Compute the mean number of customers in the buffer.*

Solution 8.1.

- *Let S be the service time and S_i be exponentially distributed with parameter μ_i. The mean service time and the utilization are*

$$\mathbf{E}(S) = (1-p)\mathbf{E}(S_1) + p\left(\mathbf{E}(S_1+S_2)\right) = (1-p)\frac{1}{\mu_2} + p\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) \;, \qquad \rho = \lambda\mathbf{E}(S) \;.$$

- *The second moment and the coefficient of variation of the service time are*

$$\begin{aligned}
\mathbf{E}(S^2) &= (1-p)\mathbf{E}(S_1^2) + p\left(\mathbf{E}((S_1+S_2)^2)\right) = \\
&= (1-p)\mathbf{E}(S_1^2) + p\left(\mathbf{E}(S_1^2) + 2\mathbf{E}(S_1)\mathbf{E}(S_2) + \mathbf{E}(S_2^2)\right) = \\
&= (1-p)\frac{2}{\mu_2^2} + p\left(\frac{2}{\mu_1^2} + \frac{2}{\mu_1\mu_2} + \frac{2}{\mu_2^2}\right) \\
\mathbf{CV}(S) &= \frac{\mathbf{E}(S^2)}{\mathbf{E}(S)^2} - 1 \;.
\end{aligned}$$

- *The mean system time of customers is*

$$\mathbf{E}(T) = \mathbf{E}(S)\left(1 + \frac{\rho}{1-\rho}\,\frac{1+\mathbf{CV}(S)}{2}\right) \;.$$

- *The mean waiting time and the mean number of customers in the buffer are*

$$\mathbf{E}(W) = \mathbf{E}(S)\,\frac{\rho}{1-\rho}\,\frac{1+\mathbf{CV}(S)}{2} \;, \qquad \mathbf{E}(L_W) = \lambda\mathbf{E}(W).$$

An alternative solution of the exercise is to recognize that the service time is PH distributed with representation

$$\beta = \{p, 1-p\}, \boldsymbol{B} = \begin{array}{|c|c|} \hline -\mu_2 & \mu_2 \\ \hline & -\mu_1 \\ \hline \end{array} .$$

Based on the PH representation of the service time we can apply the analysis of the M/PH/1 queue for which closed form expressions are available.

Exercise 8.2. *Customers arrive to a dentist according to a Poisson process with intensity λ. Arriving customers enter the dentist's surgery if it is idle, otherwise they wait in the waiting room. At the dentist's surgery there is a registration of time D (deterministic). With probability p the patient is directed to the dentist for treatment which takes an exponentially distributed time with parameter μ, with probability $1-p$ the patient is rejected.*

- *Compute the mean time of customers in the waiting room.*
- *Compute the probability that an arriving customer has to wait.*
- *Compute the mean waiting time.*

Solution 8.2.

- *Let S be the service time and S_T be the treatment time which is exponentially distributed with parameter μ. The mean service time and the utilization are*

  $$\mathbf{E}(S) = D + p\mathbf{E}(S_T) = D + p\frac{1}{\mu} , \qquad \rho = \lambda \mathbf{E}(S) .$$

  The second moment and the coefficient of variation of the service time are

  $$\mathbf{E}(S^2) = D^2 + p\mathbf{E}(S_T^2) = D^2 + p\frac{2}{\mu^2}$$
  $$\mathbf{CV}(S) = \frac{\mathbf{E}(S^2)}{\mathbf{E}(S)^2} - 1 .$$

  Based on these quantities the mean waiting time is

  $$\mathbf{E}(W) = \mathbf{E}(S)\ \frac{\rho}{1-\rho}\ \frac{1+\mathbf{CV}(S)}{2} .$$

- *The probability that an arriving customer has to wait can be computed from the utilization of the system as follows*

  $$\mathbf{P}(\textit{waiting}) = 1 - \rho .$$

- *The waiting time is the time a customer spends in the waiting room.*

Exercise 8.3. *$F_A(t)$ is the inter-arrival distribution in an G/M/1 queue whose service rate is μ. $N(t)$ is the number of customers in the system at time t and $T_1, T_2, \ldots$ denote the arrival instances of the first, second, etc. customers. The mean of the stationary number of customers is $\bar{N} = \lim_{t\to\infty} E(N(t))$ and the mean of the stationary number of customers at arrival instances is $\check{N} = \lim_{n\to\infty} E(N(T_n-))$. Compute the relation of $\bar{N}$ and $\check{N}$ if*

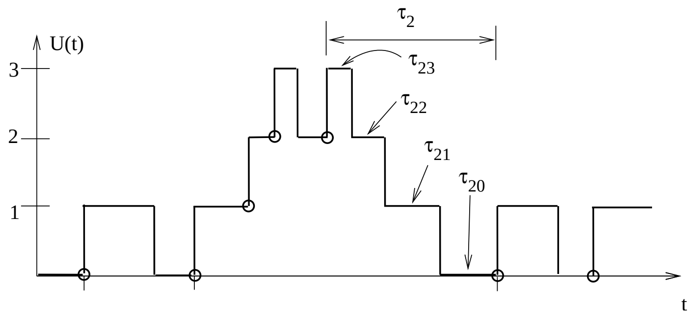
- *inter-arrival distribution is hyper-exponential ($F_A(t) = 1 - pe^{\lambda_1 t} - (1-p)e^{\lambda_2 t}$),*
- *inter-arrival distribution is deterministic,*
- *inter-arrival distribution is exponential.*

Solution 8.3. *$\bar{N}$ denotes the mean number of customers in a G/M/1 queue at a random time instant and $\check{N}$ denotes the mean number of customers right before an arrival instance. The arrival process is not a Poisson process and consequently PASTA property does not hold. That is, the distribution of the number of customers at a random time instant and the distribution of number of customers right before an arrival instance are different (in general). On page 263 we have that the number of customers right before arrivals, $N(T_n-)$, is geometrically distributed with parameter z^*, that is $P_\ell = \mathbf{P}(N(T_n-) = \ell) = z^{*\ell}(1-z^*)$, where z^* is the solution of $z^* = A^\sim(\mu - z^*\mu)$ and $A^\sim(s)$ is the Laplace Stieltjes transform of $F_A(t)$, $A^\sim(s) = \int_t e^{-st} dF_A(t)$.*

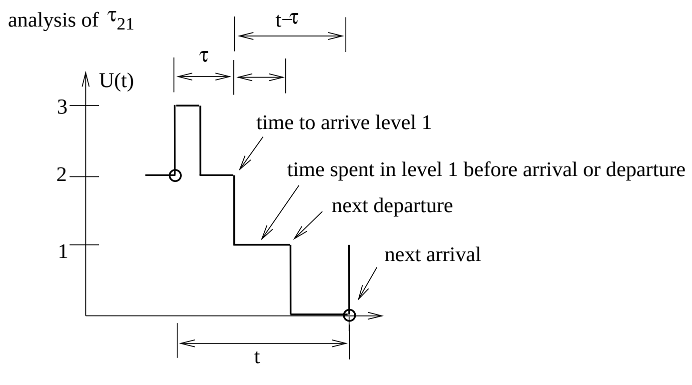
The $N(t)$ process, number of customers in the G/M/1 queue at time t, is a Markov regenerative process with regeneration instances at customer arrivals. The relation of the distribution at a regeneration instance and at a random time instance is provided on page 162. The stationary distribution (at random time) can be computed from the embedded distribution as:

$$\pi_k = \frac{\sum_j P_j \tau_{jk}}{\sum_j P_j \tau_j} \ , \qquad (8.1)$$

where τ_j is the mean time to the next embedded instance starting from state j, and τ_{jk} is the mean time spent in state k before the next embedded instance starting from state j as it is indicated in the following time diagram.



Denoting the mean arrival intensity by $\bar{\lambda}$, we have $\bar{\lambda} = 1/\int_t 1 - F_A(t)dt$. In a G/M/1 queue the embedded instances are the arrival instances, and $\tau_i = 1/\bar{\lambda}$ for all $i \geq 0$is the mean inter-arrival time. The following diagram details the stochastic process between arrival instances in order to compute τ_{ik}.



In the figure τ is the sum of $i+1-k$ service times and it has an Erlang$(i+1-k,\mu)$ distribution. Using that we have

$$\tau_{ik} = \int_{t=0}^{\infty}\int_{\tau=0}^{t}\int_{x=0}^{t-\tau} e^{-\mu x}\ dx\ f_{Erl(i+1-k)}(\tau)\ d\tau\ dA(t)$$

and

$$\tau_{i0} = \int_{t=0}^{\infty}\int_{\tau=0}^{t} (t-\tau)\ f_{Erl(i+1-k)}(\tau)\ d\tau\ dA(t)$$

Note the level independent behavior of $\tau_{i,k}$, that is $\tau_{i,k} = \tau_{i+j,k+j}$, $\forall j \geq 0$ and $\forall k > 0$.

Computing τ_{ik} and substituting into (8.1) results in

$$\pi_0 = 1-\rho \quad and \quad \pi_k = \rho(1-z^*)z^{*k-1}, \quad k \geq 1\ ,$$

where $\rho = \bar{\lambda}\frac{1}{\mu}$.

Based on the stationary behavior at random instance we can compute the mean number of customers in the queue, $\bar{N} = L$, the mean system time, T, the mean number of waiting customers, L_W, and the mean waiting time

$$\bar{N} = L = \sum_{i=0}^{\infty} i\pi_i = \frac{\rho}{1-z^*}\ , \quad T = \frac{L}{\bar{\lambda}} = \frac{1}{\mu}\ \frac{1}{1-z^*}\ ,$$

$$L_W = \sum_{i=1}^{\infty}(i-1)\pi_0 = \frac{\rho\ z^*}{1-z^*}\ , \quad W = \frac{L_W}{\bar{\lambda}} = \frac{1}{\mu}\ \frac{z^*}{1-z^*}\ ,$$

and the mean number of customers in the queue right before arrivals is

$$\check{N} = \sum_{i=0}^{\infty} iP_i = \frac{z^*}{1-z^*}$$

Special G/M/1 queues

- *exponentially distributed inter-arrival time – M/M/1 queue:*

  $A^{\sim}(s) = \dfrac{\lambda}{s+\lambda}$

  $z^* = A^{\sim}(\mu - z^*\mu) = \dfrac{\lambda}{\mu - z^*\mu + \lambda}$

  and its valid (inside the unit disk) solution is $z^* = \dfrac{\lambda}{\mu} = \rho$. $z^* = 1$ is also a solution of the equation but it is not valid.

- *Erlang$(\lambda, 2)$ distributed inter-arrival time – E_2/M/1 queue:*

  $A^{\sim}(s) = \left(\dfrac{\lambda}{s+\lambda}\right)^2$

  $\rho = \dfrac{\overline{\lambda}}{\mu} = \dfrac{\lambda}{2\mu}, \quad z^* = 2\rho + \dfrac{1}{2} - \sqrt{2\rho + \dfrac{1}{4}}$

- *deterministic inter-arrival time – D/M/1 queue:*

  $A^{\sim}(s) = e^{-sD}$

  $\rho = 1/\mu D, \qquad z^* = A^{\sim}(\mu - z^*\mu) = e^{-\mu D(1-z^*)}$

- *hyper-exponentially distributed inter-arrival time – H_2/M/1 queue:*

  $A^{\sim}(s) = \dfrac{p_1\lambda_1}{s+\lambda_1} + \dfrac{p_2\lambda_2}{s+\lambda_2}$

  Assuming $p_1 = p_2 = 0.5$, *and* $\lambda_1 = 2\lambda_2 = \lambda = 1$, *we have* $\overline{\lambda} = \dfrac{2\lambda}{3}$, $\rho = \dfrac{\overline{\lambda}}{\mu} = \dfrac{2\lambda}{3\mu}$ *and*

  $z^* = \dfrac{9\rho}{8} + \dfrac{1}{2} - \sqrt{\dfrac{9\rho^2}{64} + \dfrac{1}{4}}$

Exercise 8.4. *Find the mean value of number of customers in the system and in the waiting queue in the M/G/1 system. Let us consider the cases of of M/M/1 and M/D/1 systems.*

Solution 8.4. *The mean value of number of customers is computed on page 236. For the mean value of number of waiting customers we have*

$$\sum_{k=1}^{\infty}(k-1)p_k = \sum_{k=1}^{\infty} kp_k - \sum_{k=1}^{\infty} p_k = \rho + \frac{\lambda^2\mathbf{E}(Y^2)}{2(1-\rho)} - \rho$$

$$= \frac{\lambda^2\mathbf{E}(Y^2)}{2(1-\rho)}.$$

In case of the $M/M/1$ *system the second moment of service time is* $\mathbf{E}(Y^2) = \frac{2}{\mu^2}$. *By using this value the main queue length is*

$$\rho + \frac{\lambda^2\mathbf{E}(Y^2)}{2(1-\rho)} = \frac{\lambda}{\mu} + \frac{\lambda^2 \cdot \frac{2}{\mu^2}}{2\left(1-\frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu-\lambda}.$$

The main number of waiting customers is

$$\sum_{k=1}^{\infty}(k-1)p_k = \sum_{k=1}^{\infty} kp_k - \sum_{k=1}^{\infty} p_k = \rho + \frac{\lambda^2\mathbf{E}(Y^2)}{2(1-\rho)} - \rho$$

$$= \frac{\lambda^2\mathbf{E}(Y^2)}{2(1-\rho)} = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}.$$

These values can be computed knowing that for the $M/M/1$ *system the stationary distribution is geometrical. The mean value of number of customers is*

$$\sum_{k=1}^{\infty} k\cdot(1-\rho)\rho^k = (1-\rho)\rho\frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho}.$$

The mean value of waiting customers

$$\sum_{k=1}^{\infty}(k-1)(1-\rho)\rho^k = (1-\rho)\rho\left(\sum_{k=1}^{\infty} k\rho^{k-1} - \sum_{k=1}^{\infty}\rho^{k-1}\right)$$

$$= (1-\rho)\rho\left[\frac{1}{(1-\rho)^2} - \frac{1}{1-\rho}\right] = \frac{\rho^2}{1-\rho}.$$

Let the service time be equal to T in the $M/D/1$ system, then $\rho = \lambda T$, $\mathbf{E}(Y^2) = T^2$. The mean value of number of customers in the system is

$$\rho + \frac{\lambda^2 \mathbf{E}(Y^2)}{2(1-\rho)} = \lambda T + \frac{\lambda^2 T^2}{2(1-\lambda T)}$$

$$= \frac{2\lambda T - \lambda^2 T^2}{2(1-\lambda T)} = \frac{\rho(2-\rho)}{2(1-\rho)},$$

the mean value of waiting customers

$$\frac{\lambda^2 \mathbf{E}(Y^2)}{2(1-\rho)} = \frac{\lambda^2 T^2}{2(1-\lambda T)} = \frac{\rho^2}{2(1-\rho)}.$$

Exercise 8.5. *By using the Pollaczek-Khinchin transform equation show that in the M/M/1 system the equilibrium distribution is geometrical.*

Solution 8.5. *Use the fact that for the distribution of service time*

$$b^{\sim}(s) = \int_0^\infty e^{-sx} \mu e^{-\mu x} dx = \frac{\mu}{s+\mu}.$$

Exercise 8.6. *Let us consider the M/G/1 system with bulk arrivals, an arriving group with probability g_i consists of i customers. Show that the generating function of number of customers entering during time t is $e^{-\lambda t(1-G(z))}$, where λ is the intensity of arrivals and $G(z) = \sum_{i=1}^{\infty} g_i z^i$.*

Solution 8.6. *Let $P_i(k)$ denote the probability of event that in i groups together appear k customers. For the generating function of entering customers we have*

$$e^{-\lambda t} + \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} P_i(k) z^k$$

$$= \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} \sum_{k=i}^{\infty} P_i(k) z^k$$

$$= \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} G^i(z)$$

$$= \sum_{i=0}^{\infty} \frac{[\lambda t G(z)]^i}{i!} e^{-\lambda t} = e^{-\lambda t[1-G(z)]}.$$

Exercise 8.7. *Show that in the M/G/1 system with bulk arrivals the generating function of number of customers arriving for the service time of a customer is $b^{\sim}(\lambda(1-G(z))$, where $b^{\sim}(s)$ is the Laplace-Stieltjes transform of distribution function of this service time.*

Solution 8.7. *Let $P_i(k)$ has the same meaning as in the previous exercise. We have*

$$
\begin{aligned}
& \int_0^\infty e^{-\lambda x} dB(x) + \int_0^\infty \sum_{i=1}^\infty \sum_{k=i}^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} P_i(k) z^k dB(x) \\
= & \int_0^\infty \sum_{i=0}^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} \sum_{k=i}^\infty P_i(k) z^k dB(x) \\
= & \sum_{i=0}^\infty \frac{(\lambda x)^i}{i!} e^{-\lambda x} G^i(z) dB(x) \\
= & \int_0^\infty \sum_{i=0}^\infty \frac{[\lambda x G(z)]^i}{i!} e^{-\lambda x} dB(x) = b^\sim(\lambda(1 - G(z))) = \sum_{j=0}^\infty c_j z^j.
\end{aligned}
$$

Chapter 9

Queueing systems with structured Markov chains

Exercise 9.1. *Define a MAP representation of the departure process of an M/M/1/2 queue with arrival rate λ and service rate μ.*

Solution 9.1.

$$\boldsymbol{D_0} = \begin{array}{|c|c|c|}\hline -\lambda & \lambda & \\ \hline & -\lambda-\mu & \lambda \\ \hline & & -\mu \\ \hline\end{array} \qquad \boldsymbol{D_1} = \begin{array}{|c|c|c|}\hline & & \\ \hline \mu & & \\ \hline & \mu & \\ \hline\end{array}$$

Exercise 9.2. *Define a MAP representation of the departure process of a MAP/M/1/1 queue with arrival MAP $(\hat{\boldsymbol{D}}_{\mathbf{0}}, \hat{\boldsymbol{D}}_{\mathbf{1}})$ and service rate μ.*

Solution 9.2.

$$\boldsymbol{D_0} = \begin{array}{|c|c|}\hline \hat{\boldsymbol{D}}_{\mathbf{0}} & \hat{\boldsymbol{D}}_{\mathbf{1}} \\ \hline \mu\boldsymbol{I} & \hat{\boldsymbol{D}}_{\mathbf{0}} + \hat{\boldsymbol{D}}_{\mathbf{1}} - \mu\boldsymbol{I} \\ \hline\end{array}, \qquad \boldsymbol{D_1} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \mu\boldsymbol{I} & 0 \\ \hline\end{array}.$$

Exercise 9.3. *Define a MAP representation of the customer loss process of a MAP/M/1/1 queue with arrival MAP $(\hat{\boldsymbol{D}}_{\mathbf{0}}, \hat{\boldsymbol{D}}_{\mathbf{1}})$ and service rate μ.*

Solution 9.3.

$$\boldsymbol{D_0} = \begin{array}{|c|c|}\hline \hat{\boldsymbol{D}}_{\mathbf{0}} & \hat{\boldsymbol{D}}_{\mathbf{1}} \\ \hline \mu\boldsymbol{I} & \hat{\boldsymbol{D}}_{\mathbf{0}} - \mu\boldsymbol{I} \\ \hline\end{array}, \qquad \boldsymbol{D_1} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & \hat{\boldsymbol{D}}_{\mathbf{1}} \\ \hline\end{array}.$$

Exercise 9.4. *Compute the generator of the CTMC which describes the number of customers and the phase of the arrival PH distribution in a $PH/M/1$ queue if the representation of the PH distributed inter-arrival time is (α, A), with $\alpha = (1,0)$ and $A = \begin{pmatrix} -\alpha & \alpha/2 \\ 0 & -\gamma \end{pmatrix}$ and the service rate is μ.*

Solution 9.4.

$$\begin{array}{|cc|cc|cc|cc|}\hline -\alpha & \alpha/2 & \alpha/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma & \gamma & 0 & 0 & 0 & 0 & 0 \\ \hline \mu & 0 & -\alpha-\mu & \alpha/2 & \alpha/2 & 0 & 0 & 0 \\ 0 & \mu & 0 & -\gamma-\mu & \gamma & 0 & 0 & 0 \\ \hline 0 & 0 & \mu & 0 & -\alpha-\mu & \alpha/2 & \alpha/2 & 0 \\ 0 & 0 & 0 & \mu & 0 & -\gamma-\mu & \gamma & 0 \\ \hline & & & & \ddots & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots & \ddots \\ \hline\end{array}.$$

Exercise 9.5. *Compute the generator of the CTMC which describes the number of customers and the phase of the service PH distribution in a $M/PH/1$ queue if the arrival rate is λ and the representation of the PH distributed service time is $(\beta, \boldsymbol{B})$, with $\beta = (1/3, 2/3)$ and* $\boldsymbol{B} = \begin{pmatrix} -\mu & \mu \\ 0 & -\gamma \end{pmatrix}$.

Solution 9.5. *Solution 1: If the idle state of the queue is represented with a single state of the Markov chain then the generator is*

$$
\left|\begin{array}{c|cc|cc|cc}
\hline
-\lambda & \lambda/3 & 2\lambda/3 & 0 & 0 & 0 & 0 \\
\hline
0 & -\lambda-\mu & \mu & \lambda & 0 & 0 & 0 \\
\gamma & 0 & -\lambda-\gamma & 0 & \lambda & 0 & 0 \\
\hline
0 & 0 & 0 & -\lambda-\mu & \mu & \lambda & 0 \\
0 & \gamma/3 & 2\gamma/3 & 0 & -\lambda-\gamma & 0 & \lambda \\
\hline
0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\
\hline
\end{array}\right| .
$$

Solution 2: If the idle state of the queue is represented with two states of the Markov chain then the generator is

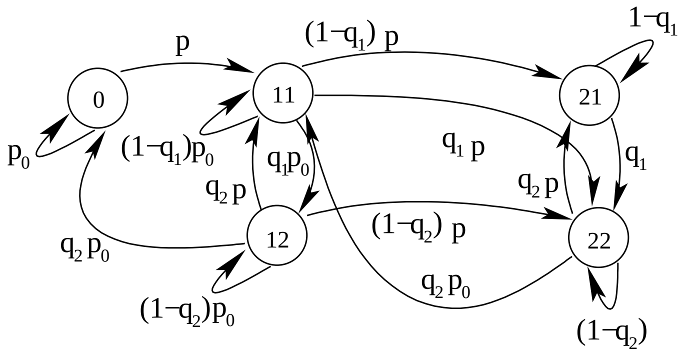
$$
\left|\begin{array}{cc|cc|cc|cc}
\hline
-\lambda & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda & 0 & \lambda & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -\lambda-\mu & \mu & \lambda & 0 & 0 & 0 \\
\gamma/3 & 2\gamma/3 & 0 & -\lambda-\gamma & 0 & \lambda & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -\lambda-\mu & \mu & \lambda & 0 \\
0 & 0 & \gamma/3 & 2\gamma/3 & 0 & -\lambda-\gamma & 0 & \lambda \\
\hline
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots \\
\hline
\end{array}\right| .
$$

Exercise 9.6. *The packet transmission is performed in two phases in a slotted time communication protocol. The first phase is the resource allocation and the second is the data transmission. The times of both phases are geometrically distributed with parameters q_1 and q_2. In every time slot one packet arrives with probability p (and no packet arrives with probability $1-p$). Compute the probability of packet loss if at most 2 packets can be in the system.*

Solution 9.6. *The service time distribution is indeed a discrete PH distribution with representation*

$$\beta = (1, 0) \text{ and } \boldsymbol{B} = \begin{pmatrix} 1-q_1 & q_1 \\ 0 & 1-q_2 \end{pmatrix} ,$$

and the queueing system is a discrete time M/PH/1/2 queue. The following DTMC characterize its behavior,



where $p_0 = 1 - p$ and state i, j indicates that there are i customers in the system and the service process of the customer in service is in phase j. The probability of packet loss is

$$p_{loss} = \frac{p_{2,1}p + p_{2,2}(1 - q_2)p}{p} = p_{2,1} + p_{2,2}(1 - q_2),$$

where $p_{i,j}$ denotes the stationary probability of state i, j.

Exercise 9.7. *Requests arrive to a computer according to a Poisson process with rate λ. The service of these requests requires first a processor operation for an exponentially distributed amount of time with parameter μ_1. After this processor operation the request leaves the system with probability p or requires a consecutive disk operation with probability $1 - p$. The time of the disk operation is exponentially distributed with parameter μ_2. After the disk operation the request requires a processor operation as it was a new one. There can be several loops of processor and disk operations. The processor is blocked during the disk operation and one request is handled at a time.*

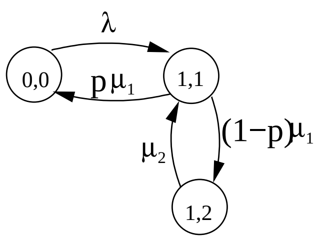
Compute the efficient utilization of the processor, and the request loss probability if there is no buffer in the system.

Compute the efficient utilization of the processor, and the system time of the requests if there is an infinite buffer in the system.

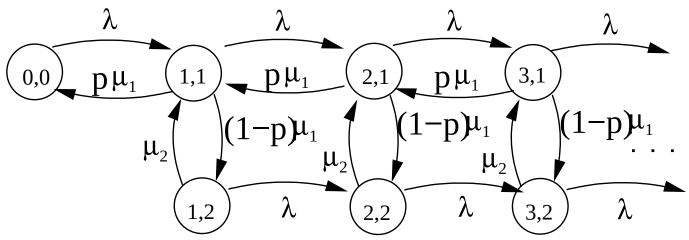
Solution 9.7. *Similar to the previous exercise the service time distribution is a continuous PH distribution with representation*

$$\beta = (1, 0) \text{ and } \boldsymbol{B} = \begin{pmatrix} -\mu_1 & \mu_1(1-p) \\ \mu_2 & -\mu_2 \end{pmatrix},$$

and the queueing system is a (continuous time) M/PH/1/1 queue if there is no buffer and an M/PH/1 queue if there is an infinite buffer. The related CTMCs are as follow.



M/PH/1/1 queue



M/PH/1 queue

State i, j indicates that there are i customers in the system and the service process of the customer in service is in phase j and $p_{i,j}$ denotes the stationary probability of state i, j.

In case of no buffer the effective utilization of the server and the loss probability are

$$\rho = p_{1,1} \text{ and } p_{loss} = p_{1,1} + p_{1,2},$$

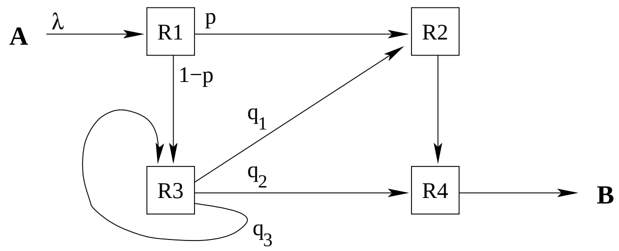
and in case of infinite buffer the effective utilization of the server and the loss probability are

$$\rho = \sum_{i=1}^{\infty} p_{i,1} \text{ and } p_{loss} = 0.$$

Chapter 10

Queueing networks

Exercise 10.1. *In the depicted queueing network the requests of input* A *are forwarded towards output* B *according to the following traffic routing probabilities* $p = 0.3, q_1 = 0.2, q_2 = 0.5, q_3 = 0.3$.



The requests from input A *arrive according to a Poisson process with rate* $\lambda = 50$. *The service times are exponentially distributed in nodes R1, R2, R3 with parameters* $\mu_1 = 90, \mu_2 = 35, \mu_3 = 100$, *respectively. The service time in R4 is composed of two phases. The first phase is exponentially distributed with parameter* $\mu_4 = 400$ *and the second phase is deterministic with* $D = 0.01$.

- *Compute the traffic load of the nodes.*
- *Compute the mean and the coefficient of variation of the service time at node R4.*
- *Compute the system time at each node.*
- *Compute* λ_{max} *at which the system is at the limit of stability.*

Solution 10.1.

- *The following traffic equations characterize the traffic load of the nodes.*

$$\lambda_1 = \lambda;\ \ \lambda_2 = p\lambda_1 + q_1\lambda_3;\ \ \lambda_3 = (1-p)\lambda_1 + q_3\lambda_3;\ \ \lambda_4 = \lambda_2 + q_2\lambda_3.$$

  With the given probabilities the solution of the traffic equations is

$$\lambda_1 = \lambda;\ \ \lambda_2 = \lambda/2;\ \ \lambda_3 = \lambda;\ \ \lambda_4 = \lambda.$$

- *The service time at node R4 is the sum of independent random variables,* $S_4 = X + D$, *where* X *is exponentially distributed and* D *is deterministic.*

$$\mathbf{E}(S_4) = \mathbf{E}(X+D) = \mathbf{E}(X) + \mathbf{E}(D) = \frac{1}{400} + \frac{1}{100} = \frac{1}{80}$$

$$\mathbf{CV}(S_4) = \frac{\mathbf{Var}(S_4)}{\mathbf{E}(S_4)^2} = \frac{\mathbf{Var}(X) + \mathbf{Var}(D)}{\mathbf{E}(X+D)^2} = \frac{(1/400)^2 + 0}{(1/400 + 1/100)^2} = \frac{1}{25}$$

- *The service time at node R1, R2, R3 is exponentially distributed with mean* $\mathbf{E}(S_1) = 1/\mu_1$, $\mathbf{E}(S_2) = 1/\mu_2$, $\mathbf{E}(S_3) = 1/\mu_3$, *respectively. Since the service time at node R4 is non-exponential we compute the system time based on the Pollaczek-Khinchin mean value formulae*

$$\mathbf{E}(T_i) = \mathbf{E}(S_i) + \frac{\rho_i}{1-\rho_i}\,\mathbf{E}(S_i)\,\frac{1+\mathbf{CV}(S_i)}{2}\ , \quad i = 1,2,3,4,$$

where $\rho_i = \lambda_i \mathbf{E}(S_i)$ *and the coefficient of variation of the exponential service time is* $\mathbf{CV}(S_1) = \mathbf{CV}(S_2) = \mathbf{CV}(S_3) = 1$.

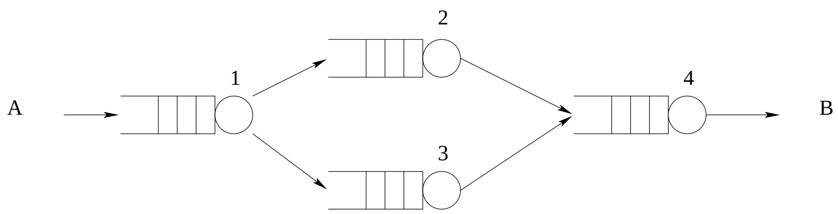
- *The utilization of the nodes as a function of* λ *are*

$$\rho_1 = \lambda_1 \mathbf{E}(S_1) = \frac{\lambda}{90}, \rho_2 = \lambda_2 \mathbf{E}(S_2) = \frac{\lambda/2}{35} = \frac{\lambda}{70},$$

$$\rho_3 = \lambda_3 \mathbf{E}(S_3) = \frac{\lambda}{100}, \rho_4 = \lambda_4 \mathbf{E}(S_4) = \frac{\lambda}{80}.$$

Consequently the limit of stability is $\lambda_{max} = 70$, *because node R2 gets instable at that load.*

Exercise 10.2. *In the depicted queueing network the requests of input* A *are forwarded towards output* B *according to the following traffic routing probabilities* $p_{12} = 0.3, p_{13} = 0.7$.



The requests from input A *arrive according to a Poisson process with rate* $\lambda = 50$. *In nodes R1, R2 and R3 there are single servers, infinite buffers and the service times are exponentially distributed with parameters* $\mu_1 = 80, \mu_2 = 45, \mu_3 = 50$, *respectively. There are two servers and two additional buffer at note R4. Both of servers can serve requests with exponentially distributed service time with parameter* $\mu_4 = 40$.

- *Characterize the nodes with the Kendall's notation.*
- *Compute the traffic load of the nodes.*
- *Compute the system time at each node.*
- *Compute the utilization of the servers at Node R4.*
- *Compute the packet loss probability.*

- *Compute the mean time of a request from* A *to* B.
- *Which node is the bottleneck of the system? Which node saturates first when* λ *increases?*

Solution 10.2.

- *There are M/M/1 queueing systems at Node R1, R2, R3, and an M/M/2/4 at Node R4.*
- $$\lambda_1 = \lambda;\ \lambda_2 = p_{12}\lambda_1;\ \lambda_3 = p_{13}\lambda_1;\ \lambda_4 = \lambda_2 + \lambda_3.$$

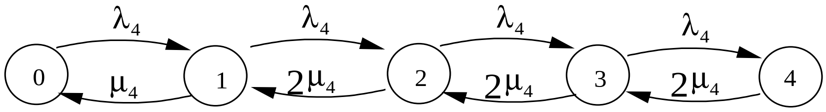
  With the given probabilities the solution of the traffic equations is

  $$\lambda_1 = \lambda;\ \lambda_2 = 0.3\lambda;\ \lambda_3 = 0.7\lambda;\ \lambda_4 = \lambda.$$

- *The system time at the M/M/1 type nodes are*

  $$\mathbf{E}(T_i) = \mathbf{E}(S_i) + \frac{\rho_i}{1-\rho_i}\mathbf{E}(S_i), \quad i = 1, 2, 3,$$

  where $\rho_i = \lambda_i \mathbf{E}(S_i) = \lambda_i/\mu_i$. *The systems time at the M/M/2/4 type node can be computed based on the stationary solution of the following CTMC (denoted as* p_i*).*

  The system time at Node R4 is

  $$\mathbf{E}(T_4) = L_4/\bar{\lambda},\ \textit{where}\ L_4 = \sum_{i=0}^{4} i p_i, \bar{\lambda} = \lambda(1-p_4).$$

- *The utilization of the servers at Node R4 is* $1 - p_0$.
- *There is no packet loss at Node R1, R2, R3. Packets are only lost at Node R4 with probability* p_4. *Due to the fact that all packets goes to Node R4, the overall packet loss probability is* p_4 *as well.*
- *A packet take the path through nodes R1, R2, R4 with probability* p_{12} *and through nodes R1, R3, R4 with probability* p_{13}, *consequently*

  $$\mathbf{E}(T) = p_{12}\left(\mathbf{E}(T_1) + \mathbf{E}(T_2) + \mathbf{E}(T_4)\right) + p_{13}\left(\mathbf{E}(T_1) + \mathbf{E}(T_3) + \mathbf{E}(T_4)\right).$$

- *Node R4 never saturates because it has finite buffer. The utilization of the other 3 nodes are*

  $$\rho_1 = \lambda_1\mathbf{E}(S_1) = \frac{\lambda}{80}, \rho_2 = \lambda_2\mathbf{E}(S_2) = \frac{0.3\lambda}{45} = \frac{\lambda}{150}, \rho_3 = \lambda_3\mathbf{E}(S_3) = \frac{0.7\lambda}{50} =\sim \frac{\lambda}{71.4}.$$

  Node R3 is the bottleneck which saturates first at around $\lambda = 71.4$.

Chapter 11

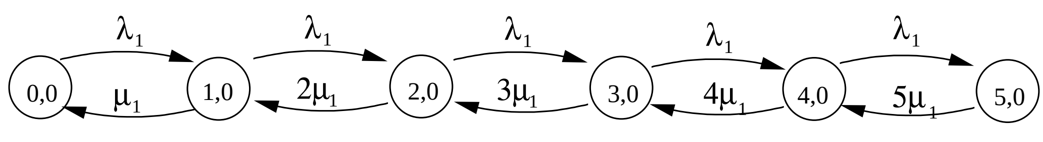
Applied queueing systems

Exercise 11.1. *A transmission link with capacity $C = 5MB/s$ serves two kinds of CBR connections. Type i connections arrive according to a Poisson process with rate λ_i and occupy c_i bandwidth of the link for an exponentially distributed amount of time with parameter μ_i ($i = 1, 2$), where $c_1 = 1MB$ and $c_2 = 2MB$.*

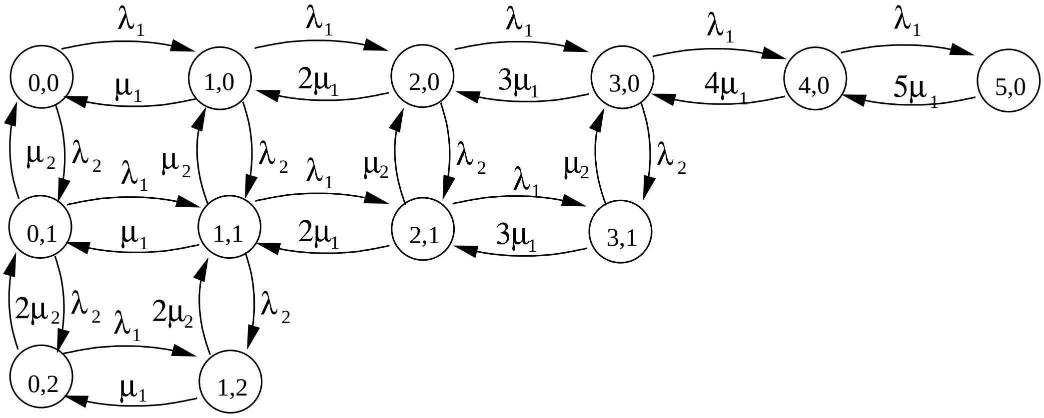
1. *Describe the system behavior with a CTMC and compute the loss probability of type 1 customers if $\lambda_2 = 0$.*
2. *Describe the system behavior with a CTMC when both λ_1 and λ_2 are positive and compute the loss probability of type 1, type 2 connections and the overall loss probability of connections.*
3. *Which loss probability is higher the one of type 1 or the one of type 2 connections? Why?*
4. *Compute the link utilization factor when both arrival intensities are positive.*
5. *Compute the link utilization of type 1 and type 2 connections.*

Solution 11.1.

- *When $\lambda_2 = 0$ we obtain an M/M/5/5 queueing system with the number of type 1 connections and the loss probability is $loss_1 = p_{5,0}$.*



- *When $\lambda_2 > 0$ we need to keep track the number of ongoing connections. The states of the Markov chain are identified by the number of ongoing type 1 and type 2 connections.*



Type 1 connections are lost in states $(1,2)$*,* $(3,1)$*,* $(5,0)$*, while type 2 connections are lost in states* $(1,2)$*,* $(3,1)$*,* $(5,0)$*,* $(0,2)$*,* $(2,1)$*,* $(4,0)$*. The related loss probabilities are*

$$loss_1 = p_{1,2} + p_{3,1} + p_{5,0}, loss_2 = p_{1,2} + p_{3,1} + p_{5,0} + p_{0,2} + p_{2,1} + p_{4,0}$$

and the overall loss probability is

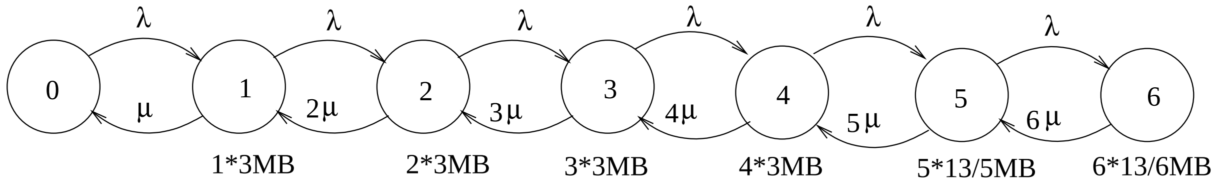
$$loss = p_{1,2} + p_{3,1} + p_{5,0} + \frac{\lambda_2}{\lambda_1 + \lambda_2}(p_{0,2} + p_{2,1} + p_{4,0}),$$

where $p_{i,j}$ *denotes the stationary probability of state* (i,j)*.*

- *The loss probability of type 2 connections is higher, because type 2 connections are lost in more states than type 1 connections.*
- *Link utilization is obtained by weighting utilized bandwidth with the associated state probabilities*
$$\rho = \frac{\sum_{i=0}^{5}\sum_{j=0}^{3} p_{i,j}(i \text{ 1MB} + j \text{ 2MB})}{\text{5MB}}.$$
- *The link utilization of type 1 and 2 connections are*
$$\rho_1 = \frac{\sum_{i=0}^{5}\sum_{j=0}^{3} p_{i,j}\ i \text{ 1MB}}{\text{5MB}} \quad , \quad \rho_2 = \frac{\sum_{i=0}^{5}\sum_{j=0}^{3} p_{i,j}\ j \text{ 2MB}}{\text{5MB}}.$$

Exercise 11.2. *There is a transmission link with capacity* $C = 13MB/s$*, which serves adaptive connections. The connections arrive according to a Poisson process with rate* λ *and their length is exponentially distributed with parameter* μ*. The minimal and maximal bandwidth of the adaptive connections are* $c_{min} = 2MB/s$ *and* $c_{max} = 3MB/s$*, respectively. Compute the average bandwidth of an adaptive connection in equilibrium.*

Solution 11.2. *The adaptive connection arrive and depart according to the number of customers in an M/M/6/6 queueing system, but the bandwidth of the active connection changes with the arrival and departure of other connections. The Markov chain indicates the number of active connections as well as the bandwidth utilization.*



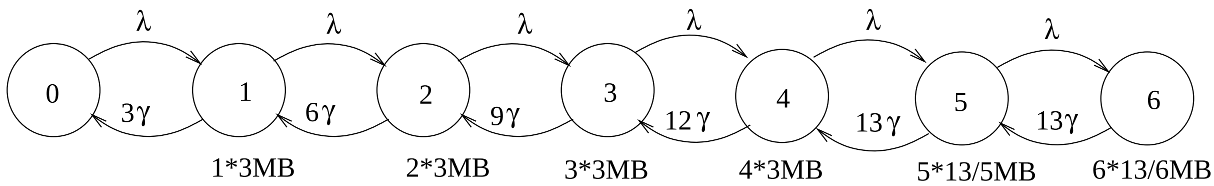
The mean bandwidth of an adaptive connection is

$$\mathbf{E}(S_A) = \sum_{i=1}^{4} p_i \; i \; 3\text{MB} + p_5 \; 5 \; 13/5\text{MB} + p_6 \; 6 \; 13/6\text{MB},$$

where p_i denotes the stationary probability of state i.

Exercise 11.3. *There is a transmission link with capacity $C = 13MB/s$, which serves elastic connections. The connections arrive according to a Poisson process with rate λ and during an elastic connection an exponentially distributed amount of data is transmitted with parameter γ. The minimal and maximal bandwidth of the elastic connections are $c_{min} = 2MB/s$ and $c_{max} = 3MB/s$, respectively. Compute the average bandwidth of an elastic connection in equilibrium. Compute the average time of an elastic connection in equilibrium.*

Solution 11.3. *The elastic connection arrive according to a Poisson process, but their departure rates depend on the bandwidth utilization. The bandwidth of the active connection also changes with the arrival and departure of other connections. The following Markov chain indicates the number of active connections as well as the bandwidth utilization.*

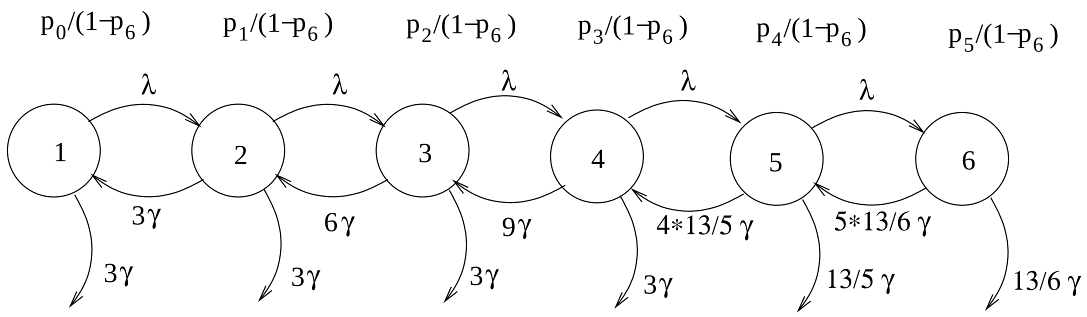


The mean bandwidth of an elastic connection is

$$\mathbf{E}(S_E) = \sum_{i=1}^{4} p_i \; i \; 3\text{MB} + p_5 \; 5 \; 13/5\text{MB} + p_6 \; 6 \; 13/6\text{MB},$$

where p_i denotes the stationary probability of state i. It seems similar to the bandwidth of the adaptive connection in the previous exercise, but the p_i probabilities differ in the two Markov chains.

The average time of an elastic connection is PH distributed with the following representation.



The number above the states indicate the associated initial probabilities an the downward arrows indicate the transitions to the absorbing state. This PH representation describes the sojourn of a randomly chosen tagged customer in the system. The arrival rate is independent of the number of connections. The probability that the tagged connection arrives to the system when there are i ($i = 0, \ldots, 5$) connections is proportional to p_i. The connections arrive when there are 6 active connections are lost and the probabilities are normalized for the states in which incoming connections are accepted. If an arrival occurs when there are i ($i = 0, \ldots, 5$) ongoing connections then after the arrival there will be $i+1$ active connections. The Markov chain of the PH distribution describes the behavior of the system up to the departure of the tagged connection. When there are i connections in the system $1/i$ portion of the utilized bandwidth is associated with the tagged connection with terminates with a rate proportional with its instantaneous bandwidth.

Exercise 11.4. *A transmission link with capacity $C = 3MB/s$ serves two kinds of elastic connections. Type 1 connections arrive according to a Poisson process with rate $\lambda_1 = 0.5$ 1/s, transmit an exponentially distributed amount of data with parameter $\gamma_1 = 4$ 1/MB. The minimal and maximal bandwidth of type 1 connections are $\check{c}_1 = 1MB/s$ and $\hat{c}_1 = 1MB/s$. Type 2 connections are characterized by $\lambda_2 = 0.1$ 1/s, $\gamma_1 = 2$ 1/MB, $\check{c}_2 = 1MB/s$ and $\hat{c}_2 = 2MB/s$.*

a. Describe the system behavior with a CTMC.

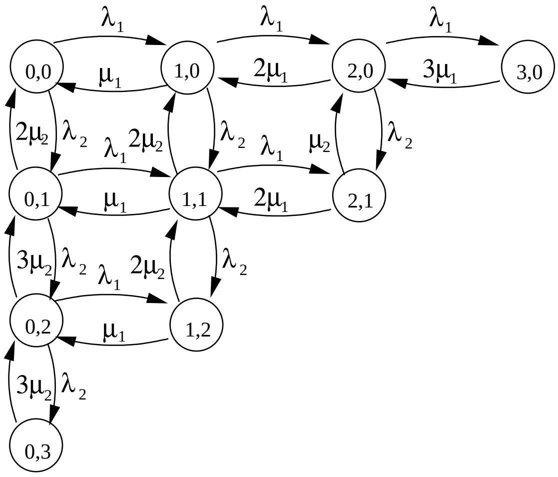
b. Compute the mean number of type 1 and type 2 connections.

c. Compute the mean channel utilization.

d. Compute the loss probability of type 1 and type 2 connections.

e. Compute the average bandwidth of type 2 connections.

Solution 11.4. *a. Describe the system behavior with a CTMC.*



b. Compute the mean number of type 1 and type 2 connections.

$$E(X_1) = \sum_i \sum_j i p_{ij} \ , \quad E(X_2) = \sum_i \sum_j j p_{ij}.$$

c. *Compute the mean channel utilization.*

$$\rho = 1 - p_{00} - 2/3p_{10} - 1/3(p_{20} + p_{01})$$

d. *Compute the loss probability of type 1 and type 2 connections.*

$$p_{loss} = p_{loss1} = p_{loss2} = p_{03} + p_{12} + p_{21} + p_{30}.$$

e. *Compute the average bandwidth of type 2 connections.*

$$\bar{C}_2 = \frac{\textit{number of connections and the bandwidth of the connections}}{\textit{number of connections}} =$$

$$= \frac{1 \cdot 2 \cdot p_{01} + 2 \cdot 1.5 \cdot p_{02} + 3 \cdot 1 \cdot p_{03} + 1 \cdot 2 \cdot p_{11} + 1 \cdot 1 \cdot p_{21} + 1 \cdot 1 \cdot p_{21}}{E(X_2)}.$$

Exercise 11.5. *A transmission link with capacity $C = 3MB/s$ serves two kinds of connections an elastic and an adaptive. Type 1 elastic connections arrive according to a Poisson process with rate λ_1 [1/s], transmit an exponentially distributed amount of data with parameter γ_1 [1/MB]. The minimal and maximal bandwidth of type 1 connections are $\check{c}_1 = 0.75MB/s$ and $\hat{c}_1 = 1.5MB/s$. Type 2 adaptive connections arrive according to a Poisson process with rate λ_2 [1/s] and stay in the system for an exponentially distributed amount of time with parameter μ_2 [1/s]. The minimal and maximal bandwidth of type 2 connections are $\check{c}_2 = 1MB/s$ and $\hat{c}_2 = 2MB/s$.*

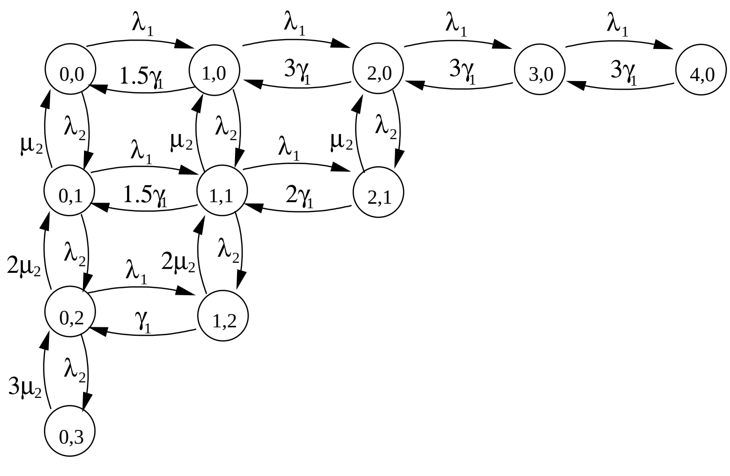
a. *Describe the system behavior with a CTMC.*

b. *Compute the loss probability of type 1 and type 2 connections.*

c. *Compute the average bandwidth of type 1 and type 2 connections.*

d. *Compute the mean number of type 1 and type 2 connections on the link.*

Solution 11.5. a. *Describe the system behavior with a CTMC.*



b. *Compute the loss probability of type 1 and type 2 connections.*

$$p_{loss1} = p_{40} + p_{21} + p_{12} + p_{03}, \qquad p_{loss2} = p_{40} + p_{30} + p_{21} + p_{12} + p_{03},$$

$$p_{loss} = \frac{(\lambda_1 + \lambda_2)(p_{40} + p_{21} + p_{12} + p_{03}) + \lambda_2 p_{40}}{\lambda_1 + \lambda_2},$$

c. Compute the average bandwidth of type 1 and type 2 connections.

$$\bar{c}_1 = 1p_{12} + 1.5(p_{10} + p_{11}) + 3(p_{20} + p_{30} + p_{40}),$$
$$\bar{c}_2 = 1p_{21} + 1.5p_{11} + 2(p_{12} + p_{01}) + 3(p_{02} + p_{03}),$$

d. Compute the mean number of type 1 and type 2 connections on the link.

$$E(X_1) = \sum_i \sum_j i p_{ij} = (p_{10} + p_{11} + p_{12}) + 2(p_{20} + p_{21}) + 3p_{30} + 4p_{40},$$

$$E(X_2) = \sum_i \sum_j j p_{ij} = (p_{01} + p_{11} + p_{21}) + 2(p_{02} + p_{12}) + 3p_{03},$$

Exercise 11.6. *Compute the mean value of waiting time in the cyclic-waiting system.*

Solution 11.6. *The generating function of waiting time (measured in cycles) is*

$$\begin{aligned} P(z) &= \left[1 - \frac{\lambda}{\mu} \frac{1 - e^{-\lambda T}}{e^{-\lambda T}(1 - e^{-\mu T})}\right] \times \\ &\times \frac{\dfrac{\mu}{\lambda + \mu} - \dfrac{\mu(1 - e^{-\lambda T})}{\lambda + \mu} \dfrac{z}{z - e^{-\lambda T}}}{1 - \dfrac{\lambda(1 - e^{-\mu T})}{\lambda + \mu} \dfrac{z}{1 - z e^{-\mu T}} - \dfrac{\mu(1 - e^{-\lambda T})}{\lambda + \mu} \dfrac{z}{z - e^{-\lambda T}}}. \end{aligned}$$

Introducing the notations

$$\begin{aligned} A(z) &= \frac{\mu}{\lambda + \mu} - \frac{\mu(1 - e^{-\lambda T})}{\lambda + \mu} \frac{z}{z - e^{-\lambda T}}, \\ B(z) &= 1 - \frac{\lambda(1 - e^{-\mu T})}{\lambda + \mu} \frac{z}{1 - z e^{-\mu T}} - \frac{\mu(1 - e^{-\lambda T})}{\lambda + \mu} \frac{z}{z - e^{-\lambda T}}, \end{aligned}$$

the mean value of number of cycles is

$$\lim_{z \to 1} \left[1 - \frac{\lambda}{\mu} \frac{1 - e^{-\lambda T}}{e^{-\lambda T}(1 - e^{-\mu T})}\right] \frac{A'B - AB'}{2B^2}.$$

By using twice l'Hospital's rule and taking into account

$$\lim_{z \to 1} \frac{A'B - AB'}{2B^2} = \lim_{z \to 1} \frac{A''B' - A'B''}{2B'^2},$$

$$\begin{aligned} A'(1) &= \frac{\mu e^{-\lambda T}}{(\lambda + \mu)(1 - e^{-\lambda T})}, \\ A''(1) &= -\frac{2\mu e^{-\lambda T}}{(\lambda + \mu)(1 - e^{-\lambda T})^2}, \\ B'(1) &= -\frac{\lambda}{(\lambda + \mu)(1 - e^{-\mu T})} + \frac{\mu e^{-\lambda T}}{(\lambda + \mu)(1 - e^{-\lambda T})}, \\ B''(1) &= -\frac{2\lambda e^{-\mu T}}{(\lambda + \mu)(1 - e^{-\mu T})^2} - \frac{2\mu e^{-\lambda T}}{(\lambda + \mu)(1 - e^{-\lambda T})^2}, \end{aligned}$$

we finally obtain

$$P'(1) = \frac{\lambda[1 - e^{-(\lambda + \mu)T}]}{(1 - e^{-\mu T})[\mu e^{-\lambda T}(1 - e^{-\mu T}) - \lambda(1 - e^{-\lambda T})]}.$$

Exercise 11.7. *Let us consider our cyclic-waiting system in case of discrete time. Divide the cycle time T into n equal parts and suppose that for an interval T/n a new customer enters with probability r (there is no entry with probability $1-r$), the service in process for such an interval is continued with probability q and completed with probability $1-q$ (i.e. the service time has geometrical distribution). The service may be started at the moment of arrival or at moments differing from it by the multiples of T.*

(a) Show that the number of customers in the system at moments t_k-0 constitute a Markov chain, find its transition probabilities.

(b) Find the generating function of number of customers in the system in equilibrium and the stability condition.

Solution 11.7. *(a) Similarly to the continuous time case we will consider two possibilities: at the beginning of service there is one customer in the system or there are at least two customers in the system.*

The case of one customer. *We begin the service of the customer and after a certain time the second one arrives. Let u be the service time and the second customer appears at time v after the beginning of service. The remaining service time is ℓ ($\ell = 0, 1, 2, \ldots$) with probability*

$$P\{u-v=\ell\} = \sum_{k=\ell+1}^{\infty} q^{k-1}(1-q)(1-r)^{k-\ell-1}r = \frac{r(1-q)q^{\ell}}{1-q(1-r)}.$$

We find the time from the entry of second customer till the beginning of its service. It is 0 if the customer arrives during the last time slice of the first customer's service, n if $u-v$ belongs to the interval $[1,n]$, $2n$ if $u-v\in[n+1,2n]$, and, generally, in if $u-v\in[(i-1)n+1,in]$. The corresponding probabilities are

$$\sum_{\ell=(i-1)n+1}^{in} \frac{r(1-q)q^{\ell}}{1-q(1-r)} = \frac{rq}{1-q(1-r)}\left(q^{(i-1)n}-q^{in}\right).$$

The generating function of number of customers arriving for a time slice is $1-r+rz$, so the generating function of customers entering for the waiting time is

$$\sum_{i=1}^{\infty} \frac{rq(1-q^n)}{1-q(1-r)} q^{(i-1)n}(1-r+rz)^{in} = \frac{rq(1-r+rz)^n(1-q^n)}{[1-q(1-r)][1-q^n(1-r+rz)^n]}.$$

Taking into account that the first customer obligatorily arrives and the waiting time may be equal to zero for the generating function of entering customers we obtain

$$A(z) = \sum_{i=0}^{\infty} a_i z^i = \frac{(1-r)(1-q)}{1-q(1-r)} + z\frac{r(1-q)}{1-q(1-r)} + z\frac{rq(1-r+rz)^n(1-q^n)}{[1-q(1-r)][1-q^n(1-r+rz)^n]},$$

where $\dfrac{(1-r)(1-q)}{1-q(1-r)}$ is the probability of event for the service of first customer no further customers arrive.

The case of at least two customers. *At the beginning of service of first customer the second customer is present, too. Let* $x = u - \left[\frac{u-1}{n}\right] n$ *([x] denote the integer part of x), and let y be the mod T interarrival time* $(1 \le y \le n)$*. The time elapsed between the starting moments of two successive customers is*

$$\left[\frac{u-1}{n}\right] n + y \quad \text{if} \quad x \le y \quad \text{and} \quad \left(\left[\frac{u-1}{n}\right] + 1\right) n + y \quad \text{if} \quad x > y.$$

Let us fix y and consider the cycle $[in+1, (i+1)n]$*. If the service of first customer ends till y (including y), then the time till the beginning of service of second customer is* $in + y$ *and the probability of this event is*

$$\sum_{j=in+1}^{in+y} q^{j-1}(1-q) = q^{in} - q^{in+y},$$

in case $x > y$ *the time is* $(i+1)n + y$ *and the probability is*

$$\sum_{j=in+y+1}^{(i+1)n} q^{j-1}(1-q) = q^{in+y} - q^{(i+1)n}.$$

i changes from 0 to ∞ *(the summation is extended for all possible values of service time), for fixed y the generating functions of entering customers in the two cases will be*

$$\sum_{i=0}^{\infty} [q^{in} - q^{in+y}](1-r+rz)^{in+y} = \frac{(1-r+rz)^y}{1-q^n(1-r+rz)^n} - \frac{(1-r+rz)^y q^y}{1-q^n(1-r+rz)^n},$$

$$\sum_{i=0}^{\infty} [q^{in+y} - q^{in+n}](1-r+rz)^{in+n+y} = \frac{q^y(1-r+rz)^{n+y}}{1-q^n(1-r+rz)^n} - \frac{q^n(1-r+rz)^{n+y}}{1-q^n(1-r+rz)^n}.$$

y has truncated geometrical distribution, it takes on the value k $(k = 0, 1, 2, \dots, n)$ *with probability* $\frac{(1-r)^k r}{1-(1-r)^n}$.

Consequently, the generating function of transition probabilities is

$$B(z) = \sum_{k=1}^{n} \frac{(1-r)^{k-1} r}{1-(1-r)^n} \frac{1}{1-q^n(1-r+rz)^n}$$

$$\times [(1-r+rz)^k - (1-r+rz)^k q^k + (1-r+rz)^{n+k} q^k - (1-r+rz)^{n+k} q^n]$$

$$= \frac{1-(1-r)^n(1-r+rz)^n}{1-(1-r)(1-r+rz)} \frac{r(1-r+rz)}{1-(1-r)^n}$$

$$+ \frac{1-q^n(1-r)^n(1-r+rz)^n}{1-q(1-r)(1-r+rz)} \frac{rq(1-r+rz)[(1-r+rz)^n - 1]}{[1-(1-r)^n][1-q^n(1-r+rz)^n]}.$$

We have seen that, as in the continuous case, the length of interval between two successive starting moments is determined by the service time of first customer and the interarrival time of first and second customers, so they are independent random variables. By using the *memoryless property of geometrical distribution, we obtain the number of customers in the system at moments just before the beginning of services constitute a Markov chain.*

(b) The system is considered at moments just before starting the services of customers. Let us denote the ergodic distribution by p_i $(i = 0, 1, 2, \ldots)$ and introduce the generating function by $P(z) = \sum_{i=0}^{\infty} p_i z^i$. For p_i we have the system of equations

$$p_j = p_0 a_j + p_1 a_j + \sum_{i=2}^{j+1} p_i b_{j-i+1},$$

from which

$$\sum_{j=0}^{\infty} p_j z^j = p_0 A(z) + p_1 A(z) + \sum_{j=0}^{\infty} \sum_{i=2}^{j+1} p_i b_{j-i+1} z^j$$

$$= \frac{1}{z} P(z) B(z) - \frac{1}{z} p_0 B(z) + p_0 A(z) + p_1 A(z) - p_1 B(z),$$

or

$$P(z) = \frac{p_0[zA(z) - B(z)] + p_1 z[A(z) - B(z)]}{z - B(z)}.$$

Since

$$p_0 = p_0 a_0 + p_1 a_0,$$

we have

$$p_1 = \frac{1 - a_0}{a_0} p_0 = \frac{r}{(1-r)(1-q)} p_0.$$

We find p_0 from the condition $P(1) = 1$

$$p_0 = \frac{1 - B'(1)}{1 - B'(1) + A'(1) + \frac{r}{(1-r)(1-q)}[A'(1) - B'(1)]}.$$

The chain is irreducible, so $p_0 > 0$.

Using the values

$$\begin{aligned} A'(1) &= \frac{r}{1 - q(1-r)} + \frac{nr^2 q}{[1 - q(1-r)](1 - q^n)}, \\ B'(1) &= 1 - \frac{nr(1-r)^n}{1 - (1-r)^n} + \frac{nr^2 q[1 - q^n(1-r)^n]}{(1-q^n)[1 - (1-r)^n][1 - q(1-r)]}, \end{aligned}$$

we obtain

$$\left(1 + \frac{r}{(1-r)(1-q)}\right) A'(1) - \frac{r}{(1-r)(1-q)} B'(1)$$

$$= \frac{nr^2 q}{(1-q^n)[1 - q(1-r)]} + \frac{nr^2(1-r)^n}{(1-r)[1 - (1-r)^n][1 - q(1-r)]} > 0,$$

so the condition $1 - B'(1) > 0$ must be fulfilled. This leads to the expression

$$\frac{nr(1-r)^n}{1 - (1-r)^n} - \frac{nr^2 q[1 - q^n(1-r)^n]}{(1-q^n)[1 - (1-r)^n][1 - q(1-r)]} > 0.$$

From it we obtain the stability condition

$$\frac{rq}{1 - q^n} \frac{1 - q^n(1-r)^n}{1 - q(1-r)} < (1-r)^n.$$

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