

Approximating the Riemann Zeta-Function by Strongly Recurrent Functions

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Abstract Bhaskar Bagchi has shown that the Riemann hypothesis holds if and only if the Riemann zeta-function $\zeta(z)$ is strongly recurrent in the strip $1/2 < \Re z < 1$. In this note we show that $\zeta(z)$ can be approximated by strongly recurrent functions sharing important properties with $\zeta(z)$.

Keywords Riemann hypothesis · Strong recurrence

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1 Introduction

In 1982, Bagchi ([1, 2]) showed a surprising equivalence between the Riemann hypothesis and a statement in topological dynamics, a subject which has its origins in the motion of particles. One is reminded of the amazing similarity between quantum dynamical systems and zeros of the Riemann zeta-function discovered by the chance encounter of Freeman Dyson and Hugh Montgomery in 1972.

For $-\infty \leq \alpha < \beta \leq +\infty$, we denote by $S = S(\alpha, \beta)$ the strip $\alpha < \Re z < \beta$ and by $\mathcal{O}(S)$ the set of functions holomorphic in S . We denote by $meas(E)$ the Lebesgue measure of a Borel subset E of \mathbb{R} . For a Borel set $E \subset \mathbb{R}$, we denote by $\overline{d}_{\mathbb{R}}(E)$ and $\underline{d}_{\mathbb{R}}(E)$ respectively the upper and lower densities of E in \mathbb{R} , defined as follows.

$$\overline{d}_{\mathbb{R}}(E) := \limsup_{T \rightarrow +\infty} \frac{meas(E \cap [-T, T])}{2T}$$

$$\underline{d}_{\mathbb{R}}(E) := \liminf_{T \rightarrow +\infty} \frac{meas(E \cap [-T, T])}{2T}.$$

A function $f \in \mathcal{O}(S)$ is said to be *strongly recurrent* if, for each compact set $K \subset S$, and for each $\epsilon > 0$, the set of $t \in \mathbb{R}$ such that $\max_K |f(z) - f(z + it)| < \epsilon$ is of positive upper density.

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Theorem 1 (Bagchi) *The Riemann hypothesis holds if and only if the Riemann zeta-function is strongly recurrent in the strip $1/2 < \Re z < 1$.*

We shall also consider a discrete form of strong recurrence. If E is a finite subset of \mathbb{Z} , we denote by $\#E$ the number of elements of E . For a subset E of \mathbb{Z} , we denote by $\overline{d}_{\mathbb{Z}}(E)$ and $\underline{d}_{\mathbb{Z}}(E)$ respectively the upper and lower densities of E in \mathbb{Z} , defined as follows.

$$\overline{d}_{\mathbb{Z}}(E) := \limsup_{N \rightarrow +\infty} \frac{\#(E \cap \{-N, \dots, N\})}{2N + 1}$$

$$\underline{d}_{\mathbb{Z}}(E) := \liminf_{N \rightarrow +\infty} \frac{\#(E \cap \{-N, \dots, N\})}{2N + 1}.$$

For $\Delta \in \mathbb{R}$, $\Delta \neq 0$ we say that a function $f \in \mathcal{O}(S)$ is *strongly recurrent modulo Δ* , if for each compact set $K \subset S$, and for each $\epsilon > 0$, the set of $k \in \mathbb{Z}$ such that $\max_K |f(z) - f(z + i(k\Delta))| < \epsilon$ is of positive upper density in \mathbb{Z} .

The following particular case of a theorem of Walter H. Gottschalk and Gustav A. Hedlund [8] allows us to pass between continuous and discrete dynamics.

Theorem 2 (Inheritance Theorem) *Let $f \in \mathcal{O}(S)$ and $\Delta \in \mathbb{R}$, $\Delta \neq 0$. Then, f is strongly recurrent if and only if f is strongly recurrent modulo Δ .*

Our main results are the following.

Theorem 3 *For each $\Delta \in \mathbb{R}$ different from 0, there exists a sequence of functions φ_n meromorphic on \mathbb{C} , each of which is strongly recurrent in $1/2 < \Re z < 1$ modulo Δ and hence strongly recurrent. Moreover:*

- (1) $\varphi_n \rightarrow \zeta$ uniformly on compact subsets of \mathbb{C} ;
- (2) φ_n has only a simple pole at $z = 1$ with residue 1;
- (3) $\varphi_n(x) \in \mathbb{R}$, $\forall x \in \mathbb{R} \setminus \{1\}$.

Theorem 4 *Let S be the fundamental strip $0 < \Re z < 1$. There is a function $h \in \mathcal{O}(S)$ which is strongly recurrent and which satisfies the functional equation of the Riemann zeta-function.*

The functional equation for the Riemann zeta-function is the following:

$$\zeta(z)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right) = \zeta(1-z)\pi^{-(1-z)/2}\Gamma\left(\frac{1-z}{2}\right)$$

and we say that a function h satisfies the functional equation for the Riemann zeta-function if the previous equation is satisfied, when ζ is replaced by h . Regarding this functional equation, we recall the famous theorem of Hans L. Hamburger [10], which states that the only function which satisfies this functional equation and has the same general character as ζ is ζ itself.

Theorem 5 (Hamburger) *If f is a Dirichlet series convergent in $\Re z > 1$ with a meromorphic continuation to \mathbb{C} as a function of finite order with only finitely many poles and, if f satisfies the functional equation for the Riemann zeta-function ζ , then $f = \zeta$.*

Theorem 3 was presented at the 2011 Winter Meeting of the Canadian Mathematical Society, in the session on Composition Operators, organized by Javad Mashreghi and Nina Zorboska. We shall give the proofs after introducing some preparatory material.

2 Symmetric Function Theory

Let $\overline{\mathbb{C}}$ denote the closed complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For a subset $E \subset \overline{\mathbb{C}}$, denote $\overline{E} = \{z : \bar{z} \in E\}$ and $\infty = \infty$. Let us say that E is *real-symmetric* if $\overline{E} = E$. For a function f defined on a set E , we denote by \bar{f} the function defined on \overline{E} by the formula $\bar{f}(z) = \overline{f(\bar{z})}$. We shall say that a function f defined on a real-symmetric set E is *real-symmetric* if $\bar{f} = f$. A meromorphic function on a real-symmetric domain meeting the real-axis is real-symmetric if and only if $f(z) \in \mathbb{R} \cup \{\infty\}$ for $z \in \mathbb{R} \cup \{\infty\}$. Some theorems in function theory remain true in the real-symmetric category.

A function is said to be meromorphic (holomorphic) on a set E if it is meromorphic (holomorphic) in an open neighborhood of E . We denote by $\mathcal{M}(E)$ and $\mathcal{O}(E)$ the family of meromorphic respectively holomorphic functions on E . If E is real-symmetric we define $\mathcal{M}_{\mathbb{R}}(E)$ to be the family of functions f meromorphic on E and real-symmetric. We denote by $\mathcal{O}_{\mathbb{R}}(E)$ the subfamily of functions in $\mathcal{M}_{\mathbb{R}}(E)$ which are holomorphic on E .

Theorem 6 (Symmetric Mittag-Leffler) *Let P be a discrete subset of \mathbb{C} with $\overline{P} = P$. For each $p \in P$, let Q_p be a non-constant polynomial and suppose $\overline{Q_p} = Q_{\bar{p}}$. Then, there is a function $h \in \mathcal{M}_{\mathbb{R}}(\mathbb{C})$ whose poles are precisely the points of P and with principal parts $Q_p(1/(z - p))$, $p \in P$.*

Proof Let h_+ be a meromorphic function on \mathbb{C} whose poles are precisely the points $p \in P$ with $\Im p > 0$ and with the prescribed principal parts. Set $h_- = \bar{h}_+$. Let ϕ be a meromorphic function on \mathbb{C} whose poles are precisely the real points of P and with the prescribed principal parts. Set $h_o = (\phi + \bar{\phi})/2$. Then, the function $h = h_+ + h_- + h_o$ has the required properties. \square

For a closed set E , we denote by $A(E)$, as usual, the family of functions continuous on E and holomorphic on the interior E° . We shall say that a closed set $E \subset \mathbb{C}$ is a *set of uniform approximation* if, for each $f \in A(E)$, and each $\epsilon > 0$, there is an entire function g such that $|f(z) - g(z)| < \epsilon$, for each $z \in E$. The following result is due to Norair U. Arakelian (see [6]).

Theorem 7 *A closed set $E \subset \mathbb{C}$ is a set of uniform approximation if and only if $\overline{\mathbb{C}} \setminus E$ is connected and locally connected.*

We may also state a real-symmetric theorem on uniform approximation.

Theorem 8 (Symmetric Approximation) *If E is a set of uniform approximation, E is real-symmetric and $f \in \mathcal{M}_{\mathbb{R}}(E)$, then, for each $\epsilon > 0$, there is a function $g \in \mathcal{M}_{\mathbb{R}}(\mathbb{C})$, whose poles in \mathbb{C} are the same as those of f on E and with same principal parts, such that*

$$|f(z) - g(z)| < \epsilon \exp(-|z|^{1/4}), \quad \text{for all } z \in E.$$

Proof By Theorem 6, there is an $h \in \mathcal{M}_{\mathbb{R}}(\mathbb{C})$ with $f - h \in \mathcal{O}_{\mathbb{R}}(E)$. By another theorem of Arakelian (see [6]), there is a $\Phi \in \mathcal{O}(\mathbb{C})$ which approximates $f - h$ as required. Then, $\phi = (\Phi + \overline{\Phi})/2$ also approximates $f - h$. Set $g = \phi + h$. Then, g has the required properties. \square

We shall require a stronger approximation than uniform approximation. A closed set $E \subset \mathbb{C}$ is a *set of tangential approximation* if, for each $f \in A(E)$, and each positive continuous function ϵ , there is an entire function g such that $|f(z) - g(z)| < \epsilon(z)$, for each $z \in E$. Of course, a set of tangential approximation is *a fortiori* a set of uniform approximation. Let us say that a family E_{α} of subsets of \mathbb{C} has no *long islands* if, for each compact $K \subset \mathbb{C}$ there is a (larger) compact set Q such that each E_{α} which intersects K is contained in Q . The following theorem gives a condition which characterizes sets of tangential approximation. I discovered this condition and proved the necessity. The sufficiency was established by Ashot H. Nersisyan (see [6]).

Theorem 9 *A set $E \subset \mathbb{C}$ of uniform approximation is a set of tangential approximation if and only if the family of components of the interior has no long islands.*

Just as for uniform approximation, there is a real-symmetric version of this theorem. Let us say that a real-symmetric set E of uniform approximation is a set of real-symmetric tangential approximation if for each real-symmetric $f \in A(E)$ and each real-symmetric positive continuous function ϵ , there is a real-symmetric entire function g such that $|f(z) - g(z)| < \epsilon(z)$, for each $z \in E$. Just as for uniform approximation, and with a similar proof, we have the following symmetric tangential approximation theorem.

Theorem 10 *A real-symmetric set $E \subset \mathbb{C}$ of uniform approximation is a set of real-symmetric tangential approximation if and only if the family of components of the interior has no long islands.*

3 Frequent Hypercyclicity

Let $T : X \rightarrow X$ be an operator from a linear space X into itself. The (forward) orbit of a vector $x \in X$ under the action of T is the set of vectors $O(x) = \{Tx, T^2x, \dots\}$, where $T^n x$ is defined inductively as $T(T^{n-1}x)$. A vector $x \in X$ is said to be a cyclic vector for the operator T if the subspace generated by the orbit $O(x)$ is dense in X and the operator T is said to be a cyclic operator if it has a cyclic vector. A vector $x \in X$ is said to be hypercyclic for T if the orbit itself $O(x)$ is dense in X . We shall say that x is hypercyclic for T in a subset $Y \subset X$ if for each $y \in Y$, there is a sequence $\{n_k\}$ in \mathbb{N} such that $T^{n_k}x \rightarrow y$. For an excellent overview of hypercyclicity, see the survey [9] by Carl-Goswin Grosse-Erdmann.

For $a \in \mathbb{C}$, let $\varphi_a : \mathbb{C} \rightarrow \mathbb{C}$ denote the translation $\varphi_a(z) = z + a$ and let C_a denote the composition operator on the space of complex-valued functions on \mathbb{C} , defined as $C_a f(z) = (f \circ \varphi_a)(z) = f(z + a)$. George D. Birkhoff showed that, for each $a \neq 0$, the composition operator C_a is hypercyclic on the space of entire functions. That is, there exists a (hypercyclic) entire function f . This means that the translates of f are dense in the space of all entire functions. More precisely, for each entire function g , there is a sequence of natural numbers $\{n_k\}$ such that $f(z + n_k a) \rightarrow g(z)$ uniformly on compact. Such a hypercyclic function f is also said to be a universal function.

My advisor, Wladimir Seidel, and Joseph L. Walsh [15] established an analog of Birkhoff's theorem in the disc, replacing translation by non-euclidian translation. There is no difficulty in extending the results of Birkhoff, Seidel and Walsh to several complex variables. Maurice Heins [11] showed the existence of a Blaschke product universal in the unit ball of $H^\infty(\mathbb{D})$, where \mathbb{D} is the unit disc. Pak-Soong Chee [5] showed the existence of a function universal in the unit ball of $H^\infty(\mathbb{B}^n)$, where \mathbb{B}^n is the unit ball in \mathbb{C}^n . XIAO Jie and I [7] showed the existence of such a universal function in $H^\infty(\mathbb{B}^n)$ which is inner.

It turns out (see [9]) that hypercyclicity is a generic phenomenon. For example, most entire functions are hypercyclic (universal). But no explicit example is known! The only known universal function in the sense of Birkhoff is the Riemann zeta-function $\zeta(s)$ (and some closely related zeta-functions). Of course, $\zeta(s)$ is not entire, but it is as close to being entire as possible. It has only one simple pole and that pole has residue 1. Let S be the strip $1/2 < \Re s < 1$, $\mathcal{O}(S)$ the set of functions holomorphic in S and $\mathcal{O}_o(S)$ the set of zero-free functions in $\mathcal{O}(S)$. The remarkable Universality Theorem of Sergei Mikhailovich Voronin (as extended by Steven M. Gonek and Bagchi) states that for any real number a , different from zero, there exists a sequence $\{t_k\}$ of real numbers such that the sequence of translates $\zeta(s + it_k a)$ comes arbitrarily close to each function in $\mathcal{O}_o(S)$. In fact, one can choose the sequence $\{t_k\}$ to be natural numbers $\{n_k\}$. Thus, the Riemann zeta-function $\zeta(s)$ is hypercyclic for the composition operator C_{ia} on the space $\mathcal{O}_o(S)$. That is, for each zero-free function g holomorphic in the strip S , there is a sequence $\{n_k\}$ of natural numbers, such that $\zeta(s + in_k a) \rightarrow g(s)$.

Frédéric Bayart and Sophie Grivaux [3] introduced the notion of *frequent* hypercyclicity [3]. For a subset E of \mathbb{N} , we denote by $\overline{d}_{\mathbb{N}}(E)$ and $\underline{d}_{\mathbb{N}}(E)$ respectively the

upper and lower densities of E in \mathbb{N} , defined as follows.

$$\overline{d}_{\mathbb{N}}(E) := \limsup_{N \rightarrow +\infty} \frac{\#(E \cap \{1, \dots, N\})}{N}$$

$$\underline{d}_{\mathbb{N}}(E) := \liminf_{N \rightarrow +\infty} \frac{\#(E \cap \{1, \dots, N\})}{N}.$$

A vector x is frequently hypercyclic for an operator T on a space X if, for each open set U in X , the set of $n \in \mathbb{N}$ for which $T^n x \in U$ has positive lower density in \mathbb{N} . If such a frequently hypercyclic vector exists for T , then the operator T is said to be frequently hypercyclic. Bayart and Grivaux [3] gave a criterion for frequent hypercyclicity. In contrast to hypercyclicity, frequent hypercyclicity is not a generic phenomenon.

The following lemma is stated in [3].

Lemma 1 *If there is a frequent hypercyclic vector in X for T , then the set of frequent hypercyclic vectors for T is dense in X .*

Proof Let x be a frequently hypercyclic vector for T and let V be a fixed open set in X . Choose a positive integer p such that $T^p x \in V$. By Theorem 6.30 in [4], x is also a frequently hypercyclic vector for T^p . Consequently, for each open set $U \subset X$,

$$\underline{d}\{m : T^m(T^p x) \in U\} \geq \underline{d}\{n : (T^p)^n x \in U\} > 0.$$

This shows that $T^p x$ is frequently hypercyclic for T . Thus, for an arbitrary open set $V \subset X$, we have found a frequently hypercyclic vector for T in V . \square

Birkhoff's theorem extends to frequent hypercyclicity by Example 2.5 in [3]. That is, for each $a \neq 0$, the composition operator C_a is frequently hypercyclic on the space of entire functions.

The above-mentioned Universality Theorem of Voronin was further refined by Reich, who showed that, for each real number a , not equal to 0, for each compact set $K \subset S$ with connected complement, for each holomorphic zero-free function on K and for each $\epsilon > 0$, the set of real t , such that $\max_K |f(s) - \zeta(s + ita)| < \epsilon$, is of positive lower density. He also showed that the t may be chosen from an arbitrary arithmetic progression $\{m\Delta\}$, $m \in \mathbb{N}$, $\Delta > 0$ with lower density taken with respect to \mathbb{N} . Thus, for each $\Delta > 0$, the Riemann zeta-function is frequently hypercyclic in $\mathcal{O}_o(S)$, for the composition operator $C_{i\Delta}$.

Markus Nieß in [12, 13], and [14] investigated approximation in a strip, by functions having universality properties *outside* the strip. The following theorem yields functions which approximate everywhere and have universality properties *inside* a strip.

Theorem 11 *Fix $-\infty \leq \alpha < \beta \leq +\infty$. Then, for each $f \in \mathcal{M}_{\mathbb{R}}(\mathbb{C})$ having no poles in $S(\alpha, \beta)$, for each compact set $L \subset \mathbb{C}$, for each $\delta > 0$, and for each $\Delta > 0$, there*

is a function $\varphi \in \mathcal{M}_{\mathbb{R}}(\mathbb{C})$ having the same poles and principal parts as f , such that φ is frequently hypercyclic in $\mathcal{O}(S(\alpha, \beta))$ for the vertical translation operator $C_{i\Delta}$ and moreover $|\varphi - f| < \delta$ on L .

Proof We may assume that $\mathbb{C} \setminus L$ is connected and $\overline{L} = L$. Let $\ell = \max\{y : z = x + iy \in L\}$. For each $p \in \mathbb{N}^+$, choose a real number $N_p > (\ell + 2p)/\Delta$ and take pairwise disjoint subsets \mathbb{N}_p of \mathbb{N} as in Lemma 6.19 in [4], each of which is of positive lower density. For each $p \in \mathbb{N}^+$ and $n \in \mathbb{N}_p$, we form the closed intervals $I_n^+ = [n\Delta - p, n\Delta + p]$ and $I_n^- = \{y : -y \in I_n^+\}$. The sets I_n^\pm , $n \in \mathbb{N}_p$, $p \in \mathbb{N}^+$, form a locally finite family of disjoint closed intervals. Moreover, these intervals are all disjoint from the closed interval $[-\ell, \ell]$.

Let $\alpha_k \searrow \alpha$ and $\beta_k \nearrow \beta$. We arrange the $n \in \mathbb{N}_p$ in an increasing sequence $\{n(k)\}$. Now fix p . For $n(k) \in \mathbb{N}_p$, let D_k^+ , D_k^- and D_k be the closed rectangles

$$\begin{aligned} D_k^+ &= \{z = x + iy : \alpha_k \leq x \leq \beta_k, y \in I_{n(k)}^+\} \\ D_k^- &= \{z = x + iy : \alpha_k \leq x \leq \beta_k, y \in I_{n(k)}^-\} \\ D_k &= \{z = x + iy : \alpha_k \leq x \leq \beta_k, |y| \leq p\}. \end{aligned}$$

Then,

$$D_k^+ - in(k)\Delta = D_k = D_k^- + in(k)\Delta.$$

Set

$$E_p = \bigcup_{n(k) \in \mathbb{N}_p} (D_k^+ \cup D_k^-).$$

We may do the same for each p and denoting the pole set of f by $f^{-1}(\infty)$, put

$$E = L \cup f^{-1}(\infty) \cup \bigcup_p E_p.$$

Then, $\overline{E} = E$ and, by Theorem 7, the set E is a set of uniform approximation.

Let \mathcal{P} be the family of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We arrange these polynomials in a sequence $\{P_p\}$ in such a way that each polynomial is repeated infinitely often. We define a function $h \in M_{\mathbb{R}}(E)$ as follows. For $z \in L \cup f^{-1}(\infty)$, we put $h = f$. If $z \in E_p$, then $z \in (D_k^+ \cup D_k^-)$, for some $n(k) \in \mathbb{N}_p$. We set $h(z) = P_p(z - in(k)\Delta)$, for $z \in D_k^+$ and $h(z) = \overline{P_p}(z + in(k)\Delta)$, for $z \in D_k^-$. By the Symmetric Approximation Theorem, there is a function $\varphi \in M_{\mathbb{R}}(\mathbb{C})$, whose poles are precisely those of h and with the same principle parts, such that

$$|\varphi(z) - h(z)| < \delta \exp(-|z|^{1/4}), \quad \text{for all } z \in E.$$

Thus, φ has the same poles as f and with the same principal parts and $|\varphi - f| < \delta$ on L .

We claim that the function φ is frequently hypercyclic for the vertical translation operator $C_{i\Delta}$, on the space $\mathcal{O}(S(\alpha, \beta))$. Indeed, suppose $g \in \mathcal{O}(S(\alpha, \beta))$,

$K \subset S(\alpha, \beta)$ is compact and $\epsilon > 0$. For all but finitely many values of p and for all but finitely many $n(k) \in \mathbb{N}_p$,

$$K \subset \{z = x + iy : \alpha_k \leq x \leq \beta_k, |y| \leq p\}.$$

We have already noted that this latter set is D_k , whenever $n(k) \in \mathbb{N}_p$. We may choose such a p for which

$$\max_K \{|g(z) - P_p(z)|\} < \epsilon/2.$$

Thus, if $n \in \mathbb{N}_p$ and $z \in K$, then $z + in\Delta \in E_p \subset E$, so

$$\begin{aligned} |\varphi(z + in\Delta) - g(z)| &\leq |\varphi(z + in\Delta) - h(z + in\Delta)| + |h(z + in\Delta) - g(z)| \\ &\leq \delta \exp(-|z + in\Delta|^{1/4}) + |P_p(z) - g(z)|. \end{aligned}$$

Hence, this is less than ϵ for all but finitely many $n \in \mathbb{N}_p$. Therefore,

$$\underline{d}\{n : |\varphi(z + in\Delta) - g(z)|_K < \epsilon\} \geq \underline{d}(\mathbb{N}_p) > 0. \quad \square$$

4 Proof of Theorem 3

Lemma 2 Fix $-\infty \leq \alpha < \beta \leq +\infty$ and $\Delta > 0$. If $\phi \in \mathcal{O}(S(\alpha, \beta))$ is frequently hypercyclic for the vertical translation operator $C_{i\Delta}$, then ϕ is strongly recurrent.

Proof Let K be a compact subset of the strip $S(\alpha, \beta)$ and $\epsilon > 0$. Set $S = S(\alpha, \beta)$ and

$$U = \left\{ f \in S : \max_{z \in K} |f(z) - \phi(z)| < \epsilon \right\}.$$

Since U is an open subset of $\mathcal{O}(S)$ and ϕ is frequently hypercyclic for the operator $C_{i\Delta}$,

$$\begin{aligned} \overline{d}_{\mathbb{Z}} \left\{ k \in \mathbb{Z} : \max_{z \in K} |\phi(z + k(i\Delta)) - \phi(z)| < \epsilon \right\} \\ \geq \underline{d}_{\mathbb{Z}} \left\{ n \in \mathbb{N} : \max_{z \in K} |\phi(z + n(i\Delta)) - \phi(z)| < \epsilon \right\} \\ = \frac{1}{2} \underline{d}_{\mathbb{N}} \{ n \in \mathbb{N} : C_{i\Delta}^n \phi \in U \} > 0. \end{aligned}$$

Thus, ϕ is strongly recurrent modulo Δ and by the Inheritance Theorem 2, it is also recurrent. \square

Now, to prove Theorem 3, we may assume that $\Delta > 0$. For $n = 1, 2, \dots$, let $L_n = \{z : |z| \leq n\}$ and choose $\{\epsilon_n\}$ decreasing to zero. For each $n = 1, 2, \dots$, we

invoke Theorem 11 for the strip $S = (1/2 < \Re z < 1)$, $f = \zeta$, $L = L_n$, and $\delta = \epsilon_n$ to obtain a function ϕ_n , frequently hypercyclic in the strip S for the translation operator $C_{i\Delta}$. By Lemma 2, the functions ϕ_n are also strongly recurrent. This concludes the proof of Theorem 3.

5 Proof of Theorem 4

It is not in general true that the product of strongly recurrent functions is strongly recurrent. It is not even true that if fg and hg are strongly recurrent, then $fghg$ is strongly recurrent, but the following lemma gives us a step in this direction.

Lemma 3 *Let S be the fundamental strip $0 < \Re z < 1$ and $g \in \mathcal{O}(S)$ be zero free. Let $\{D_k : k = 1, 2, \dots\}$ be a regular exhaustion of S by closed (filled) rectangles centered at $1/2$. Let $\{n_k\}$ be an increasing sequence in \mathbb{N} such that the rectangles $D_k^+ = D_k + in_k$ are disjoint and, for each k , the rectangles D_k and D_k^+ are also disjoint. Let $\{\epsilon_k\}$ be a sequence of positive numbers. Then, there exists $f \in \mathcal{O}(S)$, $f \not\equiv 0$, such that, for $k = 1, 2, \dots$,*

$$\max_{z \in D_k} |f(z)f(1-z)g^2(z) - f(z+in_k)f(1-z-in_k)g^2(z+in_k)| < \epsilon_k. \quad (1)$$

Proof Set $D_k^- = D_k - in_k$ and let B_k be an exhaustion of S by closed (filled) rectangles, centered at $1/2$, containing D_j , $j \leq k$ and D_j^\pm , $j < k$ but disjoint from D_j^\pm , $j \geq k$. We may assume that $\epsilon_1 < \max_{z \in B_2} |g(z)|$ and $\sum_{j>k} \epsilon_j < \epsilon_k$.

To prove (1) it is sufficient to obtain a function $f \in \mathcal{O}(S)$ satisfying

$$\begin{aligned} \max_{z \in D_k^+} \left| f(z) - \frac{f(z-in_k)f(1-z+in_k)g^2(z-in_k)}{f(1-z)g^2(z)} \right| \\ < \frac{\epsilon_k}{\max_{z \in D_k^+} |f(1-z)g^2(z)|}. \end{aligned} \quad (2)$$

We shall construct inductively a sequence f_k , $k = 0, 1, 2, \dots$ of zero-free entire functions such that, for $k = 1, 2, \dots$,

$$\max_{z \in B_k} |f_k(z) - f_{k-1}(z)| < \epsilon; \quad (3)$$

$$\begin{aligned} \max_{z \in D_k^+} \left| f_k(z) - \frac{f_k(z-in_k)f_k(1-z+in_k)g^2(z-in_k)}{f_k(1-z)g^2(z)} \right| \\ < \frac{\epsilon_k}{\max_{z \in D_k^+} |f_k(1-z)g^2(z)|}. \end{aligned} \quad (4)$$

First, we define an auxiliary function φ_1 on $D_1^- \cup B_1 \cup D_1^+$. We set $\varphi_1 = 1$ on $B_1 \cup D_1^-$. On D_1^+ we set φ_1 equal to a polynomial zero-free on D_1^+ and performing

the approximation

$$\begin{aligned} & \max_{z \in D_1^+} \left| \varphi_1(z) - \frac{\varphi_1(z - in_1)\varphi_1(1 - z + in_1)g^2(z - in_1)}{\varphi_1(1 - z)g^2(z)} \right| \\ & < \frac{\epsilon_1}{\max_{z \in D_1^+} |\varphi_1(1 - z)g^2(z)|}. \end{aligned}$$

Now, let $f_0 = 1$ and let f_1 be a zero-free entire function which approximates φ_1 so well on $D_1^- \cup B_1 \cup D_1^+$, that f_1 satisfies (3) and (4).

Now, suppose we have functions f_1, \dots, f_{k-1} satisfying (3) and (4). We define an auxiliary function φ_k on $D_k^- \cup B_k \cup D_k^+$. First, we set $\varphi_k = f_{k-1}$ on B_k . Then, we set $\varphi_k(z) = \varphi_k(z + in_k)$ on D_k^- . Finally, on D_k^+ we set φ_k equal to a polynomial zero-free on D_k^+ and performing the approximation

$$\begin{aligned} & \max_{z \in D_k^+} \left| \varphi_k(z) - \frac{\varphi_k(z - in_k)\varphi_k(1 - z + in_k)g^2(z - in_k)}{\varphi_k(1 - z)g^2(z)} \right| \\ & < \frac{\epsilon_k}{\max_{z \in D_k^+} |\varphi_k(1 - z)g^2(z)|}. \end{aligned}$$

Let f_k be a zero-free entire function which approximates φ_k so well on $D_k^- \cup B_k \cup D_k^+$, that f_k satisfies (3) and (4). By induction, we now have our sequence $f_k, k = 0, 1, \dots$

To prove the lemma, we need a single function that has the behavior of the sequence $\{f_k\}$. For this we shall employ tangential approximation. Set $E = \bigcup_k D_k^+$. Then, E is a set of tangential approximation. Thus, by Theorem 9, for an arbitrary sequence $\delta_k > 0$, there is an entire function f such that $\max_{z \in D_k^+} |f(z) - f_k(z)| < \delta_k$, for each $k = 1, 2, \dots$. Considering formula (4), we may choose δ_k sufficiently small so that f satisfies (2). Consequently, f also satisfies (1). Since $f_0 = 1$, if our approximation is sufficiently good, the function f is not identically zero. This completes the proof of the lemma. \square

Lemma 4 *Let S be the fundamental strip $0 < \Re z < 1$ and $g \in \mathcal{O}(S)$ be zero free. Then, there exists $f \in \mathcal{O}(S)$, $f \not\equiv 0$, such that $f(z)f(1 - z)g^2(z)$ is strongly recurrent.*

Proof For $p \in \mathbb{N}$ choose disjoint sets of natural numbers \mathbb{N}_p of positive lower density, as in the proof of Theorem 11 and writing $n_k = n(k)$, let D_k and D_k^+ be the corresponding sets. Note that D_k and D_k^+ are disjoint, since $N_p > 2p$. Thus, the sets D_k and D_k^+ satisfy the hypotheses of Lemma 3.

Fix a compact set $K \subset S$ and $\epsilon > 0$. Choose p so that $\max_{z \in K} |\Im z| < p$. Then, for all but finitely many $n = n_k \in \mathbb{N}_p$,

$$K \subset D_k.$$

For all but finitely many k , we have $\epsilon_k < \epsilon$. Thus, from (1),

$$\begin{aligned} & 2\overline{d}_{\mathbb{Z}} \left\{ n \in \mathbb{Z} : \max_{z \in K} |f(z)f(1-z)g^2(z) - f(z+in)f(1-z-in)g^2(z+in)| < \epsilon \right\} \\ & > \underline{d}_{\mathbb{N}} \left\{ n_k \in \mathbb{N}_p : \max_{z \in D_k} |f(z)f(1-z)g^2(z) \right. \\ & \quad \left. - f(z+in_k)f(1-z-in_k)g^2(z+in_k)| < \epsilon_k \right\} \\ & = \underline{d}_{\mathbb{N}}(\mathbb{N}_p) > 0. \end{aligned}$$

Thus, $f(z)f(1-z)g^2(z)$ is strongly recurrent modulo 1, and therefore strongly recurrent by Theorem 2. This concludes the proof of the lemma. \square

Finally we present the proof of Theorem 4. Namely, we show the existence of a strongly recurrent function f in the fundamental strip, which satisfies the functional equation of the Riemann zeta-function.

Proof Since the zeros of $\zeta(z)$ are symmetric with respect to the point $1/2$, there is an entire function $\phi(z)$ such that $\phi(z) = \phi(1-z)$ and $\zeta(z)/\phi(z)$ has no zeros in the fundamental strip S . Let $g = \sqrt{\zeta/\phi}$ be a branch of $(\zeta/\phi)^{1/2}$ in S . By Lemma 4, there is a function $f \in \mathcal{O}(S)$, $f \not\equiv 0$, such that $h(z) = f(z)f(1-z)g^2(z)$ is strongly recurrent in S . Now, set $\mu(z) = f(z)f(1-z)/\phi(z)$. Then, $h = \mu\zeta$, with $\mu(z) = \mu(1-z)$ and therefore h satisfies the functional equation of the Riemann zeta function. That is,

$$h(z)\pi^{-z/2}\Gamma\left(\frac{z}{2}\right) = h(1-z)\pi^{-(1-z)/2}\Gamma\left(\frac{1-z}{2}\right). \quad \square$$

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