Chapter 2
ε-Kronecker Sets

ε-Kronecker sets defined. Are I_0. Can be found in most infinite subsets of a discrete group. Defined by approximating ± 1. Arithmetical properties investigated. Have small sums.

2.1 Introduction

In this book we explore generalizations of Hadamard sets, both in $\mathbb{Z}$ and in discrete abelian groups other than $\mathbb{Z}$. One such generalization is the notion of an ε-Kronecker set, a set $\mathbf{E}$ with the property that for every $\varphi : \mathbf{E} \to \mathbb{T}$ there exists $x \in G$ such that $|\varphi(\gamma) - \gamma(x)| < \varepsilon$ for all $\gamma \in \mathbf{E}$.

Hadamard sets in $\mathbb{Z}$ with suitably large ratio are examples of ε-Kronecker sets (with ε depending on the ratio), but not all ε-Kronecker sets in $\mathbb{Z}$ are finite unions of Hadamard sets. Independent sets of characters (defined in the introduction) of sufficiently large order are another important class of ε-Kronecker sets. An ε-Kronecker set with $\varepsilon < \sqrt{2}$ is an I_0 set, and this is sharp. Thus, we have a particular interest in the case $\varepsilon < \sqrt{2}$. These facts and other basic properties are established in Sect. 2.2 and 2.3.

In $\mathbb{Z}$, infinite ε-Kronecker sets exist for each given $\varepsilon > 0$; just take a Hadamard set with large enough ratio. In contrast, $\mathbf{\Gamma} = \widehat{\mathbb{D}}$ contains no ε-Kronecker subsets if $\varepsilon \leq \sqrt{2}$. In Sect. 2.4 we will see that whenever $\mathbf{E}$ is an infinite set in a group that does not contain "too many" elements of order two, then $\mathbf{E}$ contains a $(1+\varepsilon)$-Kronecker subset of the same cardinality. If the torsion subgroup of $\mathbf{\Gamma}$ is finite, as with $\mathbf{\Gamma} = \mathbb{Z}$, then every infinite subset of $\mathbf{\Gamma}$ contains an ε-Kronecker subset of the same cardinality, for each $\varepsilon > 0$.

The related problem of interpolating arbitrary choices of signs, rather than all complex numbers of modulus 1, is investigated in Sect. 2.5. Conditions are given which ensure that this formally weaker interpolation implies the set is actually ε-Kronecker. Those ideas will be used in Sect. 9.3 to show that if $\mathbf{\Gamma}$

C.C. Graham and K.E. Hare, *Interpolation and Sidon Sets for Compact Groups*, CMS Books in Mathematics, DOI 10.1007/978-1-4614-5392-5_2,

does not contain too many elements of order 2, then every Sidon set in $\mathbf{\Gamma}$ is characterized by the property of being proportionally ε-Kronecker.

Provided $\varepsilon < 2$, ε-Kronecker sets have arithmetic properties similar to those possessed by Sidon sets (see Chap. 6). For instance, they do not contain long arithmetic progressions or large squares. When $\varepsilon < \sqrt{2}$, their step length must tend to infinity. Those properties are proven in Sect. 2.6. It is unknown if every ε-Kronecker set, for $\varepsilon < 2$, is Sidon [**P 4**].

ε-Kronecker sets have product properties similar to those of Hadamard sets, and these are discussed in Sect. 2.7. If $\mathbf{E}$ is ε-Kronecker, then $\mathbf{E} \cdot \mathbf{E}$ does not cluster at a continuous character, and the identity is the only continuous character at which $\mathbf{E} \cdot \mathbf{E}^{-1}$ clusters. When ε is small (depending on N) the closure of $(\mathbf{E} \cup \mathbf{E}^{-1})^N$ in the Bohr topology has zero $\overline{\mathbf{\Gamma}}$-Haar measure.

2.2 Definition and Interpolation Properties

Definition 2.2.1. Let $U \subseteq G$ and $\varepsilon > 0$. A set $\mathbf{E} \subseteq \mathbf{\Gamma}$ is ε-*Kronecker*(U) if for every $\varphi : \mathbf{E} \to \mathbb{T}$ there exists $x \in U$ such that

$$|\varphi(\gamma) - \gamma(x)| < \varepsilon \text{ for all } \gamma \in \mathbf{E}. \tag{2.2.1}$$

Weak ε-*Kronecker*(U) sets $\mathbf{E}$ are the same as ε-Kronecker(U) sets, except that the strict inequality in (2.2.1) is replaced with $\leq$. When $U = G$ we omit the writing of "(G)". The subset $\mathbf{E}$ is called *Kronecker* if it is ε-Kronecker for all $\varepsilon > 0$. By the *Kronecker constant* of $\mathbf{E}$ we mean

$$\varepsilon(\mathbf{E}) = \inf\{\varepsilon : \mathbf{E} \text{ is weak } \varepsilon\text{-Kronecker}\}.$$

A compactness argument shows that if $\mathbf{E}$ is ε-Kronecker(U) for all $\varepsilon > \varepsilon_0$, then $\mathbf{E}$ is weak ε_0-Kronecker$(\overline{U})$.

Remark 2.2.2. Every subset $\mathbf{E}$ of $\mathbf{\Gamma}$ is trivially weak 2-Kronecker. If $\mathbf{1} \in \mathbf{E}$ this cannot be improved, and thus our interest is in $\varepsilon < 2$. The case $\varepsilon < \sqrt{2}$ is particularly interesting, as will be seen in what follows.

It is often convenient to measure angular distances, that is distances in the metric space of the quotient group $\mathbb{R}/(2\pi\mathbb{Z})$, rather than the absolute value metric from $\mathbb{C}$ restricted to $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Thus, we denote by d the usual metric in the quotient space, $\mathbb{R}/(2\pi\mathbb{Z})$:

$$d(\theta, \phi) = \inf\{|\theta - \phi + 2\pi k| : k \in \mathbb{Z}\}, \text{ for } \theta, \phi \in \mathbb{R}.$$

Of course, $d(\theta, \phi) \in [0, \pi]$ for all $\theta, \phi \in \mathbb{R}$. Given $t \in \mathbb{T}$, we let $\arg(t)$ denote the element of $[0, 2\pi)$ satisfying $t = \mathrm{e}^{\mathrm{i}\arg(t)}$. This identifies $\mathbb{T}$ with $[0, 2\pi)$.

Definition 2.2.3. A set $\mathbf{E} \subseteq \mathbf{\Gamma}$ is *angular* ε-Kronecker(U) if for every $\{\theta_\gamma\}_{\gamma \in \mathbf{E}} \in [0, 2\pi)^{\mathbf{E}}$ there exists $u \in U$ such that

$$d(\arg(\gamma(u)), \theta_\gamma) < \varepsilon \text{ for all } \gamma \in \mathbf{E}.$$

It is said to be *weak angular* ε*-Kronecker(U)* if the strict inequality is replaced with $\leq$. Lastly, denote by $\alpha(\mathbf{E})$ the *angular Kronecker constant*

$$\alpha(\mathbf{E}) = \inf\{\varepsilon : \mathbf{E} \text{ is weak angular } \varepsilon\text{-Kronecker}\}.$$

It is easy to see that $\mathbf{E}$ is angular ε-Kronecker if and only if $\mathbf{E}$ is $|1 - \mathrm{e}^{\mathrm{i}\varepsilon}|$-Kronecker. Consequently, angular ε-Kronecker for some $\varepsilon < \pi$ is equivalent to ε-Kronecker for some $\varepsilon < 2$ and angular ε-Kronecker for some $\varepsilon < \pi/2$ is equivalent to ε-Kronecker for some $\varepsilon < \sqrt{2}$.

Here are some easy facts whose proofs are left for the reader.

Lemma 2.2.4. *1. If* $\mathbf{E}$ *is (weak)* ε*-Kronecker, then so is* $\mathbf{E}^{-1}$*.*
2. If $\mathbf{E}$ *is (weak)* ε*-Kronecker*(U) *and* $x \in G$*, then* $\mathbf{E}$ *is (weak)* ε*-Kronecker*$(x \cdot U)$*.*
3. $\mathbf{E} \subseteq \mathbb{Z}$ *is* ε*-Kronecker if and only if* $n\mathbf{E}$ *is* ε*-Kronecker for all* $n \neq 0$*.*

Remarks 2.2.5. (i) The class of ε-Kronecker sets is not closed under translation since a set that contains the identity element is no better than weak 2-Kronecker. On the other hand, a Baire category theorem argument (see Exercise 2.9.1 (3)) shows that if $\mathbf{\Gamma}$ is countable and has no elements of finite order, then $\mathbf{\Gamma} \setminus \{\mathbf{1}\}$ is 2-Kronecker.

(ii) An ε-Kronecker set $\mathbf{E}$ with $\varepsilon \leq \sqrt{2}$ cannot contain a character of order two since that character can take on only real values. Such an ε-Kronecker set $\mathbf{E}$ must also satisfy $\mathbf{E} \cap \mathbf{E}^{-1} = \emptyset$.

2.2.1 Examples of ε-Kronecker Sets

It is immediate from Lemma 1.3.1 that Hadamard sets are ε-Kronecker for suitable ε. Indeed, with the terminology of this chapter, Lemma 1.3.1 may be restated as:

Proposition 2.2.6. *If* $\mathbf{E} \subseteq \mathbb{Z}$ *is a Hadamard set with ratio* q*, then* $\mathbf{E}$ *is weak angular* ε*-Kronecker with* $\varepsilon = \pi/(q-1)$.

Of course, this result is only of interest if $q > 2$ and only guarantees an ε-Kronecker set with $\varepsilon < \sqrt{2}$ when $q > 3$.

Since every Hadamard set is a finite union of Hadamard sets with large ratio, it follows that every Hadamard set is a finite union of ε-Kronecker sets, whatever the choice $\varepsilon > 0$. The converse is false.

Example 2.2.7. A set $\mathbf{E} \subseteq \mathbb{N}$ with the property that for each $\varepsilon > 0$ there is a finite subset $\mathbf{F} \subseteq \mathbf{E}$ such that $\mathbf{E} \setminus \mathbf{F}$ is ε-Kronecker, but $\mathbf{E}$ is not a finite union of Hadamard sets: To construct such a set, inductively choose finite sets of positive integers, $\mathbf{E}_j$, of increasing cardinality, with Hadamard ratios tending to infinity, and an increasing sequence of positive integers m_j satisfying the conditions

$$1 < \frac{\min \mathbf{E}_{j+1}}{m_j} \to \infty \text{ and } 1 < \frac{m_j}{\max \mathbf{E}_j} \to \infty.$$

Put $\mathbf{E} = \bigcup_j (m_j + \mathbf{E}_j)$. The sets $\mathbf{F}_j = m_j + \mathbf{E}_j$ are disjoint since each element of $\mathbf{F}_{j+1}$ is greater than every element of $\mathbf{F}_j$. If $n_j = \max \mathbf{E}_j$ and $l_j = \min \mathbf{E}_j$, then

$$\frac{n_j + m_j}{l_j + m_j} \le \frac{n_j}{m_j} + 1 \to 1.$$

Thus, each Hadamard set will meet only a finite number of the sets $\mathbf{F}_j$ in more than one point. That proves $\mathbf{E}$ cannot be a finite union of Hadamard sets.

But given any number $q > 1$, the assumptions on $\mathbf{E}_j$ and m_j certainly ensure that there is an index J such that the set $\{m_j\}_{j=J}^{\infty} \cup (\bigcup_{j=J}^{\infty} \mathbf{E}_j)$ is a Hadamard set with ratio at least q. By Proposition 2.2.6, for each $\varepsilon > 0$ there exists J such that $\{m_j\}_{j=J}^{\infty} \cup (\bigcup_{j=J}^{\infty} \mathbf{E}_j)$ is an $\varepsilon/2$-Kronecker set. That $\bigcup_{j=J}^{\infty} \mathbf{F}_j$ is ε-Kronecker is now a straightforward matter of approximating 1 on the m_j's and φ on each $\mathbf{E}_j$.

A specific instance of this occurs with $\mathbf{E}_j = \{3^{2^{j^2} k} : k = 1, \ldots, j\}$ and $m_j = 3^{(j+1)2^{j^2}}$.

Other examples of ε-Kronecker sets are the independent sets, defined on p. 1. Before stating the precise Kronecker properties of independent sets, we give a very useful lemma whose proof is asked for in Exercise 2.9.4.

Lemma 2.2.8. *Let $\mathbf{E} \subseteq \mathbf{\Gamma}$, $\varepsilon > 0$ and $\Lambda \subseteq \mathbf{\Gamma}$ be a subgroup.*

1. *Let $q : \mathbf{\Gamma} \to \mathbf{\Gamma}/\Lambda$ be the quotient homomorphism. If q is one-to-one on $\mathbf{E}$ and $q(\mathbf{E})$ is (weak) ε-Kronecker, then $\mathbf{E}$ is (weak) ε-Kronecker.*
2. *If $\mathbf{E} \subseteq \Lambda$ then $\mathbf{E}$ is (weak) ε-Kronecker as a subset of $\mathbf{\Gamma}$ if and only if it is (weak) ε-Kronecker as a subset of Λ.*

Proposition 2.2.9. *Let $\mathbf{E} \subseteq \mathbf{\Gamma}$ be independent with all its elements having order at least N. Then $\mathbf{E}$ is weak angular π/N-Kronecker.*

Proof. Let $\langle \mathbf{E} \rangle$ denote the subgroup of $\mathbf{\Gamma}$ generated by $\mathbf{E}$. For each $\gamma \in \mathbf{E}$, let $\mathbf{\Gamma}_\gamma = \langle \gamma \rangle$ be the cyclic subgroup of $\mathbf{\Gamma}$ generated by γ. The independence of $\mathbf{E}$ implies that $\langle \mathbf{E} \rangle = \bigoplus_{\gamma \in E} \mathbf{\Gamma}_\gamma$. Its dual group, $\prod_{\gamma \in \mathbf{E}} \widehat{\mathbf{\Gamma}_\gamma}$, is a quotient of G. By the lemma above, there is no loss of generality in assuming $\mathbf{\Gamma} = \langle \mathbf{E} \rangle$ and that $G = \prod_{\gamma \in \mathbf{E}} \widehat{\mathbf{\Gamma}_\gamma}$.

Let $\varphi : \mathbf{E} \to [0, 2\pi)$. Since each $\gamma \in \mathbf{E}$ has order at least N, $\mathbf{\Gamma}_\gamma$ is either $\mathbb{Z}$ or $\mathbb{Z}_n$, the cyclic group of order n, for some $n \ge N$. In either case, there is some $x_\gamma \in \mathbf{\Gamma}_\gamma$ such that

$$d(\arg \gamma(x_\gamma), \varphi(\gamma)) \le \pi/N,$$

the worst case occurring when $\mathbf{\Gamma}_\gamma = \mathbb{Z}_N$. Let $x_\gamma = e$ if $\gamma \notin \mathbf{E}$ and put $x = (x_\gamma) \in G$.

For $\gamma \in \mathbf{E}$, $d(\arg\gamma(x), \varphi(\gamma)) \leq \pi/N$, and thus $\mathbf{E}$ is weak angular π/N-Kronecker. □

Corollary 2.2.10. *If* $\mathbf{E}$ *is independent and all elements of* $\mathbf{E}$ *have order at least three, then* $\mathbf{E}$ *is weak* 1*-Kronecker. If all the elements of* $\mathbf{E}$ *have infinite order, then* $\mathbf{E}$ *is* ε*-Kronecker for all* $\varepsilon > 0$*.*

Remark 2.2.11. If $G = \prod G_\alpha$ is a product of finite cyclic groups, then the set of projections onto the factor groups G_α is an independent set in $\mathbf{\Gamma}$. No subset of two or more elements in $\mathbb{Z}$ is independent.

Sets with weaker "independence-like" properties can also be ε-Kronecker, as the next example illustrates.

Example 2.2.12. A non-independent weak 1-Kronecker set: Take $G = \mathbb{Z}_2 \oplus \mathbb{Z}_3^{\mathbb{N}}$ and $\mathbf{E} = \{(\chi, \gamma_n)\} \subset \mathbf{\Gamma}$ where χ is not the identity of $\mathbb{Z}_2$ and $\{\gamma_n\}_1^\infty$ is an independent subset in the dual of $\mathbb{Z}_3^{\mathbb{N}}$ (such as the set of projections on the factors). Then $\mathbf{E}$ is not independent (for trivial reasons), but $\mathbf{E}$ is weak 1-Kronecker. See Exercise 2.9.3.

2.2.2 ε-Kronecker Sets Are ε-Kronecker(U)

We now give a very useful result.

Theorem 2.2.13. *Let* $\varepsilon > 0$ *and suppose* $\mathbf{E}$ *is a weak* ε*-Kronecker subset of* $\mathbf{\Gamma}$*. Then for each open* $U \subseteq G$ *there exists a finite set* $\mathbf{F}$ *such that* $\mathbf{E} \smallsetminus \mathbf{F}$ *is weak* ε*-Kronecker*(U)*.*

Remark 2.2.14. In the proof of Theorem 2.2.13 we will here, as elsewhere, identify the set of functions $\varphi : \mathbf{E} \to \mathbb{T}$ with the compact space $\mathbb{T}^{\mathbf{E}}$. That product space has the topology of coordinatewise convergence (see p. 209). It is convenient that this topology is "the same" as the weak* topology for (Fourier-Stieltjes transforms of) bounded subsets of measures on G. Each open $U \subseteq \mathbb{T}^{\mathbf{E}}$ contains a subset of the form $\{\psi\} \times \mathbb{T}^{\mathbf{E} \smallsetminus \mathbf{F}}$, where $\mathbf{F} \subseteq \mathbf{E}$ is finite and $\psi \in \mathbb{T}^{\mathbf{F}}$. Similar conclusions hold for the functions mapping $\mathbf{E} \to X$, where X is any of $[0, 1]$, $[-1, 1]$, $\{-1, 1\}$ or the closed unit ball in $\mathbb{C}$.

Proof (of Theorem 2.2.13). By replacing U with a smaller set we may assume U is compact. Since G is compact, there exist $x_1, \ldots, x_N \in G$ such that $G \subseteq \bigcup_{n=1}^N x_n U$. For each n let

$$X_n = \{\varphi : \mathbf{E} \to \mathbb{T} : \exists x \in x_n U \text{ such that } |\varphi(\gamma) - \gamma(x)| \leq \varepsilon \ \forall \gamma \in \mathbf{E}\}.$$

We give $\mathbb{T}^{\mathbf{E}}$ the product topology, making it a compact group, and observe that each X_n is closed in $\mathbb{T}^{\mathbf{E}}$. To see this, suppose that $\varphi_\beta \in X_n$ and that

$u_\beta \in x_n U$ satisfy $|\varphi_\beta(\gamma) - \gamma(u_\beta)| \le \varepsilon$ for all $\gamma \in \mathbf{E}$. Suppose also that $\varphi_\beta \to \varphi$ pointwise on $\mathbf{E}$ (this being the topology on $\mathbb{T}^{\mathbf{E}}$). Let $u \in x_n U$ be an accumulation point of the u_β. Passing to a subnet, we may assume $|\varphi(\gamma) - \gamma(u)| = \lim_\beta |\varphi(\gamma) - \gamma(u_\beta)| \le \varepsilon$, so φ is in X_n.

The fact that $\mathbf{E}$ is weak ε-Kronecker ensures that $\mathbb{T}^{\mathbf{E}} = \bigcup_{n=1}^{N} X_n$, and hence the Baire category theorem for compact Hausdorff spaces implies there exists n such that X_n has interior in $\mathbb{T}^{\mathbf{E}}$.

By Remark 2.2.14, X_n contains a set of the form $\{\psi\} \times \mathbb{T}^{\mathbf{E} \setminus \mathbf{F}}$ for some finite set $\mathbf{F}$ and $\psi \in \mathbb{T}^{\mathbf{F}}$. Therefore, for every $\varphi : (\mathbf{E} \setminus \mathbf{F}) \to \mathbb{T}$, there exists $u \in x_n U$ such that $|\varphi(\gamma) - \gamma(u)| \le \varepsilon$ for all $\gamma \in \mathbf{E} \setminus \mathbf{F}$. Thus, $\mathbf{E} \setminus \mathbf{F}$ is weak ε-Kronecker$(x_n U)$. By Lemma 2.2.4(2), $\mathbf{E} \setminus \mathbf{F}$ is weak ε-Kronecker(U). $\square$

As remarked earlier, the property of being ε-Kronecker, for $\varepsilon < 2$, is not preserved under translation. However, the following partial translation result is instructive.

Corollary 2.2.15. *Suppose that $\mathbf{E}$ is weak ε-Kronecker and that $\varepsilon' > \varepsilon$. Let $\gamma \in \mathbf{\Gamma}$. Then there is a finite set $\mathbf{F} \subseteq \mathbf{\Gamma}$ such that $\gamma(\mathbf{E} \setminus \mathbf{F})$ is ε'-Kronecker.*

Proof. Let $U = \{x \in G : |\gamma(x) - 1| < \varepsilon' - \varepsilon\}$. This is an open set so by Theorem 2.2.13 there is a finite set $\mathbf{F}$ such that $\mathbf{E} \setminus \mathbf{F}$ is weak ε-Kronecker(U).

Given $\varphi : \gamma(\mathbf{E} \setminus \mathbf{F}) \to \mathbb{T}$, define $\psi : \mathbf{E} \setminus \mathbf{F} \to \mathbb{T}$ by $\psi(\chi) = \varphi(\gamma\chi)$. Pick $x \in U$ such that $|\psi(\chi) - \chi(x)| \le \varepsilon$ for all $\chi \in \mathbf{E} \setminus \mathbf{F}$. It is easy to see that

$$|\varphi(\gamma\chi) - \gamma\chi(x)| \le |\psi(\chi) - \chi(x)| + |\chi(x) - \gamma\chi(x)| < \varepsilon'. \qquad \square$$

Remark 2.2.16. If γ has finite order, there is even a finite set $\mathbf{F}$ such that $\gamma(\mathbf{E} \setminus \mathbf{F})$ is weak ε-Kronecker. To see this, just take as U the open set $U = \{x : \gamma(x) = 1\}$ (see Exercise C.4.3 (2)) and repeat the argument.

It is easy to see that if $\mathbf{E} \subset \mathbb{N}$ is a Hadamard set and $m \in \mathbb{N}$, then there exists a two-element set $\mathbf{F}$ such that $(\mathbf{E} \setminus \mathbf{F}) \cup \{m\}$ is Hadamard with the same ratio as $\mathbf{E}$. We have the following analogue for ε-Kronecker sets.

Corollary 2.2.17. *Suppose that $\mathbf{E}$ is weak ε-Kronecker. Assume that $\varepsilon' > \varepsilon$ and that γ has infinite order. Then there is a finite set $\mathbf{F}$ such that $(\mathbf{E} \setminus \mathbf{F}) \cup \{\gamma\}$ is ε'-Kronecker.*

Proof. Fix $0 < \tau < \varepsilon' - \varepsilon$. Since γ has infinite order, $\gamma(G)$ is dense in $\mathbb{T}$. We pick a finite subset $X \subseteq G$ such that for every $t \in \mathbb{T}$ there exists some $x \in X$ with $|\gamma(x) - t| < \tau$. For each $x \in X$, choose a neighbourhood U_x of x such that $|\gamma(x) - \gamma(u)| < \varepsilon$ for all $u \in U_x$. By Theorem 2.2.13, for each $x \in X$, there exists a finite subset $\mathbf{F}_x$ such that $\mathbf{E} \setminus \mathbf{F}_x$ is weak ε-Kronecker(U_x). Let $\mathbf{F}$ be the finite set $\mathbf{F} = \bigcup_{x \in X} \mathbf{F}_x$.

Let $\varphi : (\mathbf{E} \smallsetminus \mathbf{F}) \cup \{\gamma\} \to \mathbb{T}$ and pick $x \in X$ such that $|\varphi(\gamma) - \gamma(x)| < \tau$. Select $u \in U_x$ such that $|\varphi(\chi) - \chi(u)| \le \varepsilon$ for all $\chi \in \mathbf{E} \smallsetminus \mathbf{F}_x$. In particular, this inequality holds for all $\chi \in \mathbf{E} \smallsetminus \mathbf{F}$. Furthermore,

$$|\varphi(\gamma) - \gamma(u)| \le |\varphi(\gamma) - \gamma(x)| + |\gamma(x) - \gamma(u)| < \tau + \varepsilon < \varepsilon',$$

and so $(\mathbf{E} \smallsetminus \mathbf{F}) \cup \{\gamma\}$ is ε'-Kronecker. □

2.3 The Relationship Between Kronecker Sets and I_0 Sets

2.3.1 ε-Kronecker Sets Are I_0 if $\varepsilon < \sqrt{2}$

The standard iteration argument, Corollary 1.3.4, immediately shows that ε-Kronecker sets with $\varepsilon < 1$ are I_0. In fact, $\varepsilon < \sqrt{2}$ will suffice, as the next result demonstrates. Example 2.3.6 shows that $\sqrt{2}$ is sharp with this property. In contrast, in Example 2.5.6, we construct an I_0 set that is not a finite union of ε-Kronecker sets for any choice of $\varepsilon < \sqrt{2}$. It is not known if such an example exists in $\mathbb{Z}$ [**P 7**].

Recall that Δ is the closed unit ball in $\mathbb{C}$.

Theorem 2.3.1. *Suppose U is a symmetric e-neighbourhood in G. Let $\varepsilon < \sqrt{2}$ and let $\mathbf{E} \subseteq \mathbf{\Gamma}$ be an ε-Kronecker(U) set. Then*

1. *$\mathbf{E}$ is I_0 and the interpolating measure can be chosen to be positive and concentrated on U.*
2. *$\mathbf{E} \cup -\mathbf{E}$ is I_0 and the interpolating measure can be chosen concentrated on U.*
3. *Furthermore, in (1) (resp., (2)) for each $\varepsilon' > 0$ there exists N, depending only on ε and ε', such that for each $\varphi : \mathbf{E} \to \Delta$ (resp., $\varphi : \mathbf{E} \cup \mathbf{E}^{-1} \to \Delta$) there exists $c_n \in [0,1]$ (resp., $c_n \in \Delta$) and $x_n \in U$ such that $|\varphi(\gamma) - \sum_1^N c_n \gamma(x_n)| < \varepsilon'$ for all $\gamma \in \mathbf{E}$ (resp., $\gamma \in \mathbf{E} \cup \mathbf{E}^{-1}$).*

Remark 2.3.2. In the terminology of Chap. 3, (1) implies that an ε-Kronecker set with $\varepsilon < \sqrt{2}$ is an $FZI_0(U)$ set and item (3) implies that it is $FZI_0(U)$ with bounded length depending only on ε.

Proof (of Theorem 2.3.1). (1) Since $\varepsilon < \sqrt{2}$, $\mathbf{E}$ is angular τ-Kronecker for some $\tau < \pi/2$. We approximate the real and imaginary parts of $\varphi : \mathbf{E} \to \Delta$ separately.

To begin, let $\varphi : \mathbf{E} \to \{-1, 1\}$ be given. The angular τ-Kronecker property gives us an $x \in U$ with $d(\arg \varphi(\gamma), \arg \gamma(x)) < \tau$ for all $\gamma \in \mathbf{E}$. Then elementary trigonometry shows $|\varphi(\gamma) - \mathfrak{Re}(\gamma(x))| < 1 - \cos \tau = \varepsilon^2/2 < 1$.

The standard iteration, Corollary 1.3.4, tells us that given any $\varphi : \mathbf{E} \to [-1,1]$ there exists $\mu \in M_d^+(U)$ with $\varphi(\gamma) = \widehat{\mu}(\gamma)$ on $\mathbf{E}$ and norm depending only on τ.

Now consider $\varphi \in \mathrm{Ball}(\ell^\infty(\mathbf{E}))$ and choose $\mu \in M_d^+(U)$ such that $\widehat{\mu} = \mathfrak{Re}\varphi$ on $\mathbf{E}$. To interpolate the imaginary $i\mathfrak{Im}\varphi(\gamma)$ we choose $u_1 \in U$ such that $d(\arg\gamma(u_1), r_\gamma) < \tau$, where $r_\gamma = \pi/2$ if $\mathfrak{Im}\varphi(\gamma) \geq 0$ and $r_\gamma = -\pi/2$ otherwise. Using the first part of the argument, obtain $\nu \in M_d^+(U)$ such that $\widehat{\nu}(\gamma) = -\mathfrak{Re}\gamma(u_1)$ on $\mathbf{E}$. Then $(\nu + \delta_{u_1^{-1}})/2 \in M_d^+(U)$ and for $\gamma \in \mathbf{E}$

$$\left|\frac{\widehat{\nu + \delta_{u_1^{-1}}}(\gamma)}{2} - i\mathfrak{Im}\varphi(\gamma)\right| = \left|\frac{\mathfrak{Im}\gamma(u_1)}{2} - \mathfrak{Im}\varphi(\gamma)\right| \leq 1 - \frac{\cos\tau}{2}$$

since $\mathfrak{Im}\gamma(u_1)/2 \in \pm[(\cos\tau)/2, 1/2]$, depending on the sign of $\mathfrak{Im}\varphi(\gamma)$. The measure $\mu_0 = \mu + (\nu + \delta_{u_1^{-1}})/2 \in M_d^+(U)$ and satisfies $|\varphi(\gamma) - \widehat{\mu_0}(\gamma)| \leq 1 - (\cos\tau)/2$ on $\mathbf{E}$. An application of Proposition 1.3.2 (1) will complete the proof.

(2) Let $\varphi \in \mathrm{Ball}(\ell^\infty(\mathbf{E} \cup \mathbf{E}^{-1}))$. We may assume that φ is real. We use the preceding to find $\mu, \nu \in M_d^+(U)$ such that

$$\widehat{\mu} = \varphi \text{ on } \mathbf{E} \text{ and } \widehat{\nu} = i\varphi \text{ on } \mathbf{E}.$$

Let $\omega_1 = \frac{1}{2}(\mu - i\nu)$. Then $\widehat{\omega_1} = \varphi$ on $\mathbf{E}$, but $\widehat{\omega_1} = 0$ on $\mathbf{E}^{-1}$. Similarly we can find $\omega_2 \in M_d(U)$ such that $\widehat{\omega_2} = \varphi$ on $\mathbf{E}^{-1}$ and $\widehat{\omega_2} = 0$ on $\mathbf{E}$. Then $\omega_1 + \omega_2 \in M_d(U)$ and $\widehat{\omega_1} + \widehat{\omega_2} = \varphi$ on $\mathbf{E} \cup \mathbf{E}^{-1}$.

(3) is Exercise 2.9.7. □

Remark 2.3.3. The norm of the interpolating measure in both parts of the theorem depends only on ε. This observation, together with Theorem 2.2.13, proves the following corollary:

Corollary 2.3.4. *Suppose $\varepsilon < \sqrt{2}$ and $\mathbf{E}$ is an ε-Kronecker set. There exists $C = C(\varepsilon)$ such that for every non-empty, open $U \subseteq G$ there is a finite set $L = L(U)$ such that every element of Ball$(\ell^\infty(\mathbf{E} \setminus L))$ can be interpolated by $\mu \in M_d^+(U)$ and $\|\mu\| \leq C(\varepsilon)$.*

2.3.2 An Example of a $\sqrt{2}$-Kronecker Set That Is a Sidon Set but Not I_0

The following lemma shows, in particular, that any two-element set of positive integers is $\sqrt{2}$-Kronecker.

Lemma 2.3.5. *Let $1 \le m < n < \infty$. Let I be a closed interval of length at least $4\pi/m$ and $w \ne z \in \mathbb{T}$. There is a closed subinterval $J \subseteq I$ of length at least $\pi/(8mn)$, such that for all $\theta \in J$ both*

$$|\mathrm{e}^{\mathrm{i}m\theta} - w| < \sqrt{2} \text{ and } |\mathrm{e}^{\mathrm{i}n\theta} - z| < \sqrt{2}.$$

Proof. Since the function $x \mapsto \mathrm{e}^{\mathrm{i}mx}$ is $2\pi/m$-periodic, the length of I ensures that there is some $y \in I$ having $\mathrm{e}^{\mathrm{i}my} = w$ and $y + \theta \in I$ whenever $\theta \in [-\pi/m, \pi/m]$. Let $\beta = \arg(\mathrm{e}^{\mathrm{i}ny}\,\overline{z}\,)$.

If $\beta \in [0, \pi/2]$, take $J = [y - \frac{4\pi}{9n}, y - \frac{\pi}{9n}] \subseteq I$. If $x \in J$, then $x = y - \theta$ for some $\theta \in [\frac{\pi}{9n}, \frac{4\pi}{9n}]$, and since $m/n \le 1$ and $n\theta < \pi/2$,

$$|\mathrm{e}^{\mathrm{i}mx} - w| \le |\mathrm{e}^{\mathrm{i}4\pi/9} - 1| < \sqrt{2}.$$

Also, since $\beta - n\theta \in (-\pi/2, \pi/2)$, it follows that

$$|\mathrm{e}^{\mathrm{i}nx} - z| = |\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{-\mathrm{i}n\theta} - 1| < \sqrt{2}.$$

If $\beta \in [\pi/2, \pi]$, take $J = \{y - \frac{\pi}{2n}(1+\theta) : \theta \in [\frac{1}{4m}, \frac{1}{2m}]\} \subseteq I$. Thus, if $x = y - \frac{\pi}{2n}(1+\theta) \in J$,

$$|\mathrm{e}^{\mathrm{i}mx} - w| = |\mathrm{e}^{-\frac{\mathrm{i}m\pi}{2n}(1+\theta)} - 1| < \sqrt{2}.$$

Since $\beta - \frac{\pi}{2}(1+\theta) \in (-\pi/2, \pi/2)$, we have $|\mathrm{e}^{\mathrm{i}nx} - z| = |\mathrm{e}^{\mathrm{i}\beta}\mathrm{e}^{-\mathrm{i}\pi(1+\theta)/2} - 1| < \sqrt{2}$. The cases $\beta \in [-\pi, -\pi/2]$ and $[-\pi/2, 0]$ are similar. □

Example 2.3.6. A $\sqrt{2}$-Kronecker set that is Sidon but not I_0: Let $\mathbf{E}$ be the set $\bigcup_{j=1}^{\infty} \{N_j, N_j + j\}$ where $N_1 = 2$ and the N_j are chosen inductively so that

$$\frac{4\pi}{N_{j+1}} < \frac{\pi}{8N_j(N_j + j)}.$$

Let $\varphi : \mathbf{E} \to \mathbb{T}$ be given, say $\varphi(N_j) = w_j$ and $\varphi(N_j + j) = z_j$. According to Lemma 2.3.5, there is a closed interval J_1 of length $\pi/(8N_1(N_1+1)) = \pi/48$ such that for all $x \in J_1$,

$$|\mathrm{e}^{\mathrm{i}N_1 x} - w_1| < \sqrt{2} \text{ and } |\mathrm{e}^{\mathrm{i}(N_1+1)x} - z_1| < \sqrt{2}.$$

Because length $J_1 > 4\pi/N_2$, there is a closed subinterval $J_2 \subseteq J_1$ of length at least $\pi/(8N_2(N_2+2))$ and such that for all $x \in J_2$, $|\mathrm{e}^{\mathrm{i}N_2 x} - w_2| < \sqrt{2}$ and $|\mathrm{e}^{\mathrm{i}(N_2+2)x} - z_2| < \sqrt{2}$. Proceed inductively to find closed nested intervals J_k such that for all $x \in J_k$,

$$|\mathrm{e}^{\mathrm{i}N_k x} - w_k| < \sqrt{2} \text{ and } |\mathrm{e}^{\mathrm{i}(N_k+k)x} - z_k| < \sqrt{2}.$$

Then every $x \in \bigcap_k J_k$ will satisfy $|\varphi(n) - \mathrm{e}^{inx}| < \sqrt{2}$ for all $n \in \mathbf{E}$, which shows that $\mathbf{E}$ is at least $\sqrt{2}$-Kronecker.

The two subsets, $\{N_j\}_j$ and $\{N_j + j\}_j$, are both Hadamard and therefore I_0 (Theorem 1.3.9), which implies Sidon. Since a finite union of Sidon sets is Sidon, as we shall see in Corollary 6.3.3, $\mathbf{E}$ is Sidon.

But $\mathbf{E}$ is not I_0 by the same reasoning as given in Example 1.5.2. The disjoint subsets $\{N_j\}$ and $\{N_j + j\}$ do not have disjoint closures in the Bohr compactification of $\mathbb{Z}$, so their union cannot be I_0 (see Corollary 3.4.3). Since it is not I_0, this also shows $\mathbf{E}$ can be no better than $\sqrt{2}$-Kronecker because of Theorem 2.3.1.

2.4 Presence of ε-Kronecker Sets

It is natural to ask which groups $\mathbf{\Gamma}$ contain large ε-Kronecker sets for any (or all) $\varepsilon < \sqrt{2}$. Of course, if $\mathbf{\Gamma}$ contains only elements of order two, then it does not contain an ε-Kronecker set with $\varepsilon \le \sqrt{2}$ since characters of order two can only take on the values ± 1. Similarly, if $\mathbf{\Gamma}$ contains infinitely many elements of order two, then not all infinite subsets will contain infinite $\sqrt{2}$-Kronecker sets.

In this section it will be shown that provided the subset $\mathbf{E}$ does not have "too many" elements of order 2 (as defined below), then $\mathbf{E}$ will contain a weak 1-Kronecker set of the same cardinality. Algebraic methods will be used to prove this.

Definition 2.4.1. Let $\mathbf{\Gamma}_2$ be the subgroup generated by the elements of order 2 in $\mathbf{\Gamma}$ and q_2 the quotient mapping $\mathbf{\Gamma} \to \mathbf{\Gamma}/\mathbf{\Gamma}_2$. The set $\mathbf{E} \subseteq \mathbf{\Gamma}$ is said to be *2-large* if $|\,q_2(\mathbf{E})| < |\,\mathbf{E}|$.

Remark 2.4.2. If $\mathbf{E} \subseteq \mathbf{\Gamma}$ is an infinite, independent set that is not 2-large, then the subset of $\mathbf{E}$ consisting of the elements of order 3 or more has the same cardinality as $\mathbf{E}$. It is weak 1-Kronecker by Corollary 2.2.10.

Here is the main existence result.

Theorem 2.4.3. *Let $\mathbf{E} \subseteq \mathbf{\Gamma}$ be infinite and not 2-large. Then there exists a weak 1-Kronecker set $\mathbf{F} \subseteq \mathbf{E}$ such that $|\,\mathbf{F}| = |\,\mathbf{E}|$.*

To prove Theorem 2.4.3, we will consider several different discrete abelian groups. The proof proper begins on p. 30.

2.4.1 Two Countable Groups: $\mathcal{C}(p^\infty)$ and $\mathbb{Q}$

Let p be a prime and denote by $\mathcal{C}(p^\infty)$ the discrete p-subgroup of $\mathbb{T}$, that is, the group of all $p^n th$-roots of unity. An important classical fact in group theory is that every abelian group is isomorphic to a subgroup of

$$\bigoplus_{\alpha} \mathbb{Q}_\alpha \oplus \bigoplus_{\beta} \mathcal{C}(p_\beta^\infty), \tag{2.4.1}$$

where $\mathbb{Q}_\alpha$ are copies of the rationals [165, Theorem 10.30]. We begin with $\mathbb{Q}$ and $\mathcal{C}(p^\infty)$.

Proposition 2.4.4. *Let $\varepsilon > 0$. Each infinite subset of $\mathcal{C}(p^\infty)$, or $\mathbb{Q}$, contains an infinite ε-Kronecker set.*

Proof. First, suppose $\mathbf{E}$ is an infinite subset of $\mathcal{C}(p^\infty)$. Note that each integer, m, defines a character on $\mathcal{C}(p^\infty)$ by the rule, $e^{2\pi i k/p^n} \to e^{2\pi i k m/p^n}$. Each element of the coset $m + p^n\mathbb{Z}$ acts in the same way on the $p^n th$-roots of unity.

Choose N so large that $|e^{2\pi i/p^N} - 1| < \varepsilon/2$. Take $\gamma_j \in \mathbf{E}$ so that $\gamma_j = e^{2\pi i k_j/p^{n_j}}$ where $n_1 \geq N$, $n_{j+1} - n_j \geq N$ for $j \geq 1$ and $1 \leq k_j < p$. We will prove that the set $\{\gamma_j\}$ is ε-Kronecker.

Let $t_j \in \mathbb{T}$. Since k_1 and p^{n_1} are coprime, the set $\{e^{2\pi i k_1 m/p^{n_1}} : m \in \mathbb{Z}\}$ consists of all the $p^{n_1} th$ roots of unity and consequently there is an integer m_1 such that

$$|e^{2\pi i k_1 m_1/p^{n_1}} - t_1| \leq |e^{2\pi i/p^{n_1}} - 1| < \varepsilon/2.$$

Of course, the same inequality is obtained if m_1 is replaced by any element of the coset $m_1 + p^{n_1}\mathbb{Z}$. By similar arguments, we can choose integer m_2 with

$$|e^{2\pi i k_2(m_1 + m_2 p^{n_1})/p^{n_2}} - t_2| \leq |e^{2\pi i k_2 m_2/p^{n_2 - n_1}} - t_2 e^{-2\pi i k_2 m_1/p^{n_2}}| < \varepsilon/2,$$

as well as

$$|e^{2\pi i k_1(m_1 + m_2 p^{n_1})/p^{n_1}} - t_1| < \varepsilon/2.$$

Again, the same inequalities are obtained if m_2 is replaced by any element of the coset $m_2 + p^{n_2}\mathbb{Z}$. Continuing in this fashion produces a sequence of integers, m_j', with

$$|e^{2\pi i k_\ell m_j'/p^{n_\ell}} - t_\ell| < \varepsilon/2$$

for all $\ell \leq j$. If g is an accumulation point of $\{m_j'\}$ in the compact dual group of $\mathcal{C}(p^\infty)$, then $|\gamma_j(g) - t_j| \leq \varepsilon/2$, and hence $\{\gamma_j\}$ is ε-Kronecker.

Now consider the case when the discrete dual group is $\mathbb{Q}$. Choose q so large that every Hadamard set in $\mathbb{Q}$ or $\mathbb{R}$, with ratio at least q is $\varepsilon/3$-Kronecker. (See Exercise 2.9.13.)

If $\mathbf{E} \subseteq \mathbb{Q}$ is an unbounded set of real numbers, then we can find a subset $\{\gamma_j\} \subseteq \mathbf{E}$ with $\gamma_{j+1}/\gamma_j \geq q$ for all j, and such a set is ε-Kronecker. Otherwise, $\mathbf{E}$ contains a sequence $\{\gamma_j\}$ with limit $r \in \mathbb{R}$, in the usual topology of $\mathbb{R}$. By passing to a further subsequence, if necessary, we can assume $(\gamma_{j+1} - r)/(\gamma_j - r) \leq 1/q$. Then each finite set $\{\gamma_n - r, \gamma_{n-1} - r, \ldots, \gamma_1 - r\} \subseteq \mathbb{R}$ is Hadamard with ratio at least q, and thus is $\varepsilon/3$-Kronecker. It is easy to see (Exercise 2.9.20) that the Kronecker constant of a set is the supremum of the Kronecker constants of its finite subsets. From that we conclude that $\mathbf{F} = \{\gamma_j - r : j \geq 1\}$ is weak $\varepsilon/3$-Kronecker in the dual, $\widehat{\mathbb{R}_d}$, of $\mathbb{R}$ with the discrete topology. By Corollary 2.2.15, there exists a finite set $\mathbf{F}'$ such that $(\mathbf{F} \setminus \mathbf{F}') + r$ is $\varepsilon/2$-Kronecker. This last set is in $\mathbb{Q}$, so it is $\varepsilon/2$-Kronecker there, as well, by Lemma 2.2.8. □

2.4.2 Proof of Theorem 2.4.3

Let $\mathbf{\Gamma}_0$ denote the torsion subgroup of $\mathbf{\Gamma}$, that is, the largest subgroup of $\mathbf{\Gamma}$ consisting only of elements of finite order, and let $q : \mathbf{\Gamma} \to \mathbf{\Gamma}/\mathbf{\Gamma}_0$ be the natural quotient homomorphism. Then $\mathbf{\Gamma}/\mathbf{\Gamma}_0$ has no non-trivial elements of finite order.

Case I: $\mathfrak{e} := |\mathbf{E}| = |q(\mathbf{E})|$

We may assume that $\mathbf{\Gamma}$ has no elements of finite order since by Lemma 2.2.8, if we find an ε-Kronecker subset of $q(\mathbf{E})$ we may lift it to an ε-Kronecker subset of $\mathbf{E}$. By (2.4.1), we may also assume that $\mathbf{\Gamma}$ is a subgroup of $\bigoplus_{\ell \in B} \mathbb{Q}_\ell$, where the $\mathbb{Q}_\ell$ are copies of the rational numbers and for every ℓ there is an element $\gamma \in \mathbf{E}$ such that the projection, $\Pi_\ell(\gamma)$, onto $\mathbb{Q}_\ell$ is non-trivial. For $\gamma \in \mathbf{E}$, let

$$B(\gamma) = \{\ell \in B : \Pi_\ell(\gamma) \neq 0\}. \tag{2.4.2}$$

Each $B(\gamma)$ is finite and $B = \bigcup_{\gamma \in \mathbf{E}} B(\gamma)$.

Case Ia: Countable E

If some $\Pi_\ell(\mathbf{E})$ is infinite, then Lemma 2.2.8 and Proposition 2.4.4 imply that, for each $\varepsilon > 0$, $\mathbf{E}$ has an ε-Kronecker subset with the same cardinality as $\mathbf{E}$. Otherwise, we may inductively find $\gamma_j \in \mathbf{E}$ such that $B(\gamma_{j+1}) \not\subseteq \bigcup_{k=1}^{j} B(\gamma_k)$. Because the γ_j have infinite order, it is immediate that $\{\gamma_j\}_{j=1}^\infty$ is an independent set and hence is ε-Kronecker for all $\varepsilon > 0$ by Corollary 2.2.10.

Case Ib: Uncountable E

Let $\mathcal{S}$ be the set of subsets of $\mathbf{E}$ that are independent and partially order $\mathcal{S}$ by inclusion. Since independence is characterized by properties of finite subsets, it follows that every chain in $\mathcal{S}$ has an upper bound, namely the union of the sets in the chain. Use Zorn's lemma to find a maximal independent subset,

$\mathbf{F}$, of $\mathbf{E}$. Such a set will contain only elements of infinite order and we claim there will be $\mathfrak{c}$ of them.

To count the elements of $\mathbf{F}$, let $\mathbf{H} = \langle \mathbf{F} \rangle$, the group generated by $\mathbf{F}$, so $|\mathbf{H}| \le \aleph_0 |\mathbf{F}|$. Observe that if $\chi \in \mathbf{E}$ is non-trivial, then by the maximality of $\mathbf{F}$, there will be a positive integer m such that $\chi^m \in \mathbf{H}$. Let $\mathbf{H}_m = \{\chi \in \mathbf{E} : \chi^m \in \mathbf{H}\}$. The map $\chi \to \chi^m$ is one-to-one, and thus $|\mathbf{H}_m| \le |\mathbf{H}| \le \aleph_0 |\mathbf{F}|$. Because $\mathbf{E} \subseteq \cup_{m=1}^{\infty} \mathbf{H}_m \cup \{\mathbf{1}\}$, it follows that $|\mathbf{E}| \le \aleph_0 |\mathbf{F}|$. Thus, $|\mathbf{F}| \ge |\mathbf{E}|$.

Corollary 2.2.10 completes the proof, as in the countable subcase.

Case II: $|\, q(\mathbf{E})| < \mathfrak{c}$

Without loss of generality, we can assume that $\mathbf{E}$ generates $\mathbf{\Gamma}$. Let $\mathbf{F}'$ be a maximal independent subset of $\mathbf{\Gamma}$ consisting of elements of infinite order. We claim that $|\mathbf{F}'| < \mathfrak{c}$. Indeed, q maps $\mathbf{F}'$ one-to-one onto an independent set of elements of infinite order since otherwise we would have elements $\gamma_j \in \mathbf{F}'$ and integers $L \ge 1$ and $\ell_j \ne 0$ such that $\prod_1^L \gamma_j^{\ell_j} \in \mathbf{\Gamma}_0$, and so for some $m > 0$, we would have $\prod_1^L \gamma_j^{m\ell_j} = \mathbf{1}$, a contradiction. If $|\mathbf{F}'| = \mathfrak{c}$, then $|\, q(\mathbf{E})| = |\, q(\mathbf{\Gamma})| \ge |\mathbf{F}'| = \mathfrak{c}$, which we have assumed is not the case.

Then cardinal arithmetic tells us that $|\mathbf{\Gamma}/\langle \mathbf{F}' \rangle| = \mathfrak{c}$, and the maximality of $\mathbf{F}'$ implies that $\mathbf{\Gamma}/\langle \mathbf{F}' \rangle$ is a torsion group. Using Lemma 2.2.8, we see that it will suffice to find a weak 1-Kronecker subset of $q(\mathbf{E})$, and so we may assume that $\mathbf{\Gamma}$ is a torsion group.

By (2.4.1), we may assume that $\mathbf{\Gamma}$ is a subgroup of $\bigoplus_{\ell \in B} \mathcal{C}(p_\ell^\infty)$, where the index set B has the property that for every $\ell \in B$ there is an element $\gamma \in \mathbf{E}$ such that the projection $\Pi_\ell(\gamma)$ onto $\mathcal{C}(p_\ell^\infty)$ is non-trivial. Furthermore, there must be $\mathfrak{c}$ indices ℓ such that $\Pi_\ell(\mathbf{E})$ contains an element of order at least 3, for otherwise (it is a routine exercise to see that) $\mathbf{E}$ would be 2-large.

We continue to use the notation of (2.4.2). We will use induction if $\mathbf{E}$ is countable and transfinite induction if $\mathbf{E}$ is uncountable. The reader will see that the argument is identical, whether the induction is transfinite or not.

Let $\mathcal{I}$ be a well-ordered index set of cardinality $|B|$, with $1, 2, \ldots$ the first elements of $\mathcal{I}$. Since $\mathbf{E}$ is not 2-large, $\mathbf{E}$ must contain an element of order at least 3. Let $\lambda_1 \in \mathbf{E}$ and $\ell(1) \in B$ be such that the order of $\Pi_{\ell(1)}(\lambda_1)$ is at least 3. That starts the induction.

Suppose $i > 1$ and that we have found $\lambda_{i'} \in \mathbf{E}$ for all $1 \le i' < i$ such that $B(\lambda_{i'}) \not\subseteq \bigcup_{k<i'} B(\lambda_k)$. If $|\{\lambda_{i'} : 1 \le i' < i\}| = \mathfrak{c}$, we stop. Otherwise, we note that $A = \bigcup_{i'<i} B(\lambda_{i'})$ also has cardinality less than $\mathfrak{c}$ and that there exist $\lambda(i) \in \mathbf{E}$ and $\ell(i) \in B$ such that $\ell(i) \notin \bigcup_{i'<i} B(\lambda_{\ell(i')})$ and $\Pi_{\ell(i)}(\lambda_i)$ has order at least 3. That completes the inductive step.

Because there are $\mathfrak{c}$ indices ℓ such that $\Pi_\ell(\mathbf{E})$ contains an element of order at least 3, the set $\mathbf{F} = \{\lambda_{\ell(i)} : i \ge 1\}$ must have the same cardinality $\mathfrak{c}$. We now claim that $\mathbf{F}$ is weak 1-Kronecker. Again we use induction, transfinite or not, depending on the cardinality of $\mathbf{E}$. It will be convenient to assume that $G = \prod_{\ell \in B} G_\ell$, where G_ℓ is the dual of $\mathcal{C}(p_\ell^\infty)$, $\ell \in B$. This assumption is justified by Lemma 2.2.8. We shall abuse notation by using $\Pi_\ell(x)$ to denote the ℓ-coordinate of x for $x \in G$.

Let $\varphi : \mathbf{F} \to \mathbb{T}$. Because $\Pi_{\ell(1)}(\lambda_1)$ has order at least 3, it is possible to choose $x_1 \in G_{\ell(1)}$ such that $|\varphi(\lambda_1) - \lambda_1(x_1)| \le 1$. Suppose now that $i > 1$ and $x_{i'} \in \prod_{k \le i'} G_{\ell(k)}$ have been chosen for $1 \le i' < i$ so that $|\varphi(\lambda_k) - \lambda_k(x_{i'})| \le 1$ and

$$\Pi_{\ell(k)}(x_k) = \Pi_{\ell(k)}(x_{i'}), \tag{2.4.3}$$

whenever $1 \le k \le i' < i$.

If i has an immediate predecessor, i', we choose $x \in G_{\ell(i)}$ such that $|\varphi(\lambda_i) - \lambda_i(x_{i'}x)| \le 1$. Set $x_i = x_{i'}x$. Then (2.4.3) holds for $1 \le k \le i' \le i$.

If i is a limit ordinal, let x_0 be the limit point of the $x_{i'}$ as $i' \to i$. Such a limit point exists because of (2.4.3). This ensures (2.4.3) holds with x_0 in place of $x_{i'}$. Now choose $x \in G_{\ell(i)}$ such that $|\varphi(\lambda_i) - \lambda_i(x_0 x)| \le 1$ and set $x_i = x_0 x$. It is clear that (2.4.3) now holds with $i' = i$. Finally, let $z = \lim_i x_i$. Then, $|\varphi(\gamma) - \gamma(z)| \le 1$, so $\mathbf{F}$ is indeed weak 1-Kronecker. □

In Case I, note that more was actually proved.

Corollary 2.4.5. *Suppose $\mathbf{\Gamma}_0$ is the torsion subgroup of $\mathbf{\Gamma}$ and $q : \mathbf{\Gamma} \to \mathbf{\Gamma}/\mathbf{\Gamma}_0$ is the quotient map. If $\mathbf{E}$ is infinite and $|\,q(\mathbf{E})| = |\,\mathbf{E}|$, then for each $\varepsilon > 0$ and neighbourhood $U \subseteq G$ there is an ε-Kronecker(U) set $\mathbf{F} \subseteq \mathbf{E}$, with the same cardinality as $\mathbf{E}$.*

Proof. The proof of the Theorem case I shows the existence of ε-Kronecker subsets, for any specified $\varepsilon > 0$, of the same cardinality as $\mathbf{E}$. To obtain the ε-Kronecker(U) set, just discard a suitable finite subset. □

Corollary 2.4.6. *Suppose the torsion subgroup of $\mathbf{\Gamma}$ is finite. If $\mathbf{E}$ is an infinite set, then, for each $\varepsilon > 0$ and neighbourhood $U \subseteq G$, there is an ε-Kronecker(U) set $\mathbf{F} \subseteq \mathbf{E}$, with the same cardinality as $\mathbf{E}$.*

2.5 Approximating Arbitrary Choices of ± 1

It can be seen from the improved standard iteration, Corollary 1.3.3, that each bounded function on $\mathbf{E}$ can be (exactly) interpolated with the Fourier–Stieltjes transform of a discrete measure provided it is possible to approximately interpolate all ± 1-valued functions on $\mathbf{E}$. But having this approximation property is clearly not sufficient to ensure that $\mathbf{E}$ is ε-Kronecker for some $\varepsilon < \sqrt{2}$; just consider, for instance, the set of Rademacher functions in $\widehat{\mathbb{D}}$.

One could ask if there are conditions (perhaps on G or $\mathbf{\Gamma}$) that would ensure that the approximation of arbitrary choices of signs is enough to guarantee the set $\mathbf{E}$ is ε-Kronecker. In this section, we give such criteria and consider related questions. Later, in Sect. 4.4, we consider the case where arbitrary signs can be interpolated exactly but outside of the ε-Kronecker context.

Let G_2 be the annihilator of the subgroup, $\mathbf{\Gamma}^{(2)}$, of all characters whose order is a power of 2. Since G_2 is the dual of the quotient group $\mathbf{\Gamma}/\mathbf{\Gamma}^{(2)}$, which has no elements of order two, Lemma C.1.15 implies every element of G_2 has a square root. This will be significant in what follows.

Theorem 2.5.1. *Let $\mathbf{E} \subseteq \mathbf{\Gamma}$ and $\varepsilon > 0$. Suppose that for all choices of signs $\{r_\gamma\}_{\gamma\in\mathbf{E}} \in \mathbb{Z}_2^{\mathbf{E}}$ there exists an element $g \in G_2$ such that $d(\arg\gamma(g), \arg r_\gamma) < \varepsilon$ for all $\gamma \in \mathbf{E}$. Then $\mathbf{E}$ is weak angular 2ε-Kronecker.*

Proof. The key step in the proof is to show that for each positive integer k and all choices of angles, $\{s_\gamma\}_{\gamma\in\mathbf{E}}$, which are the arguments of $2^k th$ roots of unity, there exists some $x_k \in G_2$ such that

$$d(\arg\gamma(x_k), s_\gamma) < (2 - 2^{-k+1})\varepsilon \text{ for all } \gamma \in \mathbf{E}. \tag{2.5.1}$$

This will be proven by an induction argument.

Suppose that (2.5.1) has been established. Fix $\varepsilon' > 2\varepsilon$ and choose k such that $\pi 2^{-k} - \varepsilon 2^{-k+1} + 2\varepsilon < \varepsilon'$. Since the angular distance between two adjacent $2^k th$ roots of unity is $2\pi/2^k$, for each selection $\{\theta_\gamma\}_{\gamma\in E} \in [0, 2\pi)^{\mathbf{E}}$ we can choose arguments of $2^k th$ roots of unity, $\{s_\gamma\}_{\gamma\in\mathbf{E}}$, such that $d(\theta_\gamma, s_\gamma) \leq \pi 2^{-k}$ for all γ. With x_k chosen by (2.5.1) and $\gamma \in \mathbf{E}$,

$$\begin{aligned} d(\arg\gamma(x_k), \theta_\gamma) &\leq d(s_\gamma, \theta_\gamma) + d(\arg\gamma(x_k), s_\gamma) \\ &< \pi 2^{-k} - \varepsilon 2^{-k+1} + 2\varepsilon < \varepsilon'. \end{aligned}$$

Consequently, $\mathbf{E}$ is weak angular 2ε-Kronecker.

It only remains to verify (2.5.1). Since ± 1 are the square roots of unity, (2.5.1) holds for $k = 1$ by the hypothesis of the theorem. Proceed by induction and assume the induction assumption is true for k. Let $\{s_\gamma\}_{\gamma\in\mathbf{E}}$ be arguments of the $2^{k+1} th$ roots of unity and consider $\{2s_\gamma\}$, the arguments of $2^k th$ roots of unity. By the induction assumption, there is some $x_k \in G_2$ such that for all $\gamma \in \mathbf{E}$, $d(\arg\gamma(x_k), 2s_\gamma) < (2 - 2^{-k+1})\varepsilon$. Since every element of G_2 is a square, there is some $y \in G_2$ such that $y^2 = x_k$. Then $\gamma(y)^2 = \gamma(x_k)$, and hence the argument of $\gamma(y)$ is either equal to $\arg\gamma(x_k)/2$ or $\arg\gamma(x_k)/2 + \pi$. Thus, for all $\gamma \in \mathbf{E}$, either

$$d(\arg\gamma(y), s_\gamma) < (1 - 2^{-k})\varepsilon$$

or

$$d(\arg\gamma(y), s_\gamma + \pi) < (1 - 2^{-k})\varepsilon.$$

In the first case, put $r_\gamma = 1$ and in the second case, put $r_\gamma = -1$. According to the hypothesis of the theorem, there is some $g \in G_2$ such that

$$d(\arg\gamma(g), \arg r_\gamma) < \varepsilon \text{ for all } \gamma \in \mathbf{E}.$$

Let $x_{k+1} = gy \in G_2$. Since $\arg r_\gamma$ is either 0 or π, it follows that either

$$\begin{aligned} d(\arg\gamma(x_{k+1}), s_\gamma) &\le d(\arg\gamma(y), s_\gamma) + d(\arg\gamma(g), 0) \\ &< (1-2^{-k})\varepsilon + \varepsilon = (2-2^{-k})\varepsilon, \end{aligned}$$

or

$$\begin{aligned} d(\arg\gamma(x_{k+1}), s_\gamma) &\le d(\arg\gamma(y), s_\gamma + \pi) + d(\arg\gamma(g), \pi) \\ &< (1-2^{-k})\varepsilon + \varepsilon = (2-2^{-k})\varepsilon, \end{aligned}$$

depending on whether it is the first or second case, and that completes the induction step. □

Remark 2.5.2. Suppose $G = \mathbb{T} = [-\pi, \pi)$ and assume the choice of $g \in G_2 = \mathbb{T}$ in the hypothesis of the theorem can be chosen from an open interval U that is centred at 0. Then $\mathbf{E}$ is weak angular 2ε-Kronecker$(\overline{2U})$. To see this, argue as in the proof of the theorem but assume in the induction step that the points x_k can be chosen belonging to $(2-2^{-k+1})U$. This is true by assumption for $k = 1$. For the induction step, note that we can simply choose $y = x_k/2 \in (1-2^{-k})U$, and then $x_{k+1} = y + g \in (1-2^{-k})U + U \subseteq (2-2^{-k})U \subseteq 2U$.

An easy corollary follows from the theorem.

Corollary 2.5.3. *Assume that $\mathbf{\Gamma}$ has no elements of order two and that for some $\tau < \pi/4$ and all choices of signs $\{r_\gamma\}_{\gamma\in\mathbf{E}}$, there exists $g \in G$ such that $d(\arg\gamma(g), \arg r_\gamma) < \tau$ for all $\gamma \in \mathbf{E}$. Then $\mathbf{E}$ is weak angular ε-Kronecker for some $\varepsilon < \pi/2$.*

Proof. Since $\mathbf{\Gamma}$ has no elements of order two, $G = G_2$. □

In particular, if G is connected, then $\mathbf{\Gamma}$ has no elements of finite order, and hence the corollary applies.

Theorem 2.5.1 can be generalized to the situation where an arbitrary choice of signs is replaced by an arbitrary choice of (fixed) two elements in $\mathbb{T}$ with angular distance π. The proof is asked for in Exercise 2.9.17.

Theorem 2.5.4. *Let $\mathbf{E} \subseteq \mathbf{\Gamma}$, $\varepsilon > 0$ and $\theta \in [0, \pi)$. Suppose that for each $\{r_\gamma\}_{\gamma\in\mathbf{E}} \subseteq \{\theta, \theta+\pi\}^{\mathbf{E}}$ there exists an element $g \in G_2$ such that $d(\arg\gamma(g), \arg r_\gamma) < \varepsilon$ for all $\gamma \in \mathbf{E}$. Then $\mathbf{E}$ is weak angular 2ε-Kronecker.*

Theorem 2.5.4 provides a geometric separation condition which ensures a set is ε-Kronecker.

Corollary 2.5.5. *Suppose there are two disjoint intervals, $I, J \subseteq \mathbb{T}$, each with arc length $l < \pi$. Assume that for each $\mathbf{F} \subseteq \mathbf{E}$ there exists an element $g \in G_2$ such that $\gamma(g) \in I$ for all $\gamma \in \mathbf{F}$ and $\gamma(g) \in J$ for all $\gamma \in \mathbf{E} \smallsetminus \mathbf{F}$. Then $\mathbf{E}$ is weak angular $(\pi - m)$-Kronecker, where m is the arc length of the smaller of the two gaps separating I and J.*

Proof. Let θ and $\theta+\pi$ be the two points of distance $\pi/2$ from the centre of the smaller of the two gaps. Since $m \le \pi$, by symmetry (and without loss of generality), the angular distance from each point in interval I (respectively, interval J) to θ (resp., $\theta+\pi$) is at most $(\pi-m)/2$. The conclusion follows from Theorem 2.5.4. □

It is not enough for the set $\mathbf{E}$, itself, to contain no elements of order two for the approximation of arbitrary choices of signs (even exactly) to ensure that $\mathbf{E}$ is ε-Kronecker for some $\varepsilon < \sqrt{2}$, as Example 2.5.6 shows. The set there is also I_0, but not a finite union of ε-Kronecker sets for any $\varepsilon < \sqrt{2}$.

Example 2.5.6. An I_0 set that is not a finite union of ε-Kronecker sets: Let $\mathbf{E} = \{(j, \pi_j) : j = 1, 2, \ldots\} \subseteq \mathbb{Z} \times \widehat{\mathbb{D}}$, where $\{\pi_j\}$ is the Rademacher set in $\widehat{\mathbb{D}}$. Notice $\mathbf{E}$ contains no elements of order two, although the subgroup it generates clearly does. Of course, it is possible to interpolate an arbitrary choice of signs, $\{r_j\}$, exactly on $\mathbf{E}$; just take the point $(0,x)$ where $\pi_j(x) = r_j$. By the standard iteration (applied to the real and imaginary parts of candidate φ's), this property is enough to ensure that $\mathbf{E}$ is I_0.

But $\mathbf{E}$ is not a finite union of ε-Kronecker sets for any $\varepsilon < \sqrt{2}$. To prove this, assume that $\mathbf{E}$ were such a union. Then one of the finitely many sets would contain a net $\{(j_\beta, \pi_{j_\beta})\}$ with $j_\beta \to 0$ in the Bohr topology on $\mathbb{Z}$. Furthermore, because the subset $\{(j_\beta, \pi_{j_\beta})\}_\beta$ is (assumed to be) ε-Kronecker, there would be (x,y) such that

$$|(j_\beta, \pi_{j_\beta})(x,y) - i| = |\mathrm{e}^{\mathrm{i}2\pi j_\beta x}\pi_{j_\beta}(y) - i| < \varepsilon < \sqrt{2} \text{ for all } \beta. \tag{2.5.2}$$

As $j_\beta \to 0$, given a $\delta > 0$, there is some β such that $|\mathrm{e}^{\mathrm{i}2\pi j_\beta x} - 1| < \delta$. Since $\pi_{j_\beta}(y)$ is either ± 1, the inequalities (2.5.2) cannot simultaneously hold for small enough δ.

However, $\mathbf{E}$ is $\sqrt{2}$-Kronecker. The argument is similar to the proof that $\mathbb{Z} \setminus \{0\}$ is 2-Kronecker; see Exercise 2.9.2.

2.6 Arithmetic Properties of ε-Kronecker Sets

2.6.1 Are ε-Kronecker Sets Sidon?

All ε-Kronecker sets with $\varepsilon < \sqrt{2}$ are I_0 and hence Sidon. The situation for $\varepsilon \in [\sqrt{2}, 2)$ is less clear. Example 2.3.6 shows that a Sidon set can be $\sqrt{2}$-Kronecker and not I_0. We do not know if *all* $\sqrt{2}$-Kronecker sets are Sidon, much less all ε-Kronecker sets with $\varepsilon \in [\sqrt{2}, 2)$ **[P 4]**. Since the non-Sidon set, $\mathbb{Z} \setminus \{0\}$, is 2-Kronecker (Remarks 2.2.5) the question is settled for $\varepsilon = 2$.

In this section, various arithmetic properties of ε-Kronecker sets, even for $\varepsilon \in [\sqrt{2}, 2)$, are established. In Sect. 6.3.2 it will be seen that Sidon sets also possess these properties, and thus the results that follow can be taken as evidence for an affirmative answer to the question of the section title, for ε in that range.

We now turn to the above-promised arithmetic properties.

Definition 2.6.1. The set $\mathbf{P}$ is called a *parallelepiped of dimension* N if $\mathbf{P} = \prod_{j=1}^{N}\{\chi_j, \gamma_j\}$, where χ_j, $\gamma_j \in \mathbf{\Gamma}$ and $|\,\mathbf{P}| = 2^N$. The characters χ_j, γ_j, for $j = 1, \ldots, N$ (which need not be distinct), are called the *generators* of $\mathbf{P}$.

An example of a parallelepiped of dimension N is an arithmetic progression in $\mathbb{Z}$ of length 2^N. Indeed, $\{a, a+d, \ldots, a+(2^N-1)d\} = \{a, a+d\} + \sum_{j=1}^{N-1}\{0, 2^j d\}$.

Like a Sidon set (Corollary 6.3.13), an ε-Kronecker set can only contain a small portion of each long arithmetic progression.

Theorem 2.6.2. *Suppose* $\mathbf{E}$ *is* ε*-Kronecker for some* $\varepsilon < 2$*. For every* $\tau > 0$ *there is a constant* $C = C(\tau, \varepsilon)$ *such that* $|\,\mathbf{E} \cap \mathbf{P}| \leq C2^{N\tau}$ *for each parallelepiped* $\mathbf{P} \subseteq \mathbf{\Gamma}$ *of dimension* N*.*

Proof. We will prove that there exists a constant $N_0 = N_0(\varepsilon, \tau)$ such that if $\mathbf{P}$ is a parallelepiped of dimension N and $\mathbf{E}_1$ is a subset of $\mathbf{E} \cap \mathbf{P}$ of cardinality $2^{N\tau}$, then $N \leq N_0(\varepsilon, \tau)$. That will prove $|\,\mathbf{E} \cap \mathbf{P}| < 2^{N\tau}$ for all parallelepipeds of dimension N when $N > N_0$ and therefore we will be able to take $C = 2^{N_0}$. Being ε-Kronecker for some $\varepsilon < 2$, $\mathbf{E}$ is angular $(\pi - \delta)$-Kronecker for some $\delta > 0$. Fix an even integer $M > \pi/\delta$ and consider X, the set of all functions mapping $\mathbf{E}_1$ to T_{MN}, the group of $MNth$ roots of unity. There are $\big(MN\big)^{|\,\mathbf{E}_1|}$ such functions. We will call a function in X *multiplicative* if it is the restriction to $\mathbf{E}_1$ of a function defined on the generators of $\mathbf{P}$ and extended by multiplicativity to $\mathbf{P}$. There are at most $(MN)^{2N}$ multiplicative functions in X.

Temporarily fix a multiplicative function Φ. If h is an arbitrary function in X and $|h(\gamma) - \Phi(\gamma)| < 2$ for every $\gamma \in \mathbf{E}_1$, then $h(\gamma) \neq -\Phi(\gamma)$ for all $\gamma \in \mathbf{E}_1$. Since M is even, $-\Phi(\gamma)$ is another $MNth$ root of unity. Hence, there can be at most $\big(MN-1\big)^{|\,\mathbf{E}_1|}$ functions in X whose distance to the function Φ is strictly less than 2 and a total of at most $\big(MN-1\big)^{|\,\mathbf{E}_1|}(MN)^{2N}$ functions in X with distance less than 2 to some multiplicative function in X.

It can be verified that if N is sufficiently large and $|\,\mathbf{E}_1| \geq 2^{\tau N}$, then

$$\big(MN\big)^{|\,\mathbf{E}_1|} > \big(MN-1\big)^{|\,\mathbf{E}_1|}(MN)^{2N}.$$

The strict inequality proves there is some function $h \in X$ whose distance to every multiplicative function in X is equal to 2.

Since $\mathbf{E}$ is angular $(\pi - \delta)$-Kronecker, there is some $x \in G$ such that $d(\arg h(\gamma), \arg \gamma(x)) < \pi - \delta$ for every $\gamma \in \mathbf{E}_1$. Define a multiplicative function Φ on the generators of $\mathbf{P}$ by choosing $\Phi(\gamma)$ to be the $MNth$ root of unity closest to $\gamma(x)$. Since every character in $\mathbf{P}$ is the product of N generators, $d(\arg\Phi(\gamma), \arg\gamma(x)) < \pi/M$ for all $\gamma \in \mathbf{P}$. But then

$$d(\arg h(\gamma), \arg\Phi(\gamma)) \leq d(\arg h(\gamma), \arg\gamma(x)) + d(\arg\Phi(\gamma), \arg\gamma(x))$$
$$< \pi - \delta + \pi/M < \pi.$$

This is a contradiction since the angular distance from h to each multiplicative function in X is π. □

Corollary 2.6.3. *Suppose* $\mathbf{E} \subseteq \mathbb{Z}$ *is an* ε*-Kronecker set and* $\mathbf{A}$ *is an arithmetic progression of length* N*. For each* $\tau > 0$ *there is a constant* C*, depending only on* ε *and* τ*, such that* $|\,\mathbf{E} \cap \mathbf{A}| \leq CN^{\tau}$*.*

A *square* in $\mathbf{\Gamma}$ is a set of the form $\mathbf{E}_1 \cdot \mathbf{E}_2 \subseteq \mathbf{\Gamma}$, where $|\,\mathbf{E}_1 \cdot \mathbf{E}_2| = |\,\mathbf{E}_1|\,|\,\mathbf{E}_2|$ and $|\,\mathbf{E}_1| = |\,\mathbf{E}_2|$. The same argument as above, viewing the characters in $\mathbf{E}_1$ and $\mathbf{E}_2$ as generators of the square, shows that ε-Kronecker sets do not contain arbitrarily large squares. The details are left to Exercise 2.9.10. That is another property possessed by Sidon sets (Proposition 6.3.12).

Proposition 2.6.4. *Suppose* $\mathbf{E}$ *is* ε*-Kronecker for some* $\varepsilon < 2$*. There is a constant* $N = N(\varepsilon)$ *such that* $\mathbf{E}$ *does not contain a square of cardinality* N^2*.*

We now turn to the sum of reciprocals of elements of $\mathbf{E}$.

Proposition 2.6.5. *Let* $\{k_j\}_{j=1}^{J} \subseteq \mathbb{N}$ *be increasing and assume* $\gamma \in \mathbf{\Gamma}$ *has order exceeding* k_J*. If* $\sum_{j=1}^{J} 1/k_j = s$*, then* $\alpha(\{\gamma^{k_j}\}_{j=1}^{J}) \geq \pi(1 - s^{-1})$*.*

The following corollary for subsets of $\mathbb{N}$ is immediate from Proposition 2.6.5 and the fact that $\log J \leq \sum_{k=1}^{J} 1/k$.

Corollary 2.6.6. $\alpha(\{1, \ldots, J\}) \geq \pi(1 - 1/\log J)$.

Proof (of Proposition 2.6.5). Let $\mathbf{E} = \{\gamma^{k_j}\}_{j=1}^{J}$ and $\delta = \pi/s$. For a positive integer k and $z \in \mathbb{T}$, consider

$$W(k, z) = \bigcup_{\ell=0}^{k-1} \big\{w \in \mathbb{T} : \, d(\arg(z e^{2\pi i \ell/k}), \arg w) \leq \delta/k\big\}.$$

Then $W(k, z)$ consists of k arcs each of (angular) length $2\delta/k$, centred at elements $z e^{2\pi i \ell/k}$, $0 \leq \ell < k$. A sketch of the unit circle may be helpful here.

Choose z_1 so that $W(k_1, z_1) \supseteq \{w : 0 \leq \arg w \leq 2\delta/k_1\}$. Choose $z_2, \ldots, z_J \in \mathbb{T}$ inductively such that

$$W(k_j, z_j) \supseteq \left\{ w : \sum_{\ell=1}^{j-1} 2\delta/k_\ell \leq \arg w \leq \sum_{\ell=1}^{j} 2\delta/k_\ell \right\} \text{ for } 2 \leq j \leq J.$$

Then the hypothesis, $\sum_1^J 1/k_j = s$, ensures $\bigcup_{j=1}^{J} W(k_j, z_j) = \mathbb{T}$.

Define $\varphi : \mathbf{E} \to \mathbb{T}$ by $\varphi(\gamma^{k_j}) = -z_j^{k_j}$ (the point antipodal to $z_j^{k_j}$) for $1 \le j \le J$. We claim $\sup_{\lambda\in\mathbf{E}} d\big(\arg(\varphi(\lambda)), \arg(\lambda(x))\big) \ge \pi - \delta$ for all $x \in G$. Indeed, for every $x \in G$ there exists $1 \le j \le J$ such that $\gamma(x) \in W(k_j, z_j)$. Then $d\big(\arg(z_j), \arg(\gamma(x))\big) \le \delta/k_j$, so $d\big(\arg(\gamma^{k_j}(x)), \arg(z_j^{k_j})\big) \le \delta$ and

$$d\big(\arg(\varphi(\gamma^{k_j})), \arg(\gamma^{k_j}(x))\big) \ge \pi - \delta. \qquad \square$$

2.7 Products of ε-Kronecker Sets Are “Small”

In this section we study the small size or “thinness” of products of an ε-Kronecker set, as was done for Hadamard sets in Chap. 1. Not clustering at a continuous character is one form of thinness, addressed in our first section. Another form of thinness is the closure of a sum of copies of $\mathbf{E}$ having zero Haar measure. That is discussed in the second section.

2.7.1 Bohr Cluster Points of Kronecker Sets and Their Products

We begin by proving that an ε-Kronecker set does not cluster in the Bohr topology (see Sect. C.1.3) at a continuous character, a special case of the fact that an I_0 set does not cluster at a continuous character (the Ryll–Nardzewski–Méla–Ramsey Theorem 3.5.1). A more elementary proof can be given for ε-Kronecker sets.

It is unknown if a Sidon set can cluster at a continuous character. That problem will be discussed in more detail in Chap. 8.

Proposition 2.7.1. *An ε-Kronecker set $\mathbf{E}$ does not cluster in the Bohr topology at any $\gamma \in \mathbf{\Gamma}$, if $\varepsilon < 2$.*

Proof. Let $\tau = (2-\varepsilon)/2 > 0$. Choose an e-neighbourhood $U \subseteq G$ such that $|1-\gamma(u)| < \tau$ for all $u \in U$. Let $\mathbf{F} \subseteq \mathbf{E}$ be finite such that $\mathbf{E} \setminus \mathbf{F}$ is weak ε-Kronecker(U) and let $u \in U$ be such that $|\chi(u) + 1| < \varepsilon$ for all $\chi \in \mathbf{E} \setminus \mathbf{F}$. Then, $|\gamma(u) - \chi(u)| \ge 2 - \varepsilon - \tau \ge (2-\varepsilon)/2$ for all $\chi \in \mathbf{E} \setminus \mathbf{F}$, so $\mathbf{E} \setminus \mathbf{F}$ does not cluster at γ in the Bohr topology. $\square$

The next proposition is an elaboration of the idea in the proof of the Kunen–Rudin Theorem 1.5.1. It applies to Hadamard sets by Theorem 1.5.1, ε-Kronecker sets with $\varepsilon < 2$ by Proposition 2.7.1 and to all I_0 sets by the Ryll–Nardzewski–Méla–Ramsey Theorem 3.5.1.

Proposition 2.7.2. *Suppose $\mathbf{E} \subseteq \mathbf{\Gamma}$ has no Bohr cluster points in $\mathbf{\Gamma}$. The following are equivalent:*

1. *The only element of* $\mathbf{\Gamma}$ *which is a Bohr cluster point of* $\mathbf{E} \cdot \mathbf{E}^{-1}$ *is* $\mathbf{1}$.
2. $\gamma\overline{\mathbf{E}} \cap \rho\overline{\mathbf{E}} \subseteq \mathbf{\Gamma}$ *for every* $\gamma \neq \rho \in \mathbf{\Gamma}$.
3. $\gamma\overline{\mathbf{E}} \cap \rho\overline{\mathbf{E}}$ *is a finite subset of* $\mathbf{\Gamma}$ *for every* $\gamma \neq \rho \in \mathbf{\Gamma}$.
4. $\big(\gamma(\overline{\mathbf{E}} \smallsetminus \mathbf{E})\big) \cap (\overline{\mathbf{E}} \smallsetminus \mathbf{E}) = \emptyset$ *for every* $\gamma \neq \mathbf{1} \in \mathbf{\Gamma}$.

Similarly, the following are equivalent:

5. *No element of* $\mathbf{\Gamma}$ *is a cluster point of* $\mathbf{E} \cdot \mathbf{E}$.
6. $\gamma\overline{\mathbf{E}} \cap \overline{\mathbf{E}}^{-1} \subseteq \mathbf{\Gamma}$ *for every* $\gamma \in \mathbf{\Gamma}$.
7. $\big(\gamma(\overline{\mathbf{E}} \smallsetminus \mathbf{E})\big) \cap (\overline{\mathbf{E}} \smallsetminus \mathbf{E})^{-1} = \emptyset$ *for every* $\gamma \in \mathbf{\Gamma}$.

Remark 2.7.3. The ρ in (2)–(3) is superfluous, but it will be convenient to have the particular formulation later.

Proof. (1) $\Rightarrow$ (2) Suppose there is some character $\lambda \in \gamma\overline{\mathbf{E}} \cap \rho\overline{\mathbf{E}}$ with $\lambda \in \overline{\mathbf{\Gamma}} \smallsetminus \mathbf{\Gamma}$. Then λ is a cluster point of nets $\{\gamma\chi_\beta\}$ and $\{\rho\psi_\beta\}$, with $\{\chi_\beta\}, \{\psi_\beta\} \subseteq \mathbf{E}$. Since multiplication is jointly continuous in $\overline{\mathbf{\Gamma}}$, $\gamma\rho^{-1} \in \mathbf{\Gamma}$ is a cluster point of the net $\{\chi_\beta^{-1}\psi_\beta\}$ and thus of $\mathbf{E} \cdot \mathbf{E}^{-1}$. By (1), $\gamma = \rho$.

(2) $\Rightarrow$ (3) If $\gamma\overline{\mathbf{E}} \cap \rho\overline{\mathbf{E}}$ were an infinite set, it would have a Bohr cluster point. That cluster point could not be a continuous character because of the first assumption of the lemma. Therefore, $\gamma\overline{\mathbf{E}} \cap \rho\overline{\mathbf{E}}$ would not be contained in $\mathbf{\Gamma}$, contradicting (2).

(3) $\Rightarrow$ (4) Suppose $\chi \in \gamma(\overline{\mathbf{E}} \smallsetminus \mathbf{E}) \cap (\overline{\mathbf{E}} \smallsetminus \mathbf{E})$. Since χ is a cluster point of $\mathbf{E}$, χ is not in $\mathbf{\Gamma}$. That ensures $\gamma\overline{\mathbf{E}} \cap \overline{\mathbf{E}}$ is not a subset of $\mathbf{\Gamma}$, and hence (3) implies $\gamma = \mathbf{1}$.

(4) $\Rightarrow$ (1) Suppose $\mathbf{E} \cdot \mathbf{E}^{-1}$ has cluster point $\gamma \in \mathbf{\Gamma} \smallsetminus \{\mathbf{1}\}$. Then there are nets $\{\lambda_\beta\}, \{\rho_\beta\} \subseteq \mathbf{E}$ such that $\lambda_\beta\rho_\beta^{-1} \to \gamma$. Without loss of generality we may assume that $\{\rho_\beta\}$ converges to a character $\zeta \in \overline{\mathbf{\Gamma}} \smallsetminus \mathbf{\Gamma}$. But then $\lambda_\beta \to \gamma\zeta$. Thus, $\zeta \in \gamma^{-1}(\overline{\mathbf{E}} \smallsetminus \mathbf{E}) \cap (\overline{\mathbf{E}} \smallsetminus \mathbf{E})$. Since $\gamma \neq \mathbf{1}$ this contradicts (4).

The proof of the equivalences of (5)–(7) is similar (Exercise 2.9.19). $\square$

The next result is an ε-Kronecker set form of the Kunen–Rudin Theorem 1.5.1.

Proposition 2.7.4. *Let* $\varepsilon < \sqrt{2}$ *and* $\mathbf{E}$ *be an* ε*-Kronecker set. Then*

1. $\mathbf{1}$ *is the only continuous character at which* $\mathbf{E} \cdot \mathbf{E}^{-1}$ *clusters.*
2. $\mathbf{E} \cdot \mathbf{E}$ *does not cluster at a continuous character.*

Proof. Let $C > 0$ be given by Corollary 2.3.4.

(1) Let $\gamma \in \mathbf{\Gamma}$, $\gamma \neq \mathbf{1}$. Then Proposition 2.7.2(1)–(4) imply it will suffice to show that $\big(\gamma(\overline{\mathbf{E}} \smallsetminus \mathbf{E})\big) \cap (\overline{\mathbf{E}} \smallsetminus \mathbf{E}) = \emptyset$.

Let $x \in G$ be such that $|1 - \gamma(x)| \geq 1$ and let $U = \{u \in G : |\gamma(u) - \gamma(x)| < 1/(4C)\}$. Now let $\text{Ł} = \text{Ł}(U)$ (whose existence is guaranteed by Corollary 2.3.4) be a finite set such that for every function $\varphi : \mathbf{E} \to \varDelta$ there exists $\mu \in M_d^+(U)$ with $\varphi = \widehat{\mu}$ on $\mathbf{E} \smallsetminus \text{Ł}$ and $\|\mu\| \leq C$.

Set $\varphi = 1$ on $\mathbf{E}$ and obtain $\mu \in M_d^+(U)$ with $\widehat{\mu} = \varphi$ on $\mathbf{E} \setminus Ł$. Since $\widehat{\mu}(\rho) = \varphi(\rho) = 1$ for every $\rho \in \mathbf{E} \setminus Ł$ we have

$$|\gamma(x) - \widehat{\mu}(\gamma\rho)| = \left| \int_U \rho(u)\big(\gamma(x) - \gamma(u)\big)\mathrm{d}\mu \right| \leq \|\mu\|/(4C) = 1/4.$$

This shows $|1 - \widehat{\mu}(\gamma\rho)| \geq 3/4$ and therefore, $\widehat{\mu}(\gamma(\mathbf{E} \setminus Ł))$ and $\widehat{\mu}(\mathbf{E} \setminus Ł)$ have disjoint closures. Consequently, $(\gamma(\overline{\mathbf{E}} \setminus \mathbf{E})) \cap (\overline{\mathbf{E}} \setminus \mathbf{E}) = \emptyset$.

(2) In view of Proposition 2.7.2(5)–(7), it will suffice to show that $\big(\gamma(\overline{\mathbf{E}} \setminus \mathbf{E})\big) \cap (\overline{\mathbf{E}} \setminus \mathbf{E})^{-1} = \emptyset$ for all $\gamma \in \mathbf{\Gamma}$. Let $U = \{x \in G : |1 - \gamma(x)| < 1/(4C)\}$ and let $Ł = Ł(U)$. Choose $\varphi = i$ on $\mathbf{E}$ and $\mu \in M_d^+(U)$ with $\widehat{\mu} = \varphi$ on $\mathbf{E} \setminus Ł$. Then, $|i - \widehat{\mu}(\gamma\rho)| \leq C/(4C)$ for all $\rho \in \mathbf{E} \setminus Ł$. Since $\mu \in M_d^+(U)$, we have $\widehat{\mu} = -i$ on $(\mathbf{E} \setminus Ł)^{-1}$ and so $\gamma(\overline{\mathbf{E}} \setminus \mathbf{E}) \cap (\overline{\mathbf{E}} \setminus \mathbf{E})^{-1} = \emptyset$. □

Remarks 2.7.5. (i) With smaller ε, one can have more terms in the products in Proposition 2.7.4. See the section Remarks and Credits for more information.

(ii) It is essential to have $\varepsilon < \sqrt{2}$ in the theorem. Consider the set $\mathbf{E}$ of Example 2.3.6. Then $\mathbf{E} - \mathbf{E} = \mathbb{Z}$, so its Bohr cluster points include all of $\mathbb{Z}$. Replacing the $N_j + j$ terms with $-N_j + j$, one obtains (Exercise 2.9.6) a $\sqrt{2}$-Kronecker set with $\mathbf{E} + \mathbf{E}$ clustering at every point of $\mathbb{Z}$ since it contains the semigroup $\mathbb{N}$ and a compact semigroup is a group (Exercise C.4.16).

In an ordered group, such as $\mathbb{Z}$, we can measure the "step lengths" of an increasing sequence $\{n_j\}$ as the difference $n_{j+1} - n_j$. In the general case, we have the following definition.

Definition 2.7.6. The *step length of* $\mathbf{E} \subseteq \mathbf{\Gamma}$ *tends to infinity* if for every finite set $\mathbf{F}' \subseteq \mathbf{\Gamma}$, there exists a finite set $\mathbf{F} \subseteq \mathbf{E}$ such that

$$\gamma\chi^{-1} \notin \mathbf{F}' \text{ if } \gamma, \chi \in \mathbf{E} \setminus \mathbf{F}, \gamma \neq \chi.$$

Hadamard sets obviously have step length tending to infinity. On the other hand, any set which is a union of an infinite set and a translate of that set does not. It will be shown later that every Sidon set is a finite union of sets whose step length tends to infinity (Corollary 6.4.7).

With the previous results we can show that ε-Kronecker sets have step length tending to infinity. In fact, a more general result is true.

Proposition 2.7.7. *Suppose* $\mathbf{E} \subseteq \mathbf{\Gamma}$ *has no Bohr cluster points in* $\mathbf{\Gamma}$ *and that* $\gamma\overline{\mathbf{E}} \cap \overline{\mathbf{E}} \subseteq \mathbf{\Gamma}$ *for all* $\gamma \neq \mathbf{1}$. *Then* $\mathbf{E}$ *has step length tending to infinity.*

Proof. If $\mathbf{E}$ does not have step length tending to infinity, then by definition there is a finite set $\mathbf{F}'$ such that for all finite $\mathbf{F}$, $(\mathbf{E} \setminus \mathbf{F}) \cdot (\mathbf{E} \setminus \mathbf{F})^{-1} \cap \mathbf{F}' \neq \emptyset$. It follows that there exists $\chi \in \mathbf{F}'$ and distinct $\gamma_n, \rho_n \in \mathbf{E}$ such that $\gamma_n\rho_n^{-1} = \chi$

for $n \geq 1$. In particular, $\rho_n \in (\chi^{-1}\mathbf{E}) \cap \mathbf{E}$ for $n \geq 1$. Of course, every cluster point of $\{\rho_n\}$ belongs to $(\chi^{-1}\overline{\mathbf{E}}) \cap \overline{\mathbf{E}}$ and cannot be a continuous character. But that contradicts the second hypothesis of the proposition. □

A stronger version of the following corollary is in Exercise 2.9.12.

Corollary 2.7.8. *Suppose that $\varepsilon < \sqrt{2}$ and that $\mathbf{E} \subseteq \mathbf{\Gamma}$ is ε-Kronecker. Then the step length of $\mathbf{E}$ tends to infinity.*

Proof. Immediate from the preceding results of this section. □

Corollary 2.7.9. *Let $\varepsilon < \sqrt{2}$ and $\mathbf{E}$ be ε-Kronecker. Then for every finite set $\mathbf{F} \subseteq \mathbf{\Gamma}$, $\mathbf{E} \cdot \mathbf{F}$ is I_0.*

Proof. Let $\mathbf{E}, \mathbf{F}$ be as above. Proposition 2.7.2 and Proposition 2.7.4 imply that for each $\gamma \neq \rho \in \mathbf{\Gamma}$, $\gamma\overline{\mathbf{E}} \cap \rho\overline{\mathbf{E}}$ is finite. Hence, there is a finite set $\mathbf{H}_{\gamma,\rho}$ such that $\gamma\overline{(\mathbf{E} \setminus \mathbf{H}_{\gamma,\rho})} \cap \rho\overline{(\mathbf{E} \setminus \mathbf{H}_{\gamma,\rho})} = \emptyset$. Let $\mathbf{H} = \bigcup_{\gamma \neq \rho \in \mathbf{F}} \mathbf{H}_{\gamma,\rho}$. Then $\mathbf{H}$ is finite. Since the sets $\gamma\overline{(\mathbf{E} \setminus \mathbf{H})}$ and $\rho\overline{(\mathbf{E} \setminus \mathbf{H})}$ are disjoint when $\gamma \neq \rho$, the local units theorem, Theorem C.1.6, applied to those subsets of $\overline{\mathbf{\Gamma}}$ implies that there exists, for each $\gamma \in \mathbf{F}$, a discrete measure $\nu_\gamma \in M_d(G)$ such that

$$\widehat{\nu_\gamma}(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \gamma\overline{(\mathbf{E} \setminus \mathbf{H})} \text{ and} \\ 0 & \text{if } \lambda \in \rho\overline{(\mathbf{E} \setminus \mathbf{H})} \text{ for some } \rho \in \mathbf{F}, \rho \neq \gamma. \end{cases}$$

Now let $\varphi \in \ell^\infty(\mathbf{F} \cdot (\mathbf{E} \setminus \mathbf{H}))$. Since the class of I_0 sets is clearly closed under translation, $\gamma(\mathbf{E} \setminus \mathbf{H})$ is I_0 for each $\gamma \in \mathbf{F}$. Hence, there exists $\mu_\gamma \in M_d(G)$ with $\widehat{\mu_\gamma} = \varphi$ on $\gamma(\mathbf{E} \setminus \mathbf{H})$. Then $\sum_{\gamma \in \mathbf{F}} \nu_\gamma * \mu_\gamma$ has Fourier–Stieltjes transform equal to φ on $\mathbf{F} \cdot (\mathbf{E} \setminus \mathbf{H})$. Since none of the translates of $\mathbf{E} \setminus \mathbf{H}$ cluster in $\mathbf{\Gamma}$, $\mathbf{F} \cdot (\mathbf{E} \setminus \mathbf{H})$ does not cluster at an element of $\mathbf{F} \cdot \mathbf{H}$. Therefore, we again use the local identity theorem, this time to add the points in $\mathbf{F} \cdot \mathbf{H}$. □

Remark 2.7.10. Corollary 2.7.9 is an instance of Proposition 5.2.7.

2.7.2 U_0 Sets and the Closure of Products

For a compact set X of a locally compact abelian group, $M_0(X)$ denotes the set of regular, bounded, Borel measures μ supported on X, such that the Fourier–Stieltjes transform of μ vanishes at infinity on the dual group. See Lemma C.1.9 for facts about $M_0(X)$.

Definition 2.7.11. A set X is called a *set of uniqueness in the weak sense* (*U_0 set*) if $M_0(X) = \{0\}$.

The Riemann–Lebesgue lemma shows a U_0 set has zero $\overline{\mathbf{\Gamma}}$-Haar measure.

An interpretation of Proposition 1.4.3 is that there exists an angular π/M-Kronecker set $\mathbf{E} \subseteq \mathbb{Z}$ such that $(\mathbf{E} \cup \mathbf{E}^{-1})^{M+2}$ is dense in $\overline{\mathbb{Z}}$. If the Kronecker constant is halved, we obtain a strong converse, namely, the product is U_0.

Theorem 2.7.12. *Suppose that M is a positive integer and $\mathbf{E} \subseteq \mathbf{\Gamma}$ is an angular τ-Kronecker set with $\tau < \pi/2M$. Then the closure in the Bohr topology of $(\mathbf{E} \cup \mathbf{E}^{-1})^M$ is a U_0 set.*

Corollary 2.7.13. *Suppose $\mathbf{E}$ is Hadamard with ratio $q > 5$. Then $\overline{\mathbf{E}+\mathbf{E}}$ and $\overline{\mathbf{E}-\mathbf{E}}$ are U_0 sets in $\overline{\mathbb{Z}}$.*

Proof (of Theorem 2.7.12). We proceed by induction on M. To begin, let $\mu \in M_0(\overline{\mathbf{E}} \cup \overline{\mathbf{E}}^{-1})$. Since $\mu|_{\overline{\mathbf{E}}}$ and $\mu|_{\overline{\mathbf{E}}^{-1}}$ are absolutely continuous with respect to μ, Lemma C.1.9 implies it will be enough to show both $\overline{\mathbf{E}}$ and $\overline{\mathbf{E}}^{-1}$ are U_0 sets. Since there are no non-zero point mass measures in M_0, Lemma C.1.9 also implies that $\mu(\mathbf{F}) = 0$ for all finite sets $\mathbf{F}$. Finally, that lemma ensures that there is no loss of generality in assuming $\mu \in M_0(\overline{\mathbf{E}})$ is a probability measure.

For each positive integer k choose a finite set, $\mathbf{F}_k \subseteq \mathbf{E}$, of cardinality k. For each $\varphi \in \{0, \pi\}^{\mathbf{F}_k}$ use the angular τ-Kronecker property of $\mathbf{E}$ to choose $x \in G$ such that $|\arg \gamma(x)| \le \tau < \pi/2$ for all $\gamma \in \mathbf{E} \setminus \mathbf{F}_k$, or, equivalently,

$$\begin{aligned} &\mathfrak{Re}\gamma(x) \ge \cos\tau > 0 \text{ for all } \gamma \in \mathbf{E} \setminus \mathbf{F}_k \text{ and} \\ &|\arg \gamma(x) - \varphi(\gamma)| \le \tau \text{ for all } \gamma \in \mathbf{F}_k. \end{aligned}$$

Since $\overline{\mathbf{E}} = (\overline{\mathbf{E} \setminus \mathbf{F}_k}) \bigcup \mathbf{F}_k$ and μ is a probability measure supported on $\overline{\mathbf{E} \setminus \mathbf{F}_k}$, this produces 2^k distinct $x \in G$ such that

$$\mathfrak{Re}\widehat{\mu}(x) = \mathfrak{Re}\Big(\int_{\overline{\mathbf{E}}} \gamma(x)\mathrm{d}\mu(\gamma)\Big) = \int_{\overline{\mathbf{E} \setminus \mathbf{F}_k}} \mathfrak{Re}\gamma(x)\mathrm{d}\mu(\gamma) \ge \cos\tau > 0.$$

Because k is arbitrary, that proves $\widehat{\mu}$ does not vanish at infinity on G_d, a contradiction.

Now proceed inductively and assume the result holds for some $M \ge 1$. Let $\mathbf{E}$ be an angular τ-Kronecker set for $\tau < \pi/2(M+1)$. Since $(\overline{\mathbf{E}} \bigcup \overline{\mathbf{E}}^{-1})^{M+1}$ is a finite union of sets of the form

$$\overbrace{\overline{\mathbf{E}}^{\pm 1} \cdots \overline{\mathbf{E}}^{\pm 1}}^{M+1},$$

another application of Lemma C.1.9 shows it is enough to prove each of these sets is U_0. For notational convenience, we will assume all signs are $+1$; it will be clear that the choice of signs is essentially irrelevant to the proof.

So let μ be a probability measure in $M_0(\overline{\mathbf{E}}^{M+1})$. Notice that for every finite set $\mathbf{F}$,

$$\overline{\mathbf{E}}^{M+1} = \overline{(\mathbf{E} \setminus \mathbf{F})}^{M+1} \cup \left(\bigcup_{j=1}^{M} \overline{\mathbf{E}}^{j} \cdot \mathbf{F}^{M-j+1}\right) \cup \mathbf{F}^{M+1}.$$

By the inductive assumption, $\nu(\overline{\mathbf{E}}^j) = 0$ for all $0 \le j \le M$ whenever $\nu \in M_0$. Since the translate of a measure in M_0 is again in M_0, it follows that $\mu(\rho\overline{\mathbf{E}}^j) = 0$ for all $0 \le j \le M$ and characters ρ. Thus, μ is supported on $\overline{(\mathbf{E} \smallsetminus \mathbf{F})}^{M+1}$.

Now we argue as in the case $M = 1$. For each positive integer k choose a finite set, $\mathbf{F}_k \subseteq \mathbf{E}$, of cardinality k. For each $\varphi \in \{0, \pi\}^{\mathbf{F}_k}$ use the angular τ-Kronecker property of $\mathbf{E}$ to choose $x \in G$ such that

$$|\arg\gamma(x)| \le \tau < \pi/2(M+1) \text{ for all } \gamma \in \mathbf{E} \smallsetminus \mathbf{F}_k \text{ and}$$
$$|\arg\gamma(x) - \varphi(\gamma)| \le \tau \text{ for all } \gamma \in \mathbf{F}_k.$$

This produces 2^k distinct $x \in G$ such that

$$|\arg\rho(x)| \le \tau(M+1) < \pi/2 \text{ for all } \rho \in \left(\overline{\mathbf{E} \smallsetminus \mathbf{F}_k}\right)^{M+1}.$$

Thus

$$\begin{aligned}\mathfrak{Re}\widehat{\mu}(x) &= \mathfrak{Re}\Big(\int_{\overline{\mathbf{E}}^{M+1}} \gamma(x)\mathrm{d}\mu(\gamma)\Big) \\ &= \int_{(\overline{\mathbf{E} \smallsetminus \mathbf{F}_k})^{M+1}} \mathfrak{Re}\gamma(x)\mathrm{d}\mu(\gamma) \ge \cos\tau > 0.\end{aligned}$$

We derive the same contradiction as before. □

2.8 Remarks and Credits

Definition and Properties. Unless otherwise indicated, the results in this chapter are from [51–53, 55, 59, 61].

ε-Kronecker sets seem to have first appeared in Kahane's exposition [102, p. 226] of Varopoulos's tensor algebra work, though they are not named. The term "ε-Kronecker set" appears first in Varopoulos [187], and such sets were studied by Givens and Kunen [46], who used the term "ε-free set". Kronecker sets (and, more generally, independent sets) have generated an extensive literature; cf. [56].

ε-Kronecker sets can be defined for non-discrete $\mathbf{\Gamma}$. Most of the results of this chapter (with obvious modifications) hold when $\mathbf{\Gamma}$ is metrizable (not merely discrete), but difficulties arise when $\mathbf{\Gamma}$ is not metrizable. See Sect. A.1 for details.

Kronecker's classical approximation theorem states that if $h_1, \ldots, h_n$ are rationally independent real numbers and $\theta_1, \ldots \theta_n$ are arbitrary real numbers, then given any $\varepsilon > 0$ there is some real number t such that $|h_j t - \theta_j| < \varepsilon \mod 1$ for all $j = 1, \ldots, n$. Thus, any finite set of rationally independent real numbers is Kronecker. For further discussion, see [87, pp. 435–436].

Example 2.2.7 is a descendent of one in [89, 178, 179], which can also be found in [167, 5.7.6]. Other examples may be found in [59, 64, 66].

A Hadamard set with ratio q is angular τ-Kronecker with $\tau \leq \pi/(q-1)$. This is not much use for small q and we do not know if every Hadamard set is ε-Kronecker for some $\varepsilon < 2$ **[P 3]**, though our examples show that this information, by itself, would be of limited use.

ε-Kronecker and I_0 Sets. That 1-Kronecker sets are I_0 was certainly known to Kahane when he wrote [102], if not to Varopoulos. That ε-Kronecker sets are I_0 for $\varepsilon < \sqrt{2}$ is in [52].

Lemma 2.3.5 is an example of a more general phenomena. Finite sets of positive integers are always ε-Kronecker for some $\varepsilon < 2$. In [76] an extensive investigation is made of the Kronecker constants of finite subsets of $\mathbb{Z}$ and an algorithm is given for calculating these constants. For a two integer set, $\{m, n\}$, the angular Kronecker constant is $\pi \gcd(m,n)/(|m|+|n|)$. For sets of three or more elements the answers are surprisingly complicated. For instance, asymptotically, the angular Kronecker constant of $\{m, m, n+m\}$ is $\pi/3$, but the exact value depends on the congruence mod 3 of $m+2n$. It is also shown in [76] that the angular Kronecker constant of a finite $\mathbf{E} \subseteq \mathbb{Z}$ is always a rational multiple of π. The exact Kronecker constant of most sets is unknown, in particular, that of $\{1, \ldots, N\}$ **[P 5]**.

Existence of ε-Kronecker Sets. Theorem 2.4.3 is a simplified version of [59, Theorem 2.3]. Theorem 2.4.3 also improves upon [53, Theorem 4.4] (when $\boldsymbol{\Gamma}$ is not 2-large). The 2-large case is addressed in Theorem 4.5.2.

Motivated in part by [184], Galindo and Herńandez, in [41] and [42], used topological methods to prove the existence of large ε-Kronecker sets in very abstract settings. A discrete abelian group satisfying their hypothesis is isomorphic to a subgroup of a direct sum of copies of $\mathbb{Q}$ together with a finite group. Thus, the existence of large ε-Kronecker sets for all $\varepsilon > 0$ in their setting follows from Corollary 2.4.5. [59] uses their methods to prove Theorem 2.4.3. The proof is shorter, but perhaps less illuminating, than the one here.

For another approach to the topological method, see Givens and Kunen [46]. Yet another existence theorem for ε-Kronecker sets is [60, Theorem 3.1].

Approximating ± 1. The results of Sect. 2.5 are adapted from [55]. They will be used in Sect. 9.3 in establishing a characterization of Sidon sets as proportional ε-Kronecker.

Arithmetic Properties. Historically, arithmetic properties were established first for Sidon sets (see Chap. 6 for more detailed references) and those results motivated the study of arithmetic properties of ε-Kronecker sets.

Products of ε-Kronecker Sets. See Sect. 1.5 and [116, Theorem 2.3] for Hadamard set versions of the results of Sect. 2.7.

A stronger version of Proposition 2.7.4 shows that if $M \geq 1$, $\varepsilon < 2\sin(\pi/4M)$ and $\mathbf{E} \subseteq \boldsymbol{\Gamma}$ is ε-Kronecker, then $\mathbf{E}^{2M}$ has no cluster points in $\boldsymbol{\Gamma}$ and the cluster points of $(\mathbf{E} \cdot \mathbf{E}^{-1})^M$ in $\boldsymbol{\Gamma}$ are exactly the elements of $(\mathbf{E} \cdot \mathbf{E}^{-1})^{M-1}$. We refer the reader to [61, Theorem 4.5] for details and related results.

Corollary 2.7.8 appears in [52] with a different proof. The (longer) proof there gives other conclusions when ε is small. For example, if $\mathbf{E} \subseteq \mathbf{\Gamma}$ is 1-Kronecker, then $\mathbf{E}$ cannot contain 50 distinct elements $\gamma_1, \ldots, \gamma_{50}$ such that for some choice of signs,

$$\gamma_1^{\pm 1}\,\gamma_2^{\pm 1} = \cdots = \gamma_{49}^{\pm 1}\,\gamma_{50}^{\pm 1}. \tag{2.8.1}$$

See Exercise 2.9.12.

There is a vast literature on sets of uniqueness and multiplicity; see [56] and its references. An approach to U_0 sets different from [56] is through descriptive set theory [111].

Exercises. Exercise 2.9.14 (1) is [46, Lemma 3.8].

2.9 Exercises

Exercise 2.9.1. 1. Show that every finite subset $\mathbf{E} \subseteq \mathbb{Z} \smallsetminus \{0\}$ is ε-Kronecker for some $\varepsilon < 2$.
2. Show that $\mathbb{Z} \smallsetminus \{0\}$ is exactly 2-Kronecker.
3. More generally, show that if $\mathbf{E} \subset \mathbf{\Gamma}$ is countable, $\mathbf{E}$ has no elements of finite order and $\mathbf{1} \notin \mathbf{E}$, then $\mathbf{E}$ is 2-Kronecker (or better).
4. Compute $\varepsilon(\mathbb{Z}_3 \smallsetminus \{\mathbf{1}\})$.

Exercise 2.9.2. 1. Show that if an ε-Kronecker set contains an element γ of order 2, then $\varepsilon > \sqrt{2}$.
2. Show that the set $\mathbf{E} = \{(j, \pi_j) : j = 1, 2, \ldots\} \subseteq \mathbb{Z} \times \widehat{\mathbb{D}}$, where $\{\pi_j\}$ is the Rademacher set in $\widehat{\mathbb{D}}$, is $\sqrt{2}$-Kronecker.

Exercise 2.9.3. 1. Suppose $\mathbf{E} = \{(\chi, \gamma_n\}_{n=1}^{\infty} \subseteq \mathbf{\Gamma}$, where $G = \mathbb{Z}_2 \oplus \mathbb{Z}_k^{\mathbb{N}}$, $\{\gamma_n\}_n$ is a set of independent characters and $k \geq 3$ is odd. Show that $\mathbf{E}$ is weak 1-Kronecker.
2. Suppose $\mathbf{E} = \chi_1 \oplus \cdots \oplus \chi_n \subseteq \mathbf{\Gamma} = \bigoplus \widehat{\mathbf{\Gamma}_n}$. Show that if each χ_n has order at least 3, then $\mathbf{E}$ is weak 1-Kronecker.

Exercise 2.9.4. 1. Prove Lemma 2.2.8.
2. Give an example of an ε-Kronecker set $\mathbf{E}$ and quotient mapping q that is one-to-one on $\mathbf{E}$, but $q(\mathbf{E})$ is not ε-Kronecker.

Exercise 2.9.5. State and prove an analogue of Corollary 2.2.17 assuming, instead, γ has finite order.

Exercise 2.9.6. Adapt the argument of Example 2.3.6 to show that the set $\{N_j, -N_j + j\}_{j=1}^{\infty}$ is $\sqrt{2}$-Kronecker.

Exercise 2.9.7. Prove Theorem 2.3.1 (3).

Exercise 2.9.8. Suppose $m < n$ are positive integers. Show that if I is an interval of length at least $3\pi/m$, then $\{m, n\}$ is ε-Kronecker(I) for some $\varepsilon < \sqrt{2}$.

Exercise 2.9.9. Let $\mathbf{E} \subseteq \mathbb{Z}$ be ε-Kronecker for some $\varepsilon < 2$. Show that $\mathbf{E}$ has upper density zero, meaning

$$\limsup_{N\to\infty} \frac{|\,\mathbf{E} \cap [-N, N]|}{2N} = 0.$$

Exercise 2.9.10. Suppose M is an even integer and $\mathbf{E}$ is angular $\pi(1 - 1/M)$-Kronecker. Prove that if $N \log M(M-1) \geq 2 \log M$, then $\mathbf{E}$ does not contain a square of cardinality N^2. (This proves a stronger statement than Proposition 2.6.4.)

Exercise 2.9.11. Let d and N be positive integers. A *d-perturbed arithmetic progression of length* N is a set of the form $\{\gamma_0\gamma^{k_1}, \gamma_0\gamma^{k_2}, \ldots, \gamma_0\gamma^{k_N}\}$ where $k_j \in [(j-1)d, jd)$, $1 \leq j \leq N$, $\gamma_0, \gamma \in \mathbf{\Gamma}$ and γ has order greater than k_N. (When $d = 1$ these are arithmetic progressions.) Let $\tau > 0$. Show that there exists $N = N(d, \tau)$ such that if $\mathbf{E} \subseteq \mathbf{\Gamma}$ is a d-perturbed arithmetic progression of length at least N, then $\alpha(\mathbf{E}) \geq \pi - \tau$.

Exercise 2.9.12. Let $\mathbf{E}$ be ε-Kronecker for $\varepsilon < \sqrt{2}$. Show that there exists an integer $N = N(\varepsilon)$, depending only on ε, such that $\mathbf{E}$ cannot contain $2N$ distinct elements, $\{\gamma_j\}$, satisfying $\gamma_1\,\gamma_2^{\pm 1} = \cdots = \gamma_{2N-1}\,\gamma_{2N}^{\pm 1}$, (whatever the choice of signs).

Exercise 2.9.13. Let $\mathbb{R}_d$ be $\mathbb{R}$ with the discrete topology.

1. Show that if $x_j \in (0, \infty)$ and $x_j/x_{j+1} \geq 4$, then $\{x_j\}$ is weak angular $\pi/(q-1)$-Kronecker in $\mathbb{R}_d$.
2. Let $\varepsilon > 0$ and let $\mathbf{E} = \{\gamma_n\} \subseteq \mathbb{R}_d$ be an infinite sequence. Show that $\mathbf{E}$ has an infinite ε-Kronecker subset.
3. Show that every infinite $\mathbf{E} \subset \mathbb{R}_d$ has a subset $\mathbf{E}'$, with $|\mathbf{E}'| = |\mathbf{E}|$ and having the property that for each $\varepsilon > 0$ there is a finite set $\mathbf{F}$ such that $\mathbf{E}' \setminus \mathbf{F}$ is ε-Kronecker.

Exercise 2.9.14. A discrete abelian group $\mathbf{\Gamma}$ is said to have *infinite exponent* if for every N there is some character in $\mathbf{\Gamma}$ with order at least N.

1. Prove that $\mathbf{\Gamma}$ has infinite exponent if and only if $\mathbf{\Gamma}$ contains an infinite ε-Kronecker set for each $\varepsilon > 0$.
2. Suppose $\mathbf{\Gamma}$ has infinite exponent. Does every infinite subset of $\mathbf{\Gamma}$ contain an ε-Kronecker subset of the same cardinality for each $\varepsilon > 0$? If not, give examples.

Exercise 2.9.15. Let $\mathbf{F} \subseteq \mathbf{\Gamma}$ be a maximal independent subset of $\mathbf{\Gamma}$. Show that $\mathbf{\Gamma}/\langle \mathbf{F} \rangle$ is a torsion group.

Exercise 2.9.16. Let $\mathbf{E} \subseteq \bigoplus_{\ell \in B} \mathcal{C}(p_\ell^\infty)$ be such that for every $\ell \in B$ there is an element $\gamma \in \mathbf{E}$ such that the projection, $\Pi_\ell(\gamma)$, onto $\mathcal{C}(p_\ell^\infty)$ is non-trivial. Show that $\mathbf{E}$ is 2-large if and only if

$$|\{\ell : \Pi_\ell(\mathbf{E}) \text{ contains an element of order } \geq 3\}| < |\mathbf{E}|.$$

Exercise 2.9.17. Generalize Theorem 2.5.1 to the situation where an arbitrary choice of signs is replaced by an arbitrary choice of (fixed) two elements in $\mathbb{T}$ with angular distance π.

Exercise 2.9.18.
1. Let $a \in \mathbb{N}$ and $\mathbf{E} = \{a^k : k \in \mathbb{N}\}$. Show that for each n, the closure (in $\overline{\mathbb{Z}}$) of $\mathbf{E} \pm \cdots \pm \mathbf{E}$ (n terms) is U_0.
2. Suppose $k_j \in \mathbb{N}$ and $k_j \to \infty$. Let $\mathbf{E} = \{k_1, k_1k_2, k_1k_2k_3, \dots\}$. Show that for each n, the closure (in $\overline{\mathbb{Z}}$) of $\mathbf{E} \pm \cdots \pm \mathbf{E}$ (n terms) is U_0.

Exercise 2.9.19. Prove the equivalences of Proposition 2.7.2(5)–(7).

Exercise 2.9.20. Let $\mathbf{E} \subseteq \mathbf{\Gamma}$. Show $\varepsilon(\mathbf{E}) = \sup\{\varepsilon(\mathbf{F}) : |\mathbf{F}| < \infty, \mathbf{F} \subseteq \mathbf{E}\}$.

http://www.springer.com/978-1-4614-5391-8

Interpolation and Sidon Sets for Compact Groups
Graham, C.; Hare, K.E.
2013, XVII, 249 p., Hardcover
ISBN: 978-1-4614-5391-8