Chapter 2
Tools: Presentations and Their Calculus

Let G be a group and X a set. A map $\vartheta : X \to G$ is called a *generation map* if $gp(X\vartheta) = G$. In such a case, ϑ extends to a homomorphism θ from the free group F on X onto G. This is known as the universal mapping property of free groups.

We write

$$G = \langle X;\, R\rangle$$

and term the right-hand side a *presentation* for G if the subset R of F generates the kernel of θ as a normal subgroup of F, that is, $ker\theta = gp_F(R)$. The group G is termed *finitely generated* if X can be chosen finite, *finitely presented* or *finitely presentable* if both X and R can be chosen finite. We also refer to θ as a *presentation map* (for G) and call R a set of *defining relators*.

We sometimes use the notation

$$G = \langle X;\, \{r = 1 \mid r \in R\}\rangle$$

in place of $G = \langle X;\, R\rangle$, and we refer to the expressions $r = 1$ $(r \in R)$ as *defining relations* for G.

The reader should note that there is a comparable notion of a relatively free group and corresponding relative presentation for certain classes of groups; such as, for example, the metabelian groups. If F is an absolutely free group on a set X, then F/F'' is metabelian and, within the class of metabelian groups, it possesses the universal mapping property. We call F/F'' the *free metabelian group* on X. Every metabelian group is a quotient of a free metabelian group.

The following observation appears quite simple, yet it is a surprisingly handy tool for dealing with groups given by generators and defining relations.

Lemma 2.1 (von Dyck). *Let $X = \{x_i \mid i \in I\}$ and suppose that*

$$G = \langle X;\, R\rangle\,.$$

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Let $\theta : X \to G$ be the map that comes with this presentation, and put $\theta(x_i) = y_i \in G$. Let $Y = \{y_i \mid i \in I\}$, H another group, and $\phi : Y \to H$ a set map. If for every word $r = (x_{i_1}, \ldots, x_{i_n}) \in R$,

$$r\,(\phi(y_{i_1}), \ldots, \phi(y_{i_n})) = 1$$

in H, then ϕ can be extended to a homomorphism φ from G to H.

Due to its utility, von Dyck's valuable lemma is used repeatedly throughout these notes, often without explicit mention. We encourage the reader to recognize all those instances.

Whenever we deal with groups in terms of presentations, we bear in mind that writing

$$G = \langle X;\, R \rangle$$

assumes a presentation map θ and a homomorphism from the free group $F = F(X)$ onto the group G.

Different-looking presentations may well be describing isomorphic groups. In particular, a finitely presented group may have other presentations that need not use finitely many relators or finitely many generators. It can be convenient to be able to switch between presentations of the same group. The following theorem and lemma dealing with the "calculus of presentations" describe some "rules" for permissible manipulations that leave the group of the presentation unchanged (up to isomorphism). Both are important results for finitely presentable groups.

Theorem 2.1 (Tietze).

- **T1** *Suppose that $G = \langle X;\, R \rangle$. Let Y be a set, disjoint from X, and consider a set of X-words w_y for each $y \in Y$. Then*

$$G = \left\langle X \cup Y;\, R \cup y^{-1} w_y \ (y \in Y) \right\rangle.$$

- **T1′** *Suppose that $G = \left\langle X \cup Y;\, R \cup y^{-1} w_y \ (y \in Y) \right\rangle$, where Y is a set disjoint from X, each w_y is an X-word, and the elements of R are X-words. Then*

$$G = \langle X;\, R \rangle.$$

- **T2** *Suppose that $G = \langle X;\, R \rangle$ and $S \subseteq gp_F(R)$ in the free group F on X. Then*

$$G = \langle X;\, R \cup S \rangle.$$

- **T2′** *Suppose that $G = \langle X;\, R \cup S \rangle$, where $S \subseteq gp_F(R)$ in the free group F on X. Then*

$$G = \langle X;\, R \rangle.$$

The moves Theorem 2.1 specifies are called Tietze transformations. Appendix B contains the proof of Tietze's theorem.

Lemma 2.2 (B. H. Neumann). *Suppose the group G is known to be finitely presentable and a presentation*

$$G = \langle x_1, \ldots, x_m;\ r_1, r_2, \ldots \rangle$$

with a finite set of generators and a (possibly infinite) countable set of relators is given for G. Then for some positive integer n, G can also be presented by the finite presentation

$$G = \langle x_1, \ldots, x_m;\ r_{i_1}, r_{i_2}, \ldots, r_{i_n} \rangle .$$

In other words, if a group G is finitely presentable, then in every presentation of G that involves a finite set of generators, some finite subset of the given relators will suffice to present G.

The proof is a straightforward application of Tietze transformations.

Example 2.1. To illustrate the difficulty of recognizing isomorphic groups, we offer a presentation of a group G that turns out to be a trivial group:

$$G = \left\langle a, b;\ a^{-1}ba = b^2;\ b^{-1}ab = a^2 \right\rangle .$$

To see that G is trivial, consider $a^{-1}ba = b^2$. This leads to $b^{-1}a^{-1}ba = b$. Now $b^{-1}a^{-1}b = (b^{-1}ab)^{-1}$ which implies $a^{-2}a = b$ and $a^{-1} = b$. Next, substituting into $bba = b^2$ results in $a = 1$ and so $a^{-1} = 1$ which implies $b = 1$.

The natural question one can ask is whether there is an algorithm deciding if any given presentation represents the trivial group. In fact, this triviality problem is algorithmically undecidable.

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