Chapter 2
Matched Asymptotic Expansions

2.1 Introduction

The ideas underlying an asymptotic approximation appeared in the early 1800s when there was considerable interest in developing formulas to evaluate special functions. An example is the expansion of Bessel's function, given in (1.15), that was derived by Poisson in 1823. It was not until later in the century that the concept of an asymptotic solution of a differential equation took form, and the most significant efforts in this direction were connected with celestial mechanics. The subject of this chapter, what is traditionally known as matched asymptotic expansions, appeared somewhat later. Its early history is strongly associated with fluid mechanics and, specifically, aerodynamics. The initial development of the subject is credited to Prandtl (1905), who was concerned with the flow of a fluid past a solid body (such as an airplane wing). The partial differential equations for viscous fluid flow are quite complicated, but he argued that under certain conditions the effects of viscosity are concentrated in a narrow layer near the surface of the body. This happens, for example, with air flow across an airplane wing, and a picture of this situation is shown in Fig. 2.1. This observation allowed Prandtl to go through an order-of-magnitude argument and omit terms he felt to be negligible in the equations. The result was a problem that he was able to solve. This was a brilliant piece of work, but it relied strongly on his physical intuition. For this reason there were numerous questions about his reduction that went unresolved for decades. For example, it was unclear how to obtain the correction to his approximation, and it is now thought that Prandtl's derivation of the second term is incorrect (Lagerstrom, 1988). This predicament was resolved when Friedrichs (1941) was able to show how to systematically reduce a boundary-layer problem. In analyzing a model problem (Exercise 2.1) he used a stretching transformation to match inner and outer

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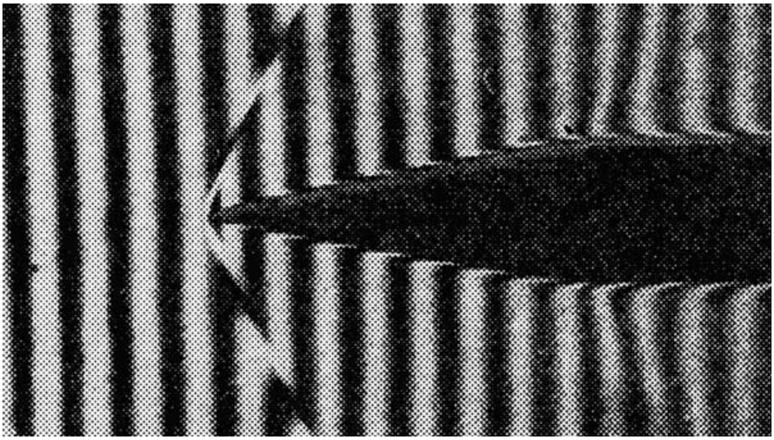

Figure 2.1 Supersonic air flow, at Mach 1.4, over a wedge. The high speed flow results in a shock layer in front of the wedge across which the pressure undergoes a rapid transition. Because of its position in the flow, the shock is an example of an interior layer (Sect. 2.5). There are also boundary layers present (Sect. 2.2). These can be seen near the surface of the wedge; they are thin regions where the flow drops rapidly to zero (which is the speed of the wedge). From Bleakney et al. (1949)

solutions, which is the basis of the method that is discussed in this chapter. This procedure was not new, however, as demonstrated by the way in which Gans (1915) used some of these ideas to solve problems in optics.

The golden age for matched asymptotic expansions was in the 1950s, and it was during this period that the method was refined and applied to a wide variety of physical problems. A short historical development of the method is presented in O'Malley (2010). The popularity of matched asymptotic expansions was also greatly enhanced with the appearance of two very good books, one by Cole (1968), the other by Van Dyke (1975). The method is now one of the cornerstones of applied mathematics. At the same time it is still being extended, both in the type of problems it is used to resolve as well as in the theory.

2.2 Introductory Example

The best way to explain the method of matched asymptotic expansions is to use it to solve a problem. The example that follows takes several pages to complete because it is used to introduce the ideas and terminology. As the procedure becomes more routine, the derivations will become much shorter.

The problem we will study is

$$\varepsilon y'' + 2y' + 2y = 0, \quad \text{for } 0 < x < 1, \tag{2.1}$$

where

$$y(0) = 0 \tag{2.2}$$

and

$$y(1) = 1. \tag{2.3}$$

This equation is similar to that used in Sect. 1.7 to discuss uniform and nonuniform approximations. The difference is that we will now derive the approximation directly from the problem rather than from a formula for the solution (Exercise 2.7).

An indication that this problem is not going to be as straightforward as the differential equations solved in Sect. 1.6 is that if $\varepsilon = 0$, then the problem is no longer second order. This leads to what is generally known as a singular perturbation problem, although singularity can occur for other reasons. In any case, to construct a first-term approximation of the solution for small ε, we will proceed in four steps. The fifth step will be concerned with the derivation of the second term in the expansion.

2.2.1 Step 1: Outer Solution

To begin, we will assume that the solution can be expanded in powers of ε. In other words,

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots . \tag{2.4}$$

Substituting this into (2.1) we obtain

$$\varepsilon(y_0'' + \varepsilon y_1'' + \cdots) + 2(y_0' + \varepsilon y_1' + \cdots) + 2(y_0 + \varepsilon y_1 + \cdots) = 0.$$

The $O(1)$ equation is therefore

$$y_0' + y_0 = 0, \tag{2.5}$$

and the general solution of this is

$$y_0(x) = a\mathrm{e}^{-x}, \tag{2.6}$$

where a is an arbitrary constant. Looking at the solution in (2.6) we have a dilemma because there is only one arbitrary constant but two boundary conditions – (2.2), (2.3). What this means is that the solution in (2.6) and the expansion in (2.4) are incapable of describing the solution over the entire interval $0 \leq x \leq 1$. At the moment we have no idea which boundary condition, if any, we should require $y_0(x)$ to satisfy, and the determination of this will have to come later. This leads to the question of what to do next. Well, (2.6) is similar to using (1.71) to approximate the solution of (1.68). In looking at the comparison in Fig. 1.14, it is a reasonable working hypothesis to assume that (2.6) describes the solution over most of the interval, but there is a boundary layer at either $x = 0$ or $x = 1$, where a different approximation must be used. Assuming for the moment that it is at $x = 0$, and in looking at the reduction from (1.70) to (1.71), then we are probably missing a term like $\mathrm{e}^{-\beta x/\varepsilon}$. The derivation of this term is the objective of the next step. Because we are going to end up with approximations of the solution over different regions, we will refer to (2.6) as the first term in the expansion of the *outer solution.*

2.2.2 Step 2: Boundary Layer

Based on the assumption that there is a boundary layer at $x = 0$, we introduce a *boundary-layer coordinate* given as

$$\bar{x} = \frac{x}{\varepsilon^{\alpha}}, \tag{2.7}$$

where $\alpha > 0$. From our earlier discussion it is expected that $\alpha = 1$, and this will be demonstrated conclusively subsequently. After changing variables from x to $\bar{x}$ we will take $\bar{x}$ to be fixed when expanding the solution in terms of ε. This has the effect of stretching the region near $x = 0$ as ε becomes small. Because of this, (2.7) is sometimes referred to as a *stretching transformation.*

From the change of variables in (2.7), and from the chain rule, we have that

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\bar{x}} = \frac{1}{\varepsilon^{\alpha}}\frac{\mathrm{d}}{\mathrm{d}\bar{x}}.$$

Letting $Y(\bar{x})$ denote the solution of the problem when using this boundary-layer coordinate, (2.1) transforms to

$$\varepsilon^{1-2\alpha}\frac{\mathrm{d}^2Y}{\mathrm{d}\bar{x}^2} + 2\varepsilon^{-\alpha}\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}} + 2Y = 0, \tag{2.8}$$

where, from (2.2),

$$Y(0) = 0. \tag{2.9}$$

The boundary condition at $x = 0$ has been included here because the boundary layer is at the left end of the interval.

The appropriate expansion for the *boundary-layer solution* is now

$$Y(\bar{x}) \sim Y_0(\bar{x}) + \varepsilon^{\gamma}Y_1(\bar{x}) + \cdots, \tag{2.10}$$

where $\gamma > 0$. As stated previously, in this expansion $\bar{x}$ is held fixed as ε goes to zero (in the same way that x is held fixed in the outer expansion). Substituting the expansion in (2.10) into (2.8) we get that

$$\underset{①}{\varepsilon^{1-2\alpha}\frac{\mathrm{d}^2}{\mathrm{d}\bar{x}^2}(Y_0+\cdots)} + \underset{②}{2\varepsilon^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\bar{x}}(Y_0+\cdots)} + \underset{③}{2(Y_0+\cdots)} = 0. \tag{2.11}$$

Just as with the algebraic equations studied in Sect. 1.5, it is now necessary to determine the correct balancing in (2.11). The balance between terms ② and ③ was considered in Step 1, and so the following possibilities remain (also see Exercise 2.6):

(i) ① $\sim$ ③ and ② is higher order.
The condition ① $\sim$ ③ requires that $1-2\alpha=0$, and so $\alpha=\frac{1}{2}$. With this we have that ①, ③ $= O(1)$ and ② $= O(\varepsilon^{-1/2})$. This violates our assumption that ② is higher order (i.e., ② $\ll$ ①), so this case is not possible.

(ii) ① $\sim$ ② and ③ is higher order.
The condition ① $\sim$ ② requires that $1-2\alpha=-\alpha$, and so $\alpha=1$. With this we have that ①, ② $= O(\varepsilon^{-1})$ and ③ $= O(1)$. In this case, the conclusion is consistent with the original assumptions, and so this is the balancing we are looking for.

With this we have the following problem to solve:

$O(\frac{1}{\varepsilon})$ $\quad Y_0'' + 2Y_0' = 0$ for $0 < \bar{x} < \infty$,
$Y_0(0) = 0$.

The general solution of this problem is

$$Y_0(\bar{x}) = A(1-\mathrm{e}^{-2\bar{x}}), \tag{2.12}$$

where A is an arbitrary constant. It should be observed that the differential equation for Y_0 contains at least one term of the outer-layer Eq. (2.5). This is important for the successful completion of Step 3.

The boundary-layer expansion in (2.10) is supposed to describe the solution in the immediate vicinity of the endpoint $x=0$. It is therefore not unreasonable to expect that the outer solution (2.6) applies over the remainder of the interval (this is assuming there are no other layers). This means that the outer solution should satisfy the boundary condition at $x=1$. From (2.6) and (2.3) one finds that

$$y_0(x) = \mathrm{e}^{1-x}. \tag{2.13}$$

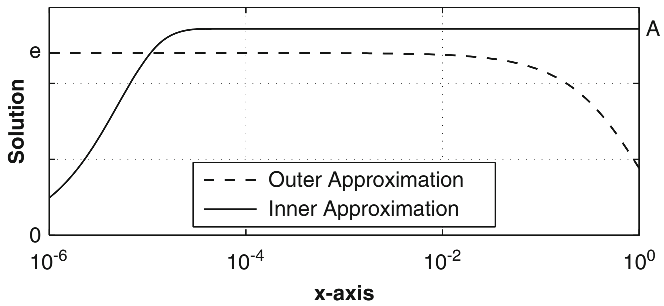

Figure 2.2 Sketch of inner solution, (2.12), and outer solution, (2.13). Note the overlap region along the x-axis where both solutions are essentially constant. Since these approximations are supposed to be describing the same continuous function, it must be that these constants are the same. Hence, $A = \mathrm{e}^1$

2.2.3 Step 3: Matching

It remains to determine the constant A in the first-term approximation of the boundary-layer solution (2.12). To do this, the approximations we have constructed so far are summarized in Fig. 2.2. The important point here is that both the inner and outer expansions are approximations of the same function. Therefore, in the transition region between the inner and outer layers we should expect that the two expansions will give the same result. This is accomplished by requiring that the value of Y_0 as one comes out of the boundary layer (i.e., as $\bar{x} \to \infty$) is equal to the value of y_0 as one comes into the boundary layer (i.e., as $x \to 0$). In other words, we require that

$$\lim_{\bar{x}\to\infty} Y_0 = \lim_{x\to 0} y_0. \tag{2.14}$$

In this text, the preceding equation will usually be written in the more compact form of $Y_0(\infty) = y_0(0^+)$. However it is expressed, this is an example of a matching condition, and from it we find that $A = \mathrm{e}^1$. With this (2.12) becomes

$$Y_0(\bar{x}) = \mathrm{e}^1 - \mathrm{e}^{1-2\bar{x}}. \tag{2.15}$$

This completes the derivation of the inner and outer approximations of the solution of (2.1). The last step is to combine them into a single expression.

Before moving on to the construction of the composite expansion, a word of caution is needed about the matching condition given in (2.14). Although we will often use this condition, it is limited in its applicability, and for more complex problems a more sophisticated matching procedure is often required. This will be discussed in more detail once the composite expansion is calculated.

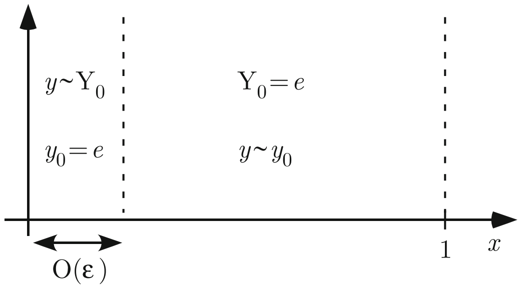

Figure 2.3 Sketch of inner and outer regions and values of approximations in those regions

2.2.4 Step 4: Composite Expansion

Our description of the solution consists of two pieces, one that applies near $x = 0$, the other that works everywhere else. Because neither can be used over the entire interval, they are not uniformly valid for $0 \leq x \leq 1$. The question we consider now is whether it is possible to combine them to produce a uniform approximation, that is, one that works over the entire interval. The position we are in is summarized in Fig. 2.3. The inner and outer solutions are constant outside their intervals of applicability, and the constant is the same for both solutions. The value of the constant can be written as either $y_0(0)$ or $Y_0(\infty)$, and the fact that they are equal is a consequence of the matching condition (2.14). This observation can be used to construct a uniform approximation, namely, we just add the approximations together and then subtract the part that is common to both. The result is

$$\begin{aligned} y &\sim y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - y_0(0) \\ &\sim \mathrm{e}^{1-x} - \mathrm{e}^{1-2x/\varepsilon}. \end{aligned} \tag{2.16}$$

The fact that the composite expansion gives a very good approximation of the solution over the entire interval is shown in Fig. 2.4. Note, however, that it satisfies the boundary condition at $x = 0$ exactly, but the one at $x = 1$ is only satisfied asymptotically. This is not of particular concern since the expansion also satisfies the differential equation in an asymptotic sense. However, an alternative expansion that satisfies both boundary conditions is developed in Exercise 2.14.

2.2.5 Matching Revisited

Because of the importance of matching, the procedure needs to be examined in more detail. The fact is that, even though the matching condition in

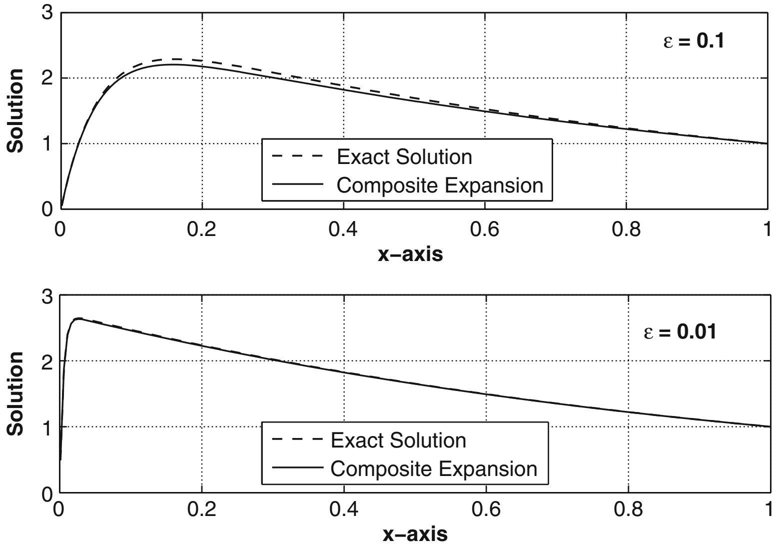

Figure 2.4 Graph of exact solution of (2.1) and composite expansion given in (2.16) in the case where $\varepsilon = 10^{-1}$ and where $\varepsilon = 10^{-2}$. Note the appearance of the boundary layer, as well as the convergence of the composite expansion to the solution of the problem, as ε decreases

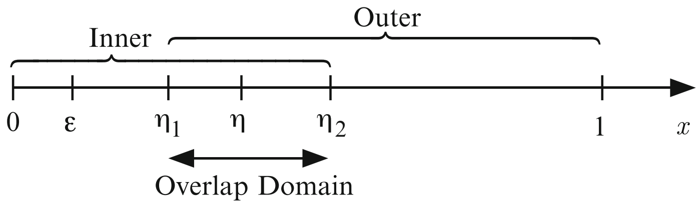

Figure 2.5 Schematic of domains of validity of inner and outer expansions as assumed in matching procedure. The intermediate variable is to be located within the overlap region

(2.14) is often used when calculating the first term in the expansion, there are situations where it is inapplicable. One of the more common examples of this occurs when either the inner or outer expansions are unbounded functions of their respective variable, so its limit does not exist. Examples of this will appear later in the chapter (e.g., Sect. 2.6), as well in succeeding chapters (e.g., Sect. 4.3). Another complication arises when constructing the second term in an expansion. What this all means is that we need a more sophisticated matching procedure, and finding one is the objective of this section.

The fundamental idea underlying matching concerns the overlap, or transition, region shown in Fig. 2.2. To connect the expansions on either side of this region, we introduce an intermediate variable $x_\eta = x/\eta(\varepsilon)$ that is positioned within this region. In particular, it is between the $O(1)$ coordinate of the outer layer and the $O(\varepsilon)$ coordinate of the inner layer (Fig. 2.5). This means that $\eta(\varepsilon)$ satisfies $\varepsilon \ll \eta \ll 1$. To match the expansions, the inner and outer approximations must give the same result when they are expressed in this transition layer coordinate.

The precise conditions imposed on $\eta(\varepsilon)$ and on the expansions are stated explicitly in the following matching procedure:

(i) Change variables in the outer expansion (from x to x_η) to obtain y_{outer}. It is assumed that there is an $\eta_1(\varepsilon)$ such that y_{outer} still provides a first-term expansion of the solution for any $\eta(\varepsilon)$ that satisfies $\eta_1(\varepsilon) \ll \eta(\varepsilon) \leq 1$.

(ii) Change variables in the inner expansion (from $\bar{x}$ to x_η) to obtain y_{inner}. It is assumed that there is an $\eta_2(\varepsilon)$ such that y_{inner} still provides a first-term expansion of the solution for any $\eta(\varepsilon)$ that satisfies $\varepsilon \leq \eta(\varepsilon) \ll \eta_2$.

(iii) It is assumed that the domains of validity of the expansions for y_{outer} and y_{inner} overlap, that is, $\eta_1 \ll \eta_2$. In this overlap region, for the expansions to match, it is required that the first terms from y_{outer} and y_{inner} be equal.

The assumptions contained in (i) and (ii) can be proved under fairly mild conditions; they are the essence of Kaplun's extension theorem (Lagerstrom, 1988). Actually, one is seldom interested in determining η_1 or η_2 but only that there is an interval for $\eta(\varepsilon)$ such that y_{inner} and y_{outer} match. It is important, however, that the matching not depend on the specific choice of $\eta(\varepsilon)$. For example, if one finds that matching can only occur if $\eta(\varepsilon) = \varepsilon^{1/2}$, then there is no overlap domain, and the procedure is not applicable. In comparison to the situation for (i) and (ii), the assumption on the existence of an overlap domain in (iii) is a different matter, and a satisfactory proof has never been given. For this reason it has become known as Kaplun's hypothesis on the domain of validity (Lagerstrom, 1988).

Examples

1. To use the foregoing matching procedure on our example problem, we introduce the intermediate variable x_η, defined as

$$x_\eta = \frac{x}{\varepsilon^\beta}\,, \tag{2.17}$$

where $0 < \beta < 1$. This interval for β comes from the requirement that the scaling for the intermediate variable must lie between the outer scale, $O(1)$, and the inner scale, $O(\varepsilon)$. Actually, it may be that in carrying out the matching of y_{inner} and y_{outer} we must reduce this interval for β. To see if the expansions match, note that the inner solution, from (2.10) and (2.12), becomes

$$
\begin{aligned}
y_{\text{inner}} &\sim A(1 - \mathrm{e}^{-2x_\eta/\varepsilon^{1-\beta}}) + \cdots \\
&\sim A + \cdots ,
\end{aligned}
\tag{2.18}
$$

and the outer solution, from (2.4) and (2.13), becomes

$$
\begin{aligned}
y_{\text{outer}} &\sim \mathrm{e}^{1-x_\eta\varepsilon^\beta} + \cdots \\
&\sim \mathrm{e}^1 + \cdots .
\end{aligned}
\tag{2.19}
$$

The expansions in (2.18) and (2.19) are supposed to agree to the first term in the overlap domain, and therefore $A = \mathrm{e}^1$. ■

2. Suppose that

$$
y = \sqrt{1 + x + \frac{\varepsilon}{\varepsilon + x}} \, , \tag{2.20}
$$

where $0 \le x \le 1$. The outer expansion is found by fixing x, with $0 < x \le 1$, and expanding for small ε to obtain

$$
y \sim \sqrt{1+x} + \frac{\varepsilon}{2x\sqrt{1+x}} + \cdots . \tag{2.21}
$$

The boundary-layer expansion is found by setting $\bar{x} = x/\varepsilon$ and expanding to obtain

$$
Y \sim \sqrt{\frac{2+\bar{x}}{1+\bar{x}}} + \frac{1}{2}\varepsilon\bar{x}\sqrt{\frac{1+\bar{x}}{2+\bar{x}}} + \cdots . \tag{2.22}
$$

Given that (2.21) and (2.22) are expansions for a known function, we do not use matching to determine an unknown constant as in the previous example. Rather, the objectives here are to demonstrate how to use an intermediate variable to match the first two terms in an expansion and to provide an example that shows that condition (2.14), although very useful, has limited applicability. Its limitations are evident in this example by looking at what happens to the second term in the expansions when letting $x \to 0$ in (2.21) and letting $\bar{x} \to \infty$ in (2.22). Because both terms become unbounded, it is necessary to use a more refined matching method than a simple limit condition. To verify the matching principle, we substitute (2.17) into (2.21) to obtain

$$
\begin{aligned}
y_{\text{outer}} &\sim \sqrt{1 + \varepsilon^\beta x_\eta} + \frac{\varepsilon^{1-\beta}}{2x_\eta\sqrt{1+\varepsilon^\beta x_\eta}} + \cdots \\
&\sim 1 + \frac{1}{2}\varepsilon^\beta x_\eta + \cdots + \varepsilon^{1-\beta}\frac{1}{2x_\eta} + \cdots .
\end{aligned}
$$

Similarly, from (2.22) we get

$$y_{\text{inner}} \sim 1 + \varepsilon^{1-\beta}\frac{1}{2x_\eta} + \cdots + \frac{1}{2}\varepsilon^\beta x_\eta + \cdots .$$

Comparing these two expansions it is evident that they match. Another observation is that there is a strong coupling between the various terms in the expansions. For example, the first term from y_1 is what matches with the second term coming from Y_0. This coupling can cause difficulties in determining if expansions match, particularly when computing several terms in an expansion. A hint of this will be seen when calculating the second term in the next example. ■

The interested reader may consult Lagerstrom (1988) for a more extensive discussion of the subtleties of matching using an intermediate variable. Also, there are other ways to match, and a quite popular one, due to Van Dyke (1975), is discussed in Exercise 2.12. His procedure is relatively simple to use but can occasionally lead to incorrect results (Fraenkel, 1969).

As the final comment about matching, it is a common mistake to think that it is equivalent to the requirement that y_0 and Y_0 must intersect. To show that this is incorrect, note that $Y_0 = 1 - \mathrm{e}^{-x/\varepsilon}$ and $y_0 = 1 + x$ are inner and outer approximations, respectively, of $y = 1 + x - \mathrm{e}^{-x/\varepsilon}$. However, even though $Y_0(\infty) = y_0(0^+)$, the two function never intersect.

2.2.6 Second Term

Generally, to illustrate a method, we will only derive the first term in an expansion. However, the second term is important as it gives a measure of the error. The procedure to find the second term is very similar to finding the first, so only the highlights will be given here.

Substituting the outer expansion (2.4) into the problem and collecting the $O(\varepsilon)$ terms one finds that $y_1' + y_1 = -\frac{1}{2}y_0''$, with $y_1(1) = 0$. The solution of this problem is

$$y_1 = \frac{1}{2}(1-x)\mathrm{e}^{1-x}.$$

Similarly, from the boundary-layer Eq. (2.11) we get that $\gamma = 1$ and $Y_1'' + 2Y_1' = -2Y_0$, with $Y_1(0) = 0$. The general solution of this is

$$Y_1 = B(1 - \mathrm{e}^{-2\bar{x}}) - \bar{x}\mathrm{e}^1(1 + \mathrm{e}^{-2\bar{x}}),$$

where B is an arbitrary constant. To match the expansions, we use the intermediate variable given in (2.17). The outer expansion in this case takes the form

$$
\begin{aligned}
y_{\text{outer}} &\sim \mathrm{e}^{1-x_\eta\varepsilon^\beta} + \frac{\varepsilon}{2}(1 - x_\eta\varepsilon^\beta)\mathrm{e}^{1-x_\eta\varepsilon^\beta} + \cdots \\
&\sim \mathrm{e}^1 - \varepsilon^\beta x_\eta \mathrm{e}^1 + \frac{1}{2}\varepsilon\mathrm{e}^1 + \frac{1}{2}\varepsilon^{2\beta}\mathrm{e}^1 x_\eta^2 + \cdots \,, \qquad (2.23)
\end{aligned}
$$

and, setting $\xi = -2x_\eta/\varepsilon^{1-\beta}$, the boundary-layer expansion becomes

$$
\begin{aligned}
y_{\text{inner}} &\sim \mathrm{e}^1(1 - \mathrm{e}^\xi) + \varepsilon\left[B(1 - \mathrm{e}^\xi) - \frac{x_\eta \mathrm{e}^1}{\varepsilon^{1-\beta}}(1 + \mathrm{e}^\xi)\right] + \cdots \\
&\sim \mathrm{e}^1 - \varepsilon^\beta x_\eta \mathrm{e}^1 + B\varepsilon + \cdots \,. \qquad (2.24)
\end{aligned}
$$

Matching these we get that $B = \frac{1}{2}\mathrm{e}^1$. Note, however, that these expansions do not appear to agree since (2.23) contains a $O(\varepsilon^{2\beta})$ term that (2.24) does not have. To understand why this occurs, note that both expansions produce a $O(\varepsilon^\beta)$ term that does not contain an arbitrary constant. If this term is not identical for both expansions, then there is no way the expansions will match. In the outer expansion this term comes from the $O(1)$ problem, and in the boundary layer it comes from the $O(\varepsilon)$ solution. In a similar manner, one finds that the x_η^2 term in (2.23) also comes from the first term. However, for the boundary layer it comes from the $O(\varepsilon^2)$ problem (the verification of this is left as an exercise). Therefore, the expansions match.

It is now possible to construct a two-term composite expansion. The basic idea is to add expansions and then subtract the common part. This yields the following result:

$$
\begin{aligned}
y &\sim y_0 + \varepsilon y_1 + Y_0 + \varepsilon Y_1 - \left(\mathrm{e}^1 - x_\eta \mathrm{e}^1\sqrt{\varepsilon} + \frac{\varepsilon}{2}\mathrm{e}^1\right) \\
&\sim \mathrm{e}^{1-x} - (1 + x)\mathrm{e}^{1-2x/\varepsilon} + \frac{\varepsilon}{2}\left[(1 - x)\mathrm{e}^{1-x} - \mathrm{e}^{1-2x/\varepsilon}\right].
\end{aligned}
$$

Note that the common part in this case contains the terms in (2.23) and (2.24) except for the x_η^2 term in (2.23).

Occasionally it happens that an expansion (inner or outer) produces a term of an order, or form, that the other does not have. A typical example of this occurs when trying to expand in powers of ε. It can happen that to be able to match one expansion with another expansion from an adjacent layer, it is necessary to include other terms such as those involving $\ln(\varepsilon)$. This process of having to insert scales into an expansion because of what is happening in another layer is called *switchbacking*. Some of the more famous examples of this involve logarithmic scales; these are discussed in Lagerstrom (1988). We will come across switchbacking in Sects. 2.4 and 6.9 when including transcendentally small terms in the expansions.

2.2.7 Discussion

The importance of matching cannot be overemphasized. Numerous assumptions went into the derivation of the inner and outer approximations, and matching is one of the essential steps that supports these assumptions. If they had not matched, it would have been necessary to go back and determine where the error had occurred. The possibilities when this happens are almost endless, but it would be helpful to start looking in the following places.

1. The boundary layer is at $x = 1$, not at $x = 0$. In this case the boundary-layer coordinate is

$$\bar{x} = \frac{x - 1}{\varepsilon^\alpha} \,. \tag{2.25}$$

   This will be considered in Sect. 2.3. For certain problems it may be necessary to replace the denominator with a function $\mu(\varepsilon)$, where $\mu(\varepsilon)$ is determined from balancing or matching. A heuristic argument to help determine the location of the boundary layer is given in Sect. 2.5.
2. There are boundary layers at both ends of an interval. See Sect. 2.3.
3. There is an interior layer. In this case the stretching transformation is

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha} \,, \tag{2.26}$$

   where x_0 is the location of the layer (this may depend on ε) (Sects. 2.5 and 2.6).
4. The form of the expansion is incorrect. For example, the outer expansion may have the form $y \sim \xi(\varepsilon) y_0(x) + \cdots$, where ξ is a function determined from the balancing or matching in the problem; see Exercise 2.2(b) and Sect. 2.6.
5. The solution simply does not have a layer structure and other methods need to be used (Exercise 2.4 and Chap. 3).

Occasionally it happens that it is so unclear how to proceed that one may want to try to solve the problem numerically to get an insight into the structure of the solution. The difficulty with this is that the presence of boundary or interior layers can make it hard, if not nearly impossible, to obtain an accurate numerical solution. Nice illustrations of this can be found in Exercises 2.29 and 2.40, and by solving (2.105) with $k = 1$. Another way to help guide the analysis occurs when the problem originates from an application and one is able to use physical intuition to determine the locations of the layers. As an example, in solving the problem associated with Fig. 2.1 one would expect a boundary layer along the surface of the wedge and an interior layer at the location of the shock.

Exercises

2.1. The Friedrichs model problem for a boundary layer in a viscous fluid is (Friedrichs, 1941)

$$\varepsilon y'' = a - y' \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$, and a is a given positive constant with $a \neq 1$.
(a) After finding the first term of the inner and outer expansions, derive a composite expansion of the solution of this problem.
(b) Derive a two-term composite expansion of the solution of this problem.

2.2. Find a composite expansion of the solution of the following problems:
(a) $\varepsilon y'' + 2y' + y^3 = 0$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1/2$.
(b) $\varepsilon y'' + e^x y' + \varepsilon y = 1$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$.
(c) $\varepsilon y'' + y(y' + 3) = 0$ for $0 < x < 1$, where $y(0) = 1$ and $y(1) = 1$.
(d) $\varepsilon y'' = f(x) - y'$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$. Also, $f(x)$ is continuous.
(e) $\varepsilon y'' + (1 + 2x)y' - 2y = 0$ for $0 < x < 1$, where $y(0) = \varepsilon$ and $y(1) = \sin(\varepsilon)$.
(f) $\varepsilon y'' + y' + y = \int_0^1 K(\varepsilon x, s)y(s)\mathrm{d}s$ for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$. Also, $K(x, s) = e^{-s(1+x)}$.
(g) $\varepsilon y'' = e^{\varepsilon y'} + y$ for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$.
(h) $\varepsilon y'' - y^3 = -1 - 7x^2$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 2$.

2.3. Consider the problem of solving

$$\varepsilon^2 y'' + ay' = x^2 \quad \text{for } 0 < x < 1,$$

where $y'(0) = \lambda$, $y(1) = 2$, and a and λ are positive constants.
(a) Find a first-term composite expansion for the solution. Explain why the approximation does not depend on λ.
(b) Find the second terms in the boundary layer and outer expansions and match them. Be sure to explain your reasoning in matching the two expansions.

2.4. A small parameter multiplying the highest derivative does not guarantee that the solution will have a boundary layer for small values of ε. As demonstrated in this problem, this can be due to the form of the differential equation or the particular boundary conditions used in the problem.
(a) After solving each of the following problems, explain why the solution does not have a boundary layer.

(i) $\varepsilon y'' + 2y' + 2y = 2(1 + x)$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$.
(ii) $\varepsilon^2 y'' + \omega^2 y = 0$ for $0 < x < 1$ and $\omega > 0$.

(b) Consider the equation $\varepsilon^2 y'' - xy' = 0$ for $0 < x < 1$. From the exact solution, show that there is no boundary layer if the boundary conditions are $y(0) = y(1) = 2$, while there is a boundary layer if the boundary conditions are $y(0) = 1$ and $y(1) = 2$.

2.5. It is possible for a solution to have boundary-layer-like properties, but the form of the expansions is by no means obvious. The following examples illustrate such situations.

(a) $\varepsilon^2 y'' = y'$ for $0 < x < 1$, where $y'(0) = -1$ and $y(1) = 0$. Solve this problem and explain why there is a boundary layer at $x = 1$ but the expansion for the outer region is not given by (2.4).

(b) $\varepsilon y' = (x-1)y$ for $0 < x$, where $y(0) = 1$. There is a boundary layer at $x = 0$. Use the methods of this section to derive a composite expansion of the solution. Find the exact solution and explain why the approximation you derived does not work.

2.6.(a) For (2.11) consider the balance of ① $\gg$ ②, ③. This case is not a distinguished limit because the order (α) is not unique. Explain why the solutions from this region are contained in (2.12).

(b) Discuss the case of ② $\gg$ ①, ③ in conjunction with the outer solution (this also is not a distinguished limit).

2.7. The exact solution of (2.1)–(2.3) is

$$y(x) = \frac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}},$$

where $\varepsilon r_\pm = -1 \pm \sqrt{1 - 2\varepsilon}$. Obtain the inner, outer, and composite expansions directly from this formula.

2.8. Consider the problem

$$\varepsilon y'' + y' + xy = 0 \quad \text{for } \alpha(\varepsilon) < x < \beta(\varepsilon),$$

where $y(\alpha) = 1$ and $y(\beta) = 0$. One way to deal with this ε-dependent interval is to change coordinates and let $s = (x - \alpha)/(\beta - \alpha)$. This fixes the domain and puts the problem into a Lagrange-like viewpoint.

(a) Find the transformed problem.

(b) Assuming $\alpha \sim \varepsilon\alpha_1 + \cdots$ and $\beta \sim 1 + \varepsilon\beta_1$, find a first-term composite expansion of the solution of the transformed problem. Transform back to the variable x and explain why the first-term composite expansion is unaffected by the perturbed domain.

(c) Find the second term in the composite expansion of the solution of the transformed problem. Transform back to the variable x and explain how the two-term composite expansion is affected by the perturbed domain.

2.9. Consider the problem

$$\varepsilon y'' + p(x)y' + q(x)y = f(x) \quad \text{for } 0 < x < 1,$$

where $y(0) = \alpha$ and $y(1) = \beta$. Assume $p(x)$, $q(x)$, and $f(x)$ are continuous and $p(x) > 0$ for $0 \le x \le 1$.

(a) In the case where $f = 0$, show that

$$y \sim \beta \exp\left(\int_x^1 \frac{q(s)}{p(s)}\,\mathrm{d}s\right) + \left[\alpha - \beta \exp\left(\int_0^1 \frac{q(s)}{p(s)}\,\mathrm{d}s\right)\right] h(x),$$

where $h(x) = \mathrm{e}^{-p(0)x/\varepsilon}$.

(b) In the case where $f = 0$, but using the WKB method [see Exercise 4.3(b)], one obtains the result in part (a) except that

$$h(x) = \frac{p(0)}{p(x)} \exp\left(\int_0^x \frac{q(s)}{p(s)}\,\mathrm{d}s - \frac{1}{\varepsilon}\int_0^x p(s)\,\mathrm{d}s\right).$$

In Ou and Wong (2003) it is stated that this, and not the expression in part (a), is the "correct asymptotic approximation." Comment on this statement. (Hint: If they are correct, then the material covered in this section can be ignored.)

(c) Find a composite expansion of the solution in the case where $f(x)$ is not zero.

(d) Suppose $p(x) < 0$ for $0 \le x \le 1$. Show that the transformation $\hat{x} = 1 - x$ and the result from part (a) can be used to obtain a composite expansion of the solution.

2.10. Consider the problem

$$\varepsilon y'' + 6\sqrt{x}y' - 3y = -3 \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$ and $y(1) = 3$.

(a) Find a composite expansion of the problem.

(b) Find a two-term composite expansion.

2.11. This problem is concerned with the integral equation

$$\varepsilon y(x) = -q(x)\int_0^x [y(s) - f(s)]s\mathrm{d}s \quad \text{for } 0 \le x \le 1,$$

where $f(x)$ is smooth and positive.

(a) Taking $q(x) = 1$ find a composite expansion of the solution $y(x)$.

(b) Find a composite expansion of the solution in the case where $q(x)$ is positive and continuous but not necessarily differentiable.

2.12. Another way to match inner and outer expansions comes from Van Dyke (1975). To understand the procedure, suppose two terms have been calculated in both regions and the boundary-layer coordinate is $\bar{x} = x/\varepsilon^{\alpha}$, then do the following:

(i) Substitute x/ε^{α} for $\bar{x}$ into the inner expansion and expand the result to two terms (with x fixed).
(ii) Substitute $\varepsilon^{\alpha}\bar{x}$ for x into the outer expansion and expand the result to two terms (with $\bar{x}$ fixed).

After the results from (i) and (ii) are rewritten in terms of x, the matching condition states that the two expansions should agree exactly (to two terms).

(a) Using this matching procedure find a two-term composite expansion of the solution of (2.1).
(b) Using this matching procedure find a two-term composite expansion of the solution of

$$\varepsilon y'' = f(x) - y' \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$, and $f(x)$ is a given smooth function.

2.13. As seen in Fig. 2.4, in the boundary layer the solution of (2.1) is concave down (i.e., $y'' < 0$). This observation is useful for locating layers; this is discussed further in Sect. 2.5. However, not all boundary layers have strict concavity properties, and this problem considers such an example. The interested reader is referred to Howes (1978) for an extended discussion of this situation.
(a) Find a composite expansion for the solution of

$$\varepsilon^2 y'' = (x - y)(y - 2) \quad \text{for } 0 < x < 1,$$

where $y(0) = 3$ and $y(1) = 1$.
(b) Explain why the solution of this problem does not have a boundary layer that is strictly concave up or concave down but has one that might be identified as concave–convex.

2.14. Some consider it bothersome that a composite expansion generally does not satisfy boundary conditions exactly. One procedure that has been used to correct this situation is to note that the composite expansion for (2.1), before imposing boundary condition (2.3), is $y \sim a(\mathrm{e}^{-x} - \mathrm{e}^{-2x/\varepsilon})$. Substituting this into (2.3) we then find that $a = \mathrm{e}^1/(1 - \mathrm{e}^{-3/\varepsilon})$.
(a) This violates our assumption, as expressed in (2.4), that $y_0(x)$ is independent of ε. However, is the result still an asymptotic approximation of the solution for $0 \le x \le 1$?
(b) Use this idea to find a first-term composite expansion (that satisfies the boundary conditions exactly) for the solution of the problem

$$\varepsilon y'' = f(x) - y' \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$ and $f(x)$ is a given smooth function.

2.3 Examples Involving Boundary Layers

Almost all of the principal ideas underlying the method of matched asymptotic expansions were introduced in the example of Sect. 2.2. In the remainder of this chapter these ideas are applied, and extended, to more complicated problems. The extensions considered in this section are what happens when there are multiple boundary layers, and what can happen when the problem is nonlinear.

Example 1

To investigate an example where there is a boundary layer at each end of an interval, consider the problem of solving

$$\varepsilon^2 y'' + \varepsilon x y' - y = -\mathrm{e}^x \quad \text{for } 0 < x < 1, \tag{2.27}$$

where

$$y(0) = 2, \text{ and } y(1) = 1. \tag{2.28}$$

Unlike the example in Sect. 2.2, one of the coefficients of the preceding equation depends on x. This is not responsible for the multiple boundary layers, but, as we will see, it does result in different equations for each layer.

2.3.1 Step 1: Outer Expansion

The expansion of the solution in this region is the same as the last example, namely, $y \sim y_0 + \cdots$. From this and (2.27) one obtains the first-term approximation

$$y_0 = \mathrm{e}^x. \tag{2.29}$$

Clearly this function is incapable of satisfying either boundary condition, an indication that there are boundary layers at each end. An illustration of our current situation is given in Fig. 2.6. The solid curve is the preceding approximation, and the boundary conditions are also shown. The dashed curves are tentative sketches of what the boundary-layer solutions look like.

2.3.2 Steps 2 and 3: Boundary Layers and Matching

For the left endpoint we introduce the boundary-layer coordinate $\bar{x} = x/\varepsilon^\alpha$, in which case (2.27) becomes

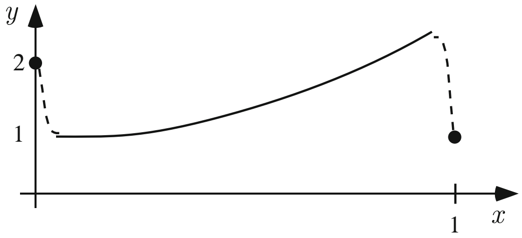

Figure 2.6 The *solid curve* is the outer approximation (2.29), and the *dashed curves* are guesses on how the boundary-layer solutions connect y_0 with the boundary conditions

$$\underset{①}{\varepsilon^{2-2\alpha}\frac{\mathrm{d}^2Y}{\mathrm{d}\bar{x}^2}} + \underset{②}{\varepsilon\bar{x}\frac{\mathrm{d}Y}{\mathrm{d}\bar{x}}} - \underset{③}{Y} = \underset{④}{-\mathrm{e}^{\varepsilon^\alpha\bar{x}}}. \tag{2.30}$$

In preparation for balancing, note that

$$\mathrm{e}^{\varepsilon^\alpha\bar{x}} \sim 1 + \varepsilon^\alpha\bar{x} + \cdots .$$

Also, as before, $Y(\bar{x})$ is used to designate the solution in this boundary-layer region. The balance in this layer is between terms ①, ③, ④, and so $\alpha = 1$. The appropriate expansion for Y is $Y \sim Y_0(\bar{x}) + \cdots$, and from (2.30) and the boundary condition at $\bar{x} = 0$ we have that

$$Y_0'' - Y_0 = -1 \quad \text{for } 0 < \bar{x} < \infty, \tag{2.31}$$

where

$$Y_0(0) = 2. \tag{2.32}$$

Note that (2.31) has at least one term in common with the equation for the outer region (which we should expect if there is to be any hope of matching the inner and outer expansions). The general solution is

$$Y_0(\bar{x}) = 1 + A\mathrm{e}^{-\bar{x}} + (1 - A)\mathrm{e}^{\bar{x}}. \tag{2.33}$$

This must match with the outer solution given in (2.29). The matching condition is $Y_0(\infty) = y_0(0)$, and so from (2.29) we have that $A = 1$.

To determine the solution in the boundary layer at the other end, we introduce the boundary-layer coordinate

$$\widetilde{x} = \frac{x - 1}{\varepsilon^\beta}. \tag{2.34}$$

In this region we will designate the solution as $\widetilde{Y}(\widetilde{x})$. Introducing (2.34) into (2.27) one obtains the equation

$$\varepsilon^{2-2\beta}\frac{\mathrm{d}^2\widetilde{Y}}{\mathrm{d}\widetilde{x}^2} + (1 + \varepsilon\widetilde{x})\varepsilon^{1-\beta}\frac{\mathrm{d}\widetilde{Y}}{\mathrm{d}\widetilde{x}} - \widetilde{Y} = -\mathrm{e}^{1+\varepsilon^\beta\widetilde{x}}. \tag{2.35}$$

The distinguished limit in this case occurs when $\beta = 1$. So the expansion $\widetilde{Y} \sim \widetilde{Y}_0(\widetilde{x})$ yields the problem

$$\widetilde{Y}_0'' + \widetilde{Y}_0' - \widetilde{Y}_0 = -e \quad \text{for } -\infty < \widetilde{x} < 0, \tag{2.36}$$

where

$$\widetilde{Y}_0(0) = 1. \tag{2.37}$$

It is important to notice that this boundary-layer equation has at least one term in common with the equation for the outer region. In solving this problem the general solution is found to be

$$\widetilde{Y}_0(\widetilde{x}) = e + B\mathrm{e}^{r_+\widetilde{x}} + (1 - e - B)\mathrm{e}^{r_-\widetilde{x}}, \tag{2.38}$$

where $2r_\pm = -1 \pm \sqrt{5}$.

The matching requirement is the same as before, which is that when approaching the boundary layer from the outer region you get the same value as when you leave the boundary layer and approach the outer region. In other words, it is required that $\widetilde{Y}_0(-\infty) = y_0(1)$. Hence, from (2.38), $B = 1 - e$. As a final comment, note that boundary-layer Eq. (2.36) differs from (2.31), and this is due to the x dependence of the terms in the original problem.

2.3.3 Step 4: Composite Expansion

The last step is to combine the three expansions into a single expression. This is done in the usual way of adding the expansions together and then subtracting the common parts. From (2.29), (2.33), and (2.38), a first-term expansion of the solution over the entire interval is

$$\begin{aligned} y &\sim y_0(x) + Y_0(\bar{x}) - Y_0(\infty) + \widetilde{Y}_0(\widetilde{x}) - \widetilde{Y}_0(-\infty) \\ &\sim \mathrm{e}^x + \mathrm{e}^{-x/\varepsilon} + (1 - e)\mathrm{e}^{-r(1-x)/\varepsilon}, \end{aligned} \tag{2.39}$$

where $2r = -1 + \sqrt{5}$. This approximation is shown in Fig. 2.7 along with the numerical solution. One can clearly see the boundary layers at the endpoints of the interval as well as the accuracy of the asymptotic approximation as ε decreases. ■

Example 2

Some interesting complications arise when using matched asymptotic expansions with nonlinear equations. For example, it is not unusual for the solution

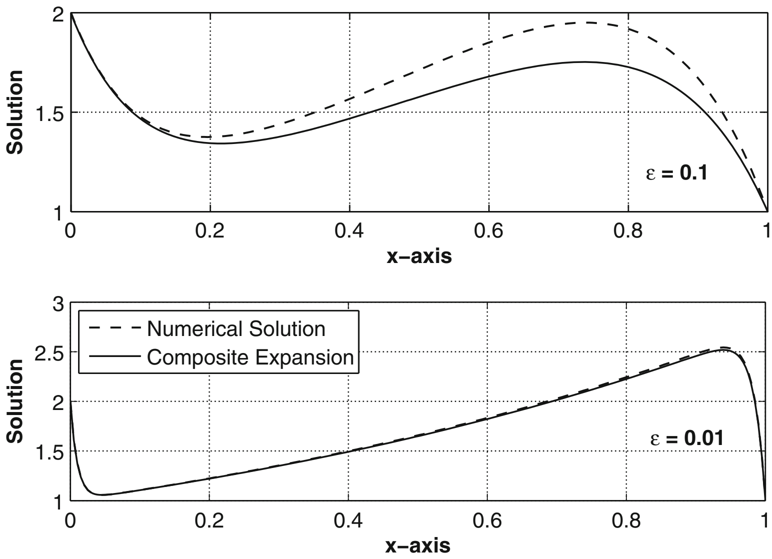

Figure 2.7 Graph of numerical solution of (2.27), (2.28) and the composite expansion given in (2.39) in the case where $\varepsilon = 10^{-1}$ and where $\varepsilon = 10^{-2}$. Note the appearance of the boundary layers, as well as the convergence of the composite expansion to the solution of the problem, as ε decreases

of a nonlinear equation to be defined implicitly. To understand this situation, consider the problem

$$\varepsilon y'' + \varepsilon y' - e^{y} = -2 - x \quad \text{for } 0 < x < 1, \tag{2.40}$$

where $y(0) = 0$ and $y(1) = 1$.

The first term in the outer expansion can be obtained by simply setting $\varepsilon = 0$ in (2.40). This yields $e^{y_0} = 2 + x$, and so $y \sim \ln(x + 2)$. Given that this does not satisfy either boundary condition, the outer expansion is assumed to hold for $0 < x < 1$. An illustration of our current situation is given in Fig. 2.8. The solid curve is the outer approximation, and the boundary conditions are also shown. The dashed curves are tentative sketches of what the boundary-layer solutions look like.

For the boundary layer at $x = 0$ the coordinate is $\bar{x} = x/\sqrt{\varepsilon}$, and one finds that $Y \sim Y_0$, where

$$Y_0'' - e^{Y_0} = -2$$

for $Y_0(0) = 0$. Multiplying the differential equation by Y_0' and integrating yields

$$\frac{1}{2}\left(\frac{d}{d\bar{x}} Y_0\right)^2 = B - 2Y_0 + e^{Y_0}. \tag{2.41}$$

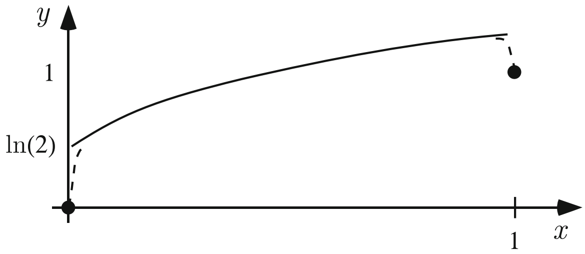

Figure 2.8 The *solid curve* is the outer approximation, and the *dashed curves* are guesses on how the boundary-layer solutions connect y_0 with the boundary conditions

We can determine the constant B if we look at how the expansions in the previous examples matched. For example, Fig. 2.2, shows that for the boundary-layer solution, $Y_0' \to 0$ and $Y_0 \to y_0(0)$ as $\bar{x} \to \infty$. Assuming this monotonic convergence occurs in the present example, then, since $y_0(0) = \ln(2)$, from (2.41) we get $B = 2[-1 + \ln(2)]$. Now, solving (2.41) for Y_0' we obtain

$$Y_0' = \pm\sqrt{2(B - 2Y_0 + \mathrm{e}^{Y_0})} .$$

To determine which sign, it is evident from Fig. 2.8 that the boundary-layer solution increases from $Y_0(0) = 0$ to $Y_0(\infty) = y_0(0) = \ln(2)$, and so we take the $+$ sign. Doing this, separating variables, and then integrating yields the following result:

$$\int_0^{Y_0} \frac{\mathrm{d}s}{\sqrt{2(B - 2s + \mathrm{e}^s)}} = \bar{x}. \tag{2.42}$$

This is the solution of the $O(1)$ problem in the boundary layer and it defines Y_0 implicitly in terms of $\bar{x}$. It is important to note that the assumptions that were made to derive this result, such as $Y_0' \to 0$ and $Y_0 \to y_0(0)$ as $\bar{x} \to \infty$, hold for this solution.

The derivation of the expansion of the solution for the boundary layer at $x = 1$ is very similar (Exercise 2.26). The boundary-layer coordinate is $\widetilde{x} = (x - 1)/\sqrt{\varepsilon}$. One finds that $\widetilde{Y} \sim \widetilde{Y}_0$, where

$$\int_1^{\widetilde{Y}_0} \frac{\mathrm{d}s}{\sqrt{2(A - 3s + \mathrm{e}^s)}} = -\widetilde{x} \tag{2.43}$$

for $A = 3[-1 + \ln(3)]$.

Even though the boundary-layer solutions are defined implicitly, it is still possible to write down a composite approximation. Adding the expansions together and then subtracting the common parts we obtain

$$
\begin{aligned}
y &\sim y_0(x) + Y_0(\bar{x}) - y_0(0) + \widetilde{Y}_0(\widetilde{x}) - y_0(1) \\
&\sim \ln\left(\frac{1}{6}(x+2)\right) + Y_0(\bar{x}) + \widetilde{Y}_0(\widetilde{x}).
\end{aligned}
$$

A few additional comments and results for this example can be found in Exercise 2.26. ■

Example 3

The ideas underlying matched asymptotic expansions are not limited to boundary-value problems. As an example, in studying the dynamics of autocatalytic reactions one comes across the problem of solving (Gray and Scott, 1994)

$$
\varepsilon \frac{\mathrm{d}u}{\mathrm{d}t} = \mathrm{e}^{-t} - uv^2 - u, \tag{2.44}
$$

$$
\frac{\mathrm{d}v}{\mathrm{d}t} = uv^2 + u - v, \tag{2.45}
$$

where $u(0) = v(0) = 1$. In this case, there is an initial layer (near $t = 0$), as well as an outer solution that applies away from $t = 0$.

For the outer solution, assuming $u \sim u_0 + \varepsilon u_1 + \cdots$ and $v \sim v_0 + \varepsilon v_1 + \cdots$ one finds that the first-order problem is

$$
\begin{aligned}
0 &= \mathrm{e}^{-t} - u_0 v_0^2 - u_0, \\
\frac{\mathrm{d}}{\mathrm{d}t} v_0 &= u_0 v_0^2 + u_0 - v_0.
\end{aligned}
$$

The solution of this system is $v_0 = (t + a)\mathrm{e}^{-t}$ and $u_0 = \mathrm{e}^{-t}/(v_0^2 + 1)$, where a is a constant determined from matching.

For the initial layer, letting $\tau = t/\varepsilon$ and assuming that $U \sim U_0(\tau) + \cdots$ and $V \sim V_0(\tau) + \cdots$ one obtains the problem of solving

$$
\begin{aligned}
U_0' + \cdots &= \mathrm{e}^{-\varepsilon\tau} - (U_0 + \cdots)(V_0 + \cdots)^2 - (U_0 + \cdots), \\
V_0' + \cdots &= \varepsilon\left[(U_0 + \cdots)(V_0 + \cdots)^2 + (U_0 + \cdots) - (V_0 + \cdots)\right].
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
U_0' &= 1 - U_0 V_0^2 - U_0, \\
V_0' &= 0.
\end{aligned}
$$

Solving these equations and using the given initial conditions that $U_0(0) = V_0(0) = 1$ one finds that $U_0 = \frac{1}{2}(1 + \mathrm{e}^{-2\tau})$ and $V_0 = 1$.

The matching conditions are simply that $u_0(0) = U_0(\infty)$ and $v_0(0) = V_0(\infty)$, from which one finds that $a = 1$. Therefore, composite expansions for the first terms are

$$u \sim \frac{1}{2}\mathrm{e}^{-2t/\varepsilon} + \frac{\mathrm{e}^{-t}}{1+(1+t)^2\mathrm{e}^{-2t}},$$
$$v \sim (1+t)\mathrm{e}^{-t}.$$

The preceding derivation was relatively straightforward, but the time variable can cause some interesting complications. For example, the time interval is usually unbounded, and this can interfere with the well-ordering of an expansion, which is discussed extensively in Chap. 3. There are also questions related to the stability, or instability, of a steady-state solution, and this is considered in Chap. 6 . ■

Example 4

It is possible to have multiple layers in a problem and have them occur on the same side of an interval. An example of this arises when solving

$$\varepsilon^3 y'' + x^3 y' - \varepsilon y = x^3, \quad \text{for } 0 < x < 1, \tag{2.46}$$

where

$$y(0) = 1, \text{ and } y(1) = 3. \tag{2.47}$$

To help explain what happens in this problem, the numerical solution is shown in Fig. 2.9. It is the hook region, near $x = 0$, that is the center of attention in this example.

The outer expansion is $y \sim x + 2$, and this holds for $0 < x \le 1$. This linear function is clearly seen in Fig. 2.9. To investigate what is happening near $x = 0$, set $\bar{x} = x/\varepsilon^\alpha$. The differential equation in this case becomes

$$\underset{①}{\varepsilon^{3-2\alpha}\frac{\mathrm{d}^2}{\mathrm{d}\bar{x}^2}Y} + \underset{②}{\varepsilon^{2\alpha}\bar{x}^3\frac{\mathrm{d}}{\mathrm{d}\bar{x}}Y} - \underset{③}{\varepsilon Y} = \underset{④}{\varepsilon^{3\alpha}\bar{x}^3}. \tag{2.48}$$

There are two distinguished limits (other than the one for the outer region). One comes when ① ∼ ③, in which case $\alpha = 1$. This is an inner–inner layer, and in Fig. 2.9 this corresponds to the monotonically decreasing portion of the curve in the immediate vicinity of $x = 0$. The other balance occurs when ② ∼ ③, in which case $\alpha = \frac{1}{2}$. This is an inner layer, and in Fig. 2.9 this corresponds to the region where the solution goes through its minimum value. Carrying out the necessary calculations, one obtains the following first-term composite expansion (Exercise 2.16):

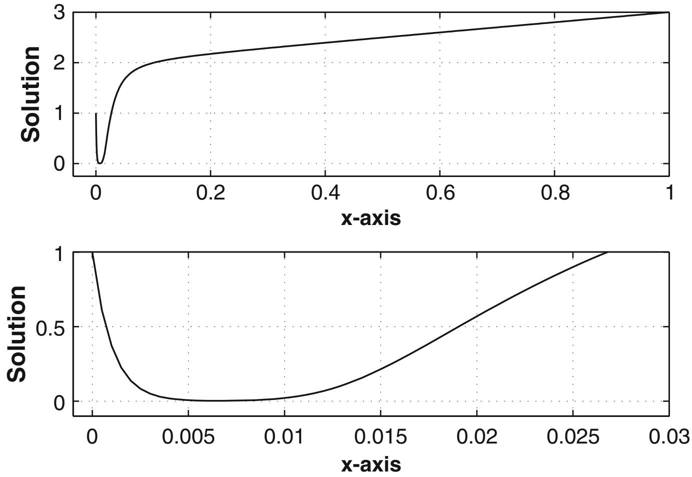

Figure 2.9 Numerical solution of (2.46) and (2.47), in the case where $\varepsilon = 10^{-3}$. The *lower plot* shows the solution in the hook region near $x = 0$

$$y \sim \begin{cases} x + \mathrm{e}^{-x/\varepsilon} + 2\mathrm{e}^{-\varepsilon/(2x^2)} & \text{for } 0 < x \leq 1, \\ 1 & \text{for } \quad x = 0. \end{cases} \tag{2.49}$$

To reiterate a point made in the earlier examples, note that the equation for the inner–inner layer has no terms in common with the equation for the outer region, but the inner layer equation shares terms with both. ■

Exercises

2.15. Find a composite expansion of the solution of the following problems and sketch the solution:

(a) $\varepsilon y'' + \varepsilon(x+1)^2 y' - y = x - 1$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = -1$.
(b) $\varepsilon y'' - y' + y^2 = 1$ for $0 < x < 1$, where $y(0) = 1/3$ and $y(1) = 1$.
(c) $\varepsilon y'' - \mathrm{e}^x y = f(x)$ for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$.
(d) $\varepsilon y'' + \varepsilon y' - y^2 = -1 - x^2$ for $0 < x < 1$, where $y(0) = 2$ and $y(1) = 2$.
(e) $\varepsilon y'' - y(y' + 1) = 0$ for $0 < x < 1$, where $y(0) = 3$ and $y(1) = 3$.
(f) $\varepsilon y'' + y(y' + 3) = 0$ for $0 < x < 1$, where $y(0) = 4$ and $y(1) = 4$.
(g) $\varepsilon y'' + y(1 - y)y' - y = 0$ for $0 < x < 1$, where $y(0) = -2$ and $y(1) = -2$.
(h) $\varepsilon^2 y'' + \varepsilon(3 - x^2)y' - 4y = 4x$ for $-1 < x < 1$, where $y(-1) = 2$ and $y(1) = 3$.
(i) $\varepsilon y'' - y(1 + y)y' - 3y = 0$ for $0 < x < 1$, where $y(0) = 2$ and $y(1) = 2$.

(j) $\varepsilon y'''' - (1+x)y' = 1 - x^2$ for $0 < x < 1$, where $y(0) = y'(0) = y(1) = y'(1) = 0$.

(k) $\varepsilon^3 y'' = \dfrac{(y-\mathrm{e}^x)^3}{(1+\varepsilon y')^2}$ for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 4$.

(l) $\varepsilon \dfrac{\mathrm{d}}{\mathrm{d}x}\left(E(x)\dfrac{u'}{1-\varepsilon u'}\right) - u = f(x)$ for $0 < x < 1$, where $u(0) = u(1) = 0$. Also, $E(x)$, $f(x)$ are known, smooth, positive functions.

(m) $\varepsilon(x^2 y')' = \dfrac{x^2(y-1)}{y}$ for $0 < x < 1$, where $y'(0) = 0$ and $y(1) = 2$.

(n) $(y^{\varepsilon})'' - y = -\mathrm{e}^x$ for $0 < x < 1$, where $y(0) = 2$ and $y(1) = 3$.

2.16. Derive (2.49).

2.17. Consider the boundary value problem

$$\varepsilon y'' - xy' - \kappa y = -1 \quad \text{for } -1 < x < 1,$$

where $y(-1) = y(1) = 0$. Assuming

$$\kappa = \int_{-1}^{1} y^2 \mathrm{d}x,$$

find the first term in an expansion of κ for small ε.

2.18. The Reynolds equation from the gas lubrication theory for slider bearings is (DiPrima, 1968; Shepherd, 1978)

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}x}(H^3 y y') = \frac{\mathrm{d}}{\mathrm{d}x}(Hy) \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 1$. Here $H(x)$ is a known, smooth, positive function with $H(0) \neq H(1)$.

(a) Find a composite expansion of the solution for small ε. Note the boundary-layer solution will be defined implicitly, but it is still possible to match the expansions.

(b) Show that if the boundary layer is assumed to be at the opposite end from what you found in part (a), then the inner and outer expansions do not match.

2.19. This exercise considers the problem of a beam with a small bending stiffness. This consists in solving (Denoel and Detournay, 2010)

$$\varepsilon y'' = (1-x)\sin y - \cos y \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = \pi/2$. In this problem, y is an angular variable that is assumed to satisfy $0 \leq y \leq \pi$, and $\sin y + (1-x)\cos y \geq 0$. Find the first term in the expansions in (i) the outer layer, (ii) the boundary layer at $x = 0$, and (iii) a composite expansion.

2.20. The Michaelis–Menten reaction scheme for an enzyme catalyzed reaction is (Holmes, 2009)

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= -s + (\mu + s)c,\\ \varepsilon\frac{\mathrm{d}c}{\mathrm{d}t} &= s - (\kappa + s)c,\end{aligned}$$

where $s(0) = 1$ and $c(0) = 0$. Here $s(t)$ is the concentration of substrate, $c(t)$ is the concentration of the chemical produced by the catalyzed reaction, and μ, κ are positive constants with $\mu < \kappa$. Find the first term in the expansions in (i) the outer layer, (ii) the initial layer, and (iii) a composite expansion.

2.21. The Poisson–Nernst–Planck model for flow of ions through a membrane consists of the following equations (Singer et al., 2008): for $0 < x < 1$,

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}x} + p\frac{\mathrm{d}\phi}{\mathrm{d}x} &= -\alpha,\\ \frac{\mathrm{d}n}{\mathrm{d}x} - n\frac{\mathrm{d}\phi}{\mathrm{d}x} &= -\beta,\\ \varepsilon^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &= -p + n.\end{aligned}$$

The boundary conditions are $\phi(0) = 1$, $\phi(1) = 0$, $p(0) = 4$, and $n(0) = 1$. In these equations, p and n are the concentrations of the ions with valency 1 and -1, respectively, and ϕ is the potential. Assume that α and β are positive constants that satisfy $\kappa < 1$, where

$$\kappa = \frac{\alpha + \beta}{2\sqrt{p(0)n(0)}}\,.$$

Also, you can assume that $\alpha \neq \beta$.

(a) Assuming there is a boundary layer at $x = 0$, derive the outer and boundary-layer approximations. Explain why, if the outer approximation for ϕ is required to satisfy $\phi(1) = 0$, the approximations you derived do not match.

(b) There is also a boundary layer at $x = 1$. Derive the resulting approximations and complete the matching you started in part (a). From this show that

$$p(1) \sim p(0)\mathrm{e}^{\phi(0)}(1-\kappa)^{2\beta/(\alpha+\beta)}$$

and

$$n(1) \sim n(0)\mathrm{e}^{-\phi(0)}(1-\kappa)^{2\alpha/(\alpha+\beta)}.$$

2.22. A modified version of the Grodsky model for insulin release is to find $y = y(t, \lambda)$, which satisfies (Carson et al., 1983)

$$\varepsilon \frac{dy}{dt} = -y + f(t) + \int_0^\infty y(t,s)e^{-\gamma s} ds \quad \text{for } 0 < t < \infty,$$

where $y = g(\lambda)$ when $t = 0$. Also, $\gamma > 1$.
(a) Find a composite expansion of the solution for small ε.
(b) Derive the composite expansion you obtained in part (a) from the exact solution, which is

$$y = \left[g(\lambda) - \gamma g_0(1 - e^{t/\kappa})\right] e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t f(\tau)e^{-(\gamma-1)(t-\tau)/\kappa} d\tau,$$

where $\kappa = \varepsilon\gamma$ and $g_0 = \int_0^\infty g(s)e^{-\gamma s} ds$. Also, what is the composite expansion when $\gamma = 1$?

2.23. The eigenvalue problem for the vertical displacement, $u(x)$, of an elastic beam that is under tension is

$$\varepsilon^2 u'''' - u'' = \lambda u \quad \text{for } 0 < x < 1,$$

where $u = u' = 0$ at $x = 0, 1$. The question is, what values of λ produce a nonzero solution of the problem? In this context, λ is the eigenvalue, and it depends on ε.
(a) Find the first term in the expansions for $u(x)$ and λ.
(b) Find the second term in the expansions for $u(x)$ and λ.

2.24. Find a composite expansion of the solution of

$$\varepsilon^2 y'' + 2\varepsilon p(x)y' - q(x)y = f(x) \quad \text{for } 0 < x < 1,$$

where $y(0) = \alpha$ and $y(1) = \beta$. The functions $p(x)$, $q(x)$, and $f(x)$ are continuous and $q(x)$ is positive for $0 \le x \le 1$.

2.25. In the study of an ionized gas confined to a bounded domain Ω, the potential $\phi(\mathbf{x})$ satisfies

$$-\nabla^2\phi + h\left(\frac{\phi}{\varepsilon}\right) = \alpha \quad \text{for } \mathbf{x} \in \Omega,$$

where conservation of charge requires

$$\int_\Omega h\left(\frac{\phi}{\varepsilon}\right) dV = \beta$$

and, assuming the exterior of the region is a conductor, $\partial_n\phi = \gamma$ on $\partial\Omega$. The function $h(s)$ is smooth and strictly increasing with $h(0) = 0$. The positive constants α and β are known (and independent of ε), and the constant γ is determined from the conservation of charge equation.

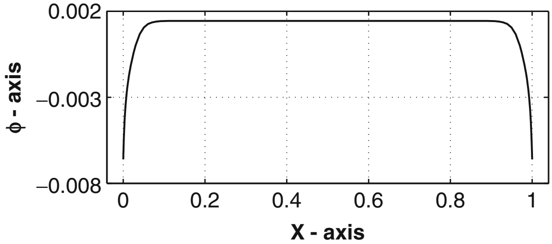

Figure 2.10 Solution of problem in Exercise 2.25 in the case where $\varepsilon = 10^{-3}$

(a) For the case of one dimension, suppose Ω is the interval $0 < x < 1$. What does the problem reduce to? Find γ in terms of α and β.
(b) Find the exact solution of the problem in part (a) when $h(s) = s$. Sketch the solution for $\gamma < 0$, and describe the boundary layers that are present.
(c) For the one-dimensional problem in part (a), find the first term in the inner and outer expansions of the solution. In doing this, take $h(s) = s^{2k+1}$, where k is a positive integer, and assume $\beta < \alpha$. For comparison the numerical solution is shown in Fig. 2.10 in the case where $k = 1$, $\alpha = 3$, $\beta = 1$, and $\varepsilon = 10^{-3}$.
(d) Discuss the steps needed to find a composite expansion involving the terms derived in part (c).
(e) For $\Omega \subset \mathbb{R}^n$, where $n > 1$, find γ.

2.26. This exercise examines Example 2 in more depth.
(a) Letting $f(s) = 2(B - 2s + \mathrm{e}^s)$, where $B = 2[-1 + \ln(2)]$, sketch f for $-\infty < s < \infty$.
(b) Writing $f(s) = (\ln(2) - s)^2 g(s)$, show that $g(s)$ is positive and $g(\ln(2)) = 2$. It is also possible to show that $g(s)$ is monotone increasing for $-\infty < s < \infty$.
(c) With part (b), (2.42) can be written as

$$\int_0^{Y_0} \frac{\mathrm{d}s}{(\ln(2) - s)\sqrt{g(s)}} = \bar{x}.$$

Use this to prove the monotonicity assumed in the derivation of (2.42).
(d) Derive (2.43).
(e) Use the ideas developed in parts (a) and (b) to show that (2.43) can be written as

$$\int_1^{\widetilde{Y}_0} \frac{\mathrm{d}s}{(\ln(3) - s)\sqrt{h(s)}} = -\widetilde{x},$$

where h is positive with $h(\ln(3)) = 3$.

2.4 Transcendentally Small Terms

Even in the simplest boundary-layer problems, a question arises about what is missing in the expansion. For example, consider the problem of solving

$$\varepsilon y'' = 2 - y' \quad \text{for } 0 < x < 1, \tag{2.50}$$

where $y(0) = 0$ and $y(1) = 1$. This problem has a boundary layer at $x = 0$. For the outer approximation, assuming the usual power series expansion

$$y \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots , \tag{2.51}$$

one finds that $y_0 = 2x - 1$ and $y_1 = y_2 = \cdots = 0$. In other words, the entire outer expansion is just $y \sim 2x - 1$. Given that $y = 2x - 1$ is not the exact solution, the expansion (2.51) is missing something. Whatever this something is, it is transcendentally small compared to the power functions. Because of this, we have ignored this part of the approximation in all of the examples and exercises in the earlier sections of this chapter. There are, however, occasional situations where transcendentally small terms in the outer region must be accounted for. The objective here is to explore how this is done. The first example demonstrates the basic ideas, even though it is not necessary to include transcendentally small terms to obtain an accurate approximation. The second example is more challenging and involves a problem where such terms are used to complete the approximation.

Example 1

The first example is (2.50). As stated earlier, when assuming an outer expansion of the form (2.51), one finds that $y_0 = 2x - 1$ and $y_1 = y_2 = \cdots = 0$. For the boundary layer at $x = 0$, the expansion is found to be

$$Y(\bar{x}) \sim A_0\left(1 - e^{-\bar{x}}\right) + \varepsilon\left[2\bar{x} + A_1\left(1 - e^{-\bar{x}}\right)\right] + \varepsilon^2 A_2\left(1 - e^{-\bar{x}}\right) + \cdots , \tag{2.52}$$

where $\bar{x} = x/\varepsilon$. Note that the preceding expansion satisfies the boundary condition at $x = 0$ but has not yet been matched with the outer expansion.

To determine what the outer expansion is missing, we replace (2.51) with the assumption that

$$y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots + z_0(x, \varepsilon) + z_1(x, \varepsilon) + \cdots , \tag{2.53}$$

where the z_i are well-ordered and transcendentally small compared to the power functions. Specifically, $z_j \ll z_i$, $\forall i < j$, and $z_i \ll \varepsilon^n$, $\forall i, n$. At this point we have no idea how z_i depends on ε, but we will determine this in the analysis to follow. It is important to note, however, that because (2.53) is the outer expansion, it holds for $\varepsilon \to 0$ with x held fixed with $0 < x \leq 1$.

Substituting (2.53) into (2.50) yields

$$\varepsilon(y_0'' + \varepsilon y_1'' + \cdots + z_0'' + z_1'' + \cdots) = 2 - (y_0' + \varepsilon y_1' + \cdots + z_0' + z_1' + \cdots).$$

The problems for the y_i are unaffected, and so as before we obtain $y_0 = 2x-1$ and $y_1 = y_2 = \cdots = 0$. To determine the equation for z_0, remember that we are not certain of how this function depends on ε. Because of this we will retain everything involving z_0, and this means that the equation is $\varepsilon z_0'' = -z_0'$. The general solution is $z_0 = a(\varepsilon) + b(\varepsilon)\exp(-x/\varepsilon)$. Imposing the boundary condition $z_0 = 0$ at $x = 1$ yields

$$z_0 = b(\varepsilon)\left(\mathrm{e}^{-x/\varepsilon} - \mathrm{e}^{-1/\varepsilon}\right).$$

It remains to match the expansions. Introducing the intermediate layer variable $x_\eta = x/\varepsilon^\beta$, where $0 < \beta < 1$, (2.53) then becomes

$$y_{\text{outer}} \sim -1 + 2x_\eta\varepsilon^\beta + \cdots + b(\varepsilon)\left(\mathrm{e}^{-x_\eta/\varepsilon^{1-\beta}} - \mathrm{e}^{-1/\varepsilon}\right) + \cdots,$$

and the boundary-layer expansion (2.52) takes the form

$$y_{\text{inner}} \sim A_0\left(1 - \mathrm{e}^{-x_\eta/\varepsilon^{1-\beta}}\right) + \varepsilon\left[2x_\eta/\varepsilon^{1-\beta} + A_1\left(1 - \mathrm{e}^{-x_\eta/\varepsilon^{1-\beta}}\right)\right] + \cdots. \tag{2.54}$$

Matching these expressions, it follows that $A_0 = -1$ and $b = -A_0 = 1$. With this, we have that

$$z_0 = \mathrm{e}^{-x/\varepsilon} - \mathrm{e}^{-1/\varepsilon}. \tag{2.55}$$

It is not hard to show that this term is transcendentally small compared to the power functions for any given value of x satisfying $0 < x \leq 1$. The resulting outer expansion is therefore

$$y \sim 2x - 1 + \cdots + \mathrm{e}^{-x/\varepsilon} - \mathrm{e}^{-1/\varepsilon} + \cdots. \tag{2.56}$$

As you would expect, the transcendentally small terms contribute very little to the numerical value of the outer solution. To illustrate, if $\varepsilon = 0.01$ and $x = 3/4$, then $y_0 = 1/2$ while $z_0 \approx 2.6 \times 10^{-33}$. This is why we have ignored this portion of the outer expansion in the earlier sections of this chapter. The fact that you cannot always do this is demonstrated in the next example. ■

The matching in the preceding example helps explain the reason for the transcendentally small terms in the outer expansion. In particular, the first transcendentally small term in the boundary-layer expansion (2.54) is what generates the need for z_0. In other words, the boundary layer causes our having to include z_0 in the outer expansion. Actually, information flows in both directions. If you calculate the higher terms in the expansions, you will find that it is also necessary to include transcendentally small terms in the boundary-layer expansion. This is explored in more depth in Exercise 2.27.

The expansion (2.53) is not very specific about how the z_i depend on ε, and we determined this dependence in the reduction of the problem. Some prefer a more explicit form of the expansion and make specific assumptions about the z_i. For example, an assumption often used to construct a composite expansion is that $z_0 = A(x)\mathrm{e}^{-g(x)/\varepsilon}$ (Latta, 1951). This will not produce the result in (2.55), and so the assumption must be modified to account for transcendentally small terms. Examples can be found in MacGillivray (1997) and Howls (2010). A somewhat different approach is explored in Exercise 2.30.

Example 2

Consider the problem of solving

$$\varepsilon y'' - xy' + \varepsilon xy = 0 \quad \text{for} \ -1 < x < 1, \tag{2.57}$$

where $y(-1) = y_{\mathrm{L}}$ and $y(1) = y_{\mathrm{R}}$ are assumed to be specified. There are boundary layers at both ends, and the analysis is very similar to Example 1 in Sect. 2.3. For the outer expansion one assumes $y \sim y_0 + \cdots$, and from the differential equation it follows that $y_0 = c$ (i.e., the first term is just a constant). For the boundary layer at the left end, one lets $\bar{x} = (x+1)/\varepsilon$ and from this finds that $Y_0'' + Y_0' = 0$. Solving this, imposing the boundary condition $Y_0(0) = y_{\mathrm{L}}$, and then matching one finds that $Y_0 = c + (y_{\mathrm{L}} - c)\mathrm{e}^{-\bar{x}}$. For the boundary layer at the right end, where $\widetilde{x} = (x-1)/\varepsilon$, one finds that $\widetilde{Y}_0 = c + (y_{\mathrm{R}} - c)\mathrm{e}^{\widetilde{x}}$. Putting the results together we have that

$$y \sim \begin{cases} c + (y_{\mathrm{L}} - c)\mathrm{e}^{-\bar{x}} & \text{boundary layer at } x = -1, \\ c & \text{outer region}, \\ c + (y_{\mathrm{R}} - c)\mathrm{e}^{\widetilde{x}} & \text{boundary layer at } x = 1. \end{cases} \tag{2.58}$$

What is unusual about this problem is that we have carried out the boundary-layer analysis but still have an unknown constant (i.e., c). How to deal with this depends on the problem. For some problems it is enough to look at the second term in the expansion (e.g., Exercise 2.57), for other linear problems a WKB type argument can be used (see Chap. 4), and for still others one can use a symmetry property (e.g., Sect. 2.5). What is going to be shown here is that the transcendentally small terms in the outer region can be used to determine c. This approach has the disadvantage of being somewhat more difficult mathematically but has the distinct advantage of being able to work on a wide range of linear and nonlinear problems (e.g., we will use this approach when studying metastability in Chap. 6).

The remedy is to use (2.53) instead of the regular power series expansion. In this case the differential equation becomes

$$\varepsilon(y_0'' + \cdots + z_0'' + \cdots) - x(y_0' + \cdots + z_0' + \cdots) + \varepsilon x(y_0 + \cdots + z_0 + \cdots) = 0.$$

The problem for y_0 is unaffected, and we only need to concentrate on z_0. In the previous example we used the dictum that everything involving z_0 is retained, which in this case means that $\varepsilon z_0'' - xz_0' + \varepsilon x z_0 = 0$. However, in this problem it is wise to think about what terms are actually needed in this equation. In fact, our construction will mimic the procedure used for (1.36), where the form of the expansion was determined in the derivation. Retaining $\varepsilon z_0''$ and xz_0' is reasonable because it is very possible that a transcendentally small term of the form $\mathrm{e}^{-x/\varepsilon}$ will occur, just as it did in the previous example. On the other hand, it is expected that $\varepsilon x z_0$ is higher order than the other two terms and for this reason contributes to the higher-order equations (e.g., the problem for z_1). This ad hoc reasoning is necessary because we have not yet determined the order for z_0, and the resulting reduction helps to simplify the analysis. What is going to be necessary, once z_0 is determined, is to check that these assumptions are correct.

Solving the equation $\varepsilon z_0'' - xz_0' = 0$, one obtains the general solution

$$z_0 = b(\varepsilon) + a(\varepsilon) \int_{-1}^{x} \mathrm{e}^{s^2/(2\varepsilon)} \mathrm{d}s.$$

The matching proceeds in the usual manner using intermediate variables.

Matching at Left End

The intermediate variable is $x_\eta = (x+1)/\varepsilon^\beta$, where $0 < \beta < 1$. The boundary-layer approximation becomes

$$y_{\text{inner}} \sim c + (y_{\mathrm{L}} - c)\mathrm{e}^{-x_\eta/\varepsilon^{1-\beta}} + \cdots . \tag{2.59}$$

Before writing down the outer expansion, note that the integral in z_0 is going to take the form

$$\int_{-1}^{-1+\varepsilon^\beta x_\eta} \mathrm{e}^{s^2/(2\varepsilon)} \mathrm{d}s.$$

The expansion of this integral can be obtained using integration by parts by writing

$$\mathrm{e}^{s^2/(2\varepsilon)} = \frac{\varepsilon}{s}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{s^2/(2\varepsilon)}.$$

In this case, setting $q = \varepsilon^\beta x_\eta$, we have that

$$\begin{aligned}
\int_{-1}^{-1+q} \mathrm{e}^{s^2/(2\varepsilon)} \mathrm{d}s &= \frac{\varepsilon}{s}\mathrm{e}^{s^2/(2\varepsilon)}\Big|_{s=-1}^{-1+q} + \int_{-1}^{-1+q} \frac{\varepsilon}{s^2}\mathrm{e}^{s^2/(2\varepsilon)} \mathrm{d}s \\
&= \frac{\varepsilon}{s}\Big(1 + \frac{\varepsilon}{s^2}\Big)\mathrm{e}^{s^2/(2\varepsilon)}\Big|_{s=-1}^{-1+q} + \int_{-1}^{-1+q} \frac{3\varepsilon^2}{s^4}\mathrm{e}^{s^2/(2\varepsilon)} \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varepsilon}{s}\left(1 + \frac{\varepsilon}{s^2} + \frac{3\varepsilon^2}{s^4} + \cdots\right) \mathrm{e}^{s^2/(2\varepsilon)}\Bigg|_{s=-1}^{-1+q} \\
&\sim \varepsilon \mathrm{e}^{1/(2\varepsilon)}\left(1 - \mathrm{e}^{-q/\varepsilon}\right).
\end{aligned}
$$

In the last step it is assumed that the β interval is reduced to $1/2 < \beta < 1$. With this the outer expansion becomes

$$
y_{\mathrm{outer}} \sim c + \cdots + b(\varepsilon) + \varepsilon a(\varepsilon) \mathrm{e}^{1/(2\varepsilon)}\left(1 - \mathrm{e}^{-q/\varepsilon}\right) + \cdots . \tag{2.60}
$$

Matching (2.59) and (2.60) it follows that $\varepsilon a(\varepsilon)\mathrm{e}^{1/(2\varepsilon)} = c - y_{\mathrm{L}}$ and $b(\varepsilon) = -\varepsilon a(\varepsilon)\mathrm{e}^{1/(2\varepsilon)}$.

Matching at Right End

The intermediate variable is $x_\eta = (x-1)/\varepsilon^\beta$, where $0 < \beta < 1$. The boundary-layer approximation becomes

$$
y_{\mathrm{outer}} \sim c + (y_{\mathrm{R}} - c)\mathrm{e}^{x_\eta/\varepsilon^{1-\beta}} + \cdots . \tag{2.61}
$$

Using an integration by parts argument similar to what was done for the layer at the left end, one finds that

$$
y_{\mathrm{inner}} \sim c + \cdots + b(\varepsilon) + \varepsilon a(\varepsilon) \mathrm{e}^{1/(2\varepsilon)}\left(1 + \mathrm{e}^{q/\varepsilon}\right) + \cdots , \tag{2.62}
$$

where $q = \varepsilon^\beta x_\eta$. Matching (2.61) and (2.62) it follows that $\varepsilon a(\varepsilon)\mathrm{e}^{1/(2\varepsilon)} = y_{\mathrm{R}} - c$ and $b(\varepsilon) = -\varepsilon a(\varepsilon)\mathrm{e}^{1/(2\varepsilon)}$.

The condition we are seeking comes from the matching conditions, which require that $\varepsilon a(\varepsilon)\mathrm{e}^{1/(2\varepsilon)} = c - y_{\mathrm{L}}$ and $\varepsilon a(\varepsilon)\mathrm{e}^{1/(2\varepsilon)} = y_{\mathrm{R}} - c$. Equating these conditions it follows that

$$
c = \frac{1}{2}\left(y_{\mathrm{L}} + y_{\mathrm{R}}\right). \tag{2.63}
$$

As an example, if $y(-1) = 2$ and $y(1) = -2$, then $c = 0$ and the outer expansion (2.51) becomes simply $y \sim 0$. We will obtain such an approximation in some of the other examples and exercises in this chapter. This does not mean the solution is zero. Rather, it means that the solution is transcendentally small compared to the power functions. ■

The last example is interesting because it shows that what appear to be inconsequential terms in an expansion can affect the value of the first-order approximation. Although this situation is not rare, it is not common. Therefore, in the remainder of this text we will mostly use power functions for the scale functions. If something remains underdetermined, as it did in Example 2, we will then entertain the idea that transcendentally small terms are needed.

Exercises

2.27. This problem concerns the higher terms in the expansions from Example 1.

(a) Verify (2.52), and, by matching with (2.51), show that $A_0 = 1$ and $A_1 = A_2 = 0$.

(b) Explain why your result from part (a) does not match with (2.56).

(c) The result from part (b) shows that the outer expansion is the reason transcendentally small terms must also be included in the boundary-layer expansion. Assume that

$$Y \sim Y_0(\bar{x}) + \varepsilon Y_1(\bar{x}) + \cdots + Z_0(\bar{x}, \varepsilon) + Z_1(\bar{x}, \varepsilon) + \cdots ,$$

where $Z_j \ll Z_i$, $\forall i < j$, and $z_i \ll \varepsilon^n$, $\forall i, n$ (with $\bar{x}$ fixed). Show that $Z_0 = B(\varepsilon)\,(1 - \mathrm{e}^{-\bar{x}})$. From matching show that $B = -\mathrm{e}^{-1/\varepsilon}$.

(d) Explain why Z_0 is the reason z_1 is needed in the outer expansion.

2.28. This problem completes some of the details in the derivation of Example 2.

(a) Use Laplace's approximation (Appendix C) to show that

$$\int_{-1}^{1} \mathrm{e}^{s^2/(2\varepsilon)} \mathrm{d}s \sim 2\varepsilon \mathrm{e}^{1/(2\varepsilon)}.$$

(b) Use part (a) to help derive (2.62).

(c) The matching shows that $b = \frac{1}{2}\,(y_{\mathrm{L}} - y_{\mathrm{R}})$, which appears to contradict the assumption that z_0 is transcendentally small compared to the power functions. Explain why there is, in fact, no contradiction in this result.

(d) Show that the assumption that $\varepsilon x z_0$ can be ignored compared to $\varepsilon z_0''$ and $x z_0'$ holds for the function z_0.

2.29. Consider the problem of solving

$$\varepsilon y'' - x y' = 0 \quad \text{for } a < x < b,$$

where $y(a) = y_{\mathrm{L}}$ and $y(b) = y_{\mathrm{R}}$. Also, assume that $a < 0$ and $b > 0$.

(a) Using the usual boundary-layer arguments show that

$$y \sim \begin{cases} c + (y_{\mathrm{L}} - c)\mathrm{e}^{-\bar{x}} & \text{boundary layer at } x = a, \\ c & \text{outer region}, \\ c + (y_{\mathrm{R}} - c)\mathrm{e}^{\widetilde{x}} & \text{boundary layer at } x = b, \end{cases}$$

where c is an arbitrary constant, $\bar{x} = (x - a)/\varepsilon$, and $\widetilde{x} = (x - b)/\varepsilon$.

(b) Find the exact solution of the problem.

(c) Using the result from part (b) and Laplace's approximation (Appendix C) show that

$$c = \begin{cases} y_{\mathrm{L}} & \text{if } |a| < b, \\ \frac{1}{2}(y_{\mathrm{L}} + y_{\mathrm{R}}) & \text{if } |a| = b, \\ y_{\mathrm{R}} & \text{if } b < |a|, \end{cases}$$

Comment on what happens to the assumed boundary layers in the problem.

(d) Sketch (by hand) the solution when $a = -2$, $b = 1$, $y_{\mathrm{L}} = 1$, and $y_{\mathrm{R}} = 3$. Comment on whether you used part (a) or part (b), and why.

2.30. There is a question whether it is possible to account for the transcendentally small terms without having to use the z_i in (2.53). One possibility is to modify the boundary layer method by first constructing a composite expansion, as described in Sect. 2.2, and then imposing the boundary conditions (Exercise 2.14). This is what is done when using the WKB method, and one of the reasons WKB has had some success in handling transcendentally small terms.

(a) Show that reversing the order for (2.50), with $y(0) = 0$ and $y(1) = 1$, results in the approximation

$$y \sim 2x + \frac{1}{1 - \mathrm{e}^{-1/\varepsilon}} \left(-1 + \mathrm{e}^{-\bar{x}} \right) .$$

Explain why this reduces to (2.56).

(b) Explain why this modified boundary-layer method does not solve the unknown-constant problem that appears in (2.58).

2.31. Consider the problem

$$\varepsilon^2 y'' + 2\varepsilon y' + 2(y - xg)^2 = \varepsilon h(x) \quad \text{for } 0 < x < 1,$$

where $y(0) = \operatorname{sech}^2(1/(2\varepsilon))$ and $y(1) = 1 + \operatorname{sech}^2(1/(2\varepsilon))$. Also, $g(x) = \mathrm{e}^{\varepsilon(x-1)}$ and $h(x) = [\varepsilon^2 + (2 + \varepsilon^2)(1 + \varepsilon x)]g(x)$.

(a) Suppose one were to argue that the exponentially small terms in the boundary conditions can be ignored and the usual power series expansion of the solution can be used. Based on this assumption, find the first two terms of a composite expansion of the solution.

(b) The exact solution of the problem is

$$y(x) = x\mathrm{e}^{\varepsilon(x-1)} + \operatorname{sech}^2\!\left(\frac{2x - 1}{2\varepsilon} \right) .$$

Discuss this solution in connection with your expansion from part (a).

2.5 Interior Layers

The rapid transitions in the solution that are characteristic of a boundary layer do not have to occur only at the boundary. When this happens, the problems tend to be somewhat harder to solve simply because the location of the layer is usually not known until after the expansions are matched. However, the expansion procedure is essentially the same as in the previous examples. To understand how the method works, consider the problem

$$\varepsilon y'' = yy' - y, \quad \text{for } 0 < x < 1, \tag{2.64}$$

where

$$y(0) = 1 \tag{2.65}$$

and

$$y(1) = -1. \tag{2.66}$$

2.5.1 Step 1: Outer Expansion

The appropriate expansion in this region is the same as it usually is, in other words,

$$y(x) \sim y_0(x) + \cdots . \tag{2.67}$$

From (2.64) one finds that

$$y_0 y_0' - y_0 = 0, \tag{2.68}$$

and so either $y_0 = 0$ or else

$$y_0 = x + a, \tag{2.69}$$

where a is an arbitrary constant. The fact that we have two possible solutions means that the matching might take somewhat longer than previously because we will have to determine which of these solutions matches to the inner expansion.

2.5.2 Step 1.5: Locating the Layer

Generally, when one first begins trying to solve a problem, it is not known where the layer is, or whether there are multiple layers. If we began this problem like the others and assumed there is a boundary layer at either one of the endpoints, we would find that the expansions do not match. This is a lot of effort for no results, but fortunately there is a simpler way to come to

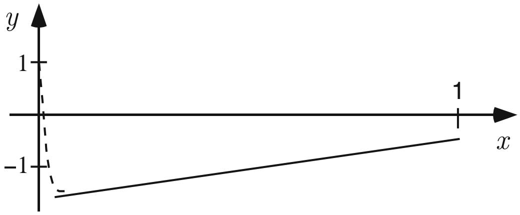

Figure 2.11 Schematic of solution if there is a (convex) boundary layer at $x = 0$ and the linear function in (2.69) is the outer solution

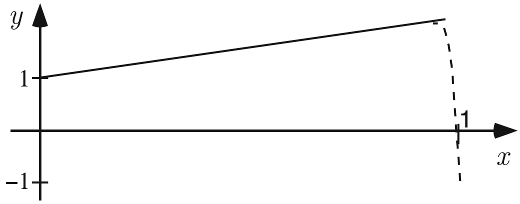

Figure 2.12 Schematic of solution if there is a concave boundary layer at $x = 1$ and the linear function in (2.69) is the outer solution

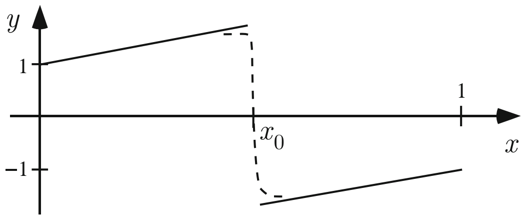

Figure 2.13 Schematic of linear functions that make up outer expansion and interior layer solution connecting them

the same conclusion. To illustrate how, suppose it is assumed that there is a boundary layer at $x = 0$ and (2.69) is the outer solution. This situation is shown in Fig. 2.11. If the solution behaves like the other example problems, then in the boundary layer it is expected that y'' is positive (i.e., y is concave up), y' is negative, and y is both positive and negative. In other words, in such a boundary layer, the right-hand side of (2.64) is positive while the left-hand side can be negative. This is impossible, and so there is not a boundary layer as indicated in Fig. 2.11.

It is possible to rule out a boundary layer at $x = 1$ in the same way. In particular, as illustrated in Fig. 2.12, in the boundary layer y'' and $y' - 1$ are negative, while y is both positive and negative. Using a similar argument one can rule out having boundary layers at both ends.

Another possibility is that the layer is interior to the interval, and this is illustrated in Fig. 2.13. To check, in the layer region to the left of x_0, $y' - 1$ is negative, y is positive, and y'' is negative. This is consistent with (2.64).

Similarly, to the right of x_0, both $y'-1$ and y are negative, and y'' is positive. This too is consistent with (2.64). It also indicates that $y(x_0) = 0$, a result we will need later to complete the derivation of the interior layer solution.

It should be pointed out that these are only plausibility arguments and do not prove anything. What they do is guide the analysis and, hopefully, reduce the work necessary to obtain the solution. A more expanded version of this analysis is explored in Exercise 2.34.

2.5.3 Steps 2 and 3: Interior Layer and Matching

Based on the preceding observations, we investigate the possibility of an interior layer. This is done by introducing the interior-layer coordinate

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha}, \tag{2.70}$$

where $0 < x_0 < 1$. The location of the layer, $x = x_0$, is not known and will be determined subsequently. Actually, the possibilities of either $x_0 = 0$ or $x_0 = 1$ could be included here, but we will not do so. Also, note that since $0 < x_0 < 1$, there are two outer regions, one for $0 \le x < x_0$, the other for $x_0 < x \le 1$ (Fig. 2.13). Now, substituting (2.70) into (2.64) yields

$$\varepsilon^{1-2\alpha} Y'' = \varepsilon^{-\alpha} Y Y' - Y. \tag{2.71}$$

The distinguished limit here occurs when $\alpha = 1$ (i.e., the first and second terms balance). Also, as in the previous examples, we are using Y to designate the solution in the layer. Expanding the interior-layer solution as

$$Y(\bar{x}) \sim Y_0(\bar{x}) + \cdots \tag{2.72}$$

it follows from (2.71) that

$$Y_0'' = Y_0 Y_0'. \tag{2.73}$$

Integrating this one obtains

$$Y_0' = \frac{1}{2} Y_0^2 + A.$$

There are three solutions of this first-order equation, corresponding to A's being positive, negative, or zero. The respective solutions are

$$Y_0 = B \frac{1 - D\mathrm{e}^{B\bar{x}}}{1 + D\mathrm{e}^{B\bar{x}}}, \tag{2.74}$$

$$Y_0 = B \tan(C - B\bar{x}/2),$$

and

$$Y_0 = \frac{2}{C - \bar{x}}\,,$$

where B, C, and D are arbitrary constants. The existence of multiple solutions makes the problem interesting, but it also means that the matching procedure is not as straightforward as it was for the linear equations studied earlier. This is because for the linear problems we relied on being able to find the general solution in each region and then determining the constants by matching. For nonlinear problems the concept of a general solution has little meaning, and because of this it can sometimes be difficult to obtain a solution that is general enough to be able to match to the outer expansion(s).

Of the solutions to (2.73), the one given in (2.74) is capable of matching to the outer expansions as $\bar{x} \to \pm\infty$. Again it should be remembered that the working hypothesis here is that $0 < x_0 < 1$. Thus, the outer expansion for $0 \le x < x_0$ should satisfy $y(0) = 1$. From this it follows that

$$y_0 = x + 1, \quad \text{for } 0 \le x < x_0. \tag{2.75}$$

Similarly, the outer region on the other side of the layer should satisfy the boundary condition at $x = 1$, and this yields

$$y_0 = x - 2, \quad \text{for } x_0 < x \le 1. \tag{2.76}$$

Now, for (2.74) to be able to match to either (2.75) or (2.76) it is going to be necessary that both B and D in (2.74) be nonzero (in fact, without loss of generality, we will take B to be positive). The requirements imposed in the matching are very similar to those obtained for boundary layers. In particular, we must have that $Y_0(\infty) = y_0(x_0^+)$ and $Y_0(-\infty) = y_0(x_0^-)$. From (2.74) and (2.75) we get that

$$B = x_0 + 1,$$

and from (2.74) and (2.76) we have that

$$-B = x_0 - 2.$$

Solving these equations one finds that $B = \frac{3}{2}$ and $x_0 = \frac{1}{2}$.

2.5.4 Step 3.5: Missing Equation

From matching we have determined the location of the layer and one of the constants in the layer solution. However, the matching procedure did not determine D in (2.74). Fortunately, from the discussion in Step 1.5, we are able to determine its value. In particular, we found that $y(x_0) = 0$, and for this to happen it must be that $D = 1$. Therefore,

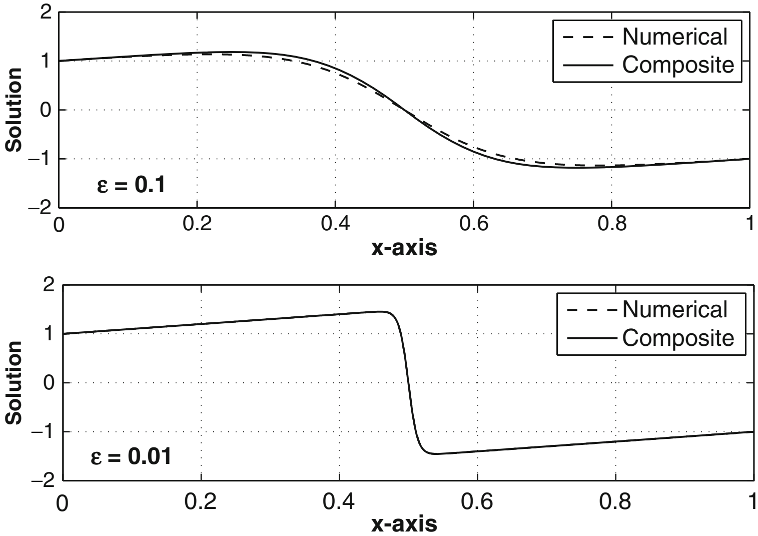

Figure 2.14 Graph of numerical solution of (2.64)–(2.66) and composite expansion given in (2.78) in the case where $\varepsilon = 10^{-1}$ and where $\varepsilon = 10^{-2}$

$$Y(\bar{x}) \sim \frac{3}{2}\frac{1 - \mathrm{e}^{3\bar{x}/2}}{1 + \mathrm{e}^{3\bar{x}/2}} \,. \tag{2.77}$$

Having an undetermined constant left after matching also occurred in the previous section (Example 2). Unlike that earlier example, we were able to determine D using the properties of the differential equation and boundary conditions. Our argument was heuristic, and those who prefer a more mathematical proof that $D = 1$ should consult Exercise 2.42. Another approach to evaluating the undetermined constant is given in Exercise 2.37.

2.5.5 Step 4: Composite Expansion

It can be more difficult to construct a composite expansion when the outer solutions are discontinuous across the interior layer like they are in this problem. What is done is to find one for the interval $0 \leq x \leq x_0$ and then another for $x_0 \leq x \leq 1$. As it turns out, for this example the expansions on either side are the same, and the result is

$$y \sim x + 1 - \frac{3}{1 + \mathrm{e}^{-3(2x-1)/4\varepsilon}} \quad \text{for } 0 \leq x \leq 1. \tag{2.78}$$

This composite expansion is graphed in Fig. 2.14 to illustrate the nature of the interior layer and how it appears as ε decreases. The rapid transition from one outer solution to the other is typical of what is sometimes called a shock solution. Also shown in Fig. 2.14 is the numerical solution, and it is clear that the composite and numerical solutions are in good agreement.

2.5.6 Kummer Functions

Interior layers can arise in linear problems. To understand this, consider the problem of solving

$$\varepsilon y'' + (3x-1)y' + xy = 0 \quad \text{for } 0 < x < 1, \tag{2.79}$$

where $y(0) = 1$ and $y(1) = 2$. A tip-off that this might have an interior layer is the fact that the coefficient of y' is zero at a point in the interval. For the moment, this observation is more of a curiosity, but it is worth pointing out that the interior layer of the last example was also located at the point where the coefficient of y' in (2.64) is zero.

There is nothing particularly unusual about this problem, so we will assume a regular expansion for the outer solution. In particular, assuming $y \sim y_0(x) + \varepsilon y_1(x) + \cdots$, the $O(1)$ equation is $(3x-1)y_0' + xy_0 = 0$. Solving this one finds that

$$y_0 = \frac{a}{(3x-1)^{1/9}}\mathrm{e}^{-x/3}. \tag{2.80}$$

Given that the denominator is zero at $x = 1/3$, it should not come as a surprise that there is an interior layer located at $x = 1/3$. This means there are two outer solutions, and we have that

$$y_0(x) = \begin{cases} \dfrac{a_\mathrm{l}}{(1-3x)^{1/9}}\mathrm{e}^{-x/3} & \text{for} \quad 0 \le x < \frac{1}{3}, \\ \dfrac{a_\mathrm{r}}{(3x-1)^{1/9}}\mathrm{e}^{-x/3} & \text{for} \quad \frac{1}{3} < x \le 1. \end{cases} \tag{2.81}$$

Satisfying the boundary conditions, one finds that $a_\mathrm{l} = 1$ and $a_\mathrm{r} = 2^{10/9}\mathrm{e}^{1/3}$.

We will use the interior-layer coordinate given in (2.70), with $x_0 = 1/3$ and $\alpha = 1/2$. The problem in this case becomes

$$Y'' + \bar{x}Y' + \left(\frac{1}{3} + \varepsilon^{1/2}\right)Y = 0, \quad \text{for } -\infty < \bar{x} < \infty. \tag{2.82}$$

The $O(1)$ equation coming from this is

$$Y_0'' + 3\bar{x}Y_0' + \frac{1}{3}Y_0 = 0, \quad \text{for } -\infty < \bar{x} < \infty. \tag{2.83}$$

This equation is why this subsection is titled "Kummer Functions"; it also provides the motivation for the next paragraph.

What we have shown is that to determine the solution in the interior layer, we must be able to solve an equation of the form

$$y'' + \alpha x y' + \beta y = 0, \quad \text{for} \quad -\infty < x < \infty,$$

where α and β are nonzero constants. This equation can be solved using a power series expansion or the Laplace transform. Doing this shows that the general solution can be written as

$$y(x) = A_0\, M\!\left(\frac{\beta}{2\alpha}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) + B_0\, x M\!\left(\frac{\alpha+\beta}{2\alpha}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right), \tag{2.84}$$

where $M(a, b, z)$ is *Kummer's function* and its definition and basic properties are given in Appendix B. As an example, if $\alpha = \beta$, then the solution is

$$y(x) = A_0 \mathrm{e}^{-\alpha x^2/2} + B_0 \frac{1}{x} \int_0^x \mathrm{e}^{\alpha(s^2 - x^2)/2}\, \mathrm{d}s.$$

For us to be able to match with the outer solutions, we need to know what happens to (2.84) when $x \to \pm\infty$. This depends on whether α is positive or negative. Using the formulas in Appendix B, for $x^2 \to \infty$,

$$y(x) \sim \sqrt{\pi}\left[\frac{A_0}{\Gamma\left(\frac{1}{2} - \kappa\right)} \pm \frac{B_0}{\sqrt{2\alpha}\,\Gamma(1-\kappa)}\right]\eta^{-\kappa} \quad \text{if } \alpha > 0 \tag{2.85}$$

and

$$y(x) \sim \sqrt{\pi}\left[\frac{A_0}{\Gamma(\kappa)} \pm \frac{B_0}{\sqrt{-2\alpha}\,\Gamma\left(\frac{1}{2} + \kappa\right)}\right](-\eta)^{\kappa - \frac{1}{2}}\mathrm{e}^{-\eta} \quad \text{if } \alpha < 0, \tag{2.86}$$

where $\kappa = \beta/(2\alpha)$ and $\eta = \frac{1}{2}\alpha x^2$. In the preceding expressions, the $+$ is taken when $x > 0$ and the $-$ when $x < 0$. Also, the arguments of the Gamma functions are assumed not to be nonpositive integers. What happens in those cases is interesting and discussed toward the end of Sect. 2.6.

Based on the preceding discussion, the general solution of the interior-layer Eq. (2.83) can be written as

$$Y_0 = A_0\, M\!\left(\frac{1}{18}, \frac{1}{2}, -\frac{3}{2}\bar{x}^2\right) + B_0\, \bar{x} M\!\left(\frac{5}{9}, \frac{3}{2}, -\frac{3}{2}\bar{x}^2\right). \tag{2.87}$$

The constants in this expression are determined by matching to the outer solution. To do this we use the intermediate variable $x_\eta = (x - x_0)/\varepsilon^\gamma$, where $0 < \gamma < 1/2$. Introducing this into the outer solution (2.81) yields

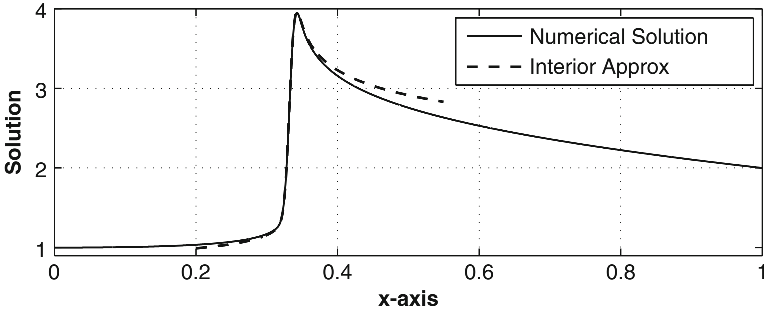

Figure 2.15 Graph of numerical solution of (2.79) and interior-layer solution given in (2.87) in the case where $\varepsilon = 10^{-4}$

$$y_{\text{outer}} \sim \begin{cases} \dfrac{a_{\text{l}}}{(3e)^{1/9}}(\varepsilon^{\gamma}|x_\eta|)^{-1/9} & \text{for} \quad x_\eta < 0, \\ \dfrac{a_{\text{r}}}{(3e)^{1/9}}(\varepsilon^{\gamma}|x_\eta|)^{-1/9} & \text{for} \quad 0 < x_\eta. \end{cases} \tag{2.88}$$

For the interior-layer solution (2.87) we use (2.85) to obtain

$$y(x) \sim \sqrt{\pi}\left[\frac{A_0}{\Gamma\left(\frac{4}{9}\right)} \pm \frac{B_0}{\sqrt{6}\,\Gamma\left(\frac{17}{18}\right)}\right]\left(\frac{2\varepsilon}{3}\right)^{1/18}(\varepsilon^{\gamma}|x_\eta|)^{-1/9}, \tag{2.89}$$

where the $+$ is taken if $x_\eta > 0$ and the $-$ if $x_\eta < 0$. Matching the inner and outer approximations, it follows that

$$A_0 = \frac{1}{2\sqrt{\pi}}(a_{\text{r}} + a_{\text{l}})\Gamma\left(\frac{4}{9}\right)\left(6\varepsilon \mathrm{e}^2\right)^{-1/18}$$

and

$$B_0 = \sqrt{\frac{3}{2\pi}}(a_{\text{r}} - a_{\text{l}})\Gamma\left(\frac{17}{18}\right)\left(6\varepsilon \mathrm{e}^2\right)^{-1/18}.$$

The coefficients have ended up depending on ε. What this means is that the original assumption that $Y \sim Y_0 + \cdots$ should have been $Y \sim \varepsilon^{-1/18}(Y_0 + \cdots)$. Because this problem is linear, the need for the multiplicative factor $\varepsilon^{-1/18}$ does not affect the validity of the asymptotic approximation.

To give a sense of how well this approximation does, the numerical solution and the interior layer approximation are plotted in Fig. 2.15. It is evident that the latter is asymptotic to the exponential functions that make up the outer solution (2.81), and it gives a very accurate approximation of the solution in the layer. What is also interesting is that, unlike the other layer examples considered so far, the interior-layer solution is not monotone. An analysis of more complex nonmonotone interior layers can be found in DeSanti (1987).

Exercises

2.32. Find a first-term expansion of the solution of each of the following problems. It should not be unexpected that for the nonlinear problems the solutions are defined implicitly or that the transition layer contains an undetermined constant.
(a) $\varepsilon y'' = -(x^2 - \frac{1}{4})y'$ for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$.
(b) $\varepsilon y'' + 2xy' + (1 + \varepsilon x^2)y = 0$ for $-1 < x < 1$, where $y(-1) = 2$ and $y(1) = -2$.
(c) $\varepsilon y'' = yy' - y^3$ for $0 < x < 1$, where $y(0) = \frac{3}{5}$ and $y(1) = -\frac{2}{3}$.
(d) $3\varepsilon y'' + 3xy' + (1 + x)y = 0$ for $-1 < x < 1$, where $y(-1) = 1$ and $y(1) = -1$.
(e) $\varepsilon y'' + y(1 + y^2)y' - \frac{1}{2}y = 0$ for $0 < x < 1$, where $y(0) = -1$ and $y(1) = 1$.
(f) $\varepsilon y'' + y(y' + 3) = 0$ for $0 < x < 1$, where $y(0) = -1$ and $y(1) = 2$.

2.33. Consider the problem

$$\varepsilon y'' = yy' \quad \text{for } 0 < x < 1,$$

where $y(0) = a$ and $y(1) = -a$. Also, a is positive.
(a) Prove that $y(\frac{1}{2}) = 0$. (Hint: Use the method described in Exercise 2.42.)
(b) Find a composite expansion of the solution.
(c) Show that the exact solution has the form

$$y = A\frac{1 - B\mathrm{e}^{Ax/\varepsilon}}{1 + B\mathrm{e}^{Ax/\varepsilon}},$$

where, for small ε, $A \sim a(1 + 2\mathrm{e}^{-a/(2\varepsilon)})$ and $B \sim \mathrm{e}^{-a/(2\varepsilon)}$. Comment on how this compares with your result from part (b).

2.34. This problem explores various possibilities for the layer solutions of (2.64). You do not need to derive the expansions to answer these questions, but you do need to know the general forms for the outer solutions as well as the general form of the layer solution (2.74).
(a) The plausibility argument used to rule out boundary layers (e.g., Fig. 2.11) did not consider the other possible outer solution, namely, $y_0 = 0$. If this is the outer solution, then there must be a boundary layer at both ends. Explain why this cannot happen.
(b) To examine how the position of the interior layer depends on the boundary conditions, suppose that $y(0) = a$ and $y(1) = b$, where $-1 < a + b < 1$ and $b < 1 + a$. What is the value of x_0 in this case? Also, state how you use the stated inequalities on a and b.
(c) What happens to the layer(s) if the boundary conditions are $y(0) = y(1) = a$?
(d) Suppose $y(0) = -1/2$ and $y(1) = 1/4$. Use the plausibility argument to show that there are multiple possible solutions.

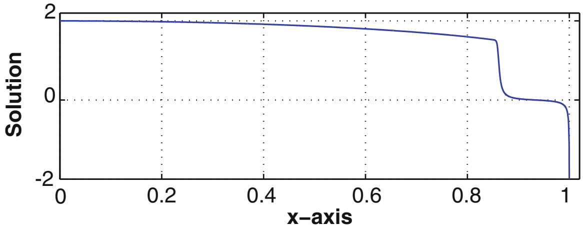

Figure 2.16 Solution of problem in Exercise 2.35

2.35. Consider the problem

$$\varepsilon y'' + y(1-y)y' - xy = 0 \quad \text{for } 0 < x < 1,$$

where $y(0) = 2$ and $y(1) = -2$. The numerical solution is shown in Fig. 2.16 in the case where $\varepsilon = 10^{-3}$. This will prove useful when deriving a first-term approximation of the solution.

(a) Find the first term in the expansion of the outer solution. Assume that this function satisfies the boundary condition at $x = 0$.
(b) Assume there is a boundary layer at $x = 1$. After finding a first-term approximation in the boundary layer show that it does not match with the outer solution you found in part (a).
(c) Assuming there is an interior layer across which the solution jumps from one outer solution to another, find a first-term approximation in the layer. From the matching show that the layer is located at $x_0 = \sqrt{3}/2$. Note that your layer solution will contain an undetermined constant.
(d) Correct the boundary-layer analysis in part (b) based on your result from part (c).

2.36. This problem examines the solution of the boundary-value problem

$$\varepsilon y'' = -f(x)y' \quad \text{for } 0 < x < 1,$$

where $y(0) = a$ and $y(1) = -b$. Assume that a and b are positive constants. Also, assume that $f(x)$ is smooth with $f'(x) > 0$ and $f(x_0) = 0$ for $0 < x_0 < 1$.

(a) Explain why there must be at least one point in the interval $0 < x < 1$ where $y(x) = 0$.
(b) Find the exact solution of the problem and then write down the equation that must be solved to determine where $y(x) = 0$. From this explain why there is exactly one solution of $y(x) = 0$.
(c) Using your result from part (b), find a two-term expansion of the solution of $y(x) = 0$. The second term will be defined implicitly and involves solving an equation of the form

$$\mathrm{erf}(\mu_0\sqrt{f'(x_0)/2}) = \frac{a-b}{a+b}$$

for μ_0. Note that Laplace's approximation (Appendix C) will be useful here.

(d) Find a two-term expansion of the solution of $y(x) = 0$ by first constructing an asymptotic expansion of the solution of the boundary-value problem.

2.37. One way to resolve the problem of having an undetermined constant after matching is to use a variational principle. To understand this approach, consider the problem

$$\varepsilon y'' + p(x,\varepsilon)y' + q(x,\varepsilon)y = 0 \quad \text{for } 0 < x < 1,$$

where $y(0) = a$ and $y(1) = b$. Associated with this is the functional

$$I(v) = \int_0^1 L(v, v')\mathrm{d}x, \text{ where } L = \frac{1}{2}[\varepsilon(v')^2 - qv^2]\mathrm{e}^{\frac{1}{\varepsilon}\int_0^x p(s,\varepsilon)\mathrm{d}s}.$$

In this variational formulation L is a Lagrangian for the equation.

(a) Show that if $\frac{\mathrm{d}}{\mathrm{d}r}I(y + ru) = 0$, at $r = 0$, for all smooth functions $u(x)$ satisfying $u(0) = u(1) = 0$, then $y(x)$ is a solution of the preceding differential equation. In other words, an extremal of the functional is a solution of the differential equation.

(b) Consider the problem

$$\varepsilon y'' - (2x-1)y' + 2y = 0 \quad \text{for } 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = -3$. The solution of this problem has a boundary layer at each end of the interval. Find a composite expansion of the solution for $0 \le x \le 1$. Your solution will contain an arbitrary constant that will be designated as k in what follows.

(c) From your result in part (b) derive an expansion for the Lagrangian L.

(d) Explain why the constant k should be such that $\frac{\mathrm{d}}{\mathrm{d}k}I(y) = 0$. From this determine k.

2.38. In the Langmuir–Hinshelwood model for the kinetics of a catalyzed reaction the following problem appears:

$$\varepsilon\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{x}F(y) \quad \text{for } 0 < x < 1,$$

where $F(y) = 2(1-y)(\alpha + y)/y$ and $y(1) = 0$. Also, $0 < \alpha < 1$. In this problem ε is the deactivation rate parameter and $y(x)$ is the concentration of the reactant (Kapila, 1983).

(a) For small ε, find a first-term expansion of the solution in the outer region and in the boundary layer.

(b) Find a composite expansion of the solution for $0 < x \le 1$.

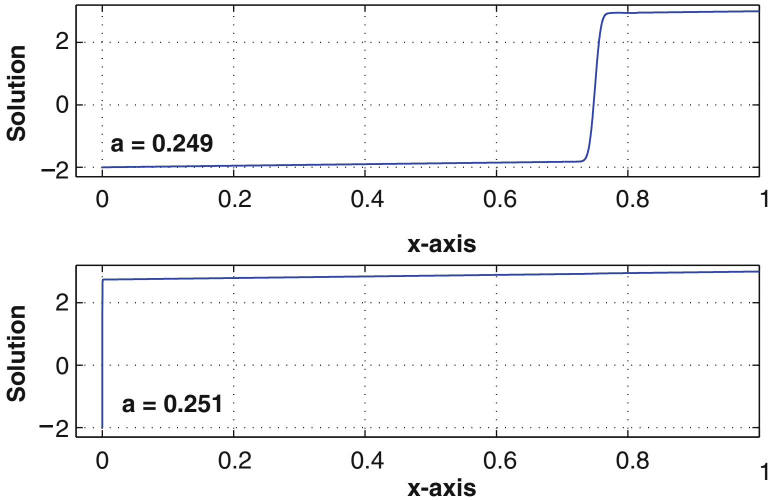

Figure 2.17 Graph of solution for Exercise 2.40 for two slightly different values of a. In this calculation, $b = 0.75$ and $\varepsilon = 10^{-4}$

2.39. This problem examines a differential-difference equation. The specific problem is (Lange and Miura, 1991)

$$\varepsilon^2 y''(x) - y(x) + q(x)y(x-1) = f(x) \quad \text{for } 0 < x < 3/2,$$

where $y(x) = 0$ for $-1 \leq x \leq 0$ and $y(3/2) = 1$. The functions $q(x)$ and $f(x)$ are assumed smooth. What is significant here is that the solution is evaluated at $x - 1$ in one of the terms of the equation. To answer the questions below, keep in mind that $y(x)$ and $y'(x)$ are continuous for $0 \leq x \leq 3/2$.

(a) There is a layer at $x = 0^+$, at $x = 1^\pm$, and at $x = 3/2$. Use this information to find a first-term approximation of the solution. To do this, you should consider $x < 1$ and $1 < x$ separately and then require smoothness at $x = 1$. Also, you will have to find the first two terms in the layer at $x = 1$ to get the expansions to match.

(b) Find the exact solution of the problem in the case where $f(x) = 0$, and compare the result with the expansion from part (a).

2.40. For some problems, locating the layer(s) can be difficult. To understand this, consider the following problem:

$$\varepsilon y'' + (x-a)(x-b)(4y'-1) = 0 \quad \text{for } 0 < x < 1,$$

where $y(0) = -2$ and $y(1) = 3$. Also, $0 < a < b < 1$. The graph of the solution of this problem is shown in Fig. 2.17.

(a) Using the plausibility argument given in the discussion for Fig. 2.11, explain why there is no boundary layer at $x = 1$ but there might be one at $x = 0$.
(b) Interior layers can appear at points where the coefficient of y' is zero. Using the plausibility argument explain why there is no layer at $x = a$ but there might be one at $x = b$.
(c) Assuming the layer is at $x = 0$, calculate the first term in the expansions. Also, assuming the layer is at $x = b$, calculate the first term in the expansions. Explain why it is not possible to determine the position for the layer from these expansions.
(d) Find the exact solution. Use this to show that the layer is at $x = 0$ if $\frac{b}{3} \leq a < b$ and at $x = b$ if $0 < a < \frac{b}{3}$. Note that Appendix C will be helpful here.

2.41. In the study of explosions of gaseous mixtures one finds a model where the (nondimensional) temperature $T(t)$ of the gas satisfies (Kassoy, 1976; Kapila, 1983)

$$T' = \varepsilon(T_\infty - T)^n \exp\left(\frac{T-1}{\varepsilon T}\right)$$

for $T(0) = 1$. Here $T_\infty > 1$ is a constant known as the adiabatic explosion temperature. Also, n is a positive integer (it is the overall reaction order). Assuming a high activation energy, the parameter ε is small.
(a) What is the steady-state temperature?
(b) Find the first two terms in a regular expansion of the temperature. This expansion satisfies the initial condition and describes the solution in what is known as the ignition period. Explain why the expansion is not uniform in time. Also, toward the end of the ignition period the solution is known to undergo a rapid transition to the steady state. Use your expansion to estimate when this occurs.
(c) To understand how the solution makes the transition from the rapid rise in the transition layer to the steady state, let

$$\tau = \frac{t - t_0}{\mu(\varepsilon)},$$

where t_0 is the time where the transition takes place and $\mu(\varepsilon)$ is determined from balancing in the layer. Assuming that $T \sim T_\infty - \varepsilon T_1(\tau) + \cdots$, find μ and T_1. Although T_1 is defined implicitly, use its direction field to determine what happens when $\tau \to \infty$ and $\tau \to -\infty$.
It is worth pointing out that there is a second internal layer in this problem, and it is located between the one for the ignition region and the layer you found in part (c). The matching of these various layers is fairly involved; the details can be found in Kapila (1983) for the case where $n = 1$. Also, it is actually possible to solve the original problem in closed form, although the solution is not simple (Parang and Jischke, 1975).

2.42. This problem outlines a proof, using a symmetry argument, that $D = 1$ in (2.74). Basically, the proof is based on the observation that in Fig. 2.13, if the solution were flipped around $y = 0$ and then flipped around $x = 1/2$, one would get the solution back again. In this problem, instead of (2.64), (2.65), suppose the boundary conditions are $y(0) = a$ and $y(1) = b$. In this case the solution can be written as $y = f(x, a, b)$.

(a) Change variables and let $s = 1-x$ (which produces a flip around $x = 1/2$) and $z = -y$ (which produces a flip around $y = 0$) to obtain

$$\varepsilon z'' = zz' - z \quad \text{for } 0 < s < 1,$$

where $z(0) = -b$ and $z(1) = -a$.

(b) Explain why the solution of the problem in part (a) is $z = f(s, -b, -a)$.

(c) Use part (b) to show that $y = -f(1-x, -b, -a)$, and from this explain why $f(x, a, b) = -f(1-x, -b, -a)$.

(d) Use part (c) to show that in the case where $a = 1$ and $b = -1$, $y(\frac{1}{2}) = 0$. It follows from this that $D = 1$.

2.6 Corner Layers

One of the distinguishing features of the problems we have studied in this chapter is the rapid changes in the solution in the layer regions. The problems we will now investigate are slightly different because the rapid changes will be in the slope, or derivatives of the solution, and not in the value of the solution itself. To illustrate this, we consider the following problem:

$$\varepsilon y'' + \left(x - \frac{1}{2}\right)p(x)y' - p(x)y = 0 \quad \text{for } 0 < x < 1, \tag{2.90}$$

where

$$y(0) = 2 \tag{2.91}$$

and

$$y(1) = 3. \tag{2.92}$$

The function $p(x)$ is assumed to be smooth and positive, with $p(1/2) = 1$. For example, one could take $p(x) = x + 1/2$ or $p(x) = \mathrm{e}^{2x-1}$. It should also be noted that the coefficient of y' is zero at $x = 1/2$. Because of this, given the observations of the previous section, it should not come as a surprise that the layer in this example is located at $x = 1/2$.

2.6.1 Step 1: Outer Expansion

The solution in this region is expanded in the usual power series as follows:

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots . \tag{2.93}$$

From (2.90) one then finds that

$$y_0 = a\left(x - \frac{1}{2}\right), \tag{2.94}$$

where a is an arbitrary constant. As usual, we are faced with having to satisfy two boundary conditions with only one integration constant.

2.6.2 Step 2: Corner Layer

We begin by determining whether there are boundary layers. These can be ruled out fairly quickly using the plausibility argument presented in the previous section. For example, if there is a boundary layer at $x = 0$ and (2.94) is the outer solution, then we have a situation similar to that shown in Fig. 2.11. In the boundary layer $y'' > 0$, $y' < 0$, and there is a portion of the curve where $y < 0$. This means that $\varepsilon y'' > 0 > -(x - \frac{1}{2})p(x)y' + p(x)y$. It is therefore impossible to satisfy the differential Eq. (2.90) and have a boundary layer as indicated in Fig. 2.11. Using a similar argument, and the fact that the coefficient of the y' term changes sign in the interval, one can also argue that there is not a boundary layer at the other end.

It therefore appears that there is an interior layer. To investigate this, we introduce the stretched variable

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha} . \tag{2.95}$$

With an interior layer there is an outer solution for $0 \le x < x_0$ and one for $x_0 < x \le 1$. Using boundary conditions (2.91), (2.92) and the general solution given in (2.94), we have that the outer solution is

$$y \sim \begin{cases} -4(x - \frac{1}{2}) & \text{if} \quad 0 \le x < x_0, \\ 6(x - \frac{1}{2}) & \text{if} \quad x_0 < x \le 1. \end{cases} \tag{2.96}$$

It will make things easier if we can determine x_0 before undertaking the layer analysis. If $0 < x_0 < \frac{1}{2}$ or if $\frac{1}{2} < x_0 < 1$, then the outer solution (2.96) is discontinuous at x_0. Using a plausibility argument similar to that presented in Sect. 2.5, one can show that neither case is possible (Exercise 2.46). In other words, $x_0 = \frac{1}{2}$. With this we get that the outer solution is continuous,

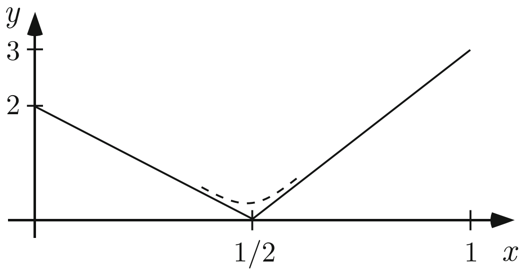

Figure 2.18 Schematic of corner layer formed by outer solution in (2.96). The solution from the corner-layer region should provide a smooth transition between these linear functions

but it is not differentiable at $x = \frac{1}{2}$. This situation is shown in Fig. 2.18. It is for this reason that we have what is called a *corner region*, or derivative layer, at $x_0 = \frac{1}{2}$.

Now, substituting (2.95) into (2.90), and letting Y designate the solution in this region, we get

$$\varepsilon^{1-2\alpha} Y'' + \bar{x}\, p\left(\frac{1}{2} + \varepsilon^{\alpha}\bar{x}\right) Y' - p\left(\frac{1}{2} + \varepsilon^{\alpha}\bar{x}\right) Y = 0. \tag{2.97}$$

To determine the distinguished limit note that

$$\begin{aligned} p\left(\frac{1}{2} + \varepsilon^{\alpha}\bar{x}\right) &= p\left(\frac{1}{2}\right) + \varepsilon^{\alpha}\bar{x}p'\left(\frac{1}{2}\right) + \cdots \\ &= 1 + O(\varepsilon^{\alpha}). \end{aligned}$$

With this, from balancing the terms in (2.97) we obtain $\alpha = \frac{1}{2}$. Unlike what we assumed in previous examples, we now take

$$Y \sim y_0(x_0) + \varepsilon^{\gamma} Y_0 + \cdots . \tag{2.98}$$

For this example $y_0(x_0) = 0$. Also, the multiplicative factor ε^{γ} is needed to be able to match to the outer solution (the constant γ will be determined from the matching). Thus, substituting this into (2.97) yields

$$Y_0'' + \bar{x}\, Y_0' - Y_0 = 0, \quad \text{for } -\infty < \bar{x} < \infty. \tag{2.99}$$

It is possible to solve this equation using power series methods. However, a simpler way is to notice that $Y_0 = \bar{x}$ is a solution, and so using the method of reduction of order, one finds that the general solution is

$$Y_0 = A\bar{x} + B\left[e^{\bar{x}^2/2} + \bar{x}\int_0^{\bar{x}} e^{-s^2/2} ds\right]. \tag{2.100}$$

2.6.3 Step 3: Matching

In the examples from the previous two sections, the layer analysis and matching were carried out in a single step. This is not done here because the matching in this problem is slightly different and is worth considering in more detail. To do the matching, we introduce the intermediate variable

$$x_\eta = \frac{x - 1/2}{\varepsilon^\kappa}, \tag{2.101}$$

where $0 < \kappa < \frac{1}{2}$. Rewriting the outer solution (2.96) in this variable yields

$$y \sim \begin{cases} -4\varepsilon^\kappa x_\eta & \text{if} \quad x_\eta < 0, \\ \ \ 6\varepsilon^\kappa x_\eta & \text{if} \quad 0 < x_\eta. \end{cases} \tag{2.102}$$

Also, using the fact that

$$\int_0^\infty \mathrm{e}^{-s^2/2}\mathrm{d}s = \sqrt{\frac{\pi}{2}},$$

it follows from (2.100) that

$$Y \sim \begin{cases} \varepsilon^{\gamma+\kappa-1/2} x_\eta (A - B\sqrt{\frac{\pi}{2}}\,) & \text{if} \quad x_\eta < 0, \\ \varepsilon^{\gamma+\kappa-1/2} x_\eta (A + B\sqrt{\frac{\pi}{2}}\,) & \text{if} \quad 0 < x_\eta. \end{cases} \tag{2.103}$$

To be able to match (2.102) and (2.103) we must have $\gamma = \frac{1}{2}$. In this case,

$$A - B\sqrt{\frac{\pi}{2}} = -4$$

and

$$A + B\sqrt{\frac{\pi}{2}} = 6,$$

from which it follows that $A = 1$ and $B = 5\sqrt{2/\pi}$.

2.6.4 Step 4: Composite Expansion

Even though the situation is slightly more complicated than before, the construction of a composite expansion follows the same rules as in the earlier examples. For example, for $0 \le x \le \frac{1}{2}$,

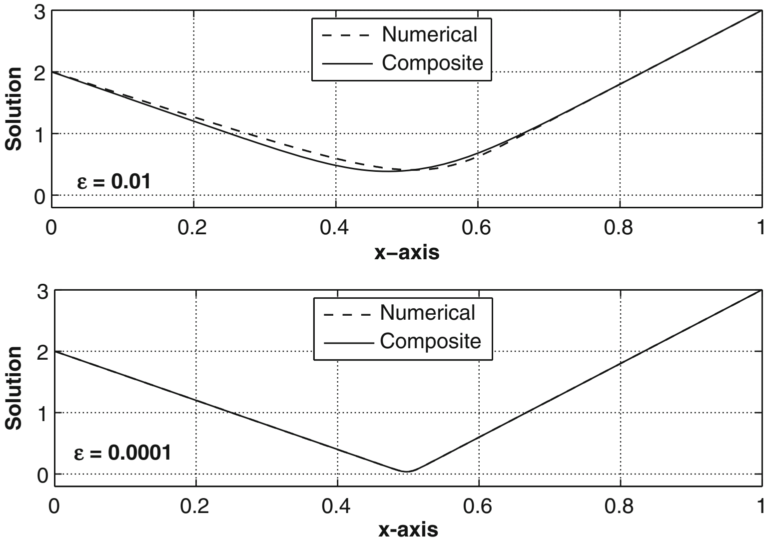

Figure 2.19 Composite expansion (2.104) and the numerical solution of (2.90)–(2.92) when $\varepsilon = 10^{-2}$ and when $\varepsilon = 10^{-4}$. Also, $p(x) = \mathrm{e}^{5(2x-1)}$

$$\begin{aligned} y &\sim -4\left(x - \frac{1}{2}\right) + \varepsilon^{1/2}\left[\bar{x} + 5\sqrt{\frac{2}{\pi}}\left(\mathrm{e}^{-\bar{x}^2/2} + \bar{x}\int_0^{\bar{x}} \mathrm{e}^{-s^2/2}\mathrm{d}s\right)\right] + 4\varepsilon^{\kappa}x_\eta \\ &= \left(x - \frac{1}{2}\right)\left[1 + 5\,\mathrm{erf}\left(\frac{x - \frac{1}{2}}{\sqrt{2\varepsilon}}\right)\right] + 5\sqrt{\frac{2\varepsilon}{\pi}}\,\mathrm{e}^{-(2x-1)^2/(8\varepsilon)}, \end{aligned} \tag{2.104}$$

where erf(·) is the error function. One finds that this is also the composite expansion for $\frac{1}{2} \le x \le 1$. Thus, (2.104) is a composite expansion over the entire interval. This function is shown in Fig. 2.19, and it is clear that it is in very good agreement with the numerical solution. In fact, when $\varepsilon = 10^{-3}$, the two curves are indistinguishable.

The equation studied in this section is related to one that has received a great deal of attention because the solution has been found to have some rather interesting properties. To understand the situation, consider

$$\varepsilon y'' - \left(x - \frac{1}{2}\right)y' + ky = 0 \quad \text{for } 0 < x < 1, \tag{2.105}$$

where k is a constant. Assume that $y(0) = y(1) = 1$; the solution is then

$$y(x) = \frac{M\left(-\frac{k}{2}, \frac{1}{2}, \frac{1}{8\varepsilon}(2x-1)^2\right)}{M\left(-\frac{k}{2}, \frac{1}{2}, \frac{1}{8\varepsilon}\right)}\,, \tag{2.106}$$

where M is Kummer's function (Appendix B). It is assumed here that the value of ε is such that the denominator in (2.106) is nonzero. Using the known asymptotic properties of M one finds that for small ε and $x \neq 1/2$,

$$y(x) \sim \begin{cases} e^{-x(1-x)/2\varepsilon} & \text{for} \quad k \neq 0, 2, 4, \ldots, \\ (2x-1)^k & \text{for} \quad k = 0, 2, 4, \ldots. \end{cases}$$

This shows that there is a boundary layer at each endpoint for all but a discrete set of values for the constant k. When there are boundary layers, the solution in the outer region is transcendentally small and goes to zero as $\varepsilon \downarrow 0$. What is significant is that for $k = 0, 2, 4, \ldots$ this does not happen. This behavior at a discrete set of points for the parameter k is reminiscent of resonance, and this has become known as Ackerberg–O'Malley resonance. Those interested in pursuing this topic further are referred to the original paper by Ackerberg and O'Malley (1970) and a later study by De Groen (1980).

Exercises

2.43. Find a composite expansion of the solutions of the following problems:
(a) $\varepsilon y'' + (y')^2 - 1 = 0$ for $0 < x < 1$, where $y(0) = 1$, and $y(1) = 1$.
(b) $\varepsilon y'' + (y')^2 - 1 = 0$ for $0 < x < 1$, where $y(0) = 1$, and $y(1) = 1/2$.
(c) $\varepsilon y'' = 9 - (y')^2$ for $0 < x < 1$, where $y(0) = 0$, and $y(1) = 1$.
(d) $\varepsilon y'' + 2xy' - (2 + \varepsilon x^2)y = 0$ for $-1 < x < 1$, where $y(-1) = 2$ and $y(1) = -2$.

2.44. Consider the problem

$$\varepsilon y'' = x^2[1 - (y')^2] \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 1$.
(a) Assuming there is a corner-layer solution, explain why there are two possible outer solutions. Each one is piecewise linear, much like the outer solution in (2.96). Use the plausibility argument to rule out one of them.
(b) After finding the corner-layer solution, construct a composite expansion.

2.45. Consider the problem

$$\varepsilon y'' + xp(x)y' - q(x)y = 0 \quad \text{for } -1 < x < 1,$$

where $y(-1) = a$ and $y(1) = b$. The functions $p(x)$ and $q(x)$ are continuous, $p(x) \neq 0$, and $q(x) > 0$ for $-1 \leq x \leq 1$.
(a) If $p(0) < 0$, then there is a boundary layer at each end. Find a composite expansion of the solution.
(b) If $p(0) > 0$, then there is an interior layer. Find the approximations in the layer and outer regions.

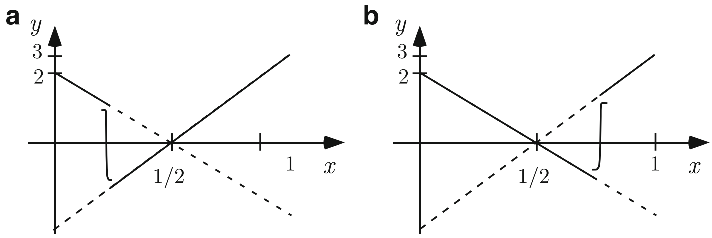

Figure 2.20 Schematic of situations considered in Exercise 2.46. The outer solution (*solid lines*) is determined from (2.94)

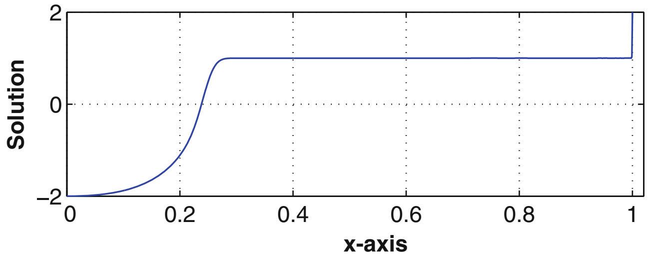

Figure 2.21 Solution of problem in Exercise 2.47

2.46. This problem demonstrates that $x_0 = \frac{1}{2}$ in (2.96).
(a) Use the plausibility argument given in Sect. 2.5 to show that it is not possible that $0 < x_0 < \frac{1}{2}$. This situation is shown in Fig. 2.20a.
(b) Use the plausibility argument given in Sect. 2.5 to show that it is not possible that $\frac{1}{2} < x_0 < 1$. This situation is shown in Fig. 2.20b.

2.47. Consider the problem

$$\varepsilon y'' - (x-a)(x-b)y' - x(y-1) = 0 \quad \text{for } 0 < x < 1,$$

where $y(0) = -2$ and $y(1) = 2$. The numerical solution is shown in Fig. 2.21 in the case where $a = 1/4$, $b = 3/4$, and $\varepsilon = 10^{-4}$. Based on this information derive a first-term approximation of the solution for arbitrary $0 < a < b < 1$.

2.48. Corner layers can occur within a boundary layer. As an example of this, consider the problem

$$\varepsilon y'' + \tanh(y') - y = -1 \quad \text{for } 0 < x < 1,$$

where $y(0) = 3$ and $y(1) = 5$. The numerical solution of the problem is shown in Fig. 2.22 when $\varepsilon = 10^{-4}$.
(a) Find a first-term expansion of the solution. You do not need to solve the problem for the corner layer but you do need to explain why the solution matches with the neighboring layer solutions.

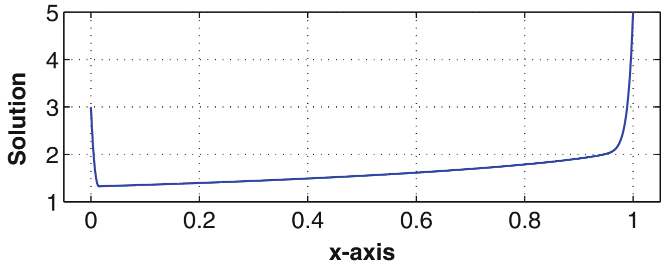

Figure 2.22 Solution of problem in Exercise 2.48

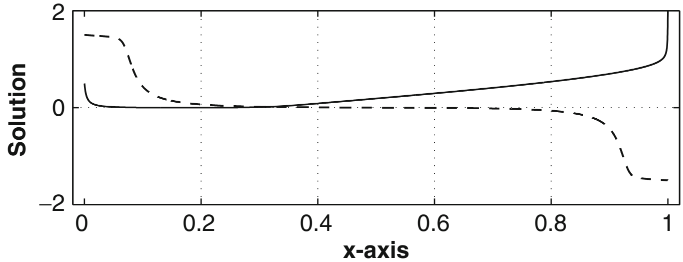

Figure 2.23 Solution of problem in Exercise 2.49

(b) As a test of the effectiveness of the numerical solvers currently available, find the numerical solution and compare it to your result in part (a) in the case where $\varepsilon = 10^{-4}$.

2.49. Consider the problem (Lorentz, 1982)

$$\varepsilon y'' + y(1 - y^2)y' - y = 0 \quad \text{for } 0 < x < 1,$$

where $y(0) = \alpha$ and $y(1) = \beta$. Find a first-term approximation of the solution in the following cases.

(a) $\alpha = 1/2$ and $\beta = 2$. The numerical solution of the problem is shown in Fig. 2.23 (solid curve), when $\varepsilon = 0.0008$. You do not need to solve the problem for the corner layer (which is located at $x_0 = 1/3$).

(b) $\alpha = 3/2$ and $\beta = -3/2$. The numerical solution of the problem is in Fig. 2.23 (dashed curve), when $\varepsilon = 0.08$.

(c) $\alpha = 3/2$ and $\beta = 2$.

2.50. This exercise concerns the shock wave produced by a cylinder that is expanding with a constant radial velocity of εa_0, where a_0 is the velocity of sound in still air (Lighthill, 1949). The velocity potential in this case has the form $\phi(r, t) = a_0^2 t f(\eta)$, where r is the radial distance from the center of the expanding cylinder, $\eta = r/(a_0 t)$, and $f(\eta)$ is determined below. Also, the radius of the cylinder is $r = \varepsilon a_0 t$, and the radius of the shock wave is $r = \alpha a_0 t$, where α is a constant determined below. The problem that determines $f(\eta)$ and α is

$$\left[1-(\gamma-1)\left(f-\eta f'+\frac{1}{2}(f')^2\right)\right](f'+\eta f'') \\ = \eta(\eta-f')^2 f'' \quad \text{for } \varepsilon<\eta<\alpha,$$

where $f'(\varepsilon)=\varepsilon$, $f(\alpha)=0$ and

$$f'(\alpha)=\frac{2(\alpha-1/\alpha)}{\gamma+1}\,.$$

In this problem γ is a positive constant called the adiabatic index. Show that, for small ε, $\alpha \sim 1+\frac{3}{8}(\gamma+1)^2\varepsilon^4$.

2.7 Partial Differential Equations

The subject of boundary and interior layers and how they appear in the solutions of partial differential equations is enormous. We will examine a couple of examples that lend themselves to matched asymptotic expansions.

2.7.1 Elliptic Problem

The first example concerns finding the function $u(x,y)$ that is the solution of the following boundary-value problem:

$$\varepsilon\nabla^2 u+\alpha\partial_x u+\beta\partial_y u+u=f(x,y) \quad \text{for } (x,y)\in\Omega, \tag{2.107}$$

where

$$u=g(x,y), \quad \text{for } (x,y)\in\partial\Omega. \tag{2.108}$$

The domain Ω is assumed to be bounded, simply connected, and have a smooth boundary $\partial\Omega$. The coefficients α and β are constant with at least one of them nonzero. The functions $f(x,y)$ and $g(x,y)$ are assumed to be continuous.

To illustrate what a solution of (2.107) looks like, consider the special case where $f(x,y)=a$ and $g(x,y)=b$, where a and b are constants. Taking Ω to be the unit disk and using polar coordinates $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, the problem can be solved using separation of variables. The resulting solution is

$$u(\rho,\varphi)=a+\frac{1}{F(\rho,\varphi)}\sum_{n=-\infty}^{\infty} I_n(\chi\rho)[a_n\sin(n\varphi)+b_n\cos(n\varphi)]/I_n(\chi), \tag{2.109}$$

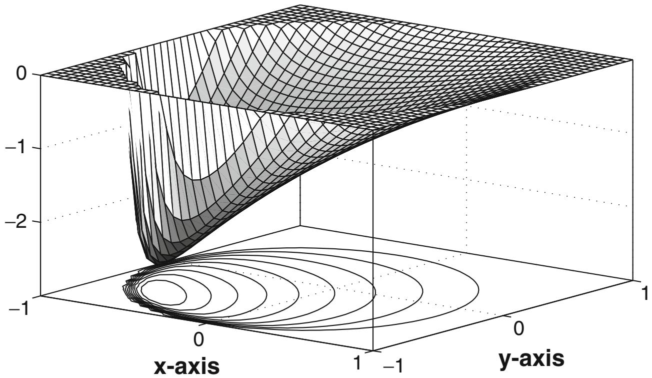

Figure 2.24 Plot of solution (2.109) and its associated contour map when $\alpha = \beta = a = 1$, $b = 0$, and $\varepsilon = 0.05$. The domain for the problem is $x^2 + y^2 < 1$, but to generate the plot, the solution was extended to the unit square $0 \le x, y \le 1$ by setting $u = 0$ for $x^2 + y^2 \ge 1$. It is apparent there is a boundary layer, and its presence is particularly pronounced in the region of the boundary near $x = y = -1/\sqrt{2}$

where $F(\rho, \varphi) = \exp(\rho(\alpha \cos\varphi + \beta \sin\varphi)/(2\varepsilon))$, $\chi = \sqrt{\alpha^2 + \beta^2 - 4\varepsilon}/(2\varepsilon)$, $I_n(z)$ is a modified Bessel function,

$$a_n = \frac{b-a}{2\pi} \int_0^{2\pi} F(1, \varphi) \sin(n\varphi) \mathrm{d}\varphi,$$

and

$$b_n = \frac{b-a}{2\pi} \int_0^{2\pi} F(1, \varphi) \cos(n\varphi) \mathrm{d}\varphi.$$

This solution is shown in Fig. 2.24. A few observations can be made from this figure that will make the derivation of the asymptotic approximation easier to follow. First, a boundary layer is clearly evident in the solution. For example, if one starts at $x = y = -1/\sqrt{2}$ and then moves into the domain, then the solution undergoes a rapid transition to what appears to be an outer solution. It is also seen that the layer is not present around the entire boundary $\partial\Omega$ but only over a portion of $\partial\Omega$. Moreover, if one follows the solution around the edge of the boundary, then it is not easy to identify exactly where the solution switches between the boundary layer and the outer region. However, wherever it is, the transition is relatively smooth. This latter observation will be useful later when deciding on the importance of what are called tangency points.

Before jumping into the derivation of the asymptotic expansion, it is worth noting that when $\varepsilon = 0$, the differential equation in (2.107) reduces to a first-order hyperbolic equation. This change in type, from an elliptic to a hyperbolic equation, has important consequences for the analysis, and it helps explain some of the steps that are taken below.

2.7.2 Outer Expansion

The procedure that we will use to find an asymptotic approximation of the solution is very similar to that used for ordinary differential equations. The first step is to find the outer solution. To do this, assume

$$u \sim u_0(x, y) + \varepsilon u_1(x, y) + \cdots . \qquad (2.110)$$

Substituting this into (2.107) yields the following O(1) problem:

$$\alpha \partial_x u_0 + \beta \partial_y u_0 + u_0 = f(x, y) \quad \text{in } \Omega, \qquad (2.111)$$

where

$$u_0 = g(x, y) \text{ on } \partial\Omega_{\text{o}}. \qquad (2.112)$$

When constructing an asymptotic approximation it is not always immediately clear which boundary condition, if any, the outer solution should satisfy. In (2.112) the portion of the boundary where u_0 satisfies the original boundary condition has been identified as $\partial\Omega_{\text{o}}$. This is presently unknown and, in fact, could turn out to be empty (e.g., Exercise 2.52).

To solve (2.111), we change coordinates from (x, y) to (r, s) , where s is the characteristic direction for (2.111). Specifically, we let $x = \alpha s + \xi(r)$ and $y = \beta s + \eta(r)$. The functions ξ and η can be chosen in a number of different ways, and our choice is based on the desire to have a simple coordinate system. In particular, we take

$$x = \alpha s + \beta r \quad \text{and} \quad y = \beta s - \alpha r. \qquad (2.113)$$

With this $\partial_s = \alpha \partial_x + \beta \partial_y$, and so (2.111) becomes

$$\partial_s u_0 + u_0 = f(\alpha s + \beta r, \beta s - \alpha r). \qquad (2.114)$$

This is easy to solve, and the general solution is

$$u_0 = a_0(r)\mathrm{e}^{-s} + \int^s \mathrm{e}^{\tau - s} f(\alpha\tau + \beta r, \beta\tau - \alpha r)\mathrm{d}\tau, \qquad (2.115)$$

where $a_0(r)$ is arbitrary. Since we have not yet determined exactly what portion of the boundary condition, if any, the outer solution should satisfy, we are not yet in a position to convert back to (x, y) coordinates.

Before investigating the boundary layer, it is instructive to consider the change of coordinates given in (2.113). Fixing r and letting s increase, one obtains the directed lines shown in Figs. 2.25 and 2.26. So, starting at boundary point P, as the variable s increases, one crosses the domain Ω and arrives at boundary point P_*. In terms of our first-term approximation, we need to know what value u_0 starts with at P and what value it has when the point P_* is reached. However, we only have one integration constant in (2.115), so there is going to be a boundary layer at either P or P_*. As we will see below,

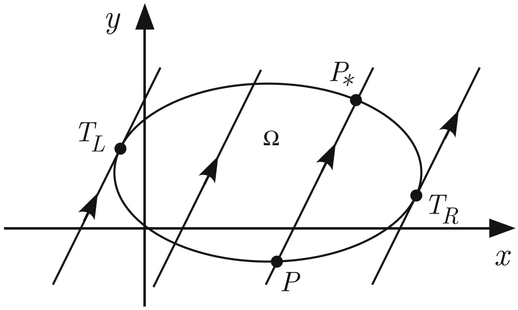

Figure 2.25 Schematic drawing of characteristic curves obtained from the problem for the outer solution. These curves are determined from (2.113) and are directed straight lines with an orientation determined by the direction of increasing s. As drawn, the coefficients α, β are assumed positive. Also note the tangency points T_{r}, T_{L}

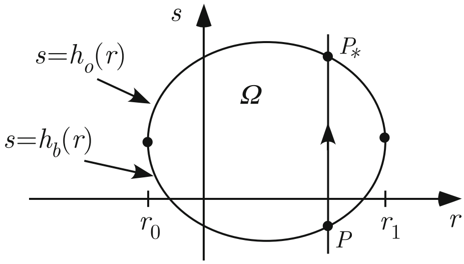

Figure 2.26 Schematic drawing of transformed region coming from the domain in Fig. 2.25

for our problem, the outer solution will satisfy the boundary condition at P_* and there will be a boundary layer of width ε at P. Based on these comments, we will assume that the points of the boundary can be separated into three disjoint sets:

- $\partial\Omega_{\mathrm{o}}$. These are the boundary points where the characteristics leave Ω (like P_* in Figs. 2.25 and 2.26). In Fig. 2.26 these points make up the curve $s = h_{\mathrm{o}}(r)$ for $r_0 < r < r_1$.
- $\partial\Omega_{\mathrm{b}}$. There are the boundary points where the characteristics enter Ω (like P in Figs. 2.25 and 2.26). In Fig. 2.26 these points make up the curve $s = h_{\mathrm{b}}(r)$ for $r_0 < r < r_1$.
- $\partial\Omega_{\mathrm{t}}$. There are the tangency points. In Fig. 2.25 there are two such points, T_{L} and T_{R}, and in Fig. 2.26 they occur when $r = r_0$ and $r = r_1$. These can cause real headaches when constructing an asymptotic approximation and will be left until the end.

To keep the presentation simple, we will assume that the situation is as pictured in Fig. 2.25, i.e., the domain is convex. This means that if a

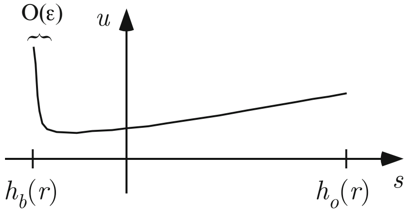

Figure 2.27 Schematic of solution as a function of the characteristic coordinate s as defined in (2.113). There is a boundary layer of width $O(\varepsilon)$ at the left end of the interval. Note that, in connection with Fig. 2.26, the point $s = h_{\mathrm{b}}(r)$ can be thought of as corresponding to the point P, and $s = h_{\mathrm{o}}(r)$ can be thought of as corresponding to the point P_*

characteristic curve enters the domain, then the only other time it intersects the boundary is when it leaves the domain. Moreover, $\partial\Omega_{\mathrm{o}}$ and $\partial\Omega_{\mathrm{b}}$ are assumed to be smooth curves.

Now that we know (or think we know) where the boundary layer is located, we are in a position to complete the specification of the outer solution given in (2.115). To do this, assume $\partial\Omega_{\mathrm{o}}$ can be described as $s = h_{\mathrm{o}}(r)$ for $r_0 < r < r_1$. In this case, using (2.112), the outer solution in (2.115) becomes

$$u_0 = g(\alpha h_{\mathrm{o}} + \beta r, \beta h_{\mathrm{o}} - \alpha r)\mathrm{e}^{h_{\mathrm{o}} - s} - \int_s^{h_{\mathrm{o}}(r)} \mathrm{e}^{\tau - s} f(\alpha\tau + \beta r, \beta\tau - \alpha r)\mathrm{d}\tau. \quad (2.116)$$

This solution does not apply along $\partial\Omega_{\mathrm{b}}$ or on $\partial\Omega_t$. To be more specific, suppose the curve $s = h_{\mathrm{b}}(r)$ for $r_0 < r < r_1$ describes $\partial\Omega_{\mathrm{b}}$. In this case (2.116) holds for $r_0 < r < r_1$ and $h_{\mathrm{b}}(r) < s \le h_{\mathrm{o}}(r)$.

2.7.3 Boundary-Layer Expansion

To find out what goes on in a boundary layer, we introduce the boundary coordinate (Fig. 2.27)

$$S = \frac{s - h_{\mathrm{b}}(r)}{\varepsilon}. \quad (2.117)$$

This coordinate, by necessity, depends on r since the boundary depends on r. This complicates the calculations in making the change of variables from (r, s) to (r, S). One finds using the chain rule that the derivatives transform as follows:

$$\partial_s \to \frac{1}{\varepsilon}\partial_S, \qquad \partial_r \to -\frac{h_{\mathrm{b}}'}{\varepsilon}\partial_S + \partial_r,$$
$$\partial_s^2 \to \frac{1}{\varepsilon^2}\partial_S^2, \qquad \partial_r^2 \to \frac{(h_{\mathrm{b}}')^2}{\varepsilon^2}\partial_S^2 - \frac{h_{\mathrm{b}}''}{\varepsilon}\partial_S - \frac{2h_{\mathrm{b}}'}{\varepsilon}\partial_S\partial_r + \partial_r^2.$$

To convert the original problem into boundary-layer coordinates, we must first change from (x, y) to (r, s) coordinates in (2.107). This is relatively easy since from (2.113) one finds that

$$\partial_x = \frac{1}{\gamma}(\alpha\partial_s + \beta\partial_r) \quad \text{and} \quad \partial_y = \frac{1}{\gamma}(\beta\partial_s - \alpha\partial_r),$$

where $\gamma = \alpha^2 + \beta^2$. With this, (2.107) becomes

$$\varepsilon(\partial_s^2 + \partial_r^2)u + \gamma\partial_s u + \gamma u = \gamma f. \tag{2.118}$$

Now, letting $U(r, S)$ denote the solution in the boundary layer, substituting the boundary-layer coordinates into (2.118) yields

$$[\mu\partial_S^2 + \gamma\partial_S + O(\varepsilon)]U = \varepsilon\gamma f, \tag{2.119}$$

where

$$\mu(r) = 1 + (h_\mathrm{b}')^2. \tag{2.120}$$

In keeping with our usual assumptions, $h_\mathrm{b}(r)$ is taken to be a smooth function. However, note that at the tangency points shown in Fig. 2.25, $h_\mathrm{b}'(r) = \infty$. For this reason, these points will have to be dealt with separately after we finish with the boundary layer.

We are now in a position to expand the solution in the usual power series expansion, and so let

$$U(r, S) \sim U_0(r, S) + \cdots . \tag{2.121}$$

Substituting this into (2.119) yields the equation

$$\mu\partial_S^2 U_0 + \gamma\partial_S U_0 = 0.$$

The general solution of this is

$$U_0(r, S) = A(r) + B(r)\mathrm{e}^{-\gamma S/\mu}, \tag{2.122}$$

where $A(r)$ and $B(r)$ are arbitrary. Now, from the boundary and matching conditions we must have $U_0(r, 0) = g$ and $U_0(r, \infty) = u_0(r, h_\mathrm{b})$. Imposing these on our solution in (2.122) yields

$$U_0(r, S) = u_0(r, h_\mathrm{b}) + [g(\alpha h_\mathrm{b} + \beta r, \beta h_\mathrm{b} - \alpha r) - u_0(r, h_\mathrm{b})]\mathrm{e}^{-\gamma S/\mu}, \tag{2.123}$$

where

$$u_0(r, h_\mathrm{b}) = g(\alpha h_\mathrm{o} + \beta r, \beta h_\mathrm{o} - \alpha r)\mathrm{e}^{h_\mathrm{o} - h_\mathrm{b}} - \int_{h_\mathrm{b}}^{h_\mathrm{o}} \mathrm{e}^{\tau - h_\mathrm{b}(r)} f(\alpha\tau + \beta r, \beta\tau - \alpha r)\mathrm{d}\tau.$$

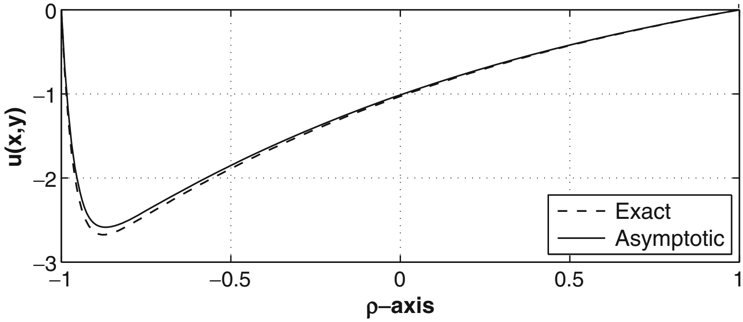

Figure 2.28 Comparison between exact solution (2.109) and composite expansion given in (2.125) in the case where $\varepsilon = 0.05$. The values of each function are given along the line $x = \rho\cos(\pi/4), y = \rho\sin(\pi/4)$ for $-1 \le \rho \le 1$

2.7.4 Composite Expansion

It is not difficult to put together a composite expansion that will give us a first-term approximation of the solution in the outer and boundary-layer regions. Adding (2.116) and (2.123) together and then subtracting their common part yields

$$u \sim g(\alpha h_{\mathrm{o}} + \beta r, \beta h_{\mathrm{o}} - \alpha r)\mathrm{e}^{h_{\mathrm{o}}-s} - \int_s^{h_{\mathrm{o}}} \mathrm{e}^{\tau-s} f(\alpha\tau + \beta r, \beta\tau - \alpha r)\mathrm{d}\tau$$
$$+ \left[g(\alpha h_{\mathrm{b}} + \beta r, \beta h_{\mathrm{b}} - \alpha r) - u_0(r, h_{\mathrm{b}})\right] \mathrm{e}^{-\gamma S/\mu}. \qquad (2.124)$$

This approximation holds for $r_0 < r < r_1$ and $h_{\mathrm{b}}(r) \le s \le h_{\mathrm{o}}(r)$. This result may not be pretty, but it does give us a first-term approximation of the solution in the boundary layer and outer domain. It is also considerably simpler than the formula for the exact solution, an example of which is given in (2.109). What is interesting is that we have been able to patch together the solution of an elliptic problem using solutions to hyperbolic and elliptic problems. Readers interested in the theoretical foundation of the approximations constructed here are referred to Levinson (1950), Eckhaus and Jager (1966), and Il'in (1992).

Example

To apply the result to the domain used in Fig. 2.24, note that the change of variables in (2.113) transforms $\partial\Omega$, which is the unit circle $x^2 + y^2 = 1$, to the circle $r^2 + s^2 = 1/(\alpha^2 + \beta^2)$. In this case, $h_{\mathrm{o}}(r)$ is the upper half of the circle, where $s > 0$, and $h_{\mathrm{b}}(r)$ is the lower half, where $s < 0$. Taking $f = a$

and $g = b$, then (2.124) reduces to

$$u \sim a + (b-a)\Big[\mathrm{e}^{h_\mathrm{o}-s} + \big(1-\mathrm{e}^{h_\mathrm{o}-h_\mathrm{b}}\big)\,\mathrm{e}^{-\gamma S/\mu}\Big]\,, \tag{2.125}$$

where $h_\mathrm{o} = \sqrt{1/\gamma - r^2}$, $h_\mathrm{b} = -h_\mathrm{o}$, $\mu = 1 + r^2/h_\mathrm{o}$, and $\gamma = \alpha^2 + \beta^2$. This can be expressed in terms of (x, y) using the formulas $r = (\beta x - \alpha y)/\gamma$ and $s = (\alpha x + \beta y)/\gamma$. With (2.125) we have a relatively simple expression that is a composite approximation of the exact solution given in (2.109). To compare them, let $f = 1$, $g = 0$, and $\alpha = \beta = 1$. These are the same values used in Fig. 2.24. The resulting approximation obtained from (2.125), along with the exact solution given in (2.109), is shown in Fig. 2.28 for a slice through the surface. Based on this graph, it seems that we have done reasonably well with our approximation. However, we are not finished as (2.124) does not hold in the immediate vicinity of the tangency points T_R and T_L, which are shown in Fig. 2.25. ■

2.7.5 Parabolic Boundary Layer

To complete the construction of a first-term approximation of the solution of (2.107), it remains to find out what happens near the tangency points shown in Figs. 2.25 and 2.26. We will concentrate on T_L. To do this, let (r_0, s_0) be its coordinates in the (r, s) system. Also, suppose the smooth curve $r = q(s)$ describes the boundary $\partial\Omega$ in this region. In this case $r_0 = q(s_0)$ and $q'(s_0) = 0$. It will be assumed here that $q''(s_0) \neq 0$. The boundary-layer coordinates are now

$$\tilde{r} = \frac{r - q(s)}{\varepsilon^\alpha} \quad \text{and} \quad \tilde{s} = \frac{s - s_0}{\varepsilon^\beta}\,. \tag{2.126}$$

The transformation formulas for the derivatives are similar to those derived earlier for the boundary layer along $\partial\Omega_\mathrm{b}$, so they will not be given. The result is that (2.118) takes the form

$$\varepsilon(\varepsilon^{-2\alpha}\partial_{\tilde{r}}^2 + \varepsilon^{-2\beta}\partial_{\tilde{s}}^2 + \cdots)\widetilde{U} \\ + \gamma(\varepsilon^{-\beta}\partial_{\tilde{s}} - \varepsilon^{\beta-\alpha}\tilde{s}q_0''\partial_{\tilde{r}} + \cdots)\widetilde{U} + \gamma\widetilde{U} = \gamma f.$$

Here $\widetilde{U}(\tilde{r}, \tilde{s})$ is the solution in this region, and we have used the Taylor series expansion $q'(s_0 + \varepsilon^\beta\tilde{s}) \sim \varepsilon^\beta\tilde{s}q_0''$ to obtain the preceding result. There are at least two balances that need to be considered. One is $\alpha = \beta = 1$, the other is $2\beta = \alpha = 2/3$. The latter is the one of interest, and assuming

$$\widetilde{U}(\tilde{r}, \tilde{s}) \sim \widetilde{U}_0(\tilde{r}, \tilde{s}) + \cdots \tag{2.127}$$

one obtains the equation

$$(\partial_{\tilde{r}}^2 - \gamma\tilde{s}q_0''\partial_{\tilde{r}} + \gamma\partial_{\tilde{s}})\widetilde{U}_0 = 0 \tag{2.128}$$

for $0 < \tilde{r} < \infty$ and $-\infty < \tilde{s} < \infty$. This is a parabolic equation, and for this reason this region is referred to as a *parabolic boundary layer*. The solution is required to match to the solutions in the adjacent regions and should satisfy the boundary condition at $\tilde{r} = 0$. The details of this calculation will not be given here but can be found in van Harten (1976). The papers by Cook and Ludford (1971, 1973) should also be consulted as they have an extensive analysis of such parabolic layers and how they appear in problems where the domain has a corner. The theory necessary to establish the uniform validity of the expansions when corners are present in the boundary, and a historical survey of this problem, can be found in Shih and Kellogg (1987).

2.7.6 Parabolic Problem

To illustrate the application of boundary-layer methods to parabolic equations, we consider the problem of solving

$$u_t + uu_x = \varepsilon u_{xx} \quad \text{for} \quad -\infty < x < \infty \text{ and } 0 < t, \tag{2.129}$$

where $u(x, 0) = \phi(x)$. It is assumed here that $\phi(x)$ is smooth and bounded except for a jump discontinuity at $x = 0$. Moreover, $\phi'(x) \leq 0$ for $x \neq 0$ and $\phi(0^-) > \phi(0^+)$.

As an example, suppose that

$$u(x, 0) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } 0 < x. \end{cases}$$

This type of initial condition generates what is known as a Riemann problem, and the resulting solution is shown in Fig. 2.29. It is a traveling wave, and the smaller the value of ε, the sharper the transition from $u = 0$ to $u = 1$. In the limit of $\varepsilon \to 0$, the transition becomes a jump, producing a solution containing a shock wave.

The nonlinear diffusion equation in (2.129) is known as *Burger's equation*. It has become the prototype problem for studying shock waves and for the use of what are called viscosity methods for finding smooth solutions to such problems. In what follows we will concentrate on the constructive aspects of finding asymptotic approximations to the solution. The theoretical underpinnings of the procedure can be found in Il'in (1992) and its extension to systems in Goodman and Xin (1992).

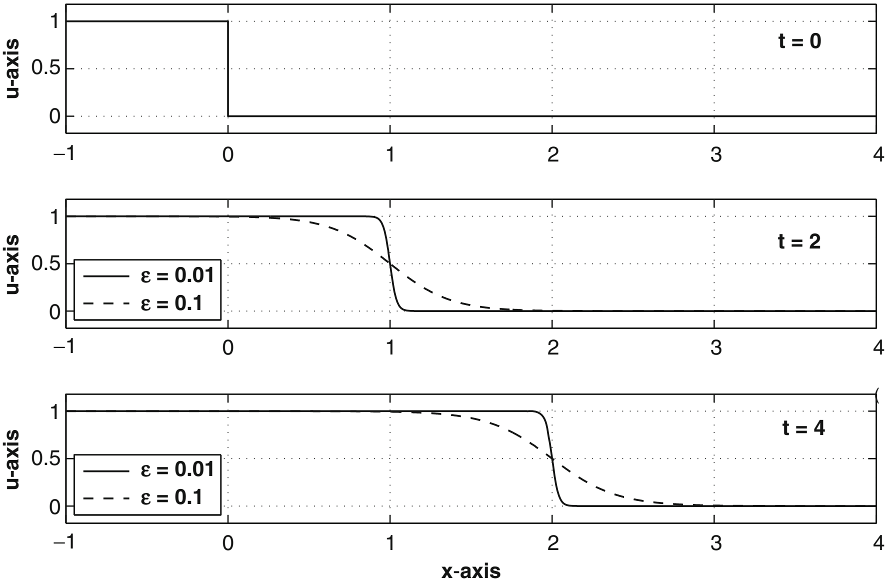

Figure 2.29 Solution of Burger's equation (2.129) for two values of ϵ, showing the traveling-wave nature of the solution as well as the sharpening of the wave as ϵ approaches zero

2.7.7 Outer Expansion

The first step is to find the outer solution. To do this, assume $u \sim u_0(x,t) + \cdots$. Substituting this into (2.129) produces the first-order hyperbolic equation

$$\partial_t u_0 + u_0 \partial_x u_0 = 0. \tag{2.130}$$

The characteristics for this equation are the straight lines $x = x_0 + \phi(x_0)t$, and the solution $u_0(x,t)$ is constant along each of these lines (Fig. 2.30). Therefore, given a point (x,t), then $u_0(x,t) = \phi(x_0)$, where x_0 is determined from the equation $x = x_0 + \phi(x_0)t$. This construction succeeds if the characteristics do not intersect and they cover the upper half-plane. For us the problem is with intersections. As illustrated in Fig. 2.30, the characteristics coming from the negative x-axis intersect with those from the positive x-axis. From the theory for nonlinear hyperbolic equations it is known that this generates a curve $x = s(t)$, known as a shock wave, across which the solution has a jump discontinuity (Holmes, 2009). The solution in this case is determined by the characteristics up to when they intersect the shock. The complication here is that the position of the interior layer is moving and centered at $x = s(t)$. As it turns out, the formula for $s(t)$ can be determined from (2.130), but we will derive it when examining the solution in the transition layer.

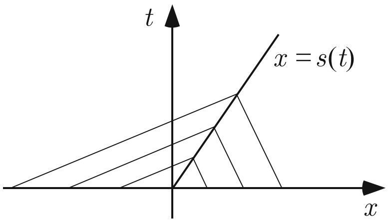

Figure 2.30 The *straight lines* are the characteristics for the reduced problem (2.130). Because ϕ has a discontinuity at $x = 0$, and $\phi' \geq 0$ for $x \neq 0$, these lines intersect along the shock curve $x = s(t)$

2.7.8 Inner Expansion

The interior layer coordinate is

$$\bar{x} = \frac{x - s(t)}{\varepsilon^{\alpha}} .$$

Letting $U(\bar{x}, t)$ denote the solution of the problem in this layer, (2.129) takes the form

$$\partial_t U - \varepsilon^{-\alpha} s'(t)\partial_{\bar{x}} U + \varepsilon^{-\alpha} U \partial_{\bar{x}} U = \varepsilon^{1-2\alpha}\partial_{\bar{x}}^2 U. \tag{2.131}$$

Balancing the terms in this equation, one finds $\alpha = 1$. Thus, the appropriate expansion of the solution is $U \sim U_0(\bar{x}, t) + \cdots$, and substituting this into (2.131) yields

$$-s'(t)\partial_{\bar{x}} U_0 + U_0 \partial_{\bar{x}} U_0 = \partial_{\bar{x}}^2 U_0.$$

Integrating this, one finds that

$$\partial_{\bar{x}} U_0 = \frac{1}{2} U_0^2 - s'(t) U_0 + A(t). \tag{2.132}$$

The boundary conditions to be used come from matching with the outer solution on either side of the layer. They are

$$\lim_{\bar{x} \to -\infty} U_0 = u_0^- \quad \text{and} \quad \lim_{\bar{x} \to \infty} U_0 = u_0^+, \tag{2.133}$$

where

$$u_0^{\pm} = \lim_{x \to s(t)^{\pm}} u_0(x, t). \tag{2.134}$$

From (2.132) it follows that $A(t) = -\frac{1}{2}(u_0^-)^2 + s'(t) u_0^-$ and

$$s'(t) = \frac{1}{2}(u_0^+ + u_0^-). \tag{2.135}$$

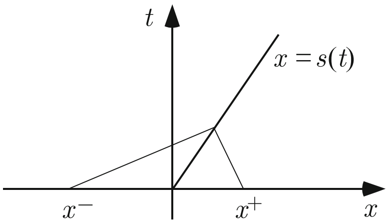

Figure 2.31 Schematic showing a shock and two characteristics that intersect on the shock

The differential equation in (2.135) determines the position of the shock and is known as the Rankine–Hugoniot condition. Its solution requires an initial condition, and, because of the assumed location of the discontinuity in $\phi(x)$, we take $s(0) = 0$.

To complete the analysis of the first-order problem in the shock layer, we separate variables in (2.132) and then integrate to obtain, for the case where $u_0^+ < u_0^-$,

$$U_0(\bar{x}, t) = \frac{u_0^+ + b(\bar{x}, t)u_0^-}{1 + b(\bar{x}, t)}, \tag{2.136}$$

where

$$b(\bar{x}, t) = B(t)\mathrm{e}^{\bar{x}(u_0^+ - u_0^-)/2} \tag{2.137}$$

and $B(t)$ is an arbitrary nonzero function.

The indeterminacy in $U_0(\bar{x}, t)$, as given by the unspecified function in (2.137), is the same difficulty we ran into when investigating interior layers in Sect. 2.5. However, for this nonlinear diffusion problem it is possible to determine $B(t)$, up to a multiplicative factor, by examining the next order problem. To state the result, note that given a point $(x, t) = (s(t), t)$ on the shock, $u_0^+ = \phi(x^+)$ and $u_0^- = \phi(x^-)$. Here $x^\pm$ are the initial points on the x-axis for the two characteristics that intersect at $(s(t), t)$ (Fig. 2.21). In this case, one finds from the $O(\varepsilon)$ problem that (Exercise 2.57)

$$B(t) = B_0\sqrt{\frac{1 + t\phi'(x^+)}{1 + t\phi'(x^-)}}, \tag{2.138}$$

where the constant B_0 is found from the initial condition.

To determine B_0 in (2.138), one must match the shock layer solution with the solution from the initial layer located near $x = x_0$. The appropriate coordinate transformation for this layer is $\bar{x} = (x - s(t))/\varepsilon$ and $\tau = t/\varepsilon$. The steps involved in this procedure are not difficult, and one finds that $B_0 = 1$ (Exercise 2.57). With this the first-term approximation of the solution in the shock layer is determined.

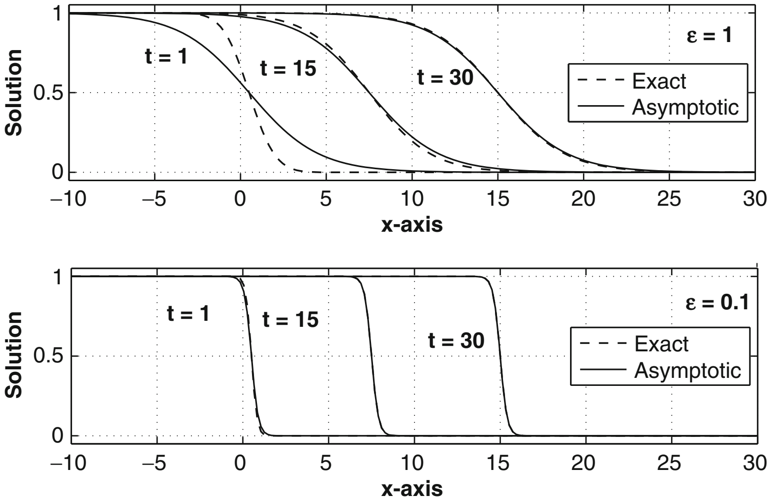

Figure 2.32 Graph of exact solution of Burger's equation and asymptotic approximation in the case where $\phi(x) = 1$ if $x < 0$ and $\phi(x) = 0$ if $x > 0$. The solutions are compared at $t = 1, 15, 30$, for both $\varepsilon = 1$ and $\varepsilon = 10^{-1}$

To demonstrate the accuracy of the asymptotic approximation, in Fig. 2.32 the approximation is shown along with the exact solution at two nonzero values of t and ε. The exact solution in this case is given in Exercise 2.56. For $\varepsilon = 0.1$ the asymptotic and exact solutions are so close that they are essentially indistinguishable in the graph. They are even in reasonable agreement when $\varepsilon = 1$, although the differences are more apparent for the smaller value of t.

In this section we have considered elliptic and parabolic problems. Matched asymptotic expansions can also be applied to hyperbolic problems, and several examples are worked out in Kevorkian and Cole (1981). However, there are better methods for wave problems, particularly when one is interested in the long time behavior of the wave. This topic will be taken up in Chaps. 3 and 4.

Exercises

2.51. A special case of (2.107) is the problem of solving

$$\varepsilon \nabla^2 u + \partial_y u = 2 \text{ in } \Omega,$$

where $u = x + y$ on $\partial\Omega$. Let Ω be the unit disk $x^2 + y^2 < 1$.

(a) Sketch the domain and characteristic curves for the outer solution (Fig. 2.25). Identify the points T_{L} and T_{R}.
(b) What is the composite expansion, as in (2.124), for this problem?
(c) What is the parabolic layer equation, as in (2.128), in this case?

2.52. Find a first-term composite expansion, for the outer region and boundary layers, of the solution of

$$\varepsilon \nabla^2 u + u = 1 \ \text{ in } \ \Omega,$$

where $u = g(x, y)$ on $\partial\Omega$. Let Ω be the unit disk $x^2 + y^2 < 1$.

2.53. In this exercise, variations of the elliptic boundary-value problem (2.107) are considered.
(a) If the coefficients α and β in (2.107) are negative, how does the composite expansion in (2.124) change?
(b) If one of the coefficients α and β in (2.107) is negative and the other is positive, how does the composite expansion in (2.124) change?

2.54. Consider the problem of solving

$$u_t = \varepsilon u_{xx} - c u_x \quad \text{for } 0 < x \text{ and } 0 < t,$$

where $u(0, t) = u_\ell$ and $u(x, 0) = u_{\mathrm{r}}$. Assume c, u_ℓ, and u_{r} are constants with c positive and $u_{\mathrm{r}} \neq u_\ell$.
(a) Find the first term in the outer expansion. Explain why this shows that there is an interior layer located at $x = ct$.
(b) Find the first term in the inner expansion. From this find a first-term composite expansion of the solution.
(c) Where is the assumption that $c > 0$ used in parts (a) or (b)? What about the assumption that $u_{\mathrm{r}} \neq u_\ell$? Note that the case where $c < 0$ is considered in Exercise 2.55.
(d) The exact solution is

$$u(x, t) = u_{\mathrm{r}} + \frac{1}{2}(u_\ell - u_{\mathrm{r}})\left[\operatorname{erfc}\left(\frac{x - ct}{2\sqrt{\varepsilon t}}\right) + \mathrm{e}^{cx/\varepsilon}\operatorname{erfc}\left(\frac{x + ct}{2\sqrt{\varepsilon t}}\right)\right].$$

Verify that this satisfies the differential equation as well as the boundary and initial conditions.
(e) Explain how your composite expansion in part (b) can be obtained from the solution in part (d).

2.55. The equation of one-dimensional heat conduction in a material with a low conductivity is (Plaschko, 1990)

$$u_t = \varepsilon u_{xx} + v(t) u_x \quad \text{for } 0 < x \text{ and } 0 < t,$$

where $u(0,t) = g(t)$, $u(\infty,t) = 0$, and $u(x,0) = h(x)$. Assume that the functions $v(t)$, $g(t)$, and $h(x)$ are smooth with $0 < v(t)$ for $0 \leq t < \infty$, $g(0) = h(0)$, and $h(\infty) = 0$.

(a) Find a first-term composite expansion of the solution.

(b) Find the second term in the composite expansion. Is the expansion uniformly valid over the interval $0 \leq t < \infty$? What conditions need to be placed on the functions $h(x)$ and $v(t)$? A method for constructing uniformly valid approximations in a case like this is the subject of Chap. 3.

2.56. Using the Cole–Hopf transformation it is possible to solve Burger's equation (2.129) (Whitham, 1974). In the case where

$$u(x,0) = \begin{cases} u_1 & \text{if} \quad x < 0, \\ u_2 & \text{if} \quad 0 < x, \end{cases}$$

where $u_1 > u_2$ are constants, one finds $u(x,t) = \frac{u_2 + K(x,t)u_1}{1+K(x,t)}$, where

$$K(x,t) = \frac{\operatorname{erfc}\left(\frac{x-u_1 t}{2\sqrt{\varepsilon t}}\right)}{\operatorname{erfc}\left(-\frac{x-u_2 t}{2\sqrt{\varepsilon t}}\right)} \mathrm{e}^{(x-v_0 t)(u_2-u_1)/2\varepsilon}$$

and $v_0 = \frac{1}{2}(u_1 + u_2)$. Compare this with the first-term approximation derived for (2.129). Make sure to comment on the possible differences for small and for large values of t.

2.57. The function $B(t)$ in (2.137) can be found by matching the second term in the inner and outer expansions. This exercise outlines the necessary steps.

(a) Show that the second term in the outer expansion is

$$u_1(x,t) = \frac{t\phi''(\xi)}{(1+t\phi'(\xi))^2},$$

where the value of ξ is determined from the equation $x = \xi + t\phi(\xi)$.

(b) By changing variables from $\bar{x}$ to $z = (1-b)/(1+b)$, where b is given in (2.137), show that the equation for U_1 becomes

$$(1-z^2)\partial_z^2 U_1 + 2U_1 = \frac{4(s'' + r'z)}{r^2(1-z^2)} - \frac{2}{r^2}\left(\frac{rB'}{B} + r' \ln\left(\frac{1-z}{B(1+z)}\right)\right),$$

where $r = \frac{1}{2}(u_0^+ - u_0^-)$.

(c) Solve the equation in part (b), and from this obtain the first two terms in the expansion of U_1 for $z \to 1$ and for $z \to -1$. (Hint: Because of the nature of this calculation, the use of a symbolic computer program is recommended.)

(d) To match the inner and outer expansions, introduce the intermediate variable $\bar{x}_\eta = (x - s(t))/\varepsilon^\eta$. With this, show that the outer expansion expands as follows:

$$u \sim \phi(\xi) + \varepsilon^\eta \frac{\bar{x}_\eta \phi'(\xi)}{1 + t\phi'(\xi)} + \varepsilon \frac{t\phi''(\xi)}{(1 + t\phi'(\xi))^2} + \cdots ,$$

where ξ is determined from the equation $s(t) = \xi + t\phi(\xi)$. Do the same for the inner expansion, and by matching the two derive the result in (2.138).

(e) To find the constant B_0, introduce the initial-layer coordinates $\bar{x} = (x - s(t))/\varepsilon$ and $\tau = t/\varepsilon$. Find the first term in this layer, and then match the result with (2.136) to show $B_0 = 1$.

2.58. This exercise involves modifications of the expansions for Burger's equation.

(a) Discuss the possibility of obtaining a composite expansion for the solution of (2.129).

(b) The center of the shock wave is where u is half-way between u_0^+ and u_0^-. If this is located at $x = X(t)$, then find the first two terms in the expansion of $X(t)$. A discussion of this, and other aspects of the problem, can be found in Lighthill (1956).

(c) Because the position of the shock is determined by the solution, then it should, presumably, depend on ε. How do things change if one allows for the possibility that $s(t) \sim s_0(t) + \varepsilon s_1(t) + \cdots$?

2.59. Consider the linear diffusion problem

$$u_t + \alpha u_x + \beta u = \varepsilon u_{xx} \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x, 0) = \phi(x)$. Assume that $\phi(x)$ has the same properties as the initial condition for (2.129) and that α and β are positive constants.

(a) Find the first terms in the inner and outer expansions of the solution.

(b) Comment on the differences between the characteristics of the shock layer for Burger's equation and the one you found in part (a).

2.60. Find the first term in the inner and outer expansions of the solution of

$$u_t + f(u)u_x = \varepsilon u_{xx} \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x, 0) = \phi(x)$. Assume $\phi(x)$ has the same properties as the initial condition for (2.129) and $f(r)$ is smooth with $f'(r) > 0$.

2.61. Consider the problem of solving

$$\frac{\varepsilon}{r}\partial_r \left[\frac{1}{r}\partial_r (r^2 w) \right] = \partial_t w + \mu w + \kappa \mathrm{e}^{-2t} \quad \text{for } 0 \le r < 1, 0 < t,$$

where $w(r, 0) = 0$, $\partial_r w(0, t) = 0$, and $w(1, t) = 0$. Also, μ and κ are positive constants with $\mu \neq 2$. This problem arises in the study of the rotation of a

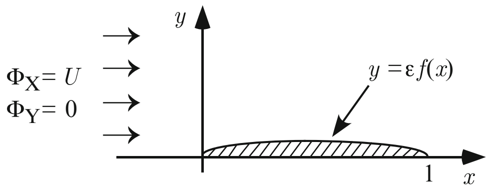

Figure 2.33 Schematic of air flow over an airplane wing as assumed in Exercise 2.62

cylindrical container filled with a dilute suspension (Ungarish, 1993). In this context, $w(r,t)$ is the angular velocity of the suspension and the boundary layer is known as a Stewartson shear layer. For small ε, find a first-term composite expansion of the solution.

2.62. The equation for the velocity potential $\phi(x,y)$ for steady air flow over an airplane wing is (Cole and Cook, 1986)

$$(a^2 - \Phi_x^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (a^2 - \Phi_y^2)\Phi_{yy} = 0,$$

where

$$a^2 = a_\infty^2 + \frac{1}{2}(\gamma - 1)(U^2 - \Phi_x^2 - \Phi_y^2).$$

Here $a_\infty > 0$, $U > 0$, and $\gamma > 1$ are constants. The wing is assumed to be described by the curve $y = \varepsilon f(x)$ for $0 < x < 1$ (Fig. 2.33). In this case, the boundary conditions are that $\phi = Ux$ as $x \to -\infty$ and

$$\Phi_y = \begin{cases} \varepsilon f'(x)\Phi_x & \text{when} \quad y = \varepsilon f(x) \text{ and } 0 < x < 1, \\ 0 & \text{when} \quad y = 0 \text{ and } -\infty < x < 0 \text{ or } 1 < x. \end{cases}$$

(a) The thickness ε of the wing is small, and this is the basis of what is known as small disturbance theory. The appropriate expansion for the potential in this case has the form

$$\Phi \sim Ux + \varepsilon^\alpha \phi_1 + \varepsilon^\beta \phi_2 + \cdots .$$

Find α, and then determine what problem ϕ_1 satisfies.

(b) Find ϕ_1 in the case where $M_\infty > 1$, where $M_\infty = U/a_\infty$.

(c) For the case where $M_\infty > 1$, find ϕ_2 and explain why for the expansion to be valid it must be that $\varepsilon \ll (M_\infty^2 - 1)^{3/2}$. (Hint: Use characteristic coordinates $\xi = x - y\sqrt{M_\infty^2 - 1}$, $\eta = x + y\sqrt{M_\infty^2 - 1}$.)

2.63. For a semiconductor to function properly, one must be concerned with the level of impurities that diffuse in from the outer surface and occupy vacant locations in the crystalline structure of the semiconductor. A problem for the concentration of impurities $c(x,t)$ and vacancies $v(x,t)$ is (King et al., 1992)

$$\left.\begin{aligned} \partial_t c &= \partial_x(v\partial_x c - c\partial_x v) \\ \partial_t v + r\partial_t c &= \varepsilon^2 \partial_x^2 v \end{aligned}\right\} \quad \text{for } 0 < x < \infty \text{ and } 0 < t,$$

where $c = 0$ and $v = 1$ when $t = 0$, $c = 1$, and $v = \mu$ when $x = 0$, and $c \to 0$ and $v \to 1$ as $x \to \infty$. Also, r and μ are positive constants. For small ε derive a composite expansion of the solution of this problem.

2.8 Difference Equations

Up until now, when discussing boundary-layer problems, we have dealt almost exclusively with differential equations. We will now expand our horizons and investigate what happens with singularly perturbed difference equations. As will be seen, many of the ideas developed in the first part of this chapter will reappear when analyzing difference equations, but there are subtle and interesting differences.

Our starting point is the boundary-value problem

$$\varepsilon y_{n+1} + \alpha_n y_n + \beta_n y_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots, N-1, \tag{2.139}$$

where

$$y_0 = a, \quad y_N = b. \tag{2.140}$$

What we have here is a second-order linear difference equation with prescribed values at the ends (where $n = 0, N$). In what follows it is assumed that N is fixed, and we will investigate how the solution behaves for small ε. It should also be pointed out that we will be assuming that the α_n's and β_n's are nonzero.

There are a couple of observations about the problem that should be made before starting the derivation of the asymptotic approximation of the solution. First, it is clear that the problem is singular for small ε since the reduced equation $\alpha_n y_n + \beta_n y_{n-1} = 0$ is first order and cannot be expected to satisfy both boundary conditions. The second observation can be made by considering an example. If $\alpha_n = 2$ and $\beta_n = a = b = 1$, then the solution of (2.139), (2.140) is

$$y_n = \left(\frac{1 - m_1^N}{m_2^N - m_1^N}\right) m_2^n - \left(\frac{1 - m_2^N}{m_2^N - m_1^N}\right) m_1^n,$$

where $m_1 = -(1 + \sqrt{1-\varepsilon}\,)/\varepsilon$ and $m_2 = -(1 - \sqrt{1-\varepsilon}\,)/\varepsilon$. For small ε this reduces to

$$y_n \sim \left(-\frac{1}{2}\right)^n + \left[1 - \left(-\frac{1}{2}\right)^N\right]\left(-\frac{\varepsilon}{2}\right)^{N-n}, \quad \text{for } n = 1, 2, \ldots, N. \tag{2.141}$$

This shows boundary-layer type of behavior near the end $n = N$ in the sense that if one starts at $n = N$ and then considers the values at $n = N-1$, $N-2$, $\ldots$, then the $O(\varepsilon^{N-n})$ term in (2.141) rapidly decays. Moreover, away from the immediate vicinity of the right end, this term is small in comparison to the other term in the expansion.

2.8.1 Outer Expansion

We now derive an asymptotic approximation of the solution of (2.139), (2.140). The easiest component to obtain is the outer expansion, and this is determined by simply assuming an expansion of the form

$$y_n \sim \bar{y}_n + \varepsilon \bar{z}_n + \cdots . \tag{2.142}$$

Substituting this into (2.139) and then equating like powers of ε, one finds that

$$\alpha_n \bar{y}_n + \beta_n \bar{y}_{n-1} = 0. \tag{2.143}$$

Based on the observations made earlier, we expect the boundary layer to be at $n = N$. Thus, we require $\bar{y}_0 = a$. Solving (2.143) and using this boundary condition, one finds that

$$\bar{y}_n = \kappa_n a \quad \text{for } n = 0, 1, 2, 3, \ldots, \tag{2.144}$$

where $\kappa_0 = 1$, and for $n \neq 0$

$$\kappa_n = \prod_{j=1}^{n} \left(-\frac{\beta_j}{\alpha_j} \right). \tag{2.145}$$

Except for special values of α_n and β_n, the solution in (2.144) does not satisfy the boundary condition at $n = N$. How to complete the construction of the approximate solution when this happens is the objective of what follows.

2.8.2 Boundary-Layer Approximation

Now the question is, how do we deal with the boundary layer at the right end? To answer this, one needs to remember that (2.144) is a first-term approximation of the solution in the outer region. As given in (2.142), the correction to this approximation in the outer region is $O(\varepsilon)$. The correction at the right end, however, is $O(1)$. This is because the exact solution satisfies $y_N = b$, and (2.144) does not do this.

Our approach to finding the boundary-layer approximation is to first rescale the problem by letting

$$y_n = \varepsilon^{\gamma(n)} Y_n. \tag{2.146}$$

Once we find $\gamma(n)$ and Y_n, the general solution of the original problem will consist of the addition of the two approximations. Specifically, the composite approximation will have the form

$$y_n \sim \bar{y}_n + \varepsilon^{\gamma(n)} \overline{Y}_n, \tag{2.147}$$

where the tilde over the variables indicates the first-term approximation from the respective region. Given that $y_N = b$, and $\bar{y}_n$ does not satisfy this boundary condition, then from (2.147) we will require that

$$\gamma(N) = 0 \tag{2.148}$$

and

$$\overline{Y}_N = b - y_N. \tag{2.149}$$

The exponent $\gamma(n)$ is determined by balancing, and to do this we substitute (2.146) into (2.139) to obtain

$$\underset{①}{\varepsilon^{1+\gamma(n+1)} Y_{n+1}} + \underset{②}{\alpha_n \varepsilon^{\gamma(n)} Y_n} + \underset{③}{\beta_n \varepsilon^{\gamma(n-1)} Y_{n-1}} = 0. \tag{2.150}$$

In the outer region, the balancing takes place between terms ② and ③. For the boundary layer we have two possibilities to investigate:

(i) ① $\sim$ ③ and ② is higher order.
The condition ① $\sim$ ③ requires that $\gamma(n+1) = \gamma(n-1) - 1$. Thus, if $n = 2k$, then $\gamma(2k) = \gamma(0) - k$, and if $n = 2k+1$, then $\gamma(2k+1) = \gamma(1) - k$. In the case where $n = 2k$, we have ①, ③ $= O(\varepsilon^{1+\gamma(1)-k})$, ② $= O(\varepsilon^{\gamma(0)-k})$, and if $n = 2k+1$, then we have ①, ③ $= O(\varepsilon^{\gamma(0)-k})$, ② $= O(\varepsilon^{\gamma(1)-k})$. To be consistent with the assumed balancing, we require that $1 + \gamma(1) \le \gamma(0)$ and $\gamma(0) \le \gamma(1)$. From this it follows that it is not possible to pick values for $\gamma(0)$ and $\gamma(1)$ that are consistent with our original assumption that ② is higher order. Thus, this balance is not possible.

(ii) ① $\sim$ ② and ③ is higher order.
The condition ① $\sim$ ② requires that $\gamma(n+1) = \gamma(n) - 1$, and so $\gamma(n+1) = \gamma(0) - n$. From this it follows that ①, ② $= O(\varepsilon^{1+\gamma(0)-n})$ and ③ $= O(\varepsilon^{2+\gamma(0)-n})$. This is consistent with the original assumption, and so this is the balancing we are looking for.

The balancing argument has shown that $\gamma(n) = 1 + \gamma(0) - n$. From this and (2.148) it follows that $\gamma(0) = N - 1$, and so

$$\gamma(n) = N - n.$$

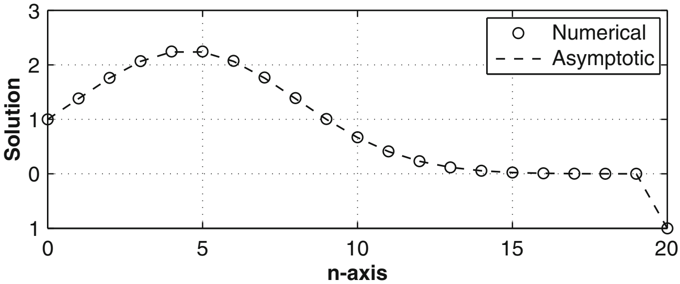

Figure 2.34 Comparison between asymptotic expansion in (2.155) and the numerical solution of (2.139), (2.140) in the case where $\alpha_n = -(1+\frac{n}{N})$, $\beta_n = \frac{3}{2} - \frac{n}{N}$, $a = 1$, $b = -1$, $N = 20$, and $\varepsilon = 10^{-2}$

In this case, (2.150) takes the form

$$Y_{n+1} + \alpha_n Y_n + \varepsilon\beta_n Y_{n-1} = 0 \quad \text{for } n = N-1, N-2, \ldots . \tag{2.151}$$

The appropriate expansion of the boundary-layer solution is

$$Y_n \sim \overline{Y}_n + \varepsilon \overline{Z}_n + \cdots . \tag{2.152}$$

Introducing this into (2.151) yields the equation

$$\overline{Y}_{n+1} + \alpha_n \overline{Y}_n = 0.$$

The solution of this that also satisfies the boundary condition (2.149) is

$$\overline{Y}_n = \lambda_{N-n}(b - \kappa_N a), \tag{2.153}$$

where $\lambda_0 = 1$, and for $k \neq 0$

$$\lambda_k = \prod_{j=1}^{k} \left(-\frac{1}{\alpha_{N-j+1}} \right). \tag{2.154}$$

Using (2.147) we have found that a composite expansion of the solution is

$$y_n \sim \kappa_n a + \varepsilon^{N-n} \lambda_{N-n}(b - \kappa_N a). \tag{2.155}$$

It is possible to prove that this does indeed give us an asymptotic approximation of the solution (Comstock and Hsiao 1976). A demonstration of the accuracy of the approximation is given in Fig. 2.34. It is seen that the numerical solution of the difference equation and the asymptotic expansion are in very close agreement.

2.8.3 Numerical Solution of Differential Equations

There is a very interesting connection between the difference equation in (2.139) and the numerical solution of an associated differential equation. To understand this, consider the boundary value problem

$$\varepsilon y'' + p(x)y' + q(x)y = 0 \quad \text{for } 0 < x < 1, \tag{2.156}$$

where $y(0) = a$ and $y(1) = b$. It is assumed that the functions $p(x)$ and $q(x)$ are continuous. Now, if $p(x) < 0$ for $0 \leq x \leq 1$, then there is a boundary layer at $x = 1$ with width $O(\varepsilon)$ (Exercise 2.10). Given this fact, suppose we want to solve the problem numerically using finite differences. The standard centered-difference approximation will be used for the second derivative, but for the first derivative we will consider using either the forward difference approximation (Holmes, 2007)

$$y'(x_n) \approx \frac{y_{n+1} - y_n}{h} \tag{2.157}$$

or the backward difference approximation

$$y'(x_n) \approx \frac{y_n - y_{n-1}}{h}\,. \tag{2.158}$$

Using the backward difference we get from (2.156) that

$$\varepsilon y_{n+1} + (\alpha_n - 2\varepsilon)y_n + (\beta_n + \varepsilon)y_{n-1} = 0, \tag{2.159}$$

where $\alpha_n = hp_n + h^2 q_n$ and $\beta_n = -hp_n$. Because this equation differs from (2.139) only in the addition of higher-order terms in the coefficients, a composite expansion of the solution is still given in (2.155). Thus, the difference equation has a boundary layer in the same location as the associated differential equation. This is good if one expects the numerical solution to have any resemblance to the solution of the original problem. What is interesting is that the forward difference (2.157) results in a difference equation with a boundary layer at $x = 0$ and not at $x = 1$ (Exercise 2.64). This observation is strong evidence that one should use (2.158) rather than (2.157) to solve this problem.

Another way to approximate the first derivative is to use a centered difference. This would seem to be a better choice because it is a $O(h^2)$ approximation while the approximations in (2.157) and (2.158) are $O(h)$. However, its major limitation is that it cannot delineate a boundary layer at either end of an interval.

Before completing the discussion of the numerical solution of (2.156), it is worth making a comment about the order of the stepsize h. Presumably h should be relatively small for the finite difference equation to be an accurate approximation of the differential equation. This introduces a second small parameter into the problem, and one must be careful about its size in comparison to ε. For example, since $\alpha_n = O(h)$ and $\beta_n = O(h)$, then it should not be unexpected that one must require $\varepsilon \ll h$ to guarantee the expansion is well ordered. We will not pursue this topic, but interested readers should consult the articles by Brown and Lorenz (1987), Farrell (1987), and Linss et al. (2000).

Exercises

2.64. (a) Find a composite expansion of the difference equation

$$\omega_n y_{n+1} + \alpha_n y_n + \varepsilon y_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots, N-1,$$

where $y_0 = a$ and $y_N = b$. Also, the ω_n are nonzero.

(b) Suppose one uses the forward difference approximation given in (2.157) to solve (2.156). Show that you get a difference equation like the one in part (a), and write down the resulting composite expansion of the solution.

(c) At which end does the difference equation you found in part (b) have a boundary layer? What condition should be placed on $p(x)$ so this numerical approximation can be expected to give an accurate approximation of the solution?

2.65. Find a composite expansion of the difference equation

$$\varepsilon y_{n+1} + \alpha_n y_n + \varepsilon\beta_n y_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots, N-1,$$

where $y_0 = a$ and $y_N = b$. It is suggested that you first solve the problem in the case where α_n and β_n are constants. With this you should be able to find the expansion of the solution of the full problem.

2.66. This problem investigates the forward and backward stability of certain difference equations. In what follows, the coefficients α_n, β_n, and ω_n are assumed to be nonzero and bounded.

(a) Consider the initial value problem

$$\varepsilon y_{n+1} + \alpha_n y_n + \beta_n y_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots,$$

where $y_0 = a$ and $y_1 = b$. Explain why the solution of this problem becomes unbounded as n increases. You can do this, if you wish, by making specific choices for the coefficients α_n and β_n.

(b) Consider the initial value problem

$$\omega_n y_{n+1} + \alpha_n y_n + \varepsilon y_{n-1} = 0 \quad \text{for } n = 1, 2, \ldots,$$

where $y_0 = a$ and $y_1 = b$. Explain why the solution of this problem is bounded as n increases. You can do this, if you wish, by making specific choices for the coefficients ω_n and α_n.

(c) Now suppose the problems in parts (a) and (b) are to be solved for $n = 0, -1, -2, -3, \ldots$. Explain why the solution of the problem in (a) is bounded as $n \to -\infty$, but the solution of the problem from (b) is unbounded as $n \to -\infty$. These observations give rise to the statement that the equation in (a) is backwardly stable, while the equation in (b) is forwardly stable. These properties are reminiscent of what is found for the heat equation.

2.67. This problem examines the use of centered-difference approximations to solve the singularly perturbed boundary value problem in (2.156).

(a) Find a first-term composite expansion of the solution of the difference equation

$$(\alpha + \varepsilon)y_{n+1} + 2(\beta - \varepsilon)y_n - (\alpha - \varepsilon)y_{n-1} = 0 \text{ for } n = 1, 2, \ldots, N-1,$$

where $y_0 = a$, $y_N = b$, and $\alpha \neq 0$. Are there any boundary layers for this problem?

(b) Suppose one uses the centered-difference approximation of the first derivative to solve (2.156). Letting $p(x)$ and $q(x)$ be constants, show that you get a difference equation like the one in part (a), and write down the resulting composite expansion of the solution. Which terms in the differential equation do not contribute to the composite expansion?

(c) The solution of (2.156) has a boundary layer at $x = 0$ if $p(x) > 0$ for $0 \leq x \leq 1$ and one at $x = 1$ if $p(x) < 0$ for $0 \leq x \leq 1$. Comment on this and the results from parts (a) and (b).

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