

## Chapter 2

# Algebraic and Data-Analytic Aspects

### 2.1 Introduction

As outlined in Chap. 1, the algebraic structure of natural interest for the dihedral analysis is the dihedral group algebra  $\mathbb{C}D_n$ , defined by the elements

$$x = \sum x_\tau \tau,$$

in one-to-one correspondence with the points  $x$  in  $\mathbb{C}^{2n}$ .

Although the algebraic aspects reviewed in this chapter are focused on the dihedral groups, many related extensions and examples can be found in [2]. Unless stated otherwise, all vector spaces are finite, and their dimensions are assumed to be evaluated over the complex field.

### 2.2 The Dihedral Groups

The dihedral groups  $D_n$  can be objectively introduced as matrix groups of planar rotations and reversals. Specifically, they can be generated by a central counter-clockwise rotation

$$R = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

of  $\phi = 2\pi/n$  radians and a line reflection (say along the horizontal axis)

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

by observing that together, the iterated rotation  $R^j$  and reversal  $R^j H$  matrices for  $n \geq 2$  multiply according to the rules

$$R^n = 1, \quad H^2 = 1, \quad HR^j = R^{-j}H, \quad j = 0, \dots, n-1. \quad (2.1)$$

We observe that the anti-commutativity  $HR = R^{-1}H$  of  $R$  and  $H$  has the effect of producing two equivalent reversal mechanisms distinguished only by the direction of the rotation mechanism.

We briefly recall that an algebraic group is a nonempty set  $G$  equipped with an associative binary operation  $G \times G \rightarrow \sigma\tau \in G$ , an (identity) element  $1 \in G$ , satisfying  $1\tau = \tau 1 = \tau$ , for all  $\tau \in G$  and such that for every  $\tau \in G$ , there is an (inverse) element  $\tau^{-1} \in G$  such that  $\tau\tau^{-1} = \tau^{-1}\tau = 1$ . The dihedral matrices introduced above, for example, give a non-commutative matrix group of order  $2n$  for  $n > 2$ . The dihedral matrix groups can also be generated by any two reflection matrices along lines with an angular separation of  $(2\pi/n)/2$  radians. These angles are often referred to as dihedral angles.

In its abstract formulation, the dihedral group  $D_n$  of order  $2n$  is the (set) orbit

$$C_n \cup C_n h$$

of

$$C_n = \{1, r, r^2, \dots, r^{n-1}\}, \quad n \geq 2,$$

by an involution  $\{1, h\}$ , with multiplication rules given by

$$r^n = 1, \quad h^2 = 1, \quad hr^j = r^{-j}h, \quad j = 0, \dots, n-1. \quad (2.2)$$

*A word about notation:* Denoting

$$\alpha : d \in \{1, -1\} \mapsto \alpha(d) = (1-d)/2 \in \{0, 1\},$$

we may occasionally write

$$r^j h^{\alpha(d)} \equiv \begin{cases} r_j & \text{if } d = 1 \\ t_j & \text{if } d = -1 \end{cases} \quad (2.3)$$

for  $j = 0, \dots, n-1$ , to indicate a generic rotation ( $d = 1$ ) or a reversal ( $d = -1$ ) in  $D_n$ . We observe that the multiplication in  $D_n$  is just the semi-direct product

$$(j, d) \rtimes (j', d') = (j + dj' \mod n, dd') \quad (2.4)$$

in  $\mathbb{Z}_n \times \mathbb{Z}_2$ . Similarly, we shall write

$$\beta_\tau = R^j H^{\alpha(d)}, \quad j = 0, \dots, n-1, \quad d = \pm 1,$$

to indicated the corresponding rotation and reversal matrices, and their corresponding harmonics

$$\beta_\tau^k = R^{jk \bmod n} H^{\alpha(d)} \quad k = 1, 2, \dots, m = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd,} \end{cases} \quad (2.5)$$

understanding that  $\beta^1 = \beta$ .

## 2.3 $D_n$ Conjugacy Orbits

Writing

$$[\eta] = \{\tau\eta\tau^{-1}, \tau \in D_n\}$$

to indicate the conjugacy orbit of  $\eta \in D_n$  and using the defining relations  $h^2=1$ ,  $hr^j = r^{-j}h$  for  $D_n$ , it directly follows that  $r^{j_1}(r^{j_2})r^{-j_1} = r^{j_2}$ , and  $r^{j_1}h(r^{j_2})hr^{-j_1} = r^{-j_2}$ , so that

$$[r^j] = \{r^j, r^{-j}\}$$

is the conjugacy orbit of  $r^j$ . In particular,  $[1] = \{1\}$ . Moreover,

$$r^{j_1}(r^j h)r^{-j_1} = r^{2j_1+j}h, \text{ and } r^{j_1}h(r^j h)hr^{-j_1} = r^{2j_1-j}h,$$

so that one obtains, for  $n$  even

$$[r^j h] = \begin{cases} \{r^{2j_1}h : j_1 = 0, \dots, n-1\}, & \text{for } j \text{ even;} \\ \{r^{2j_1+1}h : j_1 = 0, \dots, n-1\}, & \text{for } j \text{ odd,} \end{cases}$$

whereas, for  $n$  odd, the reversals are self-conjugate:

$$[r^j h] = \{r^j h : j = 0, \dots, n-1\}.$$

To illustrate, if  $n = 6$ , the conjugacy orbits are

$$\{1\}, \{r, r^5\}, \{r^2, r^4\}, \{r^3\}, \{h, r^2h, r^4h\}, \{rh, r^3h, r^5h\},$$

whereas, if  $n = 7$ , the orbits are

$$\{1\}, \{r, r^6\}, \{r^2, r^5\}, \{r^3, r^4\}, \{h, rh, \dots, r^6h\}.$$

## 2.4 Dihedral Sets and Modules

We say that a set  $S$  is a *dihedral set* if there is a mapping

$$(\tau, s) \in D_n \times S \mapsto \tau s \in S$$

such that  $1s = s$  and  $\sigma(\tau s) = (\sigma\tau)s$  for all  $\sigma, \tau \in D_n$ , and  $s \in S$ . In this case we also say that  $D_n$  acts on  $S$ . We say that the action is *opposite* if  $\sigma(\tau s) = (\tau\sigma)s$  for all  $\sigma, \tau \in D_n$ , and  $s \in S$ .

If a vector space  $V$  is a finite dihedral set and

$$\tau(x + y) = \tau x + \tau y, \quad \tau(\lambda x) = \lambda(\tau x),$$

for all  $x, y \in V$  and scalars  $\lambda$ , then we say that the dihedral action is *linear*, and that  $V$  is a *dihedral representation space* or a *dihedral module*. In this case we also say that  $x \mapsto \tau x$  is a *dihedral representation*. Its dimension is equal to the dimension of the module  $V$  as a complex vector space.

Clearly  $D_n$  is itself a dihedral set under the (left) *regular action*

$$\sigma \mapsto \tau\sigma,$$

that extends to an opposite linear action

$$\tau x = \tau \sum_{\sigma} x_{\sigma} \sigma = \sum_{\sigma} x_{\tau^{-1}\sigma} \sigma \quad (2.6)$$

on  $\mathbb{C}D_n$ , thus affording the group algebra  $\mathbb{C}D_n$  with a (left) module structure of dimension  $2n$ . We may refer to  $\mathbb{C}D_n$  as the *dihedral regular module*. The corresponding facts apply to the right regular action  $\sigma \mapsto \sigma\tau^{-1}$ , and unless otherwise stated we will assume all modules to be left modules.

From (2.6), with  $y, x \in \mathbb{C}D_n$ , we have

$$yx = \left( \sum_{\tau} y_{\tau} \tau \right) x = \sum_{\tau} y_{\tau} (\tau x) = \sum_{\tau} y_{\tau} \left( \sum_{\sigma} x_{\tau^{-1}\sigma} \sigma \right) = \sum_{\sigma} \left( \sum_{\tau} y_{\tau} x_{\tau^{-1}\sigma} \right) \sigma = \sum_{\sigma} (y * x)_{\sigma} \sigma,$$

so that the multiplication  $xy$  in  $\mathbb{C}D_n$  is given by the group convolution  $x * y$ , allowing us to write

$$xy = x * y.$$

Also from (2.6), the matrix form of the regular representation  $x \mapsto \tau x$  is a  $2n \times 2n$  permutation matrix, indicated from now on by  $\phi$ , with entries

$$(\phi_{\tau})_{\sigma\eta} = 1 \iff \tau\sigma = \eta.$$

Direct calculation then shows that, for all  $\tau, \nu \in D_n$ ,

$$\phi_{\tau}\phi_{\nu} = \phi_{\nu\tau}, \quad (2.7)$$

so that  $\phi$  is a dihedral *anti-homomorphism*, whereas its transpose  $\phi'$  gives the corresponding dihedral homomorphism. These notions are equivalent, and will generally be referred as dihedral homomorphisms.

We say that  $V$  is a *submodule* of  $\mathbb{C}D_n$  to indicate that  $V$  is a subspace of  $\mathbb{C}D_n$  where  $D_n$  acts linearly. Consequently, all submodules are *stable* subspaces in the sense that  $\tau x \in V$  for all  $x \in V$  and  $\tau \in D_n$ .

If  $V$  is a dihedral submodule in dimension of  $m$ , we write  $\rho_\tau$  to indicate the matrix form (in a given basis) of a generic dihedral (linear) action on  $V$ , so that then  $\rho_\tau$  in a point in the vector space  $\in GL_m(\mathbb{C})$  of all invertible  $m \times m$  linear mappings over  $\mathbb{C}$ . By definition, then,  $\rho$  is dihedral homomorphism.

If  $\rho$  is a dihedral homomorphism then clearly  $F\rho F^{-1}$  is also a dihedral homomorphism, for all  $F \in GL_m(\mathbb{C})$ . In that case we say that the dihedral homomorphisms  $\rho$  and  $F\rho F^{-1}$  are *equivalent*, and write  $\rho \simeq F\rho F^{-1}$  to indicate the equivalence.

Every module has at least two submodules: itself and the zero submodule  $\{0\}$ . A module with no other submodule is called a *simple* module. Clearly, one-dimensional modules are simple.

**Definition 2.1 (Dihedral linearizations).** Given a dihedral homomorphism  $\rho$  on  $GL_m(\mathbb{C})$  and  $x \in \mathbb{C}D_n$ , the evaluation

$$\langle x, \rho \rangle = \sum x_\tau \rho_\tau \in M_m(\mathbb{C})$$

is called a linearization of  $\mathbb{C}D_n$  in  $M_m(\mathbb{C})$ . In particular,  $\langle x, \phi \rangle$  is called the regular linearization of  $x$ .

**Proposition 2.1.** For all dihedral homomorphisms  $\rho, \eta, \xi$ , all  $x, y \in \mathbb{C}D_n$ , and scalars  $\lambda$ , we have:

1. If  $\rho \simeq \eta \oplus \xi$  then  $\langle x, \rho \rangle \simeq \langle x, \eta \rangle \oplus \langle x, \xi \rangle$ ;
2.  $\langle x + y, \rho \rangle = \langle x, \rho \rangle + \langle y, \rho \rangle$ ;
3.  $\lambda \langle x, \rho \rangle = \langle \lambda x, \rho \rangle$ ;
4.  $\langle xy, \rho \rangle = \langle x, \rho \rangle \langle y, \rho \rangle = \langle x * y, \rho \rangle$ .

*Proof.* The proof is by direct evaluation. For the equalities in 4, we recall that

$$xy = \sum_{\tau, \sigma} x_\tau y_\sigma \tau \sigma = \sum_{\tau} \left( \sum_{\sigma} x_\sigma y_{\sigma^{-1}\tau} \right) \tau = \sum_{\tau} (x * y)_\tau \tau = x * y,$$

as introduced earlier on page 14. □

Decompositions such as  $\langle x, \rho \rangle \simeq \langle x, \eta \rangle \oplus \langle x, \xi \rangle$  appearing in Proposition 2.1 are often referred to as product of matrix algebras, e.g. [3, p.79], [4, p.48].

We also remark that, more generally, the notation  $\rho \simeq m\eta \oplus \dots \oplus n\xi$  indicates that there is a basis in the representation space of  $\rho$  relative to which

$$\rho_\tau = \text{Diag} (I_m \otimes \eta_\tau, \dots, I_n \otimes \xi_\tau), \quad \tau \in G,$$

where  $I_r$  indicates the  $r \times r$  identity matrix.

**Proposition 2.2.** *Let, for  $n \geq 2$ , and  $(j, d) \equiv \tau \in D_n$ ,*

- $1 : (j, d) \mapsto 1$ ;
- $\alpha : (j, d) \mapsto d$ ;
- $\gamma_+ : (j, d) \mapsto (-1)^j$ , for  $n$  even;
- $\gamma_- : (j, d) \mapsto d(-1)^j$ , for  $n$  even.

*Then, for  $\xi \in \{1, \alpha, \gamma_+, \gamma_-\}$  and  $x$  in  $\mathbb{C}D_n$ , the linearizations  $\langle x, \xi \rangle$  are simple dihedral submodules of  $\mathbb{C}D_n$ .*

*Proof.* The result follows by directly showing that each  $\xi \in \{1, \alpha, \gamma_+, \gamma_-\}$  is a one-dimensional dihedral homomorphism, so that  $\langle x, \xi \rangle$  is simple.  $\square$

*Example 2.1.* The simple  $D_3$  submodules described in Proposition 2.2:

- $\langle x, 1 \rangle = \sum_\tau 1_\tau x_\tau = x_1 + x_r + x_{r^2} + x_h + x_{rh} + x_{r^2h}$ ;
- $\langle x, \alpha \rangle = \sum_\tau \alpha_\tau x_\tau = x_1 + x_r + x_{r^2} - x_h - x_{rh} - x_{r^2h}$ .

*Example 2.2.* The simple  $D_4$  submodules described in Proposition 2.2:

- $\langle x, 1 \rangle = \sum_\tau 1_\tau x_\tau = x_1 + x_r + x_{r^2} + x_{r^3} + x_h + x_{rh} + x_{r^2h} + x_{r^3h}$ ;
- $\langle x, \alpha \rangle = \sum_\tau \alpha_\tau x_\tau = x_1 + x_r + x_{r^2} + x_{r^3} - x_h - x_{rh} - x_{r^2h} - x_{r^3h}$ ;
- $\langle x, \gamma_+ \rangle = \sum_\tau \gamma_\tau^+ x_\tau = x_1 - x_r + x_{r^2} - x_{r^3} + x_h - x_{rh} + x_{r^2h} - x_{r^3h}$ ;
- $\langle x, \gamma_- \rangle = \sum_\tau \gamma_\tau^- x_\tau = x_1 - x_r + x_{r^2} - x_{r^3} - x_h + x_{rh} - x_{r^2h} + x_{r^3h}$ .

*Example 2.3.* Simple one-dimensional modules are constant over conjugacy orbits. The following is the evaluation of the simple dihedral modules described in Proposition 2.2 over the conjugacy orbits

$$\{1\}, \{r, r^5\}, \{r^2, r^4\}, \{r^3\}, \{h, r^2h, r^4h\}, \{rh, r^3h, r^5h\},$$

of  $D_6$ , from Sect. 2.3:

$$1 : \{+\}, \{+\}, \{+\}, \{+\}, \{+\}, \{+\},$$

$$\alpha : \{+\}, \{+\}, \{+\}, \{+\}, \{-\}, \{-\},$$

$$\gamma_+ : \{+\}, \{-\}, \{+\}, \{-\}, \{+\}, \{-\},$$

$$\gamma_- : \{+\}, \{-\}, \{+\}, \{-\}, \{-\}, \{+\}.$$

Similarly, over the conjugacy orbits

$$\{1\}, \{r, r^6\}, \{r^2, r^5\}, \{r^3, r^4\}, \{h, rh, \dots, r^6h\}.$$

of  $D_7$ , we obtain:

$$\begin{aligned} 1 &: \{+\}, \{+\}, \{+\}, \{+\}, \{+\}, \\ \alpha &: \{+\}, \{+\}, \{+\}, \{+\}, \{-\}. \end{aligned}$$

**Proposition 2.3 (Orbit invariance).** *If  $\rho$  is a dihedral homomorphism and  $x \in \mathbb{C}D_n$  then, for all  $\tau, \sigma \in D_n$ ,*

$$\langle \tau x, \rho \rangle = \rho_\tau \langle x, \rho \rangle, \quad \langle x\sigma, \rho \rangle = \langle x, \rho \rangle \rho_\sigma.$$

*Proof.* From (2.6), we have,

$$\langle \tau x, \rho \rangle = \sum_{\sigma} x_{\tau^{-1}\sigma} \rho_\sigma = \sum_{\gamma} x_{\gamma} \rho_{\tau\gamma} = \sum_{\gamma} x_{\gamma} \rho_{\tau} \rho_{\gamma} = \rho_{\tau} \langle x, \rho \rangle,$$

and, similarly,  $\langle x\sigma, \rho \rangle = \sum_{\tau} x_{\tau\sigma^{-1}} \rho_{\tau} = \sum_{\gamma} x_{\gamma} \rho_{\gamma\sigma} = \langle x, \rho \rangle \rho_{\sigma}.$  □

In particular

$$\langle \tau x \tau^{-1}, \rho \rangle = \rho_{\tau} \langle x, \rho \rangle \rho_{\tau^{-1}}.$$

**Proposition 2.4.** *The row (column) spaces of the linearizations  $\langle x, \rho \rangle$  are dihedral left (right) submodules of  $\mathbb{C}D_n$ .*

*Proof.* Observing that the row (column)  $j$  of a matrix product  $AB$  is a linear combination of the rows (columns) of  $B$  ( $A$ ), with coefficients the row (column)  $j$  of  $A$  ( $B$ ), the result follows from Proposition 2.3, which defines the actions

$$y\sigma = y\rho_{\sigma}, \quad \sigma \in D_n,$$

on the right for every  $y$  in the column space of  $\langle x, \rho \rangle$ , and

$$\tau y = \rho_{\tau} y, \quad \tau \in D_n$$

on the left, for every  $y$  in the row space of  $\langle x, \rho \rangle$ . □

It then follows, from Proposition 2.4 that the column (row) space of the regular linearization  $\langle x, \phi \rangle$  is a dihedral left (right) submodule of  $\mathbb{C}D_n$ .

**Proposition 2.5.**  *$\mathbb{C}D_n$  allows for  $m = n/2 - 1$  distinct two-dimensional submodules for  $n$  even and  $m = (n - 1)/2$  submodules for  $n$  odd,  $n \geq 2$ .*

*Proof.* Direct verification shows that the  $\beta^k$  defined in (2.5) are dihedral homomorphisms, so that the corresponding linearizations

$$\langle x, \beta^k \rangle, \quad k = 1, 2, \dots, m = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd, } x \in \mathbb{C}D_n, \end{cases}$$

give two-dimensional submodules according to Proposition 2.4.  $\square$

**Proposition 2.6.** *The two-dimensional submodules  $\langle x, \beta^1 \rangle, \dots, \langle x, \beta^m \rangle$  are simple.*

*Proof.* Since  $\langle \tau x, \beta^k \rangle$  reduces as  $\beta_\tau^k \langle x, \beta^k \rangle$ , it is sufficient to study the  $\mathbb{C}$ -reducibility of  $\beta^k$  in  $D_n$ . Since the eigenvectors for (all) the rotation components of  $\beta^k$  in  $D_n$  are  $(i, 1)$  and  $(-i, 1)$  for all  $k, n$ , it then follows that the only proper subspace of the (column) spaces of  $\langle x, \beta^k \rangle$  is the null space, completing the proof.  $\square$

As a consequence, referring to Proposition 2.2 on page 16, the sum of the squares of the corresponding dimensions in

$$\widehat{D}_n = \{1, \alpha, \gamma_+, \gamma_-, \beta_1, \dots, \beta_m\} \quad (2.8)$$

is equal to  $4 + (n/2 - 1)2^2 = 2n$  when  $n$  is even, and is equal to  $2 + ((n-1)/2)2^2 = 2n$  in

$$\widehat{D}_n = \{1, \alpha, \beta_1, \dots, \beta_m\} \quad (2.9)$$

when  $n$  is odd. Therefore, e.g. [2, Prop. 4.2], these are exactly the simple modules of  $\mathbb{C}D_n$ . As a result, we have:

**Proposition 2.7.**  *$\mathbb{C}D_n$  as a semi-simple module factors as*

$$\mathbb{C}D_n \simeq \langle x, 1 \rangle \oplus \langle x, \alpha \rangle \oplus \langle x, \gamma_+ \rangle \oplus \langle x, \gamma_- \rangle \bigoplus_{k=1}^{n/2-1} \langle x, \beta_k \rangle$$

when  $n$  is even and as

$$\mathbb{C}D_n \simeq \langle x, 1 \rangle \oplus \langle x, \alpha \rangle \bigoplus_{k=1}^{(n-1)/2} \langle x, \beta_k \rangle$$

when  $n$  is odd,  $n \geq 2$ .  $\square$

**Definition 2.2 (Dihedral Fourier Transforms).** Each simple module  $\langle x, \xi \rangle$  in the decomposition of  $\mathbb{C}D_n$  given by Proposition 2.7 is called the Fourier transform of  $x$  at the corresponding representation  $\xi$ .

*Additional Notation:* From now on we assume that, for  $n \geq 2$ ,

$$m = \begin{cases} \frac{n}{2} - 1 & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$



Here are a few examples of dihedral decompositions, following Proposition 2.7.

*Example 2.4.*  $\mathbb{C}D_2$  gives four simple modules all in dimension of 1:

$$\begin{aligned} \langle x, 1 \rangle &= x_1 + x_r + x_h + x_{rh}; \\ \langle x, \alpha \rangle &= x_1 + x_r - x_h - x_{rh}; \\ \langle x, \gamma_+ \rangle &= x_1 - x_r + x_h - x_{rh}; \\ \langle x, \gamma_- \rangle &= x_1 - x_r - x_h + x_{rh}. \end{aligned}$$

*Example 2.5.*  $\mathbb{C}D_3$  has two simple modules in dimension of 1:

$$\begin{aligned} \langle x, 1 \rangle &= x_1 + x_r + x_{r^2} + x_h + x_{rh} + x_{r^2h}; \\ \langle x, \alpha \rangle &= x_1 + x_r + x_{r^2} - x_h - x_{rh} - x_{r^2h}; \end{aligned}$$

and one in dimension of 2:

$$\langle x, \beta \rangle = \frac{1}{2} \begin{bmatrix} 2x_1 - x_r - x_{r^2} + 2x_h - x_{rh} - x_{r^2h} & \sqrt{3}(-x_r + x_{r^2} + x_{rh} - x_{r^2h}) \\ \sqrt{3}(x_r - x_{r^2} + x_{rh} - x_{r^2h}) & 2x_1 - x_r - x_{r^2} - 2x_h + x_{rh} + x_{r^2h} \end{bmatrix}.$$

*Example 2.6.*  $\mathbb{C}D_4$  gives four simple modules in dimension of 1 and one in dimension of 2:

$$\begin{aligned} \langle x, 1 \rangle &= x_1 + x_r + x_{r^2} + x_{r^3} + x_h + x_{rh} + x_{r^2h} + x_{r^3h}; \\ \langle x, \alpha \rangle &= x_1 + x_r + x_{r^2} + x_{r^3} - x_h - x_{rh} - x_{r^2h} - x_{r^3h}; \\ \langle x, \gamma_+ \rangle &= x_1 - x_r + x_{r^2} - x_{r^3} + x_h - x_{rh} + x_{r^2h} - x_{r^3h}; \\ \langle x, \gamma_- \rangle &= x_1 - x_r + x_{r^2} - x_{r^3} - x_h + x_{rh} - x_{r^2h} + x_{r^3h}; \\ \langle x, \beta \rangle &= \begin{bmatrix} x_1 - x_{r^2} + x_h - x_{r^2h} & -x_r + x_{r^3} + x_{rh} - x_{r^3h} \\ x_r - x_{r^3} + x_{rh} - x_{r^3h} & x_1 - x_{r^2} - x_h + x_{r^2h} \end{bmatrix}. \end{aligned}$$

*Example 2.7.*  $\mathbb{C}D_6$  allows for four simple modules in dimension of 1:

$$\begin{aligned} \langle x, 1 \rangle &= x_1 + x_r + x_{r^2} + x_{r^3} + x_{r^4} + x_{r^5} + x_h + x_{rh} + x_{r^2h} + x_{r^3h} + x_{r^4h} + x_{r^5h} \\ \langle x, \alpha \rangle &= x_1 + x_r + x_{r^2} + x_{r^3} + x_{r^4} + x_{r^5} - x_h - x_{rh} - x_{r^2h} - x_{r^3h} - x_{r^4h} - x_{r^5h} \\ \langle x, \gamma_+ \rangle &= x_1 - x_r + x_{r^2} - x_{r^3} + x_{r^4} - x_{r^5} + x_h - x_{rh} + x_{r^2h} - x_{r^3h} + x_{r^4h} - x_{r^5h} \\ \langle x, \gamma_- \rangle &= x_1 - x_r + x_{r^2} - x_{r^3} + x_{r^4} - x_{r^5} - x_h + x_{rh} - x_{r^2h} + x_{r^3h} - x_{r^4h} + x_{r^5h}; \end{aligned}$$

and two simple modules in dimension of 2. The matrix entries of  $\langle x, \beta_1 \rangle$  are:

$$\begin{aligned}
\langle x, \beta_1 \rangle_{1,1} &= \frac{1}{2} (2x_1 + x_r - x_{r^2} - 2x_{r^3} - x_{r^4} + x_{r^5} + 2x_h + x_{rh} - x_{r^2h} - 2x_{r^3h} - x_{r^4h} + x_{r^5h}) \\
\langle x, \beta_1 \rangle_{2,1} &= \frac{1}{2} \sqrt{3} (-x_r - x_{r^2} + x_{r^4} + x_{r^5} + x_{rh} + x_{r^2h} - x_{r^4h} - x_{r^5h}) \\
\langle x, \beta_1 \rangle_{1,2} &= \frac{1}{2} \sqrt{3} (x_r + x_{r^2} - x_{r^4} - x_{r^5} + x_{rh} + x_{r^2h} - x_{r^4h} - x_{r^5h}) \\
\langle x, \beta_1 \rangle_{2,2} &= \frac{1}{2} (2x_1 + x_r - x_{r^2} - 2x_{r^3} - x_{r^4} + x_{r^5} - 2x_h - x_{rh} + x_{r^2h} + 2x_{r^3h} + x_{r^4h} - x_{r^5h}),
\end{aligned}$$

whereas the matrix entries of  $\langle x, \beta_2 \rangle$  are given by

$$\begin{aligned}
\langle x, \beta_2 \rangle_{1,1} &= \frac{1}{2} (2x_1 - x_r - x_{r^2} + 2x_{r^3} - x_{r^4} - x_{r^5} + 2x_h - x_{rh} - x_{r^2h} + 2x_{r^3h} - x_{r^4h} - x_{r^5h}) \\
\langle x, \beta_2 \rangle_{2,1} &= \frac{1}{2} \sqrt{3} (-x_r + x_{r^2} - x_{r^4} + x_{r^5} + x_{rh} - x_{r^2h} + x_{r^4h} - x_{r^5h}) \\
\langle x, \beta_2 \rangle_{1,2} &= \frac{1}{2} \sqrt{3} (x_r - x_{r^2} + x_{r^4} - x_{r^5} + x_{rh} - x_{r^2h} + x_{r^4h} - x_{r^5h}) \\
\langle x, \beta_2 \rangle_{2,2} &= \frac{1}{2} (2x_1 - x_r - x_{r^2} + 2x_{r^3} - x_{r^4} - x_{r^5} - 2x_h + x_{rh} + x_{r^2h} - 2x_{r^3h} + x_{r^4h} + x_{r^5h}).
\end{aligned}$$

## 2.5 Class Functions

A point  $x \in \mathbb{C}D_n$  that is constant on the components of a dihedral conjugacy orbit is called a dihedral *class function*. More precisely,  $x$  is such that  $x_{\tau\sigma\tau^{-1}} = x_\sigma$ , for all  $\sigma, \tau \in D_n$ .

**Proposition 2.8.** *If  $x \in \mathbb{C}D_n$  is a class function then the dihedral linearizations  $\langle x, \rho \rangle$  commute with  $\rho_\tau$  for all  $\tau \in D_n$ .*

*Proof.* In fact,

$$\rho_\tau \langle x, \rho \rangle \rho_{\tau^{-1}} = \langle x, \rho_\tau \rho \rho_{\tau^{-1}} \rangle = \sum_{\sigma} x_\sigma \rho_{\tau\sigma\tau^{-1}} = \sum_{\sigma} x_{\tau\sigma\tau^{-1}} \rho_{\tau\sigma\tau^{-1}} = \langle x, \rho \rangle,$$

completing the proof.  $\square$

**Definition 2.3 (Dihedral characters).** The character of  $\xi \in \widehat{D}_n$  is the point  $\chi^\xi = \text{tr } \xi = \sum_{\tau} (\text{tr } \xi_\tau) \tau \in \mathbb{C}D_n$ .

Therefore, given  $\xi, \eta \in \widehat{D}_n$ , and since  $\chi^\xi$  is a class function, Schur's Lemma implies that

$$\langle \chi^\xi, \eta \rangle = \lambda I_{n_\eta},$$

so that taking the trace in each side of the equality and solving for  $\lambda$  we obtain

$$\langle \chi^\xi, \eta \rangle = \frac{1}{n_\eta} \langle \chi^\xi, \chi^\eta \rangle I_{n_\eta}. \quad (2.10)$$

Clearly, the dihedral characters are simply

$$\chi^\xi = \xi, \quad \text{if } n_\xi = 1$$

or else, when  $\xi = \beta^k$ ,

$$\chi_{j,d}^k = \text{tr } \beta_{j,d}^k = (1+d) \cos \frac{2\pi jk}{n}, \quad k = 1, \dots, m.$$

It immediately follows that

**Proposition 2.9 (Orthogonality of dihedral characters).** *For all  $\chi, \eta$  in  $\widehat{D}_n$  we have*

$$\frac{1}{2n} \sum_{\tau} \chi_{\tau}^{\xi} \chi_{\tau}^{\eta} = \delta_{\chi, \eta},$$

and, moreover,

$$\frac{1}{2n} \sum_{\xi} \chi_1^{\xi} \chi_{\tau}^{\xi} = \begin{cases} \sum_{\xi} \frac{n_{\xi}^2}{2n} = 1, & \text{if } \tau = 1, \\ 0, & \text{if } \tau \neq 1. \end{cases}$$

## 2.6 Dihedral Projections

Define, for  $\xi \in \widehat{D}_n$ ,

$$\pi_{\xi} = \frac{n_{\xi}}{2n} \chi^{\xi} \in \mathbb{C}D_n.$$

**Proposition 2.10.** *For all  $\xi, \eta \in \widehat{D}_n$ , the following properties hold:*

1.  $\pi_{\xi}^2 = \pi_{\xi}$  (projection);
2.  $\pi_{\xi} \pi_{\eta} = 0$ ,  $\eta \neq \xi$  (algebraic orthogonal);
3.  $\sum_{\xi} \pi_{\xi} = 1$  (partition of unity).

*Proof.* To verify the above properties we recall that, for  $x, y \in \mathbb{C}D_n$ ,

$$x y = \sum_{\tau, \sigma} x_{\tau} y_{\sigma} \tau \sigma = \sum_{\tau} \left( \sum_{\sigma} x_{\sigma} y_{\sigma^{-1} \tau} \right) \tau = \sum_{\tau} (x * y)_{\tau} \tau = x * y,$$

as introduced earlier on page 14. In each case, it is then sufficient to evaluate the convolution components  $\sum_{\sigma} x_{\sigma} y_{\sigma^{-1}\tau}$ . With the semi-direct product (2.4) notation

$$(j, d) \rtimes (j', d') = (j + dj' \mod n, dd')$$

in mind, we identify  $(j, d)_{\xi} \equiv \pi_{\tau}^{\xi}$  and write

- $(j, d)_1 = 1/2n$ ;
- $(j, d)_{\alpha} = d/2n$ ;
- $(j, d)_{\gamma_+} = (-1)^j/2n$ , for  $n$  even;
- $(j, d)_{\gamma_-} = d(-1)^j/2n$ , for  $n$  even;
- $(j, d)_{\beta k} = 2\cos\left[\frac{2jk\pi}{n}\right]/n$ ; if  $d = 1$ , and  $(j, d)_{\beta k} = 0$ , if  $d = -1$ ;  $k = 1, \dots, m$ .

Therefore

$$\begin{aligned} x_{\sigma} y_{\sigma^{-1}\tau} &= (j, d)_x [(j, d)_y^{-1} \rtimes (j', d')_y] = (j, d)_x [(-j/d, d)_y \rtimes (j', d')_y] \\ &= (j, d)_x (-j/d + dj' \mod n, dd')_y, \end{aligned}$$

and

$$\sum_{\sigma} x_{\sigma} y_{\sigma^{-1}\tau} = \sum_{j, d} (j, d)_x (-j/d + dj' \mod n, dd')_y \quad (2.11)$$

then gives the  $\tau = (j', d')$  component of  $x * y$ . The proof of the three properties follows from directly evaluating (2.11) in each of the cases. For the projection property we have:

$$(j', d')_{\pi_1^2} = \sum_{j, d} (j, d)_1 (-j/d + dj' \mod n, dd')_1 = \sum_{j, d} \frac{1}{2n} \frac{1}{2n} = \frac{1}{2n} = (j', d')_{\pi_1},$$

for all  $(j', d')$ , that is,  $\pi_1^2 = \pi_1$ . Similarly,

$$\begin{aligned} (j', d')_{\pi_{\alpha}^2} &= \sum_{j, d} (j, d)_{\alpha} (-j/d + dj' \mod n, dd')_{\alpha} \\ &= \sum_{j, d} \frac{d}{2n} \frac{dd'}{2n} = \sum_{j, d} \frac{d'}{4n^2} = \frac{d'}{2n} = (j', d')_{\pi_{\alpha}}, \end{aligned}$$

for all  $(j', d')$ , that is,  $\pi_{\alpha}^2 = \pi_{\alpha}$ ;

$$\begin{aligned} (j', d')_{\pi_{\gamma_+}^2} &= \sum_{j, d} (j, d)_{\gamma_+} (-j/d + dj' \mod n, dd')_{\gamma_+} \\ &= \sum_{j, d} \frac{(-1)^j}{2n} \frac{(-1)^{j/d + dj'}}{2n} \\ &= \frac{1}{4n^2} \left[ \sum_j (-1)^{j'} + \sum_j (-1)^{2j - j'} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4n^2} [n(-1)^{j'} + n(-1)^{-j' \pmod n}] \\
&= \frac{2n(-1)^{j'}}{2n^2} = \frac{(-1)^{j'}}{2n} = (j', d') \pi_{\gamma_+};
\end{aligned}$$

for all  $(j', d')$ , that is,  $\pi_{\gamma_+}^2 = \pi_{\gamma_+}$ ;

$$\begin{aligned}
(j', d') \pi_{\gamma_-}^2 &= \sum_{j,d} (j, d)_{\gamma_-} (-j/d + dj' \pmod n, dd')_{\gamma_-} \\
&= \sum_{j,d} \frac{d(-1)^j}{2n} \frac{dd'(-1)^{j/d+dj'}}{2n} \\
&= d' \sum_{j,d} \frac{(-1)^j}{2n} \frac{(-1)^{j/d+dj'}}{2n} = \frac{d'(-1)^{j'}}{2n} = (j', d') \pi_{\gamma_-};
\end{aligned}$$

for all  $(j', d')$ , that is,  $\pi_{\gamma_-}^2 = \pi_{\gamma_-}$ , and similarly, noting that  $(j, -1)_{\beta^k} = 0$  for all  $j = 0, \dots, n-1$ ,

$$\begin{aligned}
(j', d') \pi_{\beta^k}^2 &= \sum_j (j, 1)_{\beta^k} (-j + j' \pmod n, 1)_{\beta^k} \\
&= \sum_j \frac{4}{n^2} \text{Cos} \left[ \frac{2jk\pi}{n} \right] \text{Cos} \left[ \frac{2(j-j')k\pi}{n} \right] \\
&= \frac{2}{n} \text{Cos} \left[ \frac{2j'k\pi}{n} \right] = (j', 1)_{\beta^k},
\end{aligned}$$

and  $(j', d') \pi_{\beta^k}^2 = 0$  if  $d' = 0$ , so that  $(j', d') \pi_{\beta^k}^2 = (j', d') \pi_{\beta^k}$  for all  $j', d'$ . The derivations proving the orthogonality property are similar and are left as an exercise. To prove the partition of unit property  $\sum_{\xi} \pi_{\xi}$  we first assume that  $n$  is odd. Then, if  $(j, d) = (0, 1)$ , we have:

$$(0, 1)_1 + (0, 1)_{\alpha} + \sum_{k=1}^m (0, 1)_{\beta^k} = \frac{1}{2n} + \frac{1}{2n} + \frac{n-1}{n} = 1,$$

that is  $(\sum_{\xi} \pi_{\xi})_1 = 1$ . If  $j > 1$  and  $d = 1$  we have,

$$(j, 1)_1 + (j, 1)_{\alpha} + \sum_{k=1}^m (j, 1)_{\beta^k} = \frac{2}{2n} - \frac{1}{n} = 0,$$

whereas if  $j > 1$  and  $d = -1$  we have,

$$(j, -1)_1 + (j, -1)_{\alpha} + \sum_{k=1}^m (j, -1)_{\beta^k} = \frac{1}{2n} - \frac{1}{2n} + 0 = 0,$$

that is,  $(\sum_{\xi} \pi_{\xi})_j = 0$ , if  $j \neq 1$ , and hence  $\sum_{\xi} \pi_{\xi} = 1$ . If  $n$  is even, we have the additional terms

$$(j, d)_{\gamma_+} + (j, d)_{\gamma_-} = \begin{cases} \frac{2}{2n}, & \text{if } (j, d) = (0, 1); \\ \frac{2}{2n}(-1)^j, & \text{if } j > 1, d = 1; \\ 0 & \text{if } j > 1, d = -1, \end{cases}$$

whereas, for  $n$  even,

$$\sum_{k=1}^m (j, 1)_{\beta^k} = \begin{cases} \frac{n-2}{n} & \text{if } j = 0; \\ -\frac{2}{n} & \text{if } j > 1 \text{ is even}; \\ 0 & \text{if } j > 1 \text{ is odd}. \end{cases}$$

Therefore, if  $(j, d) = (0, 1)$ , we have:

$$\begin{aligned} \sum_{\xi} (0, 1)_{\xi} &= (0, 1)_1 + (0, 1)_{\alpha} + (j, d)_{\gamma_+} + (j, d)_{\gamma_-} + \sum_{k=1}^m (0, 1)_{\beta^k} \\ &= \frac{4}{2n} + \frac{n-2}{n} = 1, \end{aligned}$$

so that  $(\sum_{\xi} \pi_{\xi})_1 = 1$ ; If  $j > 1$  is even, we have

$$\sum_{\xi} (j, 1)_{\xi} = \frac{4}{2n} - \frac{2}{n} = 0;$$

whereas if  $j > 1$  is odd,

$$\sum_{\xi} (j, 1)_{\xi} = \frac{2}{2n} - \frac{2}{2n} = 0.$$

Therefore,  $(\sum_{\xi} \pi_{\xi})_j = 0$  for all  $j > 1$  and  $d = 1$ . Finally, if  $d = -1$ ,

$$\sum_{\xi} (j, -1)_{\xi} = \frac{1}{2n} - \frac{1}{2n} + \frac{(-1)^j}{2n} - \frac{(-1)^j}{2n} = 0,$$

and hence  $\sum_{\xi} \pi_{\xi} = 1$ , concluding the proof.  $\square$

**Proposition 2.11 (Canonical Projections).** Let  $\mathcal{P}_{\xi} = \langle \pi_{\xi}, \phi \rangle$ , where  $\phi$  is the dihedral  $(D_n)$  regular homomorphism. Then, for all  $\xi, \eta \in \widehat{D}_n$ , we have:

1.  $\mathcal{P}_{\xi}^2 = \mathcal{P}_{\xi}$ ;
2.  $\mathcal{P}_{\xi} \mathcal{P}_{\eta} = 0$ ,  $\xi \neq \eta$ ;
3.  $\sum_{\xi} \mathcal{P}_{\xi} = 1$ .

*Proof.* Equalities 1 and 2 follow from equality 4 in Proposition 2.1 on page 15 and equalities 1 and 2 of Proposition 2.10. Equality 3 follows from equality 1 in Proposition 2.1 on page 15 and equality 3 in Proposition 2.10.  $\square$

Proposition 2.11 gives a decomposition of the identity element in  $GL_m(\mathbb{C})$  into an algebraically orthogonal sum of projection matrices that decompose the identity matrix of the corresponding dimension. The many applications and interpretations of the canonical projections decomposition are presented in detail in [2].

**Proposition 2.12.** *The (algebra) homomorphism*

$$\varphi : x \in \mathbb{C}D_n \mapsto \bigoplus_{\xi} \langle x, \xi \rangle = \prod_{\xi} M_{n_{\xi}}(\mathbb{C})$$

*in an isomorphism.*

*Proof.* First note that  $\varphi(1) = \bigoplus_{\xi} \langle 1, \xi \rangle = \bigoplus_{\xi} \xi_1 = \bigoplus_{\xi} I_{n_{\xi}}$ . Suppose that  $\varphi(x) = \bigoplus_{\xi} I_{n_{\xi}}$  for some non-null  $x \in \mathbb{C}D_n$ . Then  $\langle x, \xi \rangle = I_{n_{\xi}}$  for all  $\xi \in \widehat{D}_n$ , so that, from Proposition 2.3 on page 17,

$$\langle \tau x, \xi \rangle = \xi_{\tau} \langle x, \xi \rangle = \xi_{\tau} I_{n_{\xi}} = \xi_{\tau},$$

so that

$$\frac{n_{\xi}}{2n} \langle \tau x, \xi \rangle = \frac{n_{\xi}}{2n} \xi_{\tau}.$$

Taking the trace on both sides, we have,

$$\langle \tau x, \pi^{\xi} \rangle = \frac{n_{\xi}}{2n} \chi_{\tau}^{\xi} = \frac{1}{2n} \chi_1^{\xi} \chi_{\tau}^{\xi}.$$

Summing over  $\widehat{D}_n$  we obtain, from Proposition 2.9 on page 21,

$$\sum_{\xi} \langle \tau x, \pi^{\xi} \rangle = \langle \tau x, \sum_{\xi} \pi^{\xi} \rangle = \frac{1}{2n} \sum_{\xi} \chi_1^{\xi} \chi_{\tau}^{\xi} = \begin{cases} \sum_{\xi} \frac{n_{\xi}^2}{2n} = 1, & \text{if } \tau = 1 \\ 0, & \text{if } \tau \neq 1. \end{cases}$$

From Proposition 2.10 we know that  $\sum_{\xi} \pi^{\xi} = 1$ , so that then  $\langle \tau x, 1 \rangle = \delta_{\tau 1}$ , or  $x = 1$ , concluding the proof.  $\square$

**Proposition 2.13 (Inversion Formula).**

$$x_{\tau} = \sum_{\xi} \frac{n_{\xi}}{2n} \text{tr} [\xi_{\tau} \langle x, \xi \rangle].$$

*Proof.* From Proposition 2.3 on page 17, we have

$$\frac{n_{\xi}}{2n} \langle \tau x, \xi \rangle = \frac{n_{\xi}}{2n} \xi_{\tau} \langle x, \xi \rangle,$$

so that, taking the trace on both sides,

$$\langle \tau x, \pi^\xi \rangle = \frac{n_\xi}{2n} \text{tr} [\xi_\tau \langle x, \xi \rangle],$$

summing over  $\widehat{D}_n$ , and applying Proposition 2.10, gives

$$\langle \tau x, 1 \rangle = \sum_{\xi} \frac{n_\xi}{2n} \text{tr} [\xi_\tau \langle x, \xi \rangle],$$

or

$$x_\tau = \sum_{\xi} \frac{n_\xi}{2n} \text{tr} [\xi_\tau \langle x, \xi \rangle],$$

which is the inversion formula.  $\square$

## 2.7 Fourier Bases

In data analytical applications it is often useful to have a  $\mathbb{C}^{2n}$  interpretation of Proposition 2.12, in a way that the algebra isomorphism and the resulting inversion formula appear in terms of a non-singular matrix transformation. To illustrate, consider the  $\mathbb{C}D_4$  decomposition, and write

$$\mathcal{X} = \begin{bmatrix} \langle x, 1 \rangle \\ \langle x, \alpha \rangle \\ \langle x, \gamma_+ \rangle \\ \langle x, \gamma_- \rangle \\ \langle x, \beta \rangle_{11} \\ \langle x, \beta \rangle_{21} \\ \langle x, \beta \rangle_{12} \\ \langle x, \beta \rangle_{22} \end{bmatrix} \in \mathbb{C}^8.$$

The Fourier basis for this space is defined as the normalized rows of the matrix  $F$  satisfying

$$\mathcal{X} = Fx,$$

with normalizing (row) constants  $\sqrt{n_\xi/2n}$ . In the present example, the constants are, respectively,  $\sqrt{2}/4$  for the representations in dimension of one and  $1/2$  for the representation in dimension of two. The matrix  $F$  is shown in (2.12), where the horizontal line separates the rows corresponding to the one-dimensional representations from the rows associated with (the columns of) the two-dimensional representation  $\beta$ .



$$F = \left[ \begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ \hline 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \end{array} \right]. \quad (2.12)$$

In general, we have

$$(\mathcal{F}x)_\xi = \sqrt{\frac{2n}{n_\xi}} \langle x, \xi \rangle, \quad \xi \in \widehat{D}_n, \quad (2.13)$$

where  $(\mathcal{F}x)_\xi$  indicates the partition of  $\mathcal{F}x$  corresponding to  $\xi \in \widehat{D}_n$ , with the understanding that here  $\langle x, \xi \rangle$  is written as a  $n_\xi \times 1$  vector with components given by the columns of  $\langle x, \xi \rangle$ .

In the  $D_4$  case above, we adjoined 4 blocks in dimension of  $1 \times 8$  and one block in dimension of  $2^2 \times 8$ . If  $\mathcal{F}$  is the normalized (hence orthogonal) version of  $F$  given by (2.12) and  $\phi$  the (left) regular representation of  $D_4$  then direct calculation shows that

$$\mathcal{F}\phi\mathcal{F}' = \text{Diag}(1, \alpha, \gamma_+, \gamma_-, I_2 \otimes \beta), \quad (2.14)$$

and, in general, we have:

**Proposition 2.14.** *If  $\mathcal{F}$  indicates the  $D_n$  Fourier basis and  $\phi$  the corresponding (left) regular representation, then*

$$\mathcal{F}\phi_\sigma\mathcal{F}' = \text{Diag}(\dots, I_{n_\xi} \otimes \xi_\sigma, \dots)_{\xi \in \widehat{D}_n}.$$

*Proof.* The columns of  $\mathcal{F}\phi_\sigma$  when  $\phi$  is the left regular representation are given by  $\{\sigma\tau : \tau \in D_n\}$ . Moreover, any row of  $\mathcal{F}\phi$  associated with  $\xi \in \widehat{D}_n$  is given by  $\sqrt{n_\xi/g}\{\xi_{\sigma\tau}^{if} : \tau \in D_n\}$  for some  $i, f = 1, \dots, n_\xi$ , and a column of  $\mathcal{F}'$  is a row of  $\mathcal{F}$ , or  $\sqrt{n_\eta/g}\{\eta_\tau^{jk} : \tau \in D_n\}$ , for some  $\eta \in \widehat{D}_n$ , and  $j, k = 1, \dots, n_\eta$ , and if  $\xi$  or  $\eta$  are in dimension of one we make  $i = f$  or  $j = k$ , respectively. Therefore, the  $\xi, \eta$  (block) entry of  $\mathcal{F}\phi_\sigma\mathcal{F}'$  is given by

$$(\mathcal{F}\phi_\sigma\mathcal{F}')_{if,jk} = \frac{\sqrt{n_\xi n_\eta}}{g} \sum_{\tau} \xi_{\sigma\tau}^{if} \eta_\tau^{jk},$$

whereas, writing  $\xi_{\sigma\tau}^{if} = \sum_{\ell=1}^{n_\xi} \xi_{\sigma}^{\ell\ell} \xi_\tau^{\ell f}$ , we obtain

$$(\mathcal{F}\phi_\sigma\mathcal{F}')_{if,jk} = \frac{\sqrt{n_\xi n_\eta}}{g} \sum_{\ell=1}^{n_\xi} \xi_{\sigma}^{\ell\ell} \sum_{\tau} \xi_\tau^{\ell f} \eta_\tau^{jk} = 0$$

when  $\xi \neq \eta$  e.g., [4], [2, p.81]. If  $\eta = \xi$ , then, recalling that  $\xi_\tau^{jk} = \xi_{\tau^{-1}}^{kj}$ ,

$$(\mathcal{F}\phi_\sigma\mathcal{F}')_{if,jk} = \frac{n_\xi}{g} \sum_{\ell=1}^{n_\xi} \xi_\sigma^{\ell\ell} \sum_{\tau} \xi_\tau^{\ell f} \xi_{\tau^{-1}}^{kj} = \frac{n_\xi}{g} \sum_{\ell=1}^{n_\xi} \xi_\sigma^{\ell\ell} \frac{g}{n_\xi} \delta_{\ell j} \delta_{fk} = 0$$

if  $\ell \neq j$  or  $f \neq k$ . Otherwise, when  $\ell = j$  and  $f = k$ , the  $\xi, \xi$  (block) entry of  $\mathcal{F}\phi_\sigma\mathcal{F}'$  is

$$(\mathcal{F}\phi_\sigma\mathcal{F}')_{ik,jk} = \xi_\sigma^{ij},$$

when  $k = 1, \dots, n_\xi$ , thus giving  $n_\xi$  copies of  $\xi_\sigma$ . Arranging the rows of  $\mathcal{F}$  that correspond to  $\xi$  according to  $11, \dots, n_\xi 1, \dots, 1n_\xi, \dots, n_\xi n_\xi$  expresses the  $\xi, \xi$  entry of  $\mathcal{F}\phi_\sigma\mathcal{F}'$  as  $I_{n_\xi} \otimes \xi_\sigma$ , so that, together,  $\mathcal{F}\phi_\sigma\mathcal{F}'$  is a block diagonal matrix with the diagonal components given by  $I_{n_\xi} \otimes \xi_\sigma$  for the distinct  $\xi \in \widehat{D}_n$ , concluding the proof.  $\square$

**Proposition 2.15.** *If  $\mathcal{F}$  indicates the Fourier basis of  $D_n$  and  $\mathcal{P}_\xi$  it the left regular canonical projection associated with  $\xi \in \widehat{D}_n$ , then*

$$\mathcal{F}\mathcal{P}_\xi\mathcal{F}' = \text{Diag}(0, \dots, I_{n_\xi} \otimes I_{n_\xi}, \dots, 0).$$

*Proof.* Evaluation of the regular canonical projection for  $\xi \in \widehat{D}_n$  using Propositions 2.14 on the preceding page and 2.9 on page 21, and the equality in (2.10), gives

$$\begin{aligned} \mathcal{F}\mathcal{P}_\xi\mathcal{F}' &= \frac{n_\xi}{2n} \sum_{\tau \in D_n} \chi_\tau^\xi \mathcal{F}\phi_\tau\mathcal{F}' = \frac{n_\xi}{2n} I_{n_\xi} \otimes \sum_{\tau \in D_n} \chi_\tau^\xi \xi_\tau \\ &= \frac{n_\xi}{2n} I_{n_\xi} \otimes \langle \chi^\xi, \xi \rangle = \frac{n_\xi}{2n} I_{n_\xi} \otimes \frac{2n}{n_\xi} I_{n_\xi} = I_{n_\xi} \otimes I_{n_\xi}. \end{aligned}$$

As a consequence, considering (2.13), we obtain the analysis of variance of  $x$  in terms of its spectral decomposition;

**Corollary 2.1 (Parseval's Equality).** *If  $x \in \mathbb{C}^{2n}$  then*

$$\|x\|^2 = \sum_{\xi \in \widehat{D}_n} \frac{n_\xi}{2n} \|\langle x, \xi \rangle\|^2.$$

*Example 2.8.* Consider the following two points  $x, y$  in  $\mathbb{C}D_4$ ,

$$\left( \begin{array}{c|cc} \tau & x & y \\ \hline 1 & 1 & 5 \\ r & 2 & 1 \\ r^2 & 1 & 4 \\ r^3 & 1 & 3 \\ h & 5 & 3 \\ rh & 10 & 4 \\ r^2h & 4 & 1 \\ r^3h & 5 & 5 \end{array} \right).$$

From Example 2.6 on page 19, we have,

$$\begin{aligned} \langle x, 1 \rangle &= 29, \quad \langle y, 1 \rangle = 26; \\ \langle x, \alpha \rangle &= -19, \quad \langle y, \alpha \rangle = 0; \\ \langle x, \gamma_+ \rangle &= -7, \quad \langle y, \gamma_+ \rangle = 0; \\ \langle x, \gamma_- \rangle &= -5, \quad \langle y, \gamma_- \rangle = 10; \\ \langle x, \beta \rangle &= \begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}, \quad \langle y, \beta \rangle = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}. \end{aligned}$$

Then,

$$x \cdot x = \|x\|^2 = \sum_{\tau} x_{\tau}^2 = 173,$$

whereas, equally,

$$\sum_{\xi \in \hat{D}_n} \frac{n_{\xi}}{2n} \|\langle x, \xi \rangle\|^2 = \frac{1}{8}(29^2 + 19^2 + 7^2 + 5^2) + \frac{2}{8}(1^2 + 4^2 + 6^2 + 1^2) = 173.$$

Moreover,

$$x \cdot y = \sum_{\tau} x_{\tau} y_{\tau} = 98 = \sum_{\xi \in \hat{D}_n} \frac{n_{\xi}}{2n} \langle x, \xi \rangle \langle y, \xi \rangle,$$

with the understanding that when  $\xi$  is of dimension greater than one  $\xi$  the products of the two transforms should read as the (Hadamard) inner product  $\langle x, \xi \rangle \cdot \langle y, \xi \rangle$  two matrices.

More generally, then, Corollary 2.1 read as:

**Corollary 2.2.** *If  $x, y \in \mathbb{C}^G$  then*

$$x \cdot y = \sum_{\xi \in \hat{D}_n} \frac{n_{\rho}}{2n} \langle x, \xi \rangle \langle y, \xi \rangle.$$

*Example 2.9.* Given two points

$$x = x_0 + x_1\omega + x_2\omega^2, \quad y = y_0 + y_1\omega + y_2\omega^2$$

in the group algebra of  $\mathbb{C}^3$ , then

$$\langle x, \xi_1 \rangle = x_0 + x_1 + x_2, \quad \langle x, \xi_2 \rangle = x_0 + x_1\omega + x_2\omega^2, \quad \langle x, \xi_3 \rangle = x_0 + x_1\omega^2 + x_2\omega$$

are the Fourier transforms of  $x$ , and similarly for  $y$ . Direct calculation then shows that

$$x \cdot y = \sum_{\tau} x_{\tau} \bar{y}_{\tau} = \frac{1}{3} \sum_{j=1}^3 \langle x, \xi_j \rangle \overline{\langle y, \xi_j \rangle}.$$

It is understood that in Corollaries 2.1 and 2.2 the inner products are Hermitian products.

## 2.8 The Center of $\mathbb{C}D_n$

The center of  $D_n$  is the subset of  $D_n$  whose elements commute with all points in  $D_n$ . We write  $\text{Cent. } D_n$  to indicate the dihedral center. By linearity, the definition extends to  $\mathbb{C}D_n$  and all dihedral linearizations, and clearly  $\text{Cent. } D_n = \text{Cent. } \mathbb{C}D_n$ . The reader can check that  $\text{Cent. } D_n$  is an associative algebra.  $\mathbb{C}D_n$  is also called the enveloping algebra of  $D_n$  [3, p.79].

*Example 2.10* (A basis for the center of the regular linearization of  $D_3$ ). The reader can verify, using the definition of conjugacy classes, that a basis for the center of the  $D_3$  regular linearization algebra, see page 15,

$$\{\langle x, \phi \rangle; x \in \mathbb{C}D_3\},$$

where  $\phi$  is the (left) regular representation of  $D_3$ , is given by the identity  $I$ , and the matrices

$$C_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

so that the matrix elements in the center of the regular linearization have the form

$$C = aI + bC_1 + cC_2 = \begin{bmatrix} a & b & b & c & c & c \\ b & a & b & c & c & c \\ b & b & a & c & c & c \\ c & c & c & a & b & b \\ c & c & c & b & a & b \\ c & c & c & b & b & a \end{bmatrix}.$$

Similar argument shows that the canonical projections  $\mathcal{P}_1, \mathcal{P}_\alpha, \mathcal{P}_\beta$  for the regular representation of  $D_3$  also form a basis for the center of the regular linearization algebra, with the advantage of being (algebraically) orthogonal. For example, since  $C$  is in the center of the algebra, we must have

$$C = \ell_1 \mathcal{P}_1 + \ell_\alpha \mathcal{P}_\alpha + \ell_\beta \mathcal{P}_\beta,$$

so that then

$$C\mathcal{P}_1 = \ell_1 \mathcal{P}_1, \quad C\mathcal{P}_\alpha = \ell_\alpha \mathcal{P}_\alpha, \quad C\mathcal{P}_\beta = \ell_\beta \mathcal{P}_\beta. \quad (2.15)$$

Taking the trace on both sides, we have

$$\ell_1 = \frac{\text{tr } C\mathcal{P}_1}{\text{tr } \mathcal{P}_1} = a + 2b + 3c, \quad \ell_\alpha = \frac{\text{tr } C\mathcal{P}_\alpha}{\text{tr } \mathcal{P}_\alpha} = a + 2b - 3c, \quad \ell_\beta = \frac{\text{tr } C\mathcal{P}_\beta}{\text{tr } \mathcal{P}_\beta} = a - b. \quad (2.16)$$

The reader may also verify that the characterization of the matrix elements in the center of the regular linearization algebra of  $D_3$  can be obtained by taking an arbitrary matrix

$$H = \langle a, \phi \rangle = \begin{bmatrix} a & b & c & \alpha & \beta & \gamma \\ c & a & b & \gamma & \alpha & \beta \\ b & c & a & \beta & \gamma & \alpha \\ \alpha & \gamma & \beta & a & c & b \\ \beta & \alpha & \gamma & b & a & c \\ \gamma & \beta & \alpha & c & b & a \end{bmatrix},$$

in the algebra and centralizing it

$$C \simeq \sum_{\tau \in G} \phi_\tau H \phi'_\tau.$$

The matrices  $C$  in the center of the regular linearization algebra transform according to

$$\mathcal{F}C\mathcal{F}^{-1} = \mathcal{C} = \text{Diag}(\ell_1, \ell_\alpha, \ell_\beta I_4). \quad (2.17)$$

## 2.9 Group Rings

The following definitions are relevant to the analysis of data indexed by symmetries.

**Definition 2.4.** Given a ring  $R$  and a group  $G$ , the set

$$RG = \left\{ \sum_{\tau} r_{\tau} \tau, \quad r_{\tau} \in R, \tau \in G \right\}$$

of all finite formal  $R$ -linear combinations of elements of  $G$ , together with the operations of addition (component-wise) and multiplication (induced by the group multiplication), is called the group ring of  $G$  over  $R$ .

*Example 2.11 (Visual fields).* The matrix

$$F = \begin{bmatrix} 0 & 0 & 0 & 26 & 27 & 24 & 21 & 0 & 0 & 0 \\ 0 & 0 & 23 & 23 & 26 & 28 & 25 & 22 & 0 & 0 \\ 0 & 27 & 29 & 27 & 27 & 28 & 29 & 29 & 28 & 0 \\ 28 & 29 & 29 & 29 & 31 & 20 & 20 & 28 & 28 & 29 \\ 26 & 26 & 29 & 30 & 33 & 34 & 28 & 27 & 28 & 27 \\ 28 & 30 & 30 & 31 & 32 & 33 & 33 & 4 & 27 & 28 \\ 27 & 29 & 31 & 31 & 32 & 32 & 30 & 29 & 28 & 27 \\ 0 & 30 & 31 & 31 & 28 & 29 & 32 & 30 & 30 & 0 \\ 0 & 0 & 28 & 32 & 29 & 28 & 29 & 28 & 0 & 0 \\ 0 & 0 & 0 & 26 & 27 & 28 & 29 & 0 & 0 & 0 \end{bmatrix}$$

describes the output of an automated perimetry test used in the assessment of the visual field. The entries represent the sensitivity of the retina in detecting the light stimulus and is expressed in decibel units, with a maximal possible reading of 50db. A 50db target is the dimmest target the instrument can project. The smaller the reading the lower the sensitivity at that retinal location given by reference to the rows and columns of the matrix.

The set of matrices

$$\{F_1, F_r, F_{r^2}, F_{r^3}, F_h, F_{rh}, F_{r^2h}, F_{r^3h}\}$$

shown below give a dihedral ( $D_4$ ) orbit on the visual field and each matrix field is a point in the ring of the  $10 \times 10$  matrices over the reals, indexed by  $D_4$ , so that the formal sums

$$\sum_{\tau \in D_4} F_{\tau} \tau,$$

gives an example of a point in the group ring of  $D_4$  over the ring of the real  $10 \times 10$  matrices. Each one of the following matrices is obtained from  $F$  by applying the corresponding planar (counterclockwise) rotations and reversals to its row-column

indices, in the natural way, with the x (y) axis placed between rows (columns) 5 and 6.

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 26 & 27 & 24 & 21 & 0 & 0 & 0 \\ 0 & 0 & 23 & 23 & 26 & 28 & 25 & 22 & 0 & 0 \\ 0 & 27 & 29 & 27 & 27 & 28 & 29 & 29 & 28 & 0 \\ 28 & 29 & 29 & 29 & 31 & 20 & 20 & 28 & 28 & 29 \\ 26 & 26 & 29 & 30 & 33 & 34 & 28 & 27 & 28 & 27 \\ 28 & 30 & 30 & 31 & 32 & 33 & 33 & 4 & 27 & 28 \\ 27 & 29 & 31 & 31 & 32 & 32 & 30 & 29 & 28 & 27 \\ 0 & 30 & 31 & 31 & 28 & 29 & 32 & 30 & 30 & 0 \\ 0 & 0 & 28 & 32 & 29 & 28 & 29 & 28 & 0 & 0 \\ 0 & 0 & 0 & 26 & 27 & 28 & 29 & 0 & 0 & 0 \end{bmatrix}, F_r = \begin{bmatrix} 0 & 0 & 0 & 26 & 27 & 24 & 21 & 0 & 0 & 0 \\ 0 & 0 & 23 & 23 & 26 & 28 & 25 & 22 & 0 & 0 \\ 0 & 27 & 29 & 27 & 27 & 28 & 29 & 29 & 28 & 0 \\ 28 & 29 & 29 & 29 & 31 & 20 & 20 & 28 & 28 & 29 \\ 26 & 26 & 29 & 30 & 33 & 34 & 28 & 27 & 28 & 27 \\ 28 & 30 & 30 & 31 & 32 & 33 & 33 & 4 & 27 & 28 \\ 27 & 29 & 31 & 31 & 32 & 32 & 30 & 29 & 28 & 27 \\ 0 & 30 & 31 & 31 & 28 & 29 & 32 & 30 & 30 & 0 \\ 0 & 0 & 28 & 32 & 29 & 28 & 29 & 28 & 0 & 0 \\ 0 & 0 & 0 & 26 & 27 & 28 & 29 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{r^2} = \begin{bmatrix} 0 & 0 & 0 & 29 & 28 & 27 & 26 & 0 & 0 & 0 \\ 0 & 0 & 28 & 29 & 28 & 29 & 32 & 28 & 0 & 0 \\ 0 & 30 & 30 & 32 & 29 & 28 & 31 & 31 & 30 & 0 \\ 27 & 28 & 29 & 30 & 32 & 32 & 31 & 31 & 29 & 27 \\ 28 & 27 & 4 & 33 & 33 & 32 & 31 & 30 & 30 & 28 \\ 27 & 28 & 27 & 28 & 34 & 33 & 30 & 29 & 26 & 26 \\ 29 & 28 & 28 & 20 & 20 & 31 & 29 & 29 & 29 & 28 \\ 0 & 28 & 29 & 29 & 28 & 27 & 27 & 29 & 27 & 0 \\ 0 & 0 & 22 & 25 & 28 & 26 & 23 & 23 & 0 & 0 \\ 0 & 0 & 0 & 21 & 24 & 27 & 26 & 0 & 0 & 0 \end{bmatrix}, F_{r^3} = \begin{bmatrix} 0 & 0 & 0 & 29 & 27 & 28 & 27 & 0 & 0 & 0 \\ 0 & 0 & 28 & 28 & 28 & 27 & 28 & 30 & 0 & 0 \\ 0 & 22 & 29 & 28 & 27 & 4 & 29 & 30 & 28 & 0 \\ 21 & 25 & 29 & 20 & 28 & 33 & 30 & 32 & 29 & 29 \\ 24 & 28 & 28 & 20 & 34 & 33 & 32 & 29 & 28 & 28 \\ 27 & 26 & 27 & 31 & 33 & 32 & 32 & 28 & 29 & 27 \\ 26 & 23 & 27 & 29 & 30 & 31 & 31 & 31 & 32 & 26 \\ 0 & 23 & 29 & 29 & 29 & 30 & 31 & 31 & 28 & 0 \\ 0 & 0 & 27 & 29 & 26 & 30 & 29 & 30 & 0 & 0 \\ 0 & 0 & 0 & 28 & 26 & 28 & 27 & 0 & 0 & 0 \end{bmatrix},$$

$$F_h = \begin{bmatrix} 0 & 0 & 0 & 26 & 27 & 28 & 29 & 0 & 0 & 0 \\ 0 & 0 & 28 & 32 & 29 & 28 & 29 & 28 & 0 & 0 \\ 0 & 30 & 31 & 31 & 28 & 29 & 32 & 30 & 30 & 0 \\ 27 & 29 & 31 & 31 & 32 & 32 & 30 & 29 & 28 & 27 \\ 28 & 30 & 30 & 31 & 32 & 33 & 33 & 4 & 27 & 28 \\ 26 & 26 & 29 & 30 & 33 & 34 & 28 & 27 & 28 & 27 \\ 28 & 29 & 29 & 29 & 31 & 20 & 20 & 28 & 28 & 29 \\ 0 & 27 & 29 & 27 & 27 & 28 & 29 & 29 & 28 & 0 \\ 0 & 0 & 23 & 23 & 26 & 28 & 25 & 22 & 0 & 0 \\ 0 & 0 & 0 & 26 & 27 & 24 & 21 & 0 & 0 & 0 \end{bmatrix}, F_{rh} = \begin{bmatrix} 0 & 0 & 0 & 27 & 28 & 27 & 29 & 0 & 0 & 0 \\ 0 & 0 & 30 & 28 & 27 & 28 & 28 & 28 & 0 & 0 \\ 0 & 28 & 30 & 29 & 4 & 27 & 28 & 29 & 22 & 0 \\ 29 & 29 & 32 & 30 & 33 & 28 & 20 & 29 & 25 & 21 \\ 28 & 28 & 29 & 32 & 33 & 34 & 20 & 28 & 28 & 24 \\ 27 & 29 & 28 & 32 & 32 & 33 & 31 & 27 & 26 & 27 \\ 26 & 32 & 31 & 31 & 31 & 30 & 29 & 27 & 23 & 26 \\ 0 & 28 & 31 & 31 & 30 & 29 & 29 & 29 & 23 & 0 \\ 0 & 0 & 30 & 29 & 30 & 26 & 29 & 27 & 0 & 0 \\ 0 & 0 & 0 & 27 & 28 & 26 & 28 & 0 & 0 & 0 \end{bmatrix},$$

$$F_{r^2h} = \begin{bmatrix} 0 & 0 & 0 & 21 & 24 & 27 & 26 & 0 & 0 & 0 \\ 0 & 0 & 22 & 25 & 28 & 26 & 23 & 23 & 0 & 0 \\ 0 & 28 & 29 & 29 & 28 & 27 & 27 & 29 & 27 & 0 \\ 29 & 28 & 28 & 20 & 20 & 31 & 29 & 29 & 29 & 28 \\ 27 & 28 & 27 & 28 & 34 & 33 & 30 & 29 & 26 & 26 \\ 28 & 27 & 4 & 33 & 33 & 32 & 31 & 30 & 30 & 28 \\ 27 & 28 & 29 & 30 & 32 & 32 & 31 & 31 & 29 & 27 \\ 0 & 30 & 30 & 32 & 29 & 28 & 31 & 31 & 30 & 0 \\ 0 & 0 & 28 & 29 & 28 & 29 & 32 & 28 & 0 & 0 \\ 0 & 0 & 0 & 29 & 28 & 27 & 26 & 0 & 0 & 0 \end{bmatrix}, F_{r^3h} = \begin{bmatrix} 0 & 0 & 0 & 28 & 26 & 28 & 27 & 0 & 0 & 0 \\ 0 & 0 & 27 & 29 & 26 & 30 & 29 & 30 & 0 & 0 \\ 0 & 23 & 29 & 29 & 29 & 30 & 31 & 31 & 28 & 0 \\ 26 & 23 & 27 & 29 & 30 & 31 & 31 & 31 & 32 & 26 \\ 27 & 26 & 27 & 31 & 33 & 32 & 32 & 28 & 29 & 27 \\ 24 & 28 & 28 & 20 & 34 & 33 & 32 & 29 & 28 & 28 \\ 21 & 25 & 29 & 20 & 28 & 33 & 30 & 32 & 29 & 29 \\ 0 & 22 & 29 & 28 & 27 & 4 & 29 & 30 & 28 & 0 \\ 0 & 0 & 28 & 28 & 28 & 27 & 28 & 30 & 0 & 0 \\ 0 & 0 & 0 & 29 & 27 & 28 & 27 & 0 & 0 & 0 \end{bmatrix}.$$

## 2.10 Multivariate Normal Data

The following result follows from applying Propositions 2.14 and 2.15.

**Proposition 2.16.** *Let  $x$  be a random vector with components indexed by  $D_n$  carrying a multivariate normal distribution with vector of means  $\mu$  and covariance matrix  $\Lambda$ , or  $x \sim N(\mu, \Lambda)$  and assume that  $\Lambda$  is in the center of the (left) regular  $\mathbb{C}$ -algebra of  $D_n$ . Then*

$$\Lambda = \sum_{\xi \in \widehat{D}_n} \lambda_\xi \mathcal{P}_\xi$$

where  $\mathcal{P}_\xi$  is the regular canonical projection associated with  $\xi \in \widehat{D}_n$ ,  $\lambda_\xi = \text{tr } \mathcal{P}_\xi \Lambda / \text{tr } \mathcal{P}_\xi$ , and if  $\mathcal{F}$  indicates the Fourier basis of  $D_n$ ,

$$\mathcal{F}x \sim N(\mathcal{F}\mu, \text{Diag}(\dots, \lambda_\xi I_{n_\xi^2}, \dots)_{\xi \in \widehat{D}_n}).$$

Moreover, for each  $\xi \in \widehat{D}_n$ ,

$$(\mathcal{F}x)_\xi \sim N((\mathcal{F}\mu)_\xi, \lambda_\xi I_{n_\xi^2}),$$

is the distribution of the block-component  $(\mathcal{F}x)_\xi$  of  $\mathcal{F}x$  corresponding to  $\xi$ , and these components are independently distributed (and independent) multivariate normal models.

## 2.11 Selected Dihedral Bases

In this section we derive the element components of Proposition 2.16 for the dihedral groups  $D_3, D_4, D_5, D_6$ , and  $D_7$ .

*Example 2.12 ( $D_3$ ).* The Fourier basis for  $D_3$ , following Sect. 2.5, is giving by

$$\mathcal{F}_3 = \begin{bmatrix} 0.42 & 0.42 & 0.42 & 0.42 & 0.42 & 0.42 \\ 0.42 & 0.42 & 0.42 & -0.42 & -0.42 & -0.42 \\ \hline 0.59 & -0.30 & -0.30 & 0.59 & -0.30 & -0.30 \\ 0.0 & 0.51 & -0.51 & 0.0 & 0.51 & -0.51 \\ 0.0 & -0.51 & 0.51 & 0.0 & 0.51 & -0.51 \\ \hline 0.59 & -0.30 & -0.30 & -0.59 & 0.30 & 0.30 \end{bmatrix},$$

whereas

$$\Lambda_3 = \left[ \begin{array}{ccc|ccc} a & b & b & c & c & c \\ b & a & b & c & c & c \\ b & b & a & c & c & c \\ \hline c & c & c & a & b & b \\ c & c & c & b & a & b \\ c & c & c & b & b & a \end{array} \right]$$



is the covariance structure amenable to Proposition 2.16. The first two rows of  $\mathcal{F}_3$  account for the two characters of  $D_3$  in dimension of one, and the remaining ones for the single character in dimension of two. The two row blocks correspond to rotations and reversals, respectively, and similarly for the two column blocks. This notation applies to similar matrices in this section. The matrix  $\Lambda_3$  is in the center of  $D_3$ , and hence in the center of its group algebra. Its coefficients in the basis given by the regular canonical projections are,

$$\lambda_1 = a + 2b + 3c, \quad \lambda_\alpha = a + 2b - 3c, \quad \lambda_\beta = a - b,$$

corresponding to the symmetric, signature and (single) two-dimension characters. From Proposition 2.16, corresponding to the symmetric and alternating characters, respectively,

$$(\mathcal{F}x)_1 \sim N((\mathcal{F}\mu)_1, a + 2b + 3c), \quad (\mathcal{F}x)_\alpha \sim N((\mathcal{F}\mu)_\alpha, a + 2b - 3c),$$

whereas, corresponding to the single character in dimension of two,

$$(\mathcal{F}x)_\beta \sim N((\mathcal{F}x)_\beta, (a - b)I_4).$$

*Example 2.13 ( $D_4$ ).* Similarly,

$$\mathcal{F}_4 = \begin{bmatrix} 0.36 & 0.36 & 0.36 & 0.36 & 0.36 & 0.36 & 0.36 & 0.36 \\ 0.36 & 0.36 & 0.36 & 0.36 & -0.36 & -0.36 & -0.36 & -0.36 \\ 0.36 & -0.36 & 0.36 & -0.36 & 0.36 & -0.36 & 0.36 & -0.36 \\ 0.36 & -0.36 & 0.36 & -0.36 & -0.36 & 0.36 & -0.36 & 0.36 \\ \hline 0.50 & 0 & -0.50 & 0 & 0.50 & 0 & -0.50 & 0 \\ 0 & 0.50 & 0 & -0.50 & 0 & 0.50 & 0 & -0.50 \\ 0 & -0.50 & 0 & 0.50 & 0 & 0.50 & 0 & -0.50 \\ 0.50 & 0 & -0.50 & 0 & -0.50 & 0 & 0.50 & 0 \end{bmatrix}$$

is the Fourier basis for  $D_4$ , where the first four rows are indexed by the symmetric (1), anti-symmetric ( $\alpha$ ),  $\gamma^+$ , and  $\gamma^-$  characters, all in dimension of one (their interpretation is discussed later on in Sect. 2.7). The remaining four rows account for the single character in dimension of two ( $\beta$ ). The covariance structure in the center of  $D_4$  has the pattern of

$$\Lambda_4 = \left[ \begin{array}{cccc|cccc} a & b & c & b & d & e & d & e \\ b & a & b & c & e & d & e & d \\ c & b & a & b & d & e & d & e \\ b & c & b & a & e & d & e & d \\ \hline d & e & d & e & a & b & c & b \\ e & d & e & d & b & a & b & c \\ d & e & d & e & c & b & a & b \\ e & d & e & d & b & c & b & a \end{array} \right],$$

and

$$\begin{aligned}\lambda_1 &= a + 2b + c + 2d + 2e, \quad \lambda_\alpha = a + 2b + c - 2d - 2e \\ \lambda_{\gamma^+} &= a - 2b + c + 2d - 2e, \quad \lambda_{\gamma^-} = a - 2b + c - 2d + 2e, \quad \lambda_\beta = a - c\end{aligned}$$

are its coefficients in the regular canonical projection basis.

*Example 2.14* ( $D_5$ ). Its Fourier basis

$$\mathcal{F}_5 = \begin{bmatrix} 0.31 & 0.31 & 0.31 & 0.31 & 0.31 & 0.31 & 0.31 & 0.31 & 0.31 & 0.31 \\ 0.31 & 0.31 & 0.31 & 0.31 & 0.31 & -0.31 & -0.31 & -0.31 & -0.31 & -0.31 \\ \hline 0.45 & 0.16 & -0.36 & -0.36 & 0.16 & 0.45 & 0.16 & -0.36 & -0.36 & 0.16 \\ 0.0 & 0.42 & 0.26 & -0.26 & -0.42 & 0.0 & 0.42 & 0.26 & -0.26 & -0.42 \\ 0.0 & -0.42 & -0.26 & 0.26 & 0.42 & 0.0 & 0.42 & 0.26 & -0.26 & -0.42 \\ 0.45 & 0.16 & -0.36 & -0.36 & 0.16 & -0.45 & -0.16 & 0.36 & 0.36 & -0.16 \\ \hline 0.45 & -0.36 & 0.16 & 0.16 & -0.36 & 0.45 & -0.36 & 0.16 & 0.16 & -0.36 \\ 0.0 & 0.26 & -0.42 & 0.42 & -0.26 & 0.0 & 0.26 & -0.42 & 0.42 & -0.26 \\ 0.0 & -0.26 & 0.42 & -0.42 & 0.26 & 0.0 & 0.26 & -0.42 & 0.42 & -0.26 \\ 0.45 & -0.36 & 0.16 & 0.16 & -0.36 & -0.45 & 0.36 & -0.16 & -0.16 & 0.36 \end{bmatrix}$$

is indexed by the symmetric and alternating characters, one fundamental character in dimension of two ( $\beta$ ) and its first harmonic ( $\beta^1$ ), also in dimension of two. The covariance structure with the symmetry of  $D_5$  has the pattern of

$$A_5 = \left[ \begin{array}{ccccc|ccccc} a & b & c & c & b & d & d & d & d & d \\ b & a & b & c & c & d & d & d & d & d \\ c & b & a & b & c & d & d & d & d & d \\ c & c & b & a & b & d & d & d & d & d \\ b & c & c & b & a & d & d & d & d & d \\ \hline d & d & d & d & d & a & b & c & c & b \\ d & d & d & d & d & b & a & b & c & c \\ d & d & d & d & d & c & b & a & b & c \\ d & d & d & d & d & c & c & b & a & b \\ d & d & d & d & d & b & c & c & b & a \end{array} \right],$$

and is expressed, in the canonical projection basis, in terms of the coefficients

$$\begin{aligned}\lambda_1 &= a + 2b + 2c + 5d, \quad \lambda_\alpha = a + 2b + 2c - 5d \\ \lambda_\beta &= a + 2 \cos(2/5\pi)b - 2 \cos(1/5\pi)c, \quad \lambda_{\beta^1} = a - 2 \cos(1/5\pi)b + 2 \cos(2/5\pi)c.\end{aligned}$$

*Example 2.15* ( $D_6$ ). With similar interpretations, we have:

$$\Lambda_6 = \left[ \begin{array}{c|c} a & b & c & d & c & b & e & f & e & f & e & f \\ b & a & b & c & d & c & f & e & f & e & f & e \\ c & b & a & b & c & d & e & f & e & f & e & f \\ d & c & b & a & b & c & f & e & f & e & f & e \\ c & d & c & b & a & b & e & f & e & f & e & f \\ b & c & d & c & b & a & f & e & f & e & f & e \\ \hline e & f & e & f & e & f & a & b & c & d & c & b \\ f & e & f & e & f & e & b & a & b & c & d & c \\ e & f & e & f & e & f & c & b & a & b & c & d \\ f & e & f & e & f & e & d & c & b & a & b & c \\ e & f & e & f & e & f & c & d & c & b & a & b \\ f & e & f & e & f & e & b & c & d & c & b & a \end{array} \right],$$

with coefficients

$$\begin{aligned} \lambda_1 &= a + 2b + 2c + d + 3e + 3f, \quad \lambda_\alpha = a + 2b + 2c + d - 3e - 3f \\ \lambda_{\gamma^+} &= a - 2b + 2c - d + 3e - 3f, \quad \lambda_{\gamma^-} = a - 2b + 2c - d - 3e + 3f \\ \lambda_\beta &= b + a - c - d, \quad \lambda_{\beta^1} = a - c + d - b. \end{aligned}$$

*Example 2.16* ( $D_7$ ). The matrices in its center have the form

$$\Lambda_7 = \left[ \begin{array}{c|c} a & b & c & d & d & c & b & e & e & e & e & e & e \\ b & a & b & c & d & d & c & e & e & e & e & e & e \\ c & b & a & b & c & d & d & e & e & e & e & e & e \\ d & c & b & a & b & c & d & e & e & e & e & e & e \\ d & d & c & b & a & b & c & e & e & e & e & e & e \\ c & d & d & c & b & a & b & e & e & e & e & e & e \\ b & c & d & d & c & b & a & e & e & e & e & e & e \\ \hline e & e & e & e & e & e & e & a & b & c & d & d & c \\ e & e & e & e & e & e & e & b & a & b & c & d & d \\ e & e & e & e & e & e & e & c & b & a & b & c & d \\ e & e & e & e & e & e & e & d & c & b & a & b & c \\ e & e & e & e & e & e & e & d & d & c & b & a & b \\ e & e & e & e & e & e & e & c & d & d & c & b & a \\ e & e & e & e & e & e & e & b & c & d & d & c & b \end{array} \right].$$

Its coefficients are indexed by the symmetric, alternating, the fundamental two harmonics, specifically,

$$\begin{aligned} \lambda_1 &= a + 2b + 2c + 2d + 7e, \quad \lambda_\alpha = a + 2b + 2c + 2d - 7e, \\ \lambda_\beta &= a + 2 \cos(2/7\pi)b - 2 \cos(3/7\pi)c - 2 \cos(1/7\pi)d, \\ \lambda_{\beta^1} &= a - 2 \cos(3/7\pi)b - 2 \cos(1/7\pi)c + 2 \cos(2/7\pi)d, \\ \lambda_{\beta^2} &= a - 2 \cos(3/7\pi)d + 2 \cos(2/7\pi)c - 2 \cos(1/7\pi)b. \end{aligned}$$

## 2.12 Log-Transformed Multinomial Data

Here the dihedral data  $x$  is such that the underlying dihedral group is also the support for a multinomial distribution with probability parameters  $p$  based on  $N = \langle 1, x \rangle$  observations. Let  $\ell$  indicate the corresponding log count data, with components  $\ell_\tau = \log(x_\tau/N)$ , for  $\tau \in D_n$ , and  $D_p = \text{Diag}(\dots, p_\tau, \dots)$ . Then, e.g., [5, p.494],

**Proposition 2.17.**

$$\mathcal{L}[\sqrt{N}\mathcal{F}(\ell - \lambda)] \rightarrow \mathcal{N}(0, \mathcal{F}D_p^{-1}\mathcal{F}^* - \mathcal{F}ee'\mathcal{F}^*),$$

where  $\lambda_\tau = \log p_\tau$  are the components of  $\lambda$ .

In particular, here,

$$\mathcal{F}ee'\mathcal{F}^* = \text{Diag}(g^2, 0, \dots, 0).$$

## 2.13 Additional Remarks

1. Most of the definitions introduced in this chapter are, more precisely, definitions *on the left*, with corresponding definitions obtained by dihedral actions defined on the right;
2. The regular linearizations  $\langle x, \phi \rangle$  are particular types of *linearizations* in the sense of [1]. Specifically, if a finite group  $G$  acts on a set  $S$  and  $x \in \mathbb{C}^S$ , then

$$[T(\tau)x](s) = x(\tau^{-1}s)$$

gives a linear representation of  $G$  in  $\mathbb{C}^S$ , so that when  $S = G$  the regular linearization is

$$\phi_\tau(x)\sigma = x_{\tau^{-1}\sigma};$$

3. Proposition 2.10 on page 21 is a general result for finite groups and depends essentially of the orthogonality relations among irreducible characters, e.g., [4, p.50], [6, p.473].
4. Proposition 2.12 on page 25 is the central result underlying the algebraic aspects presented in these notes and a general property of semisimple algebras e.g., [4, Sect. 6.2].

## Problems

- 2.1. Describe (1) the action of the dihedral group  $D_3$  on the distinct vertices  $\{a, b, c\}$  of a regular triangle and (2) its action on the oriented edges

$$\{(a, b), (b, c), (c, a), (b, a), (a, c), (c, b)\}$$

of the triangle. Re-evaluate (1) and (2) above when  $a$  and  $b$  are indistinguishable.

**2.2.** Describe the distinct dihedral groups  $D_2$  as permutation subgroups of  $S_4$ . For example  $D_2$  can be generated both by  $\{(12), (34)\}$  and by  $\{(12)(34), (13)(24)\}$ .

**2.3.** Describe the distinct dihedral groups  $D_3$  as permutation subgroups of  $S_3$ .

**2.4.** Describe the distinct dihedral groups  $D_4$  as permutation subgroups of  $S_4$ .

**2.5.** Verify that rotations and reversals have opposite parity.

**2.6.** Describe the  $D_4$  trace indexing  $x_\tau = \text{tr} [\beta_\tau \Sigma]$  of an arbitrary  $2 \times 2$  covariance matrix.

**2.7.** Describe the  $D_4$  trace indexing  $x_\tau = \text{tr} [(\beta_\tau \otimes \beta_\tau) \Sigma]$  of an arbitrary  $4 \times 4$  covariance matrix.

**2.8.** Functional invariance can be characterized by its commutativity with a given symmetry operator. Show that when the two-dimensional Laplace operator  $\Delta f = \partial_{xx}f + \partial_{yy}f$  is applied to a function  $f(p)$  subject to rotation  $r(p)$  of its argument, then  $\Delta f(r(p)) = r(\Delta f(p))$ . That is,  $(\Delta f)r = r(\Delta f)$ .

**2.9.** Referring to Problem 1.2 on page 7, study the correspondence between the power set of  $n$  objects and the set of all  $n$ -ary words in length of  $n$ . Endowing the power set with the inclusion-exclusion multiplication turns it into an Abelian group. Carry on the Fourier analysis and evaluate the canonical projections.

**2.10.** Determine the dihedral invariants of  $D_n$  acting (by permutation) on the power set of  $n = 2, 3, 4, 5, 6$  objects. Study how the invariants depend on the chosen realization of  $D_n$ .

**2.11.** Determine the dihedral linearizations  $\langle x, \rho \rangle$  of the permutation representations  $\rho$  of  $D_n$  acting on  $\{1, \dots, n\}$ , for  $n = 2, 3, 4, 5, 6$ . Specify the particular realization of  $D_n$  adopted in the linearization.

**2.12.** Following with the definitions introduced on page 12, show that  $\beta_{\sigma \rtimes \tau}^k = \beta_\sigma^k \beta_\tau^k$ .

**2.13.** Starting with the  $D_4$  linearization

$$\langle x, \beta \rangle = \begin{bmatrix} 1 + h - r^2 - r^2 h & -r + rh + r^3 - r^3 h \\ r + rh - r^3 - r^3 h & 1 - h - r^2 + r^2 h \end{bmatrix}$$

of  $x$  at  $\beta$ , using the short notation  $x_\tau \equiv \tau$ , let each entry  $(B_{ij}^+, B_{ij}^-)$  of

$$B = \begin{bmatrix} (1 + h, r^2 + r^2 h) & (rh + r^3, r + r^3 h) \\ (r + rh, r^3 + r^3 h) & (1 + r^2 h, h + r^2) \end{bmatrix}$$

indicate the *signed* components of  $\langle x, \beta \rangle$ . Show that  $B$  transforms as

$$\begin{aligned} rB &= \begin{bmatrix} B_{21} & B_{22} \\ B_{11}^\perp & B_{12}^\perp \end{bmatrix}, \quad r^2B = \begin{bmatrix} B_{11}^\perp & B_{12}^\perp \\ B_{21}^\perp & B_{22}^\perp \end{bmatrix}, \\ r^3B &= \begin{bmatrix} B_{21}^\perp & B_{22}^\perp \\ B_{11} & B_{12} \end{bmatrix}, \quad hB = \begin{bmatrix} B_{11} & B_{12} \\ B_{21}^\perp & B_{22}^\perp \end{bmatrix}, \\ rhB &= \begin{bmatrix} B_{21} & B_{22} \\ B_{11} & B_{12} \end{bmatrix}, \quad r^2hB = \begin{bmatrix} B_{11}^\perp & B_{12}^\perp \\ B_{21} & B_{22} \end{bmatrix}, \quad r^3hB = \begin{bmatrix} B_{21}^\perp & B_{22}^\perp \\ B_{11}^\perp & B_{12}^\perp \end{bmatrix}, \end{aligned}$$

where  $B_{ij}^\perp$  indicates the transposed entry  $(B_{ij}^-, B_{ij}^+)$  of  $B_{ij}$ . Argue that the signed components of  $B$  and their complements are stable within each column space of  $\langle x, \beta \rangle$ .

**2.14.** Prove the algebraic orthogonal property in Proposition [2.10 on page 21](#).

Dihedral Fourier Analysis

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