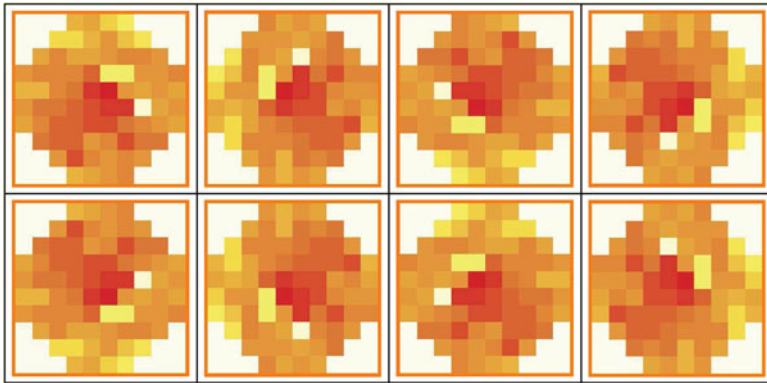


Preface

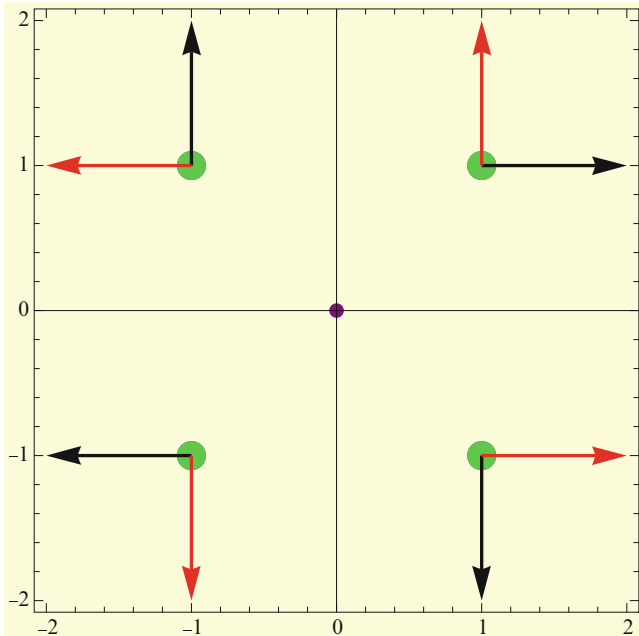
The theory and applications of dihedral Fourier analysis presented in these notes are aimed at the study of experimental data indexed by, or associated with, the points in a dihedral symmetry orbit such as the images of a retina's visual field. The images



are iterated 90° rotations and/or horizontal reflections of each other, and together, the resulting set of images gives a symmetry orbit generated by *any* one of the images. The orbit has the symmetries of the rotations $\{1, r, r^2, r^3\} = C_4$ and reversals $\{h, rh, r^2h, r^3h\} = C_4h$ that together define the dihedral group $D_4 = C_4 \cup C_4h$. Similarly, the vector field illustrated in the diagram below gives a D_4 orbit when the rotations in C_4 and reversals in C_4h are represented with the (homomorphic) substitutions

$$r \mapsto R = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}, \quad h \mapsto H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\phi = 2\pi/4$, and applied to the position (q) and the direction (p) of a given field vector (q, p), thus defining the dihedral orbit shown below



$$\mathcal{O} = \{(\tau q, \tau p); \tau \in D_4\}.$$

Here a reversal was defined as a counterclockwise rotation preceded by an x-axis reflection. Note that any two distinct-color vectors are line-reflection image of each other, whereas any two vectors of the same color are double-reflection (or rotation) image of each other. In that orbit, for example, we may want to evaluate the gradients

$$x_\tau = [\nabla f(\tau q)] \cdot \tau p, \quad \tau \in D_4,$$

of a response function f evaluated at τq in the direction of the vector τp along each field vector in the orbit \mathcal{O} , or the behavior of the center of mass

$$X_\sigma = \frac{\sum_\tau m_\tau \sigma F_\tau}{\sum_\tau m_\tau}, \quad \sigma \in D_4,$$

in the homogeneous field

$$F_\tau = (\tau q, p); \tau \in D_4,$$

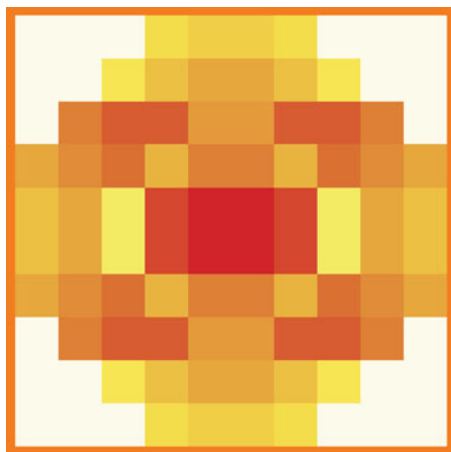
with fixed direction p and allocated masses m_τ .

The dihedral orbits generated by D_n , as planar transformations, are simply the $2n$ symmetries of an n -sided regular polygon for $n > 2$, whereas the planar transformations in $D_2 = \{1, r, h, rh\}$ describe the symmetries of a 180° rotation

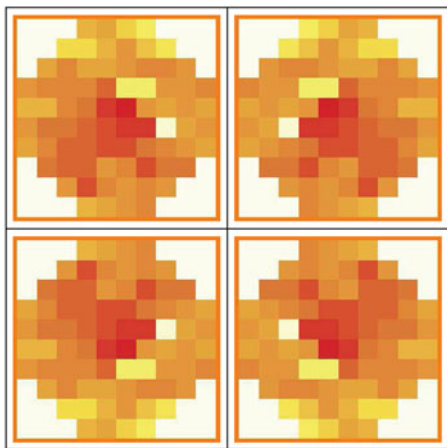
(r) and of two line reflections (h , $rh = v$) along the vertical and horizontal axes, respectively. We often write, then, $D_2 = \{1, r, h, v\}$.

The orbits illustrated above are generated by applying the symmetries $\tau \in D_4$ to any one of the eight components in the orbit, so that they are independent of the orbit's generator or initial condition.

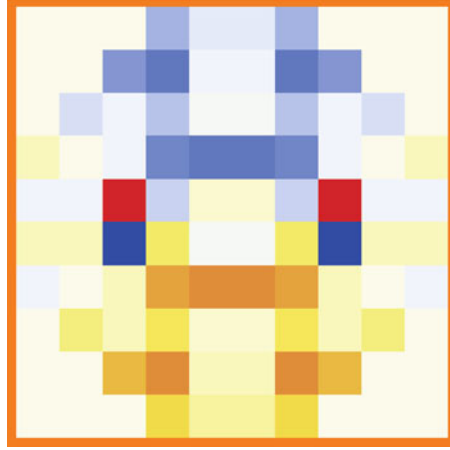
It is precisely this arbitrariness that is characteristic of the resulting *invariants* in any dihedral analysis. They are always invariant (in a way to be made precise throughout the text) to the different choices in initial conditions. Obviously, the image



resulting as the superposition $1 + r + h + v$ of the elements in the D_2 orbit



remains *the same* for all D_2 transformations, whereas the superposition $1 - r - h + v$



either stays the same, when transformed by 1 or v , or becomes *color-reversed* when transformed by r or h .

Every orbit \mathcal{O} is to be understood, in the present data-analytic context, as a framework, a set of labels for experimental observations, so that the analysis of any data indexed by a dihedral orbit consists of systematically determining the summaries of the data allowed by the orbit invariants, suggesting their statistical analysis, and proposing eventual broad-range interpretations.

The points in the data space for any dihedral analysis will often be expressed as the symbolic linear combinations

$$x = \sum_{\tau} x_{\tau} \tau, \tau \in D_n$$

in which the $2n$ coefficients x_{τ} are the data of interest along the orbit. The x_{τ} are naturally real (\mathbb{R}) scalar measurements, eventually extracted from complex coefficients. In a few cases, the x_{τ} will be points in the ring \mathcal{R} of $n \times n$ real data matrices. We write, respectively, $\mathbb{C}D_n$ or $\mathcal{R}D_n$ to indicate the dihedral (group) algebra and the dihedral (group) rings. They are complex vector spaces endowed with a multiplication induced by the dihedral multiplication. Details will be presented in Chap. 2.

As it will turn out, every dihedral algebra shall correspond to a direct sum of irreducible (matrix) linear subspaces

$$M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_m}(\mathbb{C}) \simeq \mathbb{C}D_n$$

of dimensions n_1^2, \dots, n_m^2 that is isomorphic to the original data vector space $\mathbb{C}D_n$, in dimension equal to the order $(2n)$ of D_n . In the dihedral ring case, the points in the linear subspaces are matrices over the ring \mathcal{R} .

The linear subspaces are stable under the different choices of orbit generators or initial conditions and are indexed by the set (\widehat{D}_n) of irreducible representations (ξ) of D_n . For statistical inference, a sample of these orbital data would then be obtained.

The explicit formulation of the orbit invariants, sometimes referred to as analytical properties, is the result of evaluating the dihedral Fourier transforms, to be defined as the linearizations

$$\langle x, \xi \rangle = \sum_{\tau \in D_n} x_\tau \xi_\tau \in M_{n_\xi}(\mathbb{C})$$

of x that appear as the components in the direct-sum factorization of $\mathbb{C}D_n$, one for each irreducible representation ξ in dimension of n_ξ . The isomorphic property of the factorization ensures the existence of the Fourier-inverse relation. The formulation of the Fourier transforms as direct-sum components of $\mathbb{C}D_n$ is developed in Chap. 2. In that same chapter we will show that from the full spectral information

$$\{\langle x, \xi \rangle : \xi \in \widehat{D}_n\}$$

on x , a unitary basis \mathcal{F} for $\mathbb{C}D_n$ can always be constructed, relative to which the analysis of the dihedral decomposition of $\|x\|^2$ becomes (theoretically and computationally) simplified, so that the analysis of variance

$$\|x\|^2 = \sum_{\xi} \frac{n_\xi}{2n} \|\langle x, \xi \rangle\|^2,$$

can be obtained directly from the Fourier transforms as Parseval's equalities for the dihedral groups.

Finally, we remark that when the dihedral multiplication is replaced by the additive structure of \mathbb{R}^n then

$$\langle x, \lambda \rangle = \int x_r \xi_r^\lambda dr,$$

where $\xi_r^\lambda = \exp(2\pi i \lambda \cdot r)$, for $r, \lambda \in \mathbb{R}^n$, is the usual Fourier transform of $x \in \mathbb{C}^{\mathbb{R}^n}$ evaluated at the irreducible representations ξ^λ . As suggested in [1], it is then natural to refer to the space of \widehat{D}_n as the dual space of D_n .

These lecture notes are divided as follows: Chap. 1 is an overview of the theory and methods of dihedral analysis. It introduces data sets and examples defining and connecting the algebraic notions of symmetry with those of statistical summaries and inference. Chapter 2 includes the required algebraic aspects and data-analytic results. Applications are developed in Chaps. 3–6.

Dihedral Fourier Analysis

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