Chapter 2
Mechanics Problems from the Functional Analysis Viewpoint

In the past, an engineer could calculate mechanical stresses and strains using a pencil and a logarithmic slide rule. Modern mechanical models, on the other hand, are nonlinear, and even the linear models are complicated. Numerical methods in structural dynamics cannot be applied without computers running specialized programs. However, a researcher should have a solid grasp of the equations that underlie a numerical model and the types of results that can be expected. New models appear in mechanics on a regular basis. Some of these, when written out in detail, can span multiple pages and are clearly beyond pencil-and-paper approaches. Although functional analysis does not provide a detailed picture of the results to be expected from a complicated model, it can answer questions regarding whether the problem is mathematically well-posed (e.g., whether a solution exists and is unique). It may also indicate whether the results can be obtained by a general computer program or whether a special program, based on a knowledge of the general properties of the model, is required. In other words, functional analysis can provide valuable insight even to those who rely heavily on numerical approaches.

2.1 Introduction to Sobolev Spaces

In his famous book [33], S.L. Sobolev (1908–1989) introduced some normed spaces that now bear his name; they are denoted by $W^{m,p}(\Omega)$. The norm in $W^{m,p}(\Omega)$ is

$$\|u\| = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u|^p \, d\Omega \right)^{1/p}, \tag{2.1.1}$$

where m is an integer, $p \ge 1$, and Ω is compact in $\mathbb{R}^n$. (The D^α notation was introduced on p. 19.) Indeed this is a norm on the set of functions that possess continuous derivatives up to order m on Ω, with satisfaction of axiom N3 ensured by Minkowski's inequality (1.3.6). The completion of the resulting normed space is the Banach space $W^{m,p}(\Omega)$.

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For Ω a segment $[a, b]$, the spaces $W^{m,p}(a, b)$ were introduced by Stefan Banach (1892–1945) in his dissertation. Our interest in Sobolev spaces is clear, as the elements of each of our energy spaces will belong to $W^{m,2}(\Omega)$ for some m.

Problem 2.1.1. (a) Demonstrate that if $f(x) \in W^{1,2}(a, b)$ and $g(x) \in C^{(1)}(a, b)$, then $f(x)g(x) \in W^{1,2}(a, b)$. (b) Show that $Af(x) = f(x)$ defines a bounded linear operator acting from $W^{1,2}(0, 1)$ to $L^2(0, 1)$. □

Generalized Notions of Derivative. For $u \in L^p(\Omega)$, K.O. Friedrichs (1901–1982) [12] introduced the notion of *strong derivative*, calling $v \in L^p(\Omega)$ a strong derivative $D^\alpha(u)$ if there is a sequence $\{\varphi_n\} \subset C^{(\infty)}(\Omega)$ such that

$$\int_\Omega |u(\mathbf{x}) - \varphi_n(\mathbf{x})|^p \, d\Omega \to 0 \quad \text{and} \quad \int_\Omega |v(\mathbf{x}) - D^\alpha \varphi_n(\mathbf{x})|^p \, d\Omega \to 0 \ \text{ as } n \to \infty \, .$$

Since $C^{(\infty)}(\Omega)$ is dense in any $C^{(k)}(\Omega)$, an element of $W^{m,p}(\Omega)$ has all strong derivatives up to the order m lying in $L^p(\Omega)$.

Another version of generalized derivative was proposed by Sobolev. He used, along with the classical integration by parts formula, an idea from the classical calculus of variations: if

$$\int_\Omega u(\mathbf{x}) \, \varphi(\mathbf{x}) \, d\Omega = 0$$

for all infinitely differentiable functions $\varphi(\mathbf{x})$ having compact support in an open domain Ω, then $u(\mathbf{x}) = 0$ almost everywhere (everywhere if $u(\mathbf{x})$ is to be continuous). We say that a function $\varphi(\mathbf{x})$, infinitely differentiable in Ω, has *compact support* in $\Omega \subset \mathbb{R}^n$ if the closure of the set

$$M = \{\mathbf{x} \in \Omega : \varphi(\mathbf{x}) \neq 0\}$$

is a compact set in Ω. In the sense introduced by Sobolev, $v \in L^p(\Omega)$ is called a *weak derivative* $D^\alpha u$ of $u \in L^p(\Omega)$ if for every infinitely differentiable function $\varphi(\mathbf{x})$ with compact support in Ω we have

$$\int_\Omega u(\mathbf{x}) \, D^\alpha \varphi(\mathbf{x}) \, d\Omega = (-1)^{|\alpha|} \int_\Omega v(\mathbf{x}) \, \varphi(\mathbf{x}) \, d\Omega \, . \tag{2.1.2}$$

The two definitions of generalized derivative are equivalent [33]. The proof would exceed the scope of our presentation, and the same is true for some other facts of this section.

Imbedding Theorems. The results presented next are particular cases of *Sobolev's imbedding theorem*. Later we will give proofs of some particular results when studying certain energy spaces for elastic models which turn out to be subspaces of Sobolev spaces. The theorems are formulated in terms of imbedding operators, a notion elucidated further in Sect. 2.2.

We assume that the compact set $\Omega \subset \mathbb{R}^n$ satisfies the *cone condition*. This means there is a finite circular cone in $\mathbb{R}^n$ such that any point of the boundary of Ω can be touched by the vertex of the cone while the cone lies fully inside Ω. This is the condition under which Sobolev's imbedding theorem is proved. We denote by Ω_r an r-dimensional piecewise smooth hypersurface in Ω. (This means that, at any point of smoothness, in a local coordinate system, it is described by functions having all derivatives continuous up to order m locally, if we consider $W^{m,p}(\Omega)$.)

The theory of Sobolev spaces is a substantial branch of mathematics (see Adams [1], Lions and Magenes [25], etc.). We formulate only what is needed for our purposes. This is Sobolev's imbedding theorem with some extensions:

Theorem 2.1.1 (Sobolev–Kondrashov). The imbedding operator of the Sobolev space $W^{m,p}(\Omega)$ to $L^q(\Omega_r)$ is continuous if one of the following conditions holds:

(i) $n > mp$, $r > n - mp$, $q \le pr/(n - mp)$;
(ii) $n = mp$, q is finite with $q \ge 1$.

If $n < mp$, then the space $W^{m,p}(\Omega)$ is imbedded into the Hölder space $H^\alpha(\overline{\Omega})$ when $\alpha \le (mp - n)/p$, and the imbedding operator is continuous.

The imbedding operator of $W^{m,p}(\Omega)$ to $L^q(\Omega_r)$ is compact (i.e., takes every bounded set of $W^{m,p}(\Omega)$ into a precompact set of the corresponding space[1]) if

(i) $n > mp$, $r > n - mp$, $q < pr/(n - mp)$ or
(ii) $n = mp$ and q is finite with $q \ge 1$.

If $n < mp$ then the imbedding operator is compact to $H^\alpha(\overline{\Omega})$ when $\alpha < (mp - n)/p$.

Note that this theorem provides imbedding properties not only for functions but also for their derivatives:

$$u \in W^{m,p}(\Omega) \implies D^\alpha u \in W^{m-k,p}(\Omega) \text{ for } |\alpha| = k\,.$$

Also available are stricter results on the imbedding of Sobolev spaces on Ω into function spaces given on manifolds Ω_r of dimension less than n; however, such *trace theorems* require an extended notion of Sobolev spaces.

Let us formulate some useful consequences of Theorem 2.1.1.

Theorem 2.1.2. Let γ be a piecewise differentiable curve in a compact set $\Omega \subset \mathbb{R}^2$. For any finite $q \ge 1$, the imbedding operator of $W^{1,2}(\Omega)$ to the spaces $L^q(\Omega)$ and $L^q(\gamma)$ is continuous (and compact), i.e.,

$$\max\{\|u\|_{L^q(\Omega)}, \|u\|_{L^q(\gamma)}\} \le m\,\|u\|_{W^{1,2}(\Omega)} \tag{2.1.3}$$

with a constant m which does not depend on $u(\mathbf{x})$.

Theorem 2.1.3. Let $\Omega \subset \mathbb{R}^2$ be compact. If $\alpha \le 1$, the imbedding operator of $W^{2,2}(\Omega)$ to $H^\alpha(\overline{\Omega})$ is continuous; if $\alpha < 1$, it is compact. For the first derivatives, the imbedding operator to $L^q(\Omega)$ and $L^q(\gamma)$ is continuous (and compact) for any finite $q \ge 1$.

[1] The notion of compact operator will be explored in Chap. 3.

Theorem 2.1.4. Let γ be a piecewise smooth surface in a compact set $\Omega \subset \mathbb{R}^3$. The imbedding operator of $W^{1,2}(\Omega)$ to $L^q(\Omega)$ when $1 \le q \le 6$, and to $L^p(\gamma)$ when $1 \le p \le 4$, is continuous; if $1 \le q < 6$ or $1 \le p < 4$, respectively, then it is compact.

We merely indicate how such theorems are proved. We will establish similar results for the beam problem (see (2.3.5)) and for the clamped plate problem (see (2.3.21)) via integral representations of functions from certain classes. In like manner, Sobolev's original proof is given for Ω a union of bounded star-shaped domains. A domain is called *star-shaped with respect to a ball B* if any ray with origin in B intersects the boundary of the domain only once. For a domain Ω which is bounded and star-shaped with respect to a ball B, a function $u(\mathbf{x}) \in C^{(m)}(\Omega)$ can be represented in the form

$$\begin{aligned} u(\mathbf{x}) = &\sum_{|\alpha|\le m-1} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \int_B K_\alpha(\mathbf{y})\, u(\mathbf{y})\; d\Omega \\ &+ \int_\Omega \frac{1}{|\mathbf{x}-\mathbf{y}|^{n-m}} \sum_{|\alpha|=m} K_\alpha(\mathbf{x},\mathbf{y})\, D^\alpha u(\mathbf{y})\; d\Omega_{\mathbf{y}} \end{aligned} \tag{2.1.4}$$

where $K_\alpha(\mathbf{y})$ and $K_\alpha(\mathbf{x},\mathbf{y})$ are continuous functions. Investigating properties of the integral terms on the right-hand side of the representation (2.1.4), Sobolev formulated his results; later they were extended to more general domains.

Another method is connected with the Fourier transformation of functions. In the case of $W^{m,2}(\Omega)$, it is necessary to extend functions of $C^{(m)}(\overline{\Omega})$ outside Ω in such a way that they belong to $C^m(\mathbb{R}^n)$ and $W^{m,2}(\mathbb{R}^n)$. Then using the Fourier transformation

$$\hat{u}(\mathbf{y}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\mathbf{y}} u(\mathbf{x})\, dx_1 \cdots dx_n$$

along with the facts that

$$\|u(\mathbf{x})\|_{L^2(\mathbb{R}^n)} = \|\hat{u}(\mathbf{y})\|_{L^2(\mathbb{R}^n)}$$

and

$$\widehat{D^\alpha u}(\mathbf{x}) = (iy_1)^{\alpha_1} \cdots (iy_n)^{\alpha_n}\, \hat{u}(\mathbf{y})$$

for $u \in L^2(\mathbb{R}^n)$, we can present the norm in $W^{m,2}(\Omega)$ in the form

$$\|u(\mathbf{x})\|^2_{W^{m,2}(\mathbb{R}^n)} = \sum_{|\alpha|\le m} \|y_1^{\alpha_1} \cdots y_n^{\alpha_n}\, \hat{u}(\mathbf{y})\|^2_{L^2(\mathbb{R}^n)} .$$

We can then study the properties of the weighted space $L^2_w(\mathbb{R}^n)$; this transformed problem is simpler, as many of the problems involved are algebraic estimates of Fourier images.

Moreover, we can consider $W^{p,2}(\mathbb{R}^n)$ with fractional indices p. These lead to necessary and sufficient conditions for the trace problem: given $W^{m,2}(\Omega)$, find the space $W^{p,2}(\partial\Omega)$ in which $W^{m,2}(\Omega)$ is continuously imbedded. The inverse trace problem is, given $W^{p,2}(\Omega)$, find the maximal index m such that every element $u \in W^{p,2}(\partial\Omega)$

can be extended to $\overline{\Omega}$, $u^* \in W^{m,2}(\Omega)$, in such a way that

$$||u^*||_{W^{m,2}(\Omega)} \le c\,||u||_{W^{p,2}(\partial\Omega)}\ .$$

In this way, many results from the contemporary theory of elliptic (and other types of) equations and systems are obtained. We should mention that the trace theorems are formulated mostly for smooth manifolds, hence are not applicable to practical problems involving domains with corners.

As stated in Lemma 1.16.3, all Sobolev spaces are separable. The same holds for all the energy spaces we introduce.

2.2 Operator of Imbedding

Let Ω be a Jordan measurable compact set in $\mathbb{R}^n$. Any element of $C^{(k)}(\Omega)$ also belongs to $C(\Omega)$. This correspondence — between an element of the space $C^{(k)}(\Omega)$ and the same element in the space $C(\Omega)$ — is a linear operator. Although it resembles an identity operator, its domain and range differ and it is called the *imbedding operator* from $C^{(k)}(\Omega)$ to $C(\Omega)$. Clearly, $C^{(k)}(\Omega)$ is a proper subset of $C(\Omega)$; this situation is typical for an imbedding operator that gives the correspondence between the same elements but considered in different spaces. For a function u that belongs to both spaces, we have

$$||u||_{C(\Omega)} \le ||u||_{C^{(k)}(\Omega)}\ ,$$

which shows that the imbedding operator is bounded or continuous (what can be said about its norm?). Moreover, by a consequence of Arzelà's theorem, it is compact.

We get a similar imbedding operator when considering the spaces ℓ^p and ℓ^q for $1 \le p < q$. As (1.4.3) states, any element $\mathbf{x}$ of ℓ^p belongs to ℓ^q and

$$||\mathbf{x}||_{\ell^q} \le c_1\,||\mathbf{x}||_{\ell^p}\ .$$

Here and below, the constants c_k are independent of the element of the spaces, so the imbedding operator from ℓ^p to ℓ^q is bounded.

We encounter another imbedding operator when considering the correspondence between the same equivalence classes in the spaces $L^p(\Omega)$ and $L^q(\Omega)$: if $1 \le p < q$, then any element u of $L^q(\Omega)$ belongs to $L^p(\Omega)$ and

$$||u||_{L^p(\Omega)} \le c_2\,||u||_{L^q(\Omega)}\ .$$

So the imbedding operator from $L^q(\Omega)$ to $L^p(\Omega)$ is also bounded. (Note that the relations between p and q are different in the spaces ℓ^r and $L^r(\Omega)$.)

The situation is slightly different when we consider the relation between the spaces $C(\Omega)$ and $L^p(\Omega)$. It is clear that any function u from $C(\Omega)$ has a finite norm in $L^p(\Omega)$. To use the imbedding idea, we identify the function u with the equivalence class from $L^p(\Omega)$ that contains the stationary sequence $(u, u, \ldots)$, and in this way consider u as an element of $L^p(\Omega)$. Because

$$\int_\Omega u\, d\Omega \le c_3 \|u\|_{C(\Omega)} ,$$

the imbedding operator from $C(\Omega)$ to $L^p(\Omega)$ is bounded for any $1 \le p < \infty$.

This practice of identification must be extended when we consider imbedding operators involving Sobolev spaces as treated in this book. In the imbedding from $W^{1,2}(\Omega)$ to $L^p(\Omega)$, the elements of $W^{1,2}(\Omega)$ — the equivalence classes of functions from $C^{(1)}(\Omega)$ — are different from the equivalence classes from $C(\Omega)$ that constitute $L^p(\Omega)$; however, each class from $C^{(1)}(\Omega)$ is contained by some class from $C(\Omega)$, and we identify these classes. In this sense, we say that $W^{1,2}(\Omega)$ is imbedded into $L^p(\Omega)$ and the imbedding operator is continuous (or even compact).

The imbedding of a Sobolev space into the space of continuous functions is even more complicated, although the general ideas are the same. We will consider this in more detail when dealing with the energy space for a plate. In this case we cannot directly identify an element of a Sobolev space, which is an equivalence class, with a continuous function. However, we observe that for a class of equivalent sequences in the norm of a Sobolev space, for any sequence that enters into the class, there is a limit element, a continuous function that does not differ for other representative sequences, and we identify the whole class with this function. The inequality relating the Sobolev norm of the class and the continuous norm of this function states that this identification or correspondence is a bounded operator whose linearity is obvious. Some other inequalities state that the operator is compact.

2.3 Some Energy Spaces

A Beam. Earlier we noted that the set S of all real-valued continuous functions $y(x)$ having continuous first and second derivatives on $[0, l]$ and satisfying the boundary conditions

$$y(0) = y'(0) = y(l) = y'(l) = 0 \tag{2.3.1}$$

is a metric space under the metric

$$d(y, z) = \left(\frac{1}{2}\int_0^l B(x)[y''(x) - z''(x)]^2\, dx\right)^{1/2}. \tag{2.3.2}$$

We called this an energy space for the clamped beam. We can introduce an inner product

$$(y, z) = \frac{1}{2}\int_0^l B(x)y''(x)z''(x)\, dx \tag{2.3.3}$$

and norm

$$\|y\| = \left(\frac{1}{2}\int_0^l B(x)[y''(x)]^2\, dx\right)^{1/2}$$

such that $d(y,z) = \|y - z\|$. But this space is not complete (it is clear that there are Cauchy sequences whose limits do not belong to $C^{(2)}(0,l)$; the reader should construct an example). To have a complete space, we must apply the completion theorem. The actual energy space denoted by E_B is the completion of S in the metric (2.3.2) (or, what amounts to the same thing, in the inner product (2.3.3)).

Let us consider some properties of the elements of E_B. An element $y(x) \in E_B$ is a set of Cauchy sequences equivalent in the metric (2.3.2). Let $\{y_n(x)\}$ be a representative sequence of $y(x)$. Then, according to (2.3.2),

$$\left(\frac{1}{2}\int_0^l B(x)[y_n''(x) - y_m''(x)]^2\,dx\right)^{1/2} \to 0 \ \text{ as } n, m \to \infty\,.$$

If we assume that

$$0 < m_1 \le B(x) \le m_2\,,$$

then the sequence of second derivatives $\{y_n''(x)\}$ is a Cauchy sequence in the norm of $L^2(0,l)$ because

$$m_1\int_0^l [y_n''(x) - y_m''(x)]^2\,dx \le \int_0^l B(x)[y_n''(x) - y_m''(x)]^2\,dx\,.$$

Hence $\{y_n''(x)\}$ is a representative sequence of some element of $L^2(0,l)$, and we can regard $y \in E_b$ as having a second derivative that belongs to $L^2(0,l)$.

Now consider $\{y_n'(x)\}$. For any $y(x) \in S$ we get

$$y'(x) = \int_0^x y''(t)\,dt\,.$$

So for a representative $\{y_n(x)\}$ of a class $y(x) \in E_B$ we have

$$\begin{aligned}
|y_n'(x) - y_m'(x)| &\le \int_0^x |y_n''(t) - y_m''(t)|\,dt \le \int_0^l 1\cdot|y_n''(t) - y_m''(t)|\,dt \\
&\le l^{1/2}\left(\int_0^l [y_n''(x) - y_m''(x)]^2\,dx\right)^{1/2} \\
&\le (l/m_1)^{1/2}\left(\int_0^l B(x)[y_n''(x) - y_m''(x)]^2\,dx\right)^{1/2} \\
&\to 0 \ \text{ as } n, m \to \infty\,.
\end{aligned} \tag{2.3.4}$$

It follows that $\{y_n'(x)\}$ converges uniformly on $[0,l]$; hence there exists a limit function $z(x)$ which is continuous on $[0,l]$. This function does not depend on the choice of representative sequence (Problem 2.3.1 below). The same holds for a sequence of functions $\{y_n(x)\}$: its limit is a function $y(x)$ continuous on $[0,l]$. Moreover,

$$y'(x) = z(x)\,.$$

To prove this, it is necessary to repeat the arguments of Sect. 1.11 on the differentiability of the elements of $C^{(k)}(\Omega)$, with due regard for (2.3.4). From (2.3.4) and the similar inequality for $\{y_n(x)\}$ we get

$$\max_{x\in[0,l]} (|y(x)| + |y'(x)|) \le m\left(\frac{1}{2}\int_0^l B(x)[y''(x)]^2\,dx\right)^{1/2} \tag{2.3.5}$$

for some constant m independent of $y(x) \in E_B$. So each element $y(x) \in E_B$ can be identified with an element $y(x) \in C^{(1)}(0,l)$ in such a way that (2.3.5) holds. This correspondence is an imbedding operator, and we interpret (2.3.5) as a statement that the imbedding operator from E_B to $C^{(1)}(0,l)$ is continuous (cf., Sect. 1.21). Henceforth we refer to the elements of E_B as if they were continuously differentiable functions, attaching the properties of the uniquely determined limit functions to the corresponding elements of E_B themselves.

Problem 2.3.1. Prove that the function $z(x)$, discussed above, is independent of the choice of representative sequence. □

We are interested in analyzing all terms that appear in the statement of the equilibrium problem for a body as a minimum potential energy problem. So we will consider the functional giving the work of external forces. For the beam, it is

$$A = \int_0^l F(x)y(x)\,dx\,.$$

This is well-defined on E_B if $F(x) \in L(\Omega)$; moreover, (2.3.5) shows that the work of external forces can contain terms of the form

$$\sum_k [F_k y(x_k) + M_k y'(x_k)]\,,$$

which can be interpreted as the work of point forces and point moments, respectively. This is a consequence of the continuity of the imbedding from E_B to $C^{(1)}(0,l)$.

Remark 2.3.1. Alternatively we can define E_B on a base set S_1 of smoother functions, in $C^{(4)}(0,l)$ say, satisfying (2.3.1). The result is the same, since S_1 is dense in S with respect to the norm of $C^{(2)}(0,l)$. However, sometimes such a definition is convenient. □

Remark 2.3.2. Readers familiar with the contemporary literature in this area may have noticed that Western authors usually deal with Sobolev spaces, studying the properties of operators corresponding to problems under consideration; we prefer to deal with energy spaces, studying first their properties and then those of the corresponding operators. Although these approaches lead to the same results, in our view the physics of a particular problem should play a principal role in the analysis — in this way the methodology seems simpler, clearer, and more natural. In papers devoted to the study of elastic bodies, we mainly find interest in the case of a clamped boundary. Sometimes this is done on the principle that it is better to deal solely with homogeneous Dirichlet boundary conditions, but often it is an unfortunate consequence of the use of the Sobolev spaces $H^k(\Omega)$. The theory of these spaces is well developed but is not amenable to the study of other boundary conditions. Success in the investigation of mechanics problems can be much more difficult without the benefit of the physical ideas brought out by the energy spaces. □

Remark 2.3.3. In defining the energy space of the beam, we left aside the question of smoothness of the stiffness function $B(x)$. From a mathematical standpoint this is risky since, in principle, $B(x)$ can be nonintegrable. But in the case of an actual physical beam, $B(x)$ can have no more than a finite number of discontinuities and must be differentiable everywhere else. For simplicity, we shall continue to make realistic assumptions concerning physical parameters such as stiffness and elastic constants; in particular, we shall suppose whatever degree of smoothness is required for our purposes. □

A Membrane (Clamped Edge). The subset of $C^{(1)}(\Omega)$ consisting of all functions satisfying

$$u(x,y)\big|_{\partial\Omega} = 0 \tag{2.3.6}$$

with the metric

$$d(u,v) = \left\{ \iint_\Omega \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right)^2 \right] dx\,dy \right\}^{1/2} \tag{2.3.7}$$

is an incomplete metric space. If we introduce an inner product

$$(u,v) = \iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\,dy \tag{2.3.8}$$

consistent with (2.3.7), we get an inner product space. Its completion in the metric (2.3.7) is the energy space E_{MC} for the clamped membrane, a real Hilbert space.

What can we say about the elements $U(x,y) \in E_{MC}$? If $\{u_n(x,y)\}$ is a representative of $U(x,y)$, then

$$\left\{ \iint_\Omega \left[\left(\frac{\partial u_n}{\partial x} - \frac{\partial u_m}{\partial x} \right)^2 + \left(\frac{\partial u_n}{\partial y} - \frac{\partial u_m}{\partial y} \right)^2 \right] dx\,dy \right\}^{1/2} \to 0 \text{ as } n,m \to \infty\,.$$

and we see that the sequences of first derivatives $\{\partial u_n/\partial x\}$, $\{\partial u_n/\partial y\}$ are Cauchy sequences in the norm of $L^2(\Omega)$. What about $U(x,y)$ itself? If we extend each $u_n(x,y)$ by zero outside Ω, we can write

$$u_n(x,y) = \int_0^x \frac{\partial u_n(s,y)}{\partial s}\,ds$$

(assuming, without loss of generality, that Ω is confined to the band $0 \le x \le a$). Squaring both sides and integrating over Ω, we get

$$\begin{aligned}
\iint_\Omega u_n^2(x,y)\,dx\,dy &= \iint_\Omega \left[\int_0^x \frac{\partial u_n(s,y)}{\partial s}\,ds\right]^2 dx\,dy \\
&\le \iint_\Omega \left[\int_0^a 1\cdot\left|\frac{\partial u_n(s,y)}{\partial s}\right| ds\right]^2 dx\,dy \\
&\le \iint_\Omega \left[a\int_0^a \left(\frac{\partial u_n(s,y)}{\partial s}\right)^2 ds\right] dx\,dy \\
&\le a^2 \iint_\Omega \left(\frac{\partial u_n(x,y)}{\partial x}\right)^2 dx\,dy\,.
\end{aligned}$$

This means that if $\{\partial u_n/\partial x\}$ is a Cauchy sequence in the norm of $L^2(\Omega)$, then so is $\{u_n\}$. Hence we can consider elements $U(x,y) \in E_{MC}$ to be such that $U(x,y)$, $\partial U/\partial x$, and $\partial U/\partial y$ belong to $L^2(\Omega)$.

As a consequence of the last chain of inequalities, we get *Friedrichs' inequality*

$$\iint_\Omega U^2(x,y)\,dx\,dy \le m \iint_\Omega \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2\right] dx\,dy$$

which holds for any $U(x,y) \in E_{MC}$ and a constant m independent of $U(x,y)$.

Membrane (Free Edge). Although it is natural to introduce the energy space using the energy metric (2.3.7), we cannot distinguish between two states $u_1(x,y)$ and $u_2(x,y)$ of the membrane with free edge if

$$u_2(x,y) - u_1(x,y) = c = \text{constant}\,.$$

This is the only form of "rigid" displacement possible for a membrane. We first show that no other rigid displacements (i.e., displacements associated with zero strain energy) are possible. The proof is a consequence of *Poincaré's inequality*

$$\iint_\Omega u^2\,dx\,dy \le m\left\{\left(\iint_\Omega u\,dx\,dy\right)^2 + \iint_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy\right\} \tag{2.3.9}$$

for a function $u(x,y) \in C^{(1)}(\Omega)$. The constant m does not depend on $u(x,y)$.

Proof of Poincaré's inequality [9]. We first assume that Ω is the square $[0,a]\times[0,a]$ and write down the identity

$$u(x_2,y_2) - u(x_1,y_1) = \int_{x_1}^{x_2} \frac{\partial u(s,y_1)}{\partial s}\,ds + \int_{y_1}^{y_2} \frac{\partial u(x_2,t)}{\partial t}\,dt\,.$$

Squaring both sides and then integrating over the square, first with respect to the variables x_1 and y_1 and then with respect to x_2 and y_2, we get

$$
\begin{aligned}
&\iint_{\Omega}\iint_{\Omega}\left[u^2(x_2,y_2)-2u(x_2,y_2)u(x_1,y_1)+u^2(x_1,y_1)\right]dx_1\,dy_1\,dx_2\,dy_2\\
&=\iint_{\Omega}\iint_{\Omega}\left[\int_{x_1}^{x_2}\frac{\partial u(s,y_1)}{\partial s}\,ds+\int_{y_1}^{y_2}\frac{\partial u(x_2,t)}{\partial t}\,dt\right]^2dx_1\,dy_1\,dx_2\,dy_2\\
&\le\iint_{\Omega}\iint_{\Omega}\left[\int_0^a 1\cdot\left|\frac{\partial u(s,y_1)}{\partial s}\right|ds+\int_0^a 1\cdot\left|\frac{\partial u(x_2,t)}{\partial t}\right|dt\right]^2dx_1\,dy_1\,dx_2\,dy_2\\
&\le 2a\iint_{\Omega}\iint_{\Omega}\left[\int_0^a\left(\frac{\partial u(s,y_1)}{\partial s}\right)^2ds+\int_0^a\left(\frac{\partial u(x_2,t)}{\partial t}\right)^2dt\right]dx_1\,dy_1\,dx_2\,dy_2\\
&\le 2a^4\int_0^a\int_0^a\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dx\,dy\;.
\end{aligned}
$$

Note that, along with the Schwarz inequality for integrals, we have used the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ which follows from the fact that $(a-b)^2 \ge 0$. The beginning of this chain of inequalities is

$$a^2\iint_{\Omega}u^2(x,y)\,dx\,dy-2\left(\iint_{\Omega}u(x,y)\,dx\,dy\right)^2+a^2\iint_{\Omega}u^2(x,y)\,dx\,dy$$

so

$$2a^2\iint_{\Omega}u^2\,dx\,dy\le 2\left(\iint_{\Omega}u\,dx\,dy\right)^2+2a^4\iint_{\Omega}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dx\,dy$$

and we get (2.3.9) with $m=\max(a^2,1/a^2)$. □

It can be shown that Poincaré's inequality holds on more general domains. A modification of (2.3.9) will appear in Sect. 2.12.

Let us return to the free membrane problem. Provided we consider only the membrane's state of stress, any two states are identical if they are described by functions $u_1(x,y)$ and $u_2(x,y)$ whose difference is constant. We gather all functions (such that the difference between any two is a constant) into a class denoted by $u_*(x,y)$. There is a unique representative of $u_*(x,y)$ denoted by $u_b(x,y)$ such that

$$\iint_{\Omega}u_b(x,y)\,dx\,dy=0\;. \tag{2.3.10}$$

For this *balanced representative* (or *balanced function*), Poincaré's inequality becomes

$$\iint_{\Omega}u_b^2(x,y)\,dx\,dy\le m\iint_{\Omega}\left[\left(\frac{\partial u_b}{\partial x}\right)^2+\left(\frac{\partial u_b}{\partial y}\right)^2\right]dx\,dy\;. \tag{2.3.11}$$

As the right-hand side is zero for a "rigid" displacement, so is the left-hand side and it follows that the balanced representative associated with a rigid displacement must be zero. Hence $u(x,y)=c$ is the only permissible form for a rigid displacement of a membrane.

Because (2.3.11) has the same form as Friedrichs' inequality, we can repeat our former arguments to construct the energy space E_{MF} for a free membrane using the balanced representatives of the classes $u_*(x, y)$. In what follows we shall use this space E_{MF}, remembering that its elements satisfy (2.3.10).

The condition (2.3.10) is a geometrical constraint resulting from our mathematical technique. Solving the static free membrane problem, we must remember that the formulation of the equilibrium problem does not impose this constraint — the membrane can move as a "rigid body" in the direction normal to its own surface. But if we consider only deformations and the strain energy defined by the first partial derivatives of $u(x, y)$, the results must be independent of such motions. Consider then the functional of the work of external forces

$$A = \iint_{\Omega} F(x, y)U(x, y)\, dx\, dy \ .$$

If we use the space E_{MF}, then A makes sense if $F(x, y) \in L^2(\Omega)$ (guaranteed by (2.3.11) together with the Schwarz inequality in $L^2(\Omega)$). This is the only restriction on external forces for a clamped membrane. However, in the case of equilibrium for a free membrane, the functional A must be invariant under transformations of the form $u(x, y) \mapsto u(x, y) + c$ with any constant c. This requires

$$\iint_{\Omega} F(x, y)\, dx\, dy = 0 \ . \tag{2.3.12}$$

Again, we consider the equilibrium problem where rigid motion, however, is possible. Since we did not introduce inertia forces, we have formally equated the mass of the membrane to zero. In this situation of zero mass, any forces with nonzero resultant would make the membrane as a whole move with infinite acceleration. Thus, (2.3.12) also precludes such physical nonsense.

Meanwhile, Sobolev's imbedding theorem permits us to incorporate forces ψ acting on the boundary S of Ω into the functional describing the work of external forces:

$$\iint_{\Omega} F(x, y)u(x, y)\, dx\, dy + \int_S \psi u\, ds \ . \tag{2.3.13}$$

In this case, the condition of invariance of the functional under constant motions yields the condition

$$\iint_{\Omega} F(x, y)\, dx\, dy + \int_S \psi\, ds = 0 \ . \tag{2.3.14}$$

Note that in the formulation of the Neumann problem, this quantity ψ appears in the boundary condition:

$$\left. \frac{\partial u}{\partial n} \right|_S = \psi \ .$$

The reader may ask why, if the membrane can move as a rigid body, it cannot rotate freely in the manner of an ordinary free rigid body. The answer is that in this model, unlike the other models for elastic bodies in this book, rotation of the membrane as a rigid body alters the membrane strain energy. In the membrane model, the deflection $u(x, y)$ is the deflection that is imposed on the prestressed state of the membrane. The membrane equilibrium problem is an example of a problem for a prestressed body.

There is another way to formulate the equilibrium problem for a free membrane, based on a different method of introducing the energy space. Let us return to Poincaré's inequality (2.3.9). Denote

$$\|u\|_1 = \left\{ \left(\iint_\Omega u\, dx\, dy \right)^2 + D(u) \right\}^{1/2}$$

where

$$D(u) = \iint_\Omega \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx\, dy\ ,$$

and recall that the norm in $W^{1,2}(\Omega)$ is

$$\|u\|_{W^{1,2}(\Omega)} = \left\{ \left(\iint_\Omega u^2\, dx\, dy \right)^2 + D(u) \right\}^{1/2} .$$

By (2.3.9), the norms $\|\cdot\|_1$ and $\|\cdot\|_{W^{1,2}(\Omega)}$ are equivalent on the set of continuously differentiable functions, and hence on $W^{1,2}(\Omega)$. Let us use the norm $\|\cdot\|_1$ on $W^{1,2}(\Omega)$. Now the space $W^{1,2}(\Omega)$ is the completion, with respect to the norm $\|\cdot\|_1$, of the functions continuously differentiable on $\overline{\Omega}$. Let us take an element $U(x, y) \in W^{1,2}(\Omega)$ and select from $U(x, y)$ an arbitrary representative Cauchy sequence $\{u_k(x, y)\}$. A smooth function $u_k(x, y)$ can be uniquely written in the form

$$u_k(x, y) = \tilde{u}_k(x, y) + a_k \quad \text{where} \quad \iint_\Omega \tilde{u}_k\, dx\, dy = 0\ .$$

Since

$$\|u_k - u_m\|_1^2 = \left(\iint_\Omega (a_k - a_m)\, dx\, dy \right)^2 + D(\tilde{u}_k - \tilde{u}_m) \to 0 \ \text{ as } k, m \to \infty\ ,$$

we see that $\{a_k\}$ is a numerical Cauchy sequence that has a limit corresponding to U. This means that $\{a_k\}$ belongs to c, the space of numerical sequences each of which has a limit. It is easy to show that this limit does not depend on the choice of representative sequence $\{u_k(x, y)\}$, and we can regard it as a rigid displacement of the membrane. Now, because the membrane is not geometrically fixed and can be moved through any uniform displacement with no change in energy, we place all the elements of $W^{1,2}(\Omega)$ that characterize a strained state of the membrane into the same class such that for any two elements $U'(x, y)$ and $U''(x, y)$ of the class and any representative sequences $\{u'_k(x, y)\}$ and $\{u''_k(x, y)\}$ taken from them, $\{\tilde{u}'_k(x, y)\}$ and $\{\tilde{u}''_k(x, y)\}$ are equivalent in the norm (i.e., $D(\tilde{u}'_k - \tilde{u}''_k) \to 0$) and the difference sequence $\{a'_k - a''_k\}$ is in c. This construction of the classes is equivalent to the construction of the factor space $W^{1,2}(\Omega)/c$, which can also be called the energy space for the free membrane. The zero of this energy factor space is the set of the elements of $W^{1,2}(\Omega)$ each containing as a representative an element from c. Clearly if we wish to have the energy functional defined in this space, the necessary condition for its continuity is that its linear part — the work of external forces — must be zero over any constant displacement. The latter involves the self-balance condition (2.3.12). It is seen that the principal part of any class-element of the new energy space, defined by the sequences $\{\tilde{u}_k(x,y)\}$, coincides with the corresponding sequences for the elements of the space E_{MF}; moreover, this correspondence between the two energy spaces maintains equal norms in both spaces. So we can repeat the procedure to establish the existence-uniqueness theorem in the new energy space, using the proof in E_{MF}.

The restriction (2.3.12) (or (2.3.14)) is necessary for the functional of external forces to be uniquely defined for an element $U_*(x,y)$. We shall use the same notation E_{MF} for this type of energy space since there is a one-to-one correspondence, preserving distances and inner products, between the two types of energy space for the free membrane. Moreover, we shall always make clear which version we mean.

Those familiar with the theory of the Neumann problem for Laplace's equation should note that the necessary condition for solvability that arises in mathematical physics as a mathematical consequence, i.e.,

$$\int_S \psi\, ds = 0\ ,$$

is a particular case of (2.3.14) when $F(x,y) = 0$. This means that for solvability of the problem, the external forces acting on the membrane edge should be self-balanced.

Finally, we note that Poisson's equation governs not only membranes, but also situations in electricity, magnetism, hydrodynamics, mathematical biology, and other fields. So we can consider spaces such as E_M in various other sciences. It is clear that the results will be the same.

We will proceed to introduce other energy spaces in a similar manner: they will be completions of corresponding metric (inner product) spaces consisting of smooth functions satisfying certain boundary conditions. The problem is to determine properties of the elements of those completions. As a rule, metrics must contain all the strain energy terms (we now discuss only linear systems). For example, we can consider a membrane whose edge is elastically supported; then we must include the energy of elastic support in the expression for the energy metric.

Bending a Plate. Here we begin with the work of internal forces on variations of displacements

$$-(w_1, w_2) = -\iint_\Omega D^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}(w_1)\,\rho_{\alpha\beta}(w_2)\,dx\,dy \tag{2.3.15}$$

where $w_1(x,y)$ is the normal displacement of the plate midsurface Ω, $w_2(x,y)$ can be considered as its variation, $\rho_{\alpha\beta}(u)$ are components of the change-of-curvature tensor,

$$\rho_{11}(u) = \frac{\partial^2 u}{\partial x^2}\,, \qquad \rho_{12} = \frac{\partial^2 u}{\partial x \partial y}\,, \qquad \rho_{22} = \frac{\partial^2 u}{\partial y^2}\,,$$

$D^{\alpha\beta\gamma\delta}$ are components of the tensor of elastic constants of the plate such that

$$D^{\alpha\beta\gamma\delta} = D^{\gamma\delta\alpha\beta} = D^{\beta\alpha\gamma\delta} \tag{2.3.16}$$

and, for any symmetric tensor $\rho_{\alpha\beta}$ there exists a constant $m_0 > 0$ such that

$$D^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}\,\rho_{\alpha\beta} \geq m_0 \sum_{\alpha,\beta=1}^{2} \rho_{\alpha\beta}^2\,. \tag{2.3.17}$$

We suppose the $D^{\alpha\beta\gamma\delta}$ are constants, but piecewise continuity of these parameters would be sufficient.

For the theory of shells and plates, Greek indices will assume values from the set $\{1, 2\}$ while Latin indices will assume values from the set $\{1, 2, 3\}$. The repeated index convention for summation is also in force. For example, we have

$$a^{\alpha\beta} b_{\alpha\beta} \equiv \sum_{\alpha,\beta=1}^{2} a^{\alpha\beta} b_{\alpha\beta}\,.$$

We first consider a plate with clamped edge $\partial\Omega$:

$$w\Big|_{\partial\Omega} = \frac{\partial w}{\partial n}\Big|_{\partial\Omega} = 0\,. \tag{2.3.18}$$

(Of course, the variation of w must satisfy (2.3.18) as well.) Let us show that on S_4, the subset of $C^{(4)}(\Omega)$ consisting of those functions which satisfy (2.3.18), the form (w_1, w_2) given in (2.3.15) is an inner product. We begin with the axiom P1:

$$\begin{aligned}(w, w) &= \iint_\Omega D^{\alpha\beta\gamma\delta}\rho_{\alpha\beta}(w)\,\rho_{\gamma\delta}(w)\,dx\,dy \geq m_0 \iint_\Omega \sum_{\alpha,\beta=1}^{2} \rho_{\alpha\beta}^2(w)\,dx\,dy \\ &= m_0 \iint_\Omega \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dx\,dy \geq 0\,.\end{aligned}$$

If $w = 0$ then $(w, w) = 0$. If $(w, w) = 0$ then, on Ω,

$$\frac{\partial^2 w}{\partial x^2} = 0\,, \qquad \frac{\partial^2 w}{\partial x \partial y} = 0\,, \qquad \frac{\partial^2 w}{\partial y^2} = 0\,.$$

It follows that

$$w(x, y) = a_1 + a_2 x + a_3 y\,,$$

where the a_i are constants. By (2.3.18) then, $w(x, y) = 0$. Hence P1 is satisfied. Satisfaction of P2 follows from (2.3.16), and it is evident that P3 is also satisfied.

Thus S_4 with inner product (2.3.15) is an inner product space; its completion in the corresponding metric is the energy space E_{PC} for a clamped plate.

Let us consider some properties of the elements of E_{PC}. It was shown that

$$m_0 \iint_\Omega \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right]^2 dx\, dy$$
$$\leq \iint_\Omega D^{\alpha\beta\gamma\delta} \rho_{\gamma\delta}(w)\, \rho_{\alpha\beta}(w)\, dx\, dy \equiv (w, w)\ . \qquad (2.3.19)$$

From this and the Friedrichs inequality, written first for w and then for the first derivatives of $w \in S_4$ as well, we get

$$\iint_\Omega w^2\, dx\, dy \leq m_1 \iint_\Omega \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dx\, dy$$
$$\leq m_2 \iint_\Omega \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] dx\, dy$$
$$\leq m_3 \iint_\Omega D^{\alpha\beta\gamma\delta} \rho_{\gamma\delta}(w)\, \rho_{\alpha\beta}(w)\, dx\, dy \equiv m_3\, (w, w)\ . \qquad (2.3.20)$$

Hence if $\{w_n\} \subset S_4$ is a Cauchy sequence in E_{PC}, then the sequences

$$\{w_n\}, \quad \left\{ \frac{\partial w_n}{\partial x} \right\}, \quad \left\{ \frac{\partial w_n}{\partial y} \right\}, \quad \left\{ \frac{\partial^2 w_n}{\partial x^2} \right\}, \quad \left\{ \frac{\partial^2 w_n}{\partial x \partial y} \right\}, \quad \left\{ \frac{\partial^2 w_n}{\partial y^2} \right\},$$

are Cauchy sequences in $L^2(\Omega)$. So we can say that an element W of the completion E_{PC} is such that $W(x, y)$ and all its derivatives up to order two are in $L^2(\Omega)$.

We now investigate $W(x, y)$ further. Let $w \in S_4$ and $w(x, y) \equiv 0$ outside Ω. Suppose Ω lies in the domain $\{(x, y)\colon\ x > 0, y > 0\}$. Then the representation

$$w(x, y) = \int_0^x \int_0^y \frac{\partial^2 w(s, t)}{\partial s \partial t}\, ds\, dt$$

holds. Using Hölder's inequality and (2.3.20), we get

$$|w(x, y)| \leq \int_0^x \int_0^y \left| \frac{\partial^2 w(s, t)}{\partial s \partial t} \right| ds\, dt \leq \iint_\Omega 1 \cdot \left| \frac{\partial^2 w(s, t)}{\partial s \partial t} \right| ds\, dt$$
$$\leq (\operatorname{mes} \Omega)^{1/2} \left(\iint_\Omega \left| \frac{\partial^2 w(s, t)}{\partial s \partial t} \right|^2 ds\, dt \right)^{1/2} \leq m_4\, (w, w)^{1/2}. \qquad (2.3.21)$$

This means that if $\{w_n\} \subset S_4$ is a Cauchy sequence in the metric of E_{PC}, then it converges uniformly on Ω. Hence there exists a limit function

$$w_0(x, y) = \lim_{n \to \infty} w_n(x, y)$$

which is continuous on Ω; this function is identified, as above, with the corresponding element of E_{PC} and we shall say that E_{PC} is continuously imbedded into $C(\Omega)$.

The functional describing the work of external forces

$$A = \iint_\Omega F(x,y)\, W(x,y)\, dx\, dy$$

now makes sense if $F(x,y) \in L(\Omega)$; moreover, it can contain the work of point forces

$$\sum_k F(x_k, y_k)\, w_0(x_k, y_k)$$

and line forces

$$\int_\gamma F(x,y)\, w_0(x,y)\, ds$$

where γ is a line in Ω and $w_0(x,y)$ is the corresponding limit function for $W(x,y)$.

Remark 2.3.4. Modern books on partial differential equations often require that $F(x,y) \in H^{-2}(\Omega)$. This is a complete characterization of external forces — however, it is difficult for an engineer to verify this property. □

Now let us consider a plate with free edge. In this case, we also wish to use the inner product (2.3.15) to create an energy space. As in the case of a membrane with free edge, the axiom P1 is not fulfilled: we saw that from $(w,w) = 0$ it follows that

$$w(x,y) = a_1 + a_2 x + a_3 y\,. \tag{2.3.22}$$

This admissible motion of the plate as a rigid whole is called a rigid motion, but still differs from real "rigid" motions of the plate as a three-dimensional body.

Poincaré's inequality (2.3.9) implies that the zero element of the corresponding completion consists of functions of the form (2.3.22). Indeed, taking $w(x,y) \in C^{(4)}(\Omega)$ we write down Poincaré's inequality for $\partial w/\partial x$:

$$\iint_\Omega \left(\frac{\partial w}{\partial x}\right)^2 dx\, dy \le m \left\{ \left(\iint_\Omega \frac{\partial w}{\partial x}\, dx\, dy \right)^2 + \iint_\Omega \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dx\, dy \right\},$$

and then the same inequality for $\partial w/\partial y$ with the roles of x and y interchanged. From these and (2.3.9) we get

$$\begin{aligned} \iint_\Omega \left[w^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dx\, dy \le m_1 \Bigg\{ &\left(\iint_\Omega w\, dx\, dy \right)^2 \\ &+ \left(\iint_\Omega \frac{\partial w}{\partial x}\, dx\, dy \right)^2 + \left(\iint_\Omega \frac{\partial w}{\partial y}\, dx\, dy \right)^2 \\ &+ \iint_\Omega \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right] dx\, dy \Bigg\}. \end{aligned}$$

From (2.3.19) it follows that

$$\iint_\Omega \left[w^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dx\,dy \le m_2 \left\{ \left(\iint_\Omega w\,dx\,dy \right)^2 \right.$$
$$+ \left(\iint_\Omega \frac{\partial w}{\partial x}\,dx\,dy \right)^2 + \left(\iint_\Omega \frac{\partial w}{\partial y}\,dx\,dy \right)^2$$
$$\left. + \iint_\Omega D^{\alpha\beta\gamma\delta} \rho_{\gamma\delta}(w)\,\rho_{\alpha\beta}(w)\,dx\,dy \right\}. \tag{2.3.23}$$

For any function $w(x, y) \in C^{(4)}(\Omega)$, we can take suitable constants a_i and find a function $w_b(x, y)$ of the form

$$w_b(x, y) = w(x, y) + a_1 + a_2 x + a_3 y \tag{2.3.24}$$

such that

$$\iint_\Omega w_b\,dx\,dy = 0\,, \quad \iint_\Omega \frac{\partial w_b}{\partial x}\,dx\,dy = 0\,, \quad \iint_\Omega \frac{\partial w_b}{\partial y}\,dx\,dy = 0\,. \tag{2.3.25}$$

As for the membrane with free edge, we can now consider a subset S_{4b} of $C^{(4)}(\Omega)$ consisting of balanced functions satisfying (2.3.25). We construct an energy space E_{PF} for a plate with free edge as the completion of S_{4b} in the metric induced by the inner product (2.3.15).

From (2.3.25), (2.3.23), and (2.3.19), we see that an element $W(x, y) \in E_{PF}$ is such that $W(x, y)$ and all its "derivatives" up to order two are in $L^2(\Omega)$. We could show the existence of a limit function

$$w_0(x, y) = \lim_{n\to\infty} w_n \in C(\Omega)$$

for any Cauchy sequence $\{w_n\}$, but in this case the technique is more complicated and, in what follows, we have this result as a particular case of the Sobolev imbedding theorem.

Note that (2.3.25) can be replaced by

$$\iint_\Omega w(x, y)\,dx\,dy = 0\,, \quad \iint_\Omega x\,w(x, y)\,dx\,dy = 0\,, \quad \iint_\Omega y\,w(x, y)\,dx\,dy = 0\,,$$

since these also uniquely determine the a_i for a class of functions of the form (2.3.24). (This possibility follows from Sobolev's general result [33] on equivalent norms in Sobolev spaces.)

The system (2.3.25) represents constraints that are absent in nature. For a static problem, there must be a certain invariance of some objects under transformations of the form (2.3.24) with arbitrary constants a_k. In particular, the work of external forces should not depend on the a_k if the problem is stated correctly. This leads to the necessary conditions

$$\iint_\Omega F(x,y)\,dx\,dy = 0\,, \quad \iint_\Omega x\,F(x,y)\,dx\,dy = 0\,, \quad \iint_\Omega y\,F(x,y)\,dx\,dy = 0\,. \tag{2.3.26}$$

The mechanical sense of (2.3.26) is clear: the resultant force and moments must vanish. This is the condition for a self-balanced force system.

Problem 2.3.2. What is the form of (2.3.26) if the external forces contain point and line forces? □

An energy space for a free plate, as for the membrane with free edge, can be introduced in another way: namely, we begin with an element of $W^{2,2}(\Omega)$, selecting a representative sequence $\{\tilde{w}_k(x,y)+a_k+b_kx+c_ky\}$, where $\tilde{w}$ satisfies (2.3.25) and a_k, b_k, c_k are constants. It is easy to show that the numerical sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ are Cauchy and therefore belong to the space c. Next we combine into a class-element all the elements of $W^{2,2}(\Omega)$ whose differences are a linear polynomial $a + bx + cy$, and state that the energy space is the factor space of $W^{2,2}(\Omega)$ by the space which is the completion of the space of linear polynomials with respect to the norm

$$\|a + bx + cy\| = (a^2 + b^2 + c^2)^{1/2}.$$

In the factor space, the zero is the class of all Cauchy sequences whose differences are equivalent (in the norm of $W^{2,2}(\Omega)$) to a sequence $\{a_k + b_kx + c_ky\}$ with $\{a_k\}, \{b_k\}, \{c_k\} \in c$. The norm of W, an element in the factor space, is

$$\left(\iint_\Omega D^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}(W)\,\rho_{\alpha\beta}(W)\,dx\,dy\right)^{1/2}.$$

The elements of the factor space are uniquely identified with the elements of the energy space E_{PF} whose elements satisfy (2.3.25). Moreover, the identification is isometric so, as for the membrane, we can use this factor space as the energy space and repeat the proof of the existence-uniqueness theorem in terms of the elements of the energy factor space. The self-balance condition for the external forces is a necessary condition to have the functional of the work of external forces be meaningful. The reader may also consider mixed boundary conditions: how must the treatment be modified if the plate is clamped only along a segment $AB \subset \Omega$ so that

$$w(x,y)\big|_{AB} = 0\,,$$

with the rest of the boundary free of geometrical constraints?

Linear Elasticity. We return to the problem of linear elasticity, considered in Sect. 1.5. Let us introduce a functional describing the work of internal forces on variations $\mathbf{v}(\mathbf{x})$ of the displacement field $\mathbf{u}(\mathbf{x})$:

$$-(\mathbf{u},\mathbf{v}) = -\iiint_\Omega c^{ijkl}\epsilon_{kl}(\mathbf{u})\,\epsilon_{ij}(\mathbf{v})\,d\Omega\,. \tag{2.3.27}$$

For the notation, see (1.5.7)–(1.5.9). We recall that the strain energy $\mathcal{E}_4(\mathbf{u})$ of an elastic body occupying volume Ω is related to the introduced inner product as follows:

$$(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2 = 2\mathcal{E}_4(\mathbf{u}) \, .$$

The elastic moduli c^{ijkl} may be piecewise continuous functions satisfying (1.5.8) and (1.5.9), which guarantee that all inner product axioms are satisfied by $(\mathbf{u}, \mathbf{v})$ for vector functions $\mathbf{u}, \mathbf{v}$ continuously differentiable on Ω, except P1: from $(\mathbf{u}, \mathbf{u}) = 0$ it follows that $\mathbf{u} = \mathbf{a} + \mathbf{b} \times \mathbf{x}$. Note that $(\mathbf{u}, \mathbf{v})$ is consistent with the metric (1.5.10).

Let us consider boundary conditions prescribed by

$$\mathbf{u}(\mathbf{x})\big|_{\partial\Omega} = \mathbf{0} \, . \tag{2.3.28}$$

If we use the form (2.3.27) on the set S_3 of vector-functions $\mathbf{u}(\mathbf{x})$ satisfying (2.3.28) and such that each of their components is of class $C^{(2)}(\Omega)$, then $(\mathbf{u}, \mathbf{v})$ is an inner product and S_3 with this inner product becomes an inner product space. Its completion E_{EC} in the corresponding metric (or norm) is the energy space of an elastic body with clamped boundary. To describe the properties of the elements of E_{EC}, we establish *Korn's inequality*.

Lemma 2.3.1 (Korn). For a vector function $\mathbf{u}(\mathbf{x}) \in S_3$, we have

$$\iiint_\Omega \left[|\mathbf{u}|^2 + \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2 \right] d\Omega \le m \iiint_\Omega c^{ijkl} \epsilon_{kl}(\mathbf{u}) \, \epsilon_{ij}(\mathbf{u}) \, d\Omega$$

for some constant m which does not depend on $\mathbf{u}(\mathbf{x})$.

Proof. By (1.5.9) and Friedrichs' inequality, it is sufficient to show that

$$\iiint_\Omega \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2 d\Omega \le m_1 \iiint_\Omega \sum_{\substack{i,j=1 \\ i \le j}}^{3} \epsilon_{ij}^2(\mathbf{u}) \, d\Omega \, .$$

Consider the term on the right:

$$\begin{aligned} A \equiv \iiint_\Omega \sum_{\substack{i,j=1 \\ i \le j}}^{3} \epsilon_{ij}^2(\mathbf{u}) \, d\Omega &= \frac{1}{4} \iiint_\Omega \sum_{\substack{i,j=1 \\ i \le j}}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 d\Omega \\ &= \iiint_\Omega \left\{ \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial x_i} \right)^2 + \frac{1}{4} \sum_{\substack{i,j=1 \\ i < j}}^{3} \left[\left(\frac{\partial u_i}{\partial x_j} \right)^2 + \left(\frac{\partial u_j}{\partial x_i} \right)^2 + 2 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right] \right\} d\Omega \, . \end{aligned}$$

Integrating by parts (twice) the term

$$B \equiv \frac{1}{2} \iiint_\Omega \sum_{\substack{i,j=1 \\ i < j}}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \, d\Omega = \frac{1}{2} \iiint_\Omega \sum_{\substack{i,j=1 \\ i < j}}^{3} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \, d\Omega$$

and using the elementary inequality $|ab| \le (a^2 + b^2)/2$, we get

$$|B| \le \frac{1}{4} \iiint_{\Omega} \sum_{\substack{i,j=1 \\ i<j}}^{3} \left[\left(\frac{\partial u_i}{\partial x_i} \right)^2 + \left(\frac{\partial u_j}{\partial x_j} \right)^2 \right] d\Omega = \frac{1}{2} \iiint_{\Omega} \sum_{i=1}^{3} \left(\frac{\partial u_i}{\partial x_i} \right)^2 d\Omega \, .$$

Therefore

$$A \ge \frac{1}{4} \iiint_{\Omega} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2 d\Omega \, ,$$

which completes the proof. □

By Korn's inequality, each component of an element $U \in E_{EC}$ belongs to E_{MC}, i.e., the u_i and their first derivatives belong to $L^2(\Omega)$.

Note that the construction of an energy space is the same if the boundary condition (2.3.28) is given only on some part $\partial\Omega_1$ of the boundary of Ω:

$$\mathbf{u}(\mathbf{x})\big|_{\partial\Omega_1} = \mathbf{0} \, .$$

Korn's inequality also holds, but its proof is more complicated (see, for example, [26, 11]).

If we consider an elastic body with free boundary, we encounter issues similar to those for a membrane or plate with free edge: we must circumvent the difficulty with the zero element of the energy space. The restrictions

$$\iiint_{\Omega} \mathbf{u} \, d\Omega = \mathbf{0} \, , \qquad \iiint_{\Omega} \mathbf{x} \times \mathbf{u}(\mathbf{x}) \, d\Omega = \mathbf{0} \, , \tag{2.3.29}$$

provide that the rigid motion $\mathbf{u} = \mathbf{a} + \mathbf{b} \times \mathbf{x}$ becomes zero, and that Korn's inequality remains valid for smooth vector functions satisfying (2.3.29). So by completion, we get an energy space E_{EF} with known properties: all Cartesian components of vectors pertain to the space $W^{1,2}(\Omega)$.

As for a free membrane, we can also organize an energy space of classes — a factor space — in which the zero element is the set of all elements whose differences between any representative sequences in the norm of $(W^{1,2}(\Omega))^3$ are equivalent to a sequence of the form $\{\mathbf{a}_k + \mathbf{b}_k \times \mathbf{x}\}$ such that the Cartesian components of the vectors $\{\mathbf{a}_k\}$ and $\{\mathbf{b}_k\}$ constitute some elements of the space c.

2.4 Generalized Solutions in Mechanics

We now discuss how to introduce generalized solutions in mechanics. We begin with Poisson's equation

$$-\Delta u(x, y) = F(x, y) \, , \qquad (x, y) \in \Omega \, , \tag{2.4.1}$$

where Ω is a bounded open domain in $\mathbb{R}^2$. The Dirichlet problem consists of this equation supplemented by the boundary condition

$$u\big|_{\partial\Omega} = 0\ . \tag{2.4.2}$$

Let $u(x, y)$ be its classical solution; i.e., let $u \in C^{(2)}(\overline{\Omega})$ satisfy (2.4.1) and (2.4.2). Let $\varphi(x, y)$ be a function with compact support in Ω. Again, this means that $\varphi \in C^{(\infty)}(\overline{\Omega})$ and the closure of the set $M = \{(x, y) \in \Omega\colon \varphi(x, y) \neq 0\}$ lies in Ω.

Multiplying both sides of (2.4.1) by $\varphi(x, y)$ and integrating over Ω, we get

$$-\iint_\Omega \varphi(x, y)\, \Delta u(x, y)\, dx\, dy = \iint_\Omega F(x, y)\, \varphi(x, y)\, dx\, dy\ . \tag{2.4.3}$$

If this equality holds for every infinitely differentiable function $\varphi(x, y)$ with compact support in Ω, and if $u \in C^{(2)}(\overline{\Omega})$ and satisfies (2.4.2), then, as is well known from the classical calculus of variations, $u(x, y)$ is the unique classical solution to the Dirichlet problem.

But using (2.4.3), we can pose this Dirichlet problem directly without using the differential equation (2.4.1); namely, $u(x, y)$ is a solution to the Dirichlet problem if, obeying (2.4.2), it satisfies (2.4.3) for every $\varphi(x, y)$ that is infinitely differentiable with compact support in Ω. If $F(x, y)$ belongs to $L^p(\Omega)$ then we can take, as it seems, $u(x, y)$ having second derivatives in the space $L^p(\Omega)$; such a $u(x, y)$ is not a classical solution, and it is natural to call it a generalized solution.

We can go further by applying integration by parts to the left-hand side of (2.4.3) as follows:

$$\iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y} \right) dx\, dy = \iint_\Omega F(x, y)\, \varphi(x, y)\, dx\, dy\ . \tag{2.4.4}$$

In such a case we may impose weaker restrictions on a solution $u(x, y)$ and call it the generalized solution if it belongs to E_{MC}, the energy space for a clamped membrane. Equation (2.4.4) defines this solution if it holds for every $\varphi(x, y)$ that has a compact support in Ω. Note the disparity in requirements on $u(x, y)$ and $\varphi(x, y)$.

Further integration by parts on the left-hand side of (2.4.4) yields

$$-\iint_\Omega u(x, y)\, \Delta\varphi(x, y)\, dx\, dy = \iint_\Omega F(x, y)\, \varphi(x, y)\, dx\, dy\ . \tag{2.4.5}$$

Now we can formally consider solutions from the space $L(\Omega)$ and this is a new class of generalized solutions.

This way leads to the theory of distributions originated by Schwartz [32]. He extended the notion of generalized solution to a class of linear continuous functionals, or *distributions*, defined on the set $\mathcal{D}(\Omega)$ of all functions infinitely differentiable in Ω and with compact support in Ω. For this it is necessary to introduce the convergence and other structures of continuity in $\mathcal{D}(\Omega)$. Unfortunately $\mathcal{D}(\Omega)$ is not a normed space (see, for example, Yosida [44] — it is a locally convex topological space) and its presentation would exceed our scope. This theory justifies, in particular, the use of the so-called δ-function, which was introduced in quantum mechanics via the equality

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a)\,, \tag{2.4.6}$$

valid for every continuous function $f(x)$. Physicists considered $\delta(x)$ to be a function vanishing everywhere except at $x = 0$, where its value is infinity. Any known theory of integration gave zero for the value of the integral on the left-hand side of (2.4.6), and the theory of distributions explained how to understand such strange functions. It is interesting to note that the δ-function was well known in classical mechanics, too; if we consider $\delta(x-a)$ as a unit point force applied at $x = a$, then the integral on the left-hand side of (2.4.6) is the work of this force on the displacement $f(a)$, which is indeed $f(a)$.

So we have several generalized statements of the Dirichlet problem, but which one is most natural from the viewpoint of mechanics?

From mechanics it is known that a solution to the problem is a minimizer of the total potential energy functional

$$I(u) = \iint_{\Omega} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx\, dy - 2 \iint_{\Omega} F u\, dx\, dy\,. \tag{2.4.7}$$

According to the calculus of variations, a minimizer of $I(u)$ on the subset of $C^{(2)}(\overline{\Omega})$ consisting of all functions satisfying (2.4.2), if it exists, is a classical solution to the Dirichlet problem. But we also can consider $I(u)$ on the energy space E_{MC} if $F(x, y) \in L^p(\Omega)$ for $p > 1$. Indeed, the first term in $I(u)$ is well-defined in E_{MC} and can be written in the form $\|u\|^2$; the second,

$$\Phi(u) = - \iint_{\Omega} F(x, y)\, u(x, y)\, dx\, dy\,,$$

is a linear functional with respect to $u(x, y)$. It is also continuous in E_{MC}; by virtue of Hölder's inequality with exponents p and $q = p/(p-1)$, we have

$$\begin{aligned} \left| \iint_{\Omega} F u\, dx\, dy \right| &\le \left(\iint_{\Omega} |F|^p\, dx\, dy \right)^{1/p} \left(\iint_{\Omega} |u|^q\, dx\, dy \right)^{1/q} \\ &\le m_1 \|F\|_{L^p(\Omega)} \|u\|_{W^{1,2}(\Omega)} \le m_2 \|u\|_{E_{MC}}\,. \end{aligned}$$

To show this, we have used the imbedding Theorem 2.1.2 and the Friedrichs inequality. By Theorem 1.21.1, $\Phi(u)$ is continuous in E_{MC}, and therefore so is $I(u)$.

Thus $I(u)$ is of the form

$$I(u) = \|u\|^2 + 2\Phi(u)\,. \tag{2.4.8}$$

Let $u_0 \in E_{MC}$ be a minimizer of $I(u)$, i.e.,

$$I(u_0) \le I(u) \quad \text{for all } u \in E_{MC}\,. \tag{2.4.9}$$

We try a method from the classical calculus of variations. Take $u = u_0 + \epsilon v$ where v is an arbitrary element of E_{MC}. Then

$$\begin{aligned} I(u) &= I(u_0 + \epsilon v) = \|u_0 + \epsilon v\|^2 + 2\Phi(u_0 + \epsilon v) \\ &= (u_0 + \epsilon v, u_0 + \epsilon v) + 2\Phi(u_0 + \epsilon v) \\ &= \|u_0\|^2 + 2\epsilon(u_0, v) + \epsilon^2 \|v\|^2 + 2\Phi(u_0) + 2\epsilon\Phi(v) \\ &= \|u_0\|^2 + 2\Phi(u_0) + 2\epsilon[(u_0, v) + \Phi(v)] + \epsilon^2 \|v\|^2 \,. \end{aligned}$$

From (2.4.9), we get

$$2\epsilon[(u_0, v) + \Phi(v)] + \epsilon^2 \|v\|^2 \geq 0 \,.$$

Since ϵ is an arbitrary real number (in particular it can take either sign, and the first term, if nonzero, dominates for ϵ small in magnitude), it follows that

$$(u_0, v) + \Phi(v) = 0 \,. \tag{2.4.10}$$

In other words,

$$\iint_\Omega \left(\frac{\partial u_0}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u_0}{\partial y}\frac{\partial v}{\partial y} \right) dx\,dy - \iint_\Omega F(x, y)\, v(x, y)\, dx\,dy = 0 \,. \tag{2.4.11}$$

This equality holds for every $v \in E_{MC}$ and defines the minimizer $u_0 \in E_{MC}$. Note that (2.4.11) has the same form as (2.4.4).

So we have introduced a notion of generalized solution that has an explicit mechanical background.

Definition 2.4.1. An element $u \in E_{MC}$ is called the *generalized solution* to the Dirichlet problem if u satisfies (2.4.11) for any $v \in E_{MC}$.

We can also obtain (2.4.11) from the principle of virtual displacements (work). This asserts that in the state of equilibrium, the sum of the work of internal forces (which is now the variation of the strain energy with a negative sign) and the work of external forces is zero on all virtual (admissible) displacements.

In the case under consideration, both approaches to introducing generalized energy solutions are equivalent. In general, however, this is not so, and the virtual work principle has wider applicability. If $F(x, y)$ is a nonconservative load depending on $u(x, y)$, we cannot use the principle of minimum total energy; however, (2.4.11) remains valid since it has the mathematical form of the virtual work principle. In what follows, we often use this principle to pose problems in equation form.

Since the part of the presentation from (2.4.8) up to (2.4.10) is general and does not depend on the specific form (2.4.11) of the functional $I(u)$, we can formulate

Theorem 2.4.1. Let u_0 be a minimizer of a functional $I(u) = \|u\|^2 + 2\Phi(u)$ given in an inner product (Hilbert) space H, where the functional $\Phi(u)$ is linear and continuous. Then u_0 satisfies (2.4.10) for every $v \in H$.

Equation (2.4.10) is a necessary condition for minimization of the functional $I(u)$, analogous to the condition that the first derivative of an ordinary function must vanish at a point of minimum.

We can obtain (2.4.10) formally by evaluating

$$\frac{d}{d\epsilon} I(u_0 + \epsilon v)\bigg|_{\epsilon=0} = 0\ . \tag{2.4.12}$$

This is valid for the following reason. Given u_0 and v, the functional $I(u_0 + \epsilon v)$ is an ordinary function of the numerical variable ϵ, and assumes a minimum value at $\epsilon = 0$. The left-hand side of (2.4.12) can be interpreted as a partial derivative at $u = u_0$ in the direction v, and is called the *Gâteaux derivative* of $I(u)$ at $u = u_0$ in the direction of v. We shall return to this issue later.

The Dirichlet problem for a clamped membrane is a touchstone for all static problems in continuum mechanics. In a similar way we can introduce a natural notion of generalized solution for other problems under consideration. As we said, each of them can be represented as a minimization problem for a total potential energy functional of the form (2.4.10) in an energy space. For example, equation (2.4.11), a particular form of (2.4.10) for a clamped membrane, is the same for a free membrane — we need only replace E_{MC} by E_{MF}. The quantity $\Phi(u)$, to be a continuous linear functional in E_{MF}, must be supplemented with self-balance condition (2.3.12) for the load.

Let us concretize equation (2.4.10) for each of the other problems we have under consideration.

Plate. The definition of generalized solution $w_0 \in E_P$ is given by the equation

$$\begin{aligned}\iint_\Omega D^{\alpha\beta\gamma\delta} \rho_{\gamma\delta}(w_0)\, \rho_{\alpha\beta}(w)\, dx\, dy - \iint_\Omega F(x,y)\, w(x,y)\, dx\, dy \\ - \sum_{k=1}^{m} F_k w(x_k, y_k) - \int_\gamma f(s)\, w(x,y)\, ds = 0\end{aligned} \tag{2.4.13}$$

(see the notation of Sect. 2.3) which must hold for every $w \in E_P$. The equation is the same for any kind of homogeneous boundary conditions (i.e., for usual ones) but the energy space will change from one set of boundary conditions to another. If a plate can move as a rigid whole, the requirement that

$$F(x,y) \in L(\Omega)\ , \qquad f(s) \in L(\gamma)\ ,$$

for Φ to be a continuous linear functional, must be supplemented with self-balance conditions for the load:

$$\iint_\Omega F(x,y)\, w_i(x,y)\, dx\, dy + \sum_{k=1}^{m} F_k w_i(x_k, y_k) + \int_\gamma f(s)\, w_i(x,y)\, ds = 0 \tag{2.4.14}$$

for $i = 1, 2, 3$, where $w_1(x, y) = 1$, $w_2(x, y) = x$, and $w_3(x, y) = y$. This condition is necessary if we use the space where the set of rigid plate motions is the zero of the space. If from each element of the energy space we select a representative using (2.3.25), then on the energy space of all representatives with norm $\|\cdot\|_P$ we can prove existence and uniqueness of the energy solution without self-balance condition (2.4.14). But for solvability of the initial problem for a free plate, we still should require (2.4.14) to hold as constraints (2.3.25) are absent in the problem statement.

Note that for each concrete problem we must specify the energy space. The same is true for the following problem.

Linear Elasticity. The generalized solution $\mathbf{u}$ is defined by the integro-differential equation

$$\iiint_\Omega c^{ijkl}\epsilon_{kl}(\mathbf{u})\,\epsilon_{ij}(\mathbf{v})\,d\Omega - \iiint_\Omega \mathbf{F}(x, y, z)\cdot\mathbf{v}(x, y, z)\,d\Omega$$
$$- \iint_\Gamma \mathbf{f}(x, y, z)\cdot\mathbf{v}(x, y, z)\,dS = 0\,, \tag{2.4.15}$$

where $\mathbf{F}$ and $\mathbf{f}$ are forces distributed over Ω and over some surface $\Gamma \subset \Omega$, respectively. If we consider the Dirichlet (or first) problem of elasticity, which is

$$\mathbf{u}(\mathbf{x})\big|_{\partial\Omega} = \mathbf{0}$$

where $\partial\Omega$ is the boundary of Ω, the solution $\mathbf{u}$ should belong to the space E_{EC} and equation (2.4.15) must hold for every virtual displacement $\mathbf{v} \in E_{EC}$. Note that in this case,

$$\iint_{\partial\Omega} \mathbf{f}(x, y, z)\cdot\mathbf{v}(x, y, z)\,dS = 0\,.$$

The load, thanks to Theorem 2.1.4 and Korn's inequality, is of the class

$$F_i(x, y, z) \in L^{6/5}(\Omega)\,, \quad f_i(x, y, z) \in L^{4/3}(\Gamma) \qquad (i = 1, 2, 3)\,,$$

where $F_i(x, y, z)$ and f_i are Cartesian components of $\mathbf{F}$ and $\mathbf{f}$, respectively, and Γ is a piecewise smooth surface in $\overline{\Omega}$. This provides continuity of $\Phi(w)$.

In the second problem of elasticity there are given forces distributed over the boundary:

$$c^{ijkl}\epsilon_{kl}(\mathbf{x})\,n_j(\mathbf{x})\big|_{\partial\Omega} = f_i(\mathbf{x})\,,$$

where the n_j are Cartesian components of the unit exterior normal to $\partial\Omega$. This can be written in tensor notation as

$$\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}\big|_{\partial\Omega} = \mathbf{f}(\mathbf{x})$$

where the components of stress tensor $\boldsymbol{\sigma}$ are related to the components of the strain tensor by $\sigma_{ij} = c^{ijkl}\epsilon_{kl}$.

As for equilibrium problems for free membranes and plates, in the case of a free elastic body we must require that the load be self-balanced:

$$\iiint_{\Omega} \mathbf{F}(\mathbf{x})\, d\Omega + \iint_{\Gamma} \mathbf{f}(\mathbf{x})\, dS = \mathbf{0}\,,$$

$$\iiint_{\Omega} \mathbf{x} \times \mathbf{F}(\mathbf{x})\, d\Omega + \iint_{\Gamma} \mathbf{x} \times \mathbf{f}(\mathbf{x})\, dS = \mathbf{0}\,. \tag{2.4.16}$$

The solution is required to be in E_{EF}.

We have argued that it is legitimate to introduce the generalized solution in such a way. Of course, full legitimacy will be assured when we prove that this solution exists and is unique in the corresponding space.

We emphasize once more that the definition of generalized solution arose in a natural way from the variational principle of mechanics.

2.5 Existence of Energy Solutions to Some Mechanics Problems

In Sect. 2.4 we introduced generalized solutions for several mechanics problems and reduced those problems to a solution of the abstract equation

$$(u, v) + \Phi(v) = 0 \tag{2.5.1}$$

in an energy (Hilbert) space. We obtained some restrictions on the forces to provide continuity of the linear functional $\Phi(v)$ in the energy space. The following theorem guarantees solvability of those mechanics problems in a generalized sense.

Theorem 2.5.1. Assume $\Phi(v)$ is a continuous linear functional given on a Hilbert space H. Then there is a unique element $u \in H$ that satisfies (2.5.1) for every $v \in H$.

Proof. By the Riesz representation theorem there is a unique $u_0 \in H$ such that the continuous linear functional $\Phi(v)$ is represented in the form $\Phi(v) = (v, u_0) \equiv (u_0, v)$. Hence (2.5.1) takes the form

$$(u, v) + (u_0, v) = 0\,. \tag{2.5.2}$$

We need to find $u \in H$ that satisfies (2.5.2) for every $v \in H$. Rewriting it in the form

$$(u + u_0, v) = 0\,,$$

we see that its unique solution is $u = -u_0$. □

This theorem answers the question of solvability, in the generalized sense, of the problems treated in Sect. 2.4. To demonstrate this, we rewrite Theorem 2.5.1 in concrete terms for a pair of problems.

Theorem 2.5.2. Assume $F(x, y) \in L(\Omega)$ and $f(x, y) \in L(\gamma)$ where $\Omega \subset \mathbb{R}^2$ is compact and γ is a piecewise smooth curve in Ω. The equilibrium problem for a plate with clamped edge has a unique generalized solution: there is a unique $w_0 \in E_{PC}$ which satisfies (2.4.13) for all $w \in E_{PC}$.

Changes for a plate which is free of clamping are evident: we must add the self-balance condition (2.4.14) for forces and replace the space E_{PC} by E_{PF}.

Theorem 2.5.3. Assume all Cartesian components of the volume forces $\mathbf{F}(x, y, z)$ are in $L^{6/5}(\Omega)$ and those of the surface forces $\mathbf{f}(x, y, z)$ are in $L^{4/3}(S)$, where Ω is compact in $\mathbb{R}^3$ and S is a piecewise smooth surface in Ω. Then the problem of equilibrium of an elastic body occupying Ω, with clamped boundary, has a unique generalized solution $\mathbf{u} \in E_{EC}$; namely, $\mathbf{u}(x, y, z)$ satisfies (2.4.15) for every $\mathbf{v} \in E_{EC}$.

In both theorems, the load restrictions provide continuity of the corresponding functionals Φ, the work of external forces.

Problem 2.5.1. Formulate existence theorems for the other mechanics problems discussed in Sect. 2.4. □

2.6 Operator Formulation of an Eigenvalue Problem

We have seen how to use the Riesz representation theorem to prove the existence and uniqueness of a generalized solution. Now let us consider another application of the Riesz representation theorem: how to cast a problem as an operator equation.

The eigenvalue equation for a membrane has the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\lambda u \,. \tag{2.6.1}$$

Similar to the equilibrium problem for a membrane, we can introduce a generalized solution to the eigenvalue problem for a clamped membrane by the integro-differential equation

$$\iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx\, dy = \lambda \iint_\Omega uv\, dx\, dy \,. \tag{2.6.2}$$

The eigenvalue problem is to find a nontrivial element $u \in E_{MC}$ and a corresponding number λ such that u satisfies (2.6.2) for every $v \in E_{MC}$.

First we reformulate this problem as an operator equation

$$u = \lambda K u \tag{2.6.3}$$

in the space E_{MC}. For this, consider the term

$$F(v) = \iint_\Omega uv\, dx\, dy$$

as a functional in E_{MC}, with respect to v, when u is a fixed element of E_{MC}. It is seen that $F(v)$ is a linear functional. By the Schwarz inequality

$$|F(v)| = \left| \iint_\Omega uv \, dx \, dy \right| \le \left(\iint_\Omega u^2 \, dx \, dy \right)^{1/2} \left(\iint_\Omega v^2 \, dx \, dy \right)^{1/2}$$

hence by the Friedrichs inequality

$$|F(v)| \le m \, \|u\| \, \|v\| = m_1 \|v\| \tag{2.6.4}$$

(hereafter the norm $\|\cdot\|$ and the inner product $(\cdot, \cdot)$ are taken in E_{MC}). So $F(v)$ is a continuous linear functional acting in the Hilbert space E_{MC}. By the Riesz representation theorem, $F(v)$ has the unique representation

$$F(v) \equiv \iint_\Omega uv \, dx \, dy = (v, f) = (f, v) \, . \tag{2.6.5}$$

What have we shown? For every $u \in E_{MC}$, by this representation, there is a unique element $f \in E_{MC}$. Hence the correspondence

$$u \mapsto f$$

is an operator $f = K(u)$ from E_{MC} to E_{MC}.

Let us display some properties of this operator. First we show that it is linear. Let

$$f_1 = K(u_1) \quad \text{and} \quad f_2 = K(u_2) \, .$$

Then

$$\iint_\Omega (\lambda_1 u_1 + \lambda_2 u_2) v \, dx \, dy = (K(\lambda_1 u_1 + \lambda_2 u_2), v)$$

while on the other hand,

$$\begin{aligned} \iint_\Omega (\lambda_1 u_1 + \lambda_2 u_2) v \, dx \, dy &= \lambda_1 \iint_\Omega u_1 v \, dx \, dy + \lambda_2 \iint_\Omega u_2 v \, dx \, dy \\ &= \lambda_1 (K(u_1), v) + \lambda_2 (K(u_2), v) \\ &= (\lambda_1 K(u_1) + \lambda_2 K(u_2), v) \, . \end{aligned}$$

Combining these we have

$$(K(\lambda_1 u_1 + \lambda_2 u_2), v) = (\lambda_1 K(u_1) + \lambda_2 K(u_2), v) \, ,$$

hence

$$K(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 K(u_1) + \lambda_2 K(u_2)$$

because $v \in E_{MC}$ is arbitrary. Therefore linearity is proven.

Now let us rewrite (2.6.4) in terms of this representation:

$$|(K(u), v)| \le m \, \|u\| \, \|v\| \, .$$

Take $v = K(u)$; then

$$\|K(u)\|^2 \le m\,\|u\|\,\|K(u)\|$$

and it follows that

$$\|K(u)\| \le m\,\|u\| \,. \tag{2.6.6}$$

Hence K is a continuous operator in E_{MC}.

Equation (2.6.2) can now be written in the form

$$(u, v) = \lambda(K(u), v) \,.$$

Since v is an arbitrary element of E_{MC}, this equation is equivalent to the operator equation

$$u = \lambda K(u)$$

with a continuous linear operator K.

By (2.6.6), we get

$$\|\lambda K(u) - \lambda K(v)\| = |\lambda|\,\|K(u - v)\| \le m\,|\lambda|\,\|u - v\| \,.$$

If

$$m\,|\lambda| < 1 \,,$$

then λK is a contraction operator in E_{MC} and, by the contraction mapping principle, there is a unique fixed point of λK which clearly is $u = 0$. So the set $|\lambda| < 1/m$ does not contain real eigenvalues of the problem. Further, we shall see (and this is well known in mechanics) that eigenvalues in this problem must be real. The fact that the set $|\lambda| < 1/m$ does not contain real eigenvalues, and so any eigenvalues of the problem, has a clear mechanical sense: the lowest eigenfrequency of oscillation of a bounded clamped membrane is strictly positive. Note that from (2.6.2), when $v = u$ it follows that an eigenvalue must be positive.

In a similar way, we can introduce eigenvalue problems for plates and elastic bodies. Here we can obtain corresponding equations of the form (2.6.3) with continuous linear operators and can also show that the corresponding lowest eigenvalues are strictly positive. All this we leave to the reader; later we shall consider eigenvalue problems in more detail.

In what follows, we shall see that, using the Riesz representation theorem, one can also introduce operators and operator equations for nonlinear problems of mechanics. One of them is presented in the next section.

2.7 Problem of Elastico-Plasticity; Small Deformations

Following the lines of a paper by I.I. Vorovich and Yu.P. Krasovskij [40] that was published in a sketchy form, we consider a variant of the theory of elastico-plasticity (Il'yushin [16]), and justify the *method of elastic solutions* for corresponding boundary value problems.

The system of partial differential equations describing the behavior of an elastic-plastic body occupying a bounded volume Ω is

$$\left(\frac{\nu}{\nu-2}-\frac{\omega}{3}\right)\frac{\partial\theta}{\partial x_k}+(1-\omega)\Delta u_k$$
$$-\frac{2}{3}e_I\frac{d\omega}{de_I}\sum_{s,t=1}^{3}\epsilon^*_{ks}\sum_{l=1}^{3}\epsilon^*_{lt}\frac{\partial^2 u_l}{\partial x_s\partial x_t}+\frac{F_k}{G}=0 \quad (k=1,2,3)\,, \tag{2.7.1}$$

where ν is Poisson's ratio, G is the shear modulus, $\mathbf{F}=(F_1,F_2,F_3)$ are the volume forces, and $\omega(e_i)$ is a function of the variable e_I, the intensity of the strain tensor which defines plastic properties of the material with hardening:

$$e_I=\frac{\sqrt{2}}{3}\left[(\epsilon_{11}-\epsilon_{22})^2+(\epsilon_{11}-\epsilon_{33})^2+(\epsilon_{22}-\epsilon_{33})^2+6(\epsilon_{12}^2+\epsilon_{13}^2+\epsilon_{23}^2)\right]^{1/2}.$$

The function $\omega(e_I)$ must satisfy

$$0\le\omega(e_I)\le\omega(e_I)+e_I\frac{d\omega(e_I)}{de_I}\le\lambda<1\,. \tag{2.7.2}$$

Other bits of notation are

$$\theta\equiv\theta(\mathbf{u})=\epsilon_{11}(\mathbf{u})+\epsilon_{22}(\mathbf{u})+\epsilon_{33}(\mathbf{u})\,,$$

and

$$\epsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\qquad \epsilon^*_{ks}=\begin{cases}\left(\dfrac{\partial u_k}{\partial x_s}-\dfrac{\theta}{3}\right)\dfrac{\sqrt{2}}{e_I}\,, & k=s\,,\\[2ex] \left(\dfrac{\partial u_k}{\partial x_s}+\dfrac{\partial u_s}{\partial x_k}\right)\dfrac{1}{\sqrt{2}e_I}\,, & k\ne s\,.\end{cases}$$

If $\omega(e_I)\equiv 0$ we get the equations of linear elasticity for an isotropic homogeneous body. By analogy with elasticity problems, to pose a boundary value problem for (2.7.1) we must supplement the equations with boundary conditions. We consider a mixed boundary value problem: a part S_0 of the boundary $\partial\Omega$ of a body occupying the domain Ω is fixed,

$$\mathbf{u}\big|_{S_0}=\mathbf{0}\,, \tag{2.7.3}$$

and the remainder $S_1=\partial\Omega\setminus S_0$ is subjected to surface forces $\mathbf{f}(\mathbf{x})$ (see [16]):

$$\boldsymbol{\sigma}\cdot\mathbf{n}\big|_{S_1}=\mathbf{f}\,, \tag{2.7.4}$$

where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{n}$ is the external unit normal to S_1.

When $\omega(e_I)$ is small (as it is if e_I is small) we have a nonlinear boundary value problem which is, in a certain way, a perturbation of a corresponding boundary value problem of linear elasticity. It leads to the idea of using an iterative procedure, the method of elastic solutions, to solve the former. This procedure looks like that of the contraction mapping principle if we can make the problem take the corresponding operator form. Then it remains to show that the operator of the problem is a contraction. Now we begin to carry out the program.

Let us introduce the notation

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle = \tfrac{2}{9}\{&[\epsilon_{11}(\mathbf{u}) - \epsilon_{22}(\mathbf{u})][\epsilon_{11}(\mathbf{v}) - \epsilon_{22}(\mathbf{v})] \\ &+ [\epsilon_{11}(\mathbf{u}) - \epsilon_{33}(\mathbf{u})][\epsilon_{11}(\mathbf{v}) - \epsilon_{33}(\mathbf{v})] \\ &+ [\epsilon_{22}(\mathbf{u}) - \epsilon_{33}(\mathbf{u})][\epsilon_{22}(\mathbf{v}) - \epsilon_{33}(\mathbf{v})] \\ &+ 6[\epsilon_{12}(\mathbf{u})\epsilon_{12}(\mathbf{v}) + \epsilon_{13}(\mathbf{u})\epsilon_{13}(\mathbf{v}) + \epsilon_{23}(\mathbf{u})\epsilon_{23}(\mathbf{v})]\} .\end{aligned} \tag{2.7.5}$$

If we consider the terms on the right-hand side of (2.7.5) as coordinates of vectors $\mathbf{a} = (a_1, \ldots, a_6)$, $\mathbf{b} = (b_1, \ldots, b_6)$,

$$a_i = c_i(\mathbf{u}) , \qquad b_i = c_i(\mathbf{v}) \qquad (i = 1, \ldots, 6) ,$$

$$\begin{aligned} c_1(\mathbf{w}) &= \tfrac{\sqrt{2}}{3}[\epsilon_{11}(\mathbf{w}) - \epsilon_{22}(\mathbf{w})] , & c_2(\mathbf{w}) &= \tfrac{\sqrt{2}}{3}[\epsilon_{11}(\mathbf{w}) - \epsilon_{33}(\mathbf{w})] , \\ c_3(\mathbf{w}) &= \tfrac{\sqrt{2}}{3}[\epsilon_{22}(\mathbf{w}) - \epsilon_{33}(\mathbf{w})] , & c_4(\mathbf{w}) &= \tfrac{2}{\sqrt{3}}\epsilon_{12}(\mathbf{w}) , \\ c_5(\mathbf{w}) &= \tfrac{2}{\sqrt{3}}\epsilon_{13}(\mathbf{w}) , & c_6(\mathbf{w}) &= \tfrac{2}{\sqrt{3}}\epsilon_{23}(\mathbf{w}) ,\end{aligned}$$

then the form $\langle \mathbf{u}, \mathbf{v}\rangle$ is a scalar product between $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^6$:

$$\langle \mathbf{u}, \mathbf{v}\rangle = \sum_{i=1}^{6} a_i b_i .$$

Besides,

$$\langle \mathbf{u}, \mathbf{u}\rangle = \sum_{i=1}^{6} c_i^2(\mathbf{u}) = e_I^2(\mathbf{u}) \tag{2.7.6}$$

and by the Schwarz inequality we get

$$|\langle \mathbf{u}, \mathbf{v}\rangle| = \left| \sum_{i=1}^{6} c_i(\mathbf{u}) c_i(\mathbf{v}) \right| \le e_I(\mathbf{u}) e_I(\mathbf{v}) . \tag{2.7.7}$$

On the set C_2 of vector functions satisfying the boundary condition (2.7.3) and such that each of their components is of class $C^{(2)}(\Omega)$, we introduce an inner product

$$(\mathbf{u}, \mathbf{v}) = \iiint_{\Omega} \left(\tfrac{3}{2} G\langle \mathbf{u}, \mathbf{v}\rangle + \tfrac{1}{2} K\, \theta(\mathbf{u})\, \theta(\mathbf{v}) \right) d\Omega . \tag{2.7.8}$$

This coincides with a special case of the inner product (2.3.27) in the linear theory of elasticity. So the completion of C_2 in the metric corresponding to (2.7.8) is the energy space of linear elasticity E_{EM} (M for "mixed") if we suppose that the condition (2.7.3) provides $\mathbf{u} = 0$ if

$$\|\mathbf{u}\|^2 = \iiint_\Omega \left(\tfrac{3}{2} G e_I^2(\mathbf{u}) + \tfrac{1}{2} K\theta^2(\mathbf{u})\right) d\Omega = 0 .$$

The norm of E_{EM} is equivalent to one of $W^{1,2}(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ (see Sect. 2.3 and Fichera [11]). (By $H_1 \times H_2$ we denote the *Cartesian product* of Hilbert spaces H_1 and H_2, the elements of which are pairs (x, y) for $x \in H_1$ and $y \in H_2$. The scalar product in $H_1 \times H_2$ is defined by the expression $(x_1, x_2)_1 + (y_1, y_2)_2$ where $x_1, x_2 \in H_1$ and $y_1, y_2 \in H_2$.)

By the principle of virtual displacements, the integro-differential equation of equilibrium of an elastico-plastic body is

$$(\mathbf{u}, \mathbf{v}) - \tfrac{3}{2} G \iiint_\Omega \omega(e_I(\mathbf{u}))\langle \mathbf{u}, \mathbf{v}\rangle \, d\Omega - \sum_{i=1}^{3} \iiint_\Omega F_i v_i \, d\Omega - \sum_{i=1}^{3} \iint_{S_1} f_i v_i \, dS = 0 . \tag{2.7.9}$$

This equation can be obtained using the equations (2.7.1) and the boundary conditions (2.7.3)–(2.7.4). Conversely, using the technique of the classical calculus of variations we can get (2.7.1) and the natural boundary conditions (2.7.4). Thus, in a certain way, (2.7.9) is equivalent to the above statement of the problem. So we can introduce

Definition 2.7.1. A vector function $\mathbf{u} \in E_{EM}$ is called the generalized solution of the problem of elastico-plasticity with boundary conditions (2.7.3)–(2.7.4) if it satisfies (2.7.9) for every $\mathbf{v} \in E_{EM}$.

For correctness of this definition we must impose some restrictions on external forces. It is evident that they coincide with those for linear elasticity. So we assume that

$$F_i(x_1, x_2, x_3) \in L^{6/5}(\Omega) , \qquad f_i(x_1, x_2, x_3) \in L^{4/3}(S_1) . \tag{2.7.10}$$

Consider the form

$$B[\mathbf{u}, \mathbf{v}] = \tfrac{3}{2} G \iiint_\Omega \omega(e_I(\mathbf{u}))\langle \mathbf{u}, \mathbf{v}\rangle \, d\Omega + \sum_{i=1}^{3} \iiint_\Omega F_i v_i \, d\Omega + \sum_{i=1}^{3} \iint_{S_1} f_i v_i \, dS$$

as a functional in E_{EM} with respect to $\mathbf{v}(x_1, x_2, x_3)$ when $\mathbf{u}(x_1, x_2, x_3) \in E_{EM}$ is fixed. As in linear elasticity, the load terms, thanks to (2.7.10), are continuous linear functionals with respect to $\mathbf{v} \in E_{EM}$. In accordance with (2.7.5) and (2.7.2), we get

$$\left| \tfrac{3}{2} G \iiint_\Omega \omega(e_I(\mathbf{u}))\langle \mathbf{u}, \mathbf{v}\rangle \, d\Omega \right| \le \lambda \tfrac{3}{2} G \iiint_\Omega |\langle \mathbf{u}, \mathbf{v}\rangle| \, d\Omega \le \lambda \|\mathbf{u}\| \, \|\mathbf{v}\| ,$$

so this part of the functional is also continuous.

Therefore we can apply the Riesz representation theorem to $B[\mathbf{u}, \mathbf{v}]$ and obtain

$$B[\mathbf{u}, \mathbf{v}] = (\mathbf{v}, \mathbf{g}) \equiv (\mathbf{g}, \mathbf{v}) .$$

This representation uniquely defines a correspondence

$$\mathbf{u} \mapsto \mathbf{g}$$

where $\mathbf{u}, \mathbf{g} \in E_{EM}$. We obtain an operator A acting in E_{EM} by the equality

$$\mathbf{g} = A(\mathbf{u}) \ .$$

Equation (2.7.9) is now equivalent to

$$(\mathbf{u}, \mathbf{v}) - (A(\mathbf{u}), \mathbf{v}) = 0 \tag{2.7.11}$$

or, since $\mathbf{v} \in E_{EM}$ is arbitrary,

$$\mathbf{u} = A(\mathbf{u}) \ . \tag{2.7.12}$$

The operator A is nonlinear. We shall show that it is a contraction operator. For this, take arbitrary elements $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E_{EM}$ and consider

$$(A(\mathbf{u}) - A(\mathbf{v}), \mathbf{w}) = \tfrac{3}{2} G \iiint_{\Omega} [\omega(e_I(\mathbf{u}))\langle \mathbf{u}, \mathbf{w}\rangle - \omega(e_I(\mathbf{v}))\langle \mathbf{v}, \mathbf{w}\rangle] \ d\Omega \ . \tag{2.7.13}$$

First, let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in C_2. At every point of Ω, by (2.7.7), we can estimate the integrand from (2.7.13) as follows. We have

$$\begin{aligned} \text{Int} &\equiv \left|\omega(e_I(\mathbf{u}))\,\langle \mathbf{u}, \mathbf{w}\rangle - \omega(e_I(\mathbf{v}))\,\langle \mathbf{v}, \mathbf{w}\rangle\right| \\ &= \left|\omega(e_I(\mathbf{u})) \sum_{i=1}^{6} c_i(\mathbf{u})\, c_i(\mathbf{w}) - \omega(e_I(\mathbf{v})) \sum_{i=1}^{6} c_i(\mathbf{v})\, c_i(\mathbf{w})\right| . \end{aligned}$$

Let us introduce a real-valued function $f(t)$ of a real variable t by the relation

$$f(t) = \sum_{i=1}^{6} \omega(e_I(t\mathbf{u} + (1-t)\mathbf{v}))c_i(t\mathbf{u} + (1-t)\mathbf{v})c_i(\mathbf{w}) \ .$$

It is seen that

$$\text{Int} = |f(1) - f(0)| \ .$$

As $f(t)$ is continuously differentiable, the classical mean value theorem gives

$$f(1) - f(0) = f'(z)(1-0) = f'(z) \text{ for some } z \in [0,1] \ ,$$

or, in the above terms, we get

$$\mathrm{Int} = \left| \frac{d}{dt}\left\{ \sum_{i=1}^{6} \omega(e_I(t\mathbf{u} + (1-t)\mathbf{v}))\, c_i(t\mathbf{u} + (1-t)\mathbf{v})\, c_i(\mathbf{w}) \right\}_{t=z} \right|$$

$$= \left| \left\{ \frac{d\omega(e_I(t\mathbf{u} + (1-t)\mathbf{v}))}{de_I} \frac{de_I(t\mathbf{u} + (1-t)\mathbf{v})}{dt} \cdot \right.\right.$$

$$\left.\left. \cdot \sum_{i=1}^{6} c_i(t\mathbf{u} + (1-t)\mathbf{v})\, c_i(\mathbf{w}) + \omega \sum_{i=1}^{6} c_i(\mathbf{u} - \mathbf{v})\, c_i(\mathbf{w}) \right\}_{t=z} \right| .$$

(Here we have used the linearity of $c_i(\mathbf{u})$ in $\mathbf{u}$ and, thus, in t.) Let us consider the term

$$T = \sum_{i=1}^{6} \frac{de_I(t\mathbf{u} + (1-t)\mathbf{v})}{dt}\, c_i(t\mathbf{u} + (1-t)\mathbf{v})\, c_i(\mathbf{w})$$

$$= \sum_{i=1}^{6} \frac{d}{dt}\Big(\sum_{j=1}^{6} c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \Big)^{1/2} c_i(t\mathbf{u} + (1-t)\mathbf{v})\, c_i(\mathbf{w})$$

$$= \sum_{i=1}^{6} \frac{2 \sum_{j=1}^{6} c_j(t\mathbf{u} + (1-t)\mathbf{v})\, c_j(\mathbf{u} - \mathbf{v})}{2\Big(\sum_{j=1}^{6} c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \Big)^{1/2}}\, c_i(t\mathbf{u} + (1-t)\mathbf{v})\, c_i(\mathbf{w}) .$$

Applying the Schwarz inequality, we obtain

$$|T| \le \sum_{i=1}^{6} \frac{\Big(\sum_{j=1}^{6} c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \Big)^{1/2} \Big(\sum_{j=1}^{6} c_j^2(\mathbf{u} - \mathbf{v}) \Big)^{1/2}}{\Big(\sum_{j=1}^{6} c_j^2(t\mathbf{u} + (1-t)\mathbf{v}) \Big)^{1/2}} \cdot |c_i(t\mathbf{u} + (1-t)\mathbf{v})|\, |c_i(\mathbf{w})|$$

so that by (2.7.6)

$$|T| \le \Big(\sum_{j=1}^{6} c_j^2(\mathbf{u} - \mathbf{v}) \Big)^{1/2} \sum_{i=1}^{6} |c_i(t\mathbf{u} + (1-t)\mathbf{v})|\, |c_i(\mathbf{w})|$$

$$\le e_I(\mathbf{u} - \mathbf{v}) \Big(\sum_{i=1}^{6} c_i^2(t\mathbf{u} + (1-t)\mathbf{v}) \Big)^{1/2} \Big(\sum_{i=1}^{6} c_i^2(\mathbf{w}) \Big)^{1/2}$$

$$= e_I(\mathbf{u} - \mathbf{v})\, e_I(t\mathbf{u} + (1-t)\mathbf{v})\, e_I(\mathbf{w}) .$$

Similarly,

$$\Big|\sum_{i=1}^{6} c_i(\mathbf{u}-\mathbf{v})\, c_i(\mathbf{w})\Big| \le \Big(\sum_{i=1}^{6} c_i^2(\mathbf{u}-\mathbf{v})\Big)^{1/2} \Big(\sum_{i=1}^{6} c_i^2(\mathbf{w})\Big)^{1/2} = e_I(\mathbf{u}-\mathbf{v})\, e_I(\mathbf{w})\ .$$

Combining all these, we get

$$\begin{aligned}\text{Int} &\le \left\{ \frac{d\omega(e_I(t\mathbf{u}+(1-t)\mathbf{v}))}{de_I}\, e_I(t\mathbf{u}+(1-t)\mathbf{v}))\, e_I(\mathbf{u}-\mathbf{v})\, e_I(\mathbf{w}) \right.\\ &\qquad \left. + \,\omega(e_I(t\mathbf{u}+(1-t)\mathbf{v}))\, e_I(\mathbf{u}-\mathbf{v})\, e_I(\mathbf{w}) \right\}\Big|_{t=z} \\ &= \left\{ \omega(e_I(t\mathbf{u}+(1-t)\mathbf{v})) + \frac{d\omega(e_I(t\mathbf{u}+(1-t)\mathbf{v}))}{de_I}\,\cdot \right.\\ &\qquad \left. \cdot\, e_I(t\mathbf{u}+(1-t)\mathbf{v})) \right\}\Big|_{t=z} e_I(\mathbf{u}-\mathbf{v})\, e_I(\mathbf{w})\ .\end{aligned}$$

By the condition (2.7.2), we have

$$\text{Int} \le \lambda e_I(\mathbf{u}-\mathbf{v})\, e_I(\mathbf{w}) \tag{2.7.14}$$

at every point of Ω.

Returning to (2.7.13) we have, using (2.7.14),

$$|(A(\mathbf{u})-A(\mathbf{v}),\mathbf{w})| \le \lambda \iiint_\Omega \tfrac{3}{2} G\, e_I(\mathbf{u}-\mathbf{v})\, e_I(\mathbf{w})\, d\Omega\ .$$

In accordance with the norm of E_{EM} it follows that

$$|(A(\mathbf{u})-A(\mathbf{v}),\mathbf{w})| \le \lambda\, \|\mathbf{u}-\mathbf{v}\|\, \|\mathbf{w}\|$$

or, putting $\mathbf{w} = A(\mathbf{u}) - A(\mathbf{v})$, we get

$$\|A(\mathbf{u})-A(\mathbf{v})\| \le \lambda\, \|\mathbf{u}-\mathbf{v}\|\ , \qquad \lambda = \text{constant} < 1\ . \tag{2.7.15}$$

Being obtained for $\mathbf{u},\mathbf{v},\mathbf{w} \in C_2$, this inequality holds for all $\mathbf{u},\mathbf{v},\mathbf{w} \in E_{EM}$ since in this inequality we can pass to the limit for corresponding Cauchy sequences in E_{EM}.

Inequality (2.7.15) states that A is a contraction operator in E_{EM}; hence, by the contraction mapping principle, (2.7.12) has a unique solution that can be found using the iterative procedure

$$\mathbf{u}_{k+1} = A(\mathbf{u}_k) \qquad (k = 0, 1, 2, \ldots)\ .$$

This procedure begins with an arbitrary element $\mathbf{u}_0 \in E_{EM}$; when $\mathbf{u}_0 = \mathbf{0}$, it is called the method of elastic solutions since at each step we must solve a problem of linear elasticity with some given load terms. From a practical standpoint, the method works best when the constant λ is small.

So we can formulate

Theorem 2.7.1. Assume S_0 is a piecewise smooth surface of nonzero area and that conditions (2.7.2) and (2.7.10) hold. Then a mixed boundary value problem of elastico-plasticity has a unique generalized solution in the sense of Definition 2.7.1; the iterative procedure (2.7.15) defines a sequence of successive approximations $\mathbf{u}_k \in E_{EM}$ that converges to the solution $\mathbf{u} \in E_{EM}$ and

$$\|\mathbf{u}_k - \mathbf{u}\| \le \frac{\lambda^k}{1-\lambda} \|\mathbf{u}_0 - \mathbf{u}_1\| . \tag{2.7.16}$$

It is clear that we cannot apply this theorem when, say, $S_1 = \partial\Omega$. In such a case, we must add the self-balance conditions (2.4.16). These guarantee that we can repeat the above method for a free elastic-plastic body, and so we can formulate

Theorem 2.7.2. Assume that all the requirements of Theorem 2.7.1 and the self-balance conditions (2.4.16) are met. Then there is a unique generalized solution of the boundary value problem for a bounded elastic-plastic body, and it can be found by an iterative procedure of the form (2.7.15).

Problem 2.7.1. Is an estimate of the type (2.7.16) valid in Theorem 2.7.2? □

We recommend that the reader prove Theorem 2.7.2 in detail, in order to gain experience with the technique.

Remark 2.7.1. We should call attention to the way in which we obtained the main inequality of this section: it was proved for smooth functions and then extended to the general case. This is a standard technique in the treatment of nonlinear problems of mechanics. □

2.8 Bases and Complete Systems; Fourier Series

If a linear space Y has finite dimension n, then there is a set $\{g_1, \ldots, g_n\}$ of n linearly independent elements, called a *basis* of Y, such that every $y \in Y$ has a unique representation

$$y = \sum_{k=1}^{n} \alpha_k g_k$$

where the α_k are scalars. We now consider an infinite dimensional normed space X.

Definition 2.8.1. A system of elements $\{e_k\}$ is a (*countable*) *basis* of X if any element $x \in X$ has a unique representation

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$

where the α_k are scalars.

It is clear that a basis $\{e_k\}$ is linearly independent since the equation

$$0 = \sum_{k=1}^{\infty} \alpha_k e_k$$

has the unique solution $\alpha_k = 0$ for each k.

A normed space with a countable basis is separable: a countable set of all linear combinations $\sum_{k=1}^{n} c_k e_k$ (n arbitrary) with rational coefficients c_k is dense in the space.

Problem 2.8.1. Prove this. □

We are familiar with some systems of functions which could be bases in certain spaces: for example,

$$\{g_k\} = \left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} \right\} \tag{2.8.1}$$

in $L^2(0, 2\pi)$. Later, we confirm this example.

Now we consider the system of monomials $\{x^k\}$ ($k = 1, 2, \ldots$) in $C(0, 1)$. If it is a basis, then any function $f(x) \in C(0, 1)$ could be represented in the form

$$f(x) = \sum_{k=0}^{\infty} \alpha_k x^k ,$$

where the series converges uniformly on $[0, 1]$. This means the function is analytic, but we know there are continuous functions on $[0, 1]$ that are not analytic. Hence the system $\{x^k\}$ is not a basis. On the other hand, the Weierstrass theorem states that this system possesses properties similar to those of a basis. To generalize this similarity, we introduce

Definition 2.8.2. A countable system $\{g_k\}$ of elements in a normed space X is *complete* (or *total*) in X if for any $x \in X$ and any positive number ε there is a finite linear combination $\sum_{i=1}^{n(\varepsilon)} \alpha_i g_i$ such that

$$\left\| x - \sum_{i=1}^{n(\varepsilon)} \alpha_i g_i \right\| < \varepsilon .$$

By Definition 2.8.2 and the Weierstrass theorem, the system of monomials $\{x^k\}$ is complete in $C(0, 1)$. Because $C(0, 1)$ is dense in $L^p(0, 1)$ for $p \geq 1$, this system is also complete in $L^p(0, 1)$.

Problem 2.8.2. Which systems are complete in $L^p(\Omega)$ or $W^{k,p}(\Omega)$? □

If a normed space has a countable complete system, then the space is separable. The reader should be able to name a countable dense set to verify this.

Problem 2.8.3. Name such a set. □

The problem of existence of a basis in a certain normed space is difficult, but there is a special case where it is fully solved: a separable Hilbert space. The reader will find here the theory of Fourier series largely repeated in abstract terms. We begin with

Definition 2.8.3. A system $\{x_k\}$ of elements of a Hilbert space H is *orthonormal* if for all integers m, n,

$$(x_m, x_n) = \delta_{mn}$$

where δ_{mn} is the Kronecker delta symbol.

We know that, at least for $\mathbb{R}^n$, there are some advantages in using an orthonormal system of vectors as a basis.

Suppose we have an arbitrary linearly independent system of elements of a Hilbert space H, say $\{f_1, \dots, f_n\}$, and let H_n be the subspace of H spanned by this system. We would like to use the system to construct an orthonormal system $\{g_1, \dots, g_n\}$ that is also a basis of H_n. This can be accomplished by the *Gram–Schmidt procedure*:

(1) The first element of the new system is $g_1 = f_1/\|f_1\|$, $\|g_1\| = 1$.

(2) Take $e_2 = f_2 - (f_2, g_1)g_1$; then $(e_2, g_1) = (f_2, g_1) - (f_2, g_1)\|g_1\|^2 = 0$, so the second element is $g_2 = e_2/\|e_2\|$.

(3) Take $e_3 = f_3 - (f_3, g_1)g_1 - (f_3, g_2)g_2$; then $(e_3, g_1) = 0$ and $(e_3, g_2) = 0$. Since $e_3 \neq 0$, we get the third element as $g_3 = e_3/\|e_3\|$.

$\vdots$

(i) Let $e_i = f_i - (f_i, g_1)g_1 - \cdots - (f_i, g_{i-1})g_{i-1}$. It is seen that $(e_i, g_k) = 0$ for $k = 1, \dots, i-1$, hence we set $g_i = e_i/\|e_i\|$.

This process can be continued ad infinitum since all $e_k \neq 0$ (why?). So we obtain an orthonormalized system $\{g_1, g_2, g_3, \dots\}$. The process is, however, found to be unstable for numerical computation.

As is known from linear algebra, a system $\{f_1, \dots, f_n\}$ is linearly independent in an inner product space if and only if the *Gram determinant*

$$\begin{vmatrix} (f_1, f_1) & \cdots & (f_1, f_n) \\ \vdots & \ddots & \vdots \\ (f_n, f_1) & \cdots & (f_n, f_n) \end{vmatrix}$$

is nonzero. For an orthonormal system of elements the Gram determinant, being the determinant of the identity matrix, equals $+1$; hence an orthonormal system is linearly independent.

Problem 2.8.4. Provide a more direct proof that an orthonormal system is linearly independent. □

Let $\{g_k\}$ $(k = 1, 2, \dots)$ be an orthonormal system in a complex Hilbert space H. For an element $f \in H$, the numbers α_k defined by $\alpha_k = (f, g_k)$ are called the *Fourier coefficients* of f. Now we prove

Theorem 2.8.1. A complete orthonormal system $\{g_k\}$ in a Hilbert space H is a basis of H; any $f \in H$ has the unique *Fourier series* representation

$$f = \sum_{k=1}^{\infty} \alpha_k g_k \tag{2.8.2}$$

where $\alpha_k = (f, g_k)$ are the Fourier coefficients of f.

Proof. First we consider the problem of the best approximation of an element $f \in H$ by elements of a subspace H_n spanned by $g_1, \ldots, g_n$. In Sect. 1.19 we showed that this problem has a unique solution. Now we show that it is $\sum_{k=1}^{n} \alpha_k g_k$. Indeed, consider an arbitrary linear combination $\sum_{k=1}^{n} c_k g_k$. Then

$$\begin{aligned}
\left\| f - \sum_{k=1}^{n} c_k g_k \right\|^2 &= \left(f - \sum_{k=1}^{n} c_k g_k, f - \sum_{k=1}^{n} c_k g_k \right) \\
&= \|f\|^2 - \left(f, \sum_{k=1}^{n} c_k g_k \right) - \left(\sum_{k=1}^{n} c_k g_k, f \right) + \left\| \sum_{k=1}^{n} c_k g_k \right\|^2 \\
&= \|f\|^2 - \sum_{k=1}^{n} \overline{c_k} \alpha_k - \sum_{k=1}^{n} c_k \overline{\alpha_k} + \sum_{k=1}^{n} c_k \overline{c_k} \\
&= \|f\|^2 - \sum_{k=1}^{n} |\alpha_k|^2 + \sum_{k=1}^{n} |c_k - \alpha_k|^2 .
\end{aligned}$$

Because the right-hand side takes its minimum value when $c_k = \alpha_k$, we have

$$\left\| f - \sum_{k=1}^{n} \alpha_k g_k \right\|^2 = \min_{c_1,\ldots,c_n} \left\| f - \sum_{k=1}^{n} c_k g_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{n} |\alpha_k|^2 \geq 0 ; \tag{2.8.3}$$

moreover, we obtain *Bessel's inequality*

$$\sum_{k=1}^{n} |(f, g_k)|^2 \leq \|f\|^2 . \tag{2.8.4}$$

Denote by

$$f_n = \sum_{k=1}^{n} \alpha_k g_k \tag{2.8.5}$$

the nth partial sum of the Fourier series for f. Let us show that $\{f_n\}$ is a Cauchy sequence in H. By Bessel's inequality,

$$\sum_{k=1}^{n} |\alpha_k|^2 \leq \|f\|^2 ;$$

hence

$$\|f_n - f_{n+m}\|^2 = \left\| \sum_{k=n+1}^{n+m} \alpha_k g_k \right\|^2 = \sum_{k=n+1}^{n+m} |\alpha_k|^2 \to 0 \text{ as } n \to \infty .$$

Now we show that $\{f_n\}$ converges to f. Indeed, by completeness of the system $\{g_k\}$ in H, for any $\varepsilon > 0$ we can find a number N and coefficients $c_k(\varepsilon)$ such that

$$\left\| f - \sum_{k=1}^{N} c_k(\varepsilon) g_k \right\|^2 < \varepsilon .$$

By (2.8.3),

$$\|f - f_N\|^2 = \left\| f - \sum_{k=1}^{N} \alpha_k g_k \right\|^2 \le \left\| f - \sum_{k=1}^{N} c_k(\varepsilon) g_k \right\|^2 < \varepsilon ,$$

so the sequence $\{f_N\}$ converges to f and thus

$$f = \lim_{n\to\infty} f_n . \tag{2.8.6}$$

This completes the proof. □

From (2.8.6) we can obtain *Parseval's equality*

$$\sum_{k=1}^{\infty} |(f, g_k)|^2 = \|f\|^2 , \tag{2.8.7}$$

which holds whenever $\{g_k\}$ is a complete orthonormal system in H. Indeed, by (2.8.3),

$$0 = \lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} \alpha_k g_k \right\|^2 = \lim_{n\to\infty} \left(\|f\|^2 - \sum_{k=1}^{n} |\alpha_k|^2 \right) .$$

Now we introduce

Definition 2.8.4. A system $\{e_k\}$ ($k = 1, 2, \ldots$) in a Hilbert space H is *closed in H* if from the system of equations

$$(f, e_k) = 0 \text{ for all } k = 1, 2, 3, \ldots$$

it follows that $f = 0$.

It is clear that a complete orthonormal system of elements is closed in H.

Problem 2.8.5. Provide a detailed proof. □

The converse statement holds as well. We formulate

Theorem 2.8.2. Let $\{g_k\}$ be an orthonormal system of elements in a Hilbert space H. This system is complete in H if and only if it is closed in H.

Proof. We need to demonstrate only that a closed orthonormal system in H is complete. Proving Theorem 2.8.1, we established that for any element $f \in H$ the sequence of partial Fourier sums (2.8.5) is a Cauchy sequence. By completeness of H, there exists $f^* = \lim_{n\to\infty} f_n$ that belongs to H. To complete the proof we need to show that $f = f^*$. We have

$$(f - f^*, g_m) = \lim_{n\to\infty}\left(f - \sum_{k=1}^{n} \alpha_k g_k, g_m\right) = \alpha_m - \alpha_m = 0\ .$$

By Definition 2.8.4, it follows that $f = f^*$, hence $\{g_k\}$ is complete. □

It is normally simpler to check whether a system is closed than to check whether it is complete. At the beginning of this section we established that any system of linearly independent elements in H can be transformed into an orthonormal system equivalent to the original system in a certain way. So we draw the following conclusion.

Theorem 2.8.3. A complete system $\{g_k\}$ in H is closed in H; conversely, a system closed in H is complete in H.

Problem 2.8.6. Write out a detailed proof. □

As stated above, the existence of a countable basis in a Hilbert space provides its separability. The converse statement is also valid. We formulate that as

Theorem 2.8.4. A Hilbert space H has a countable orthonormal basis if and only if H is separable.

The proof follows immediately from the previous theorem. Indeed, in H select a countable set of elements that is dense everywhere in H. Using the Gram–Schmidt procedure, produce an orthonormal system of elements from this set (removing any linearly dependent elements). Since the initial system is dense it is complete and thus, as a result of the Gram–Schmidt procedure, we get an orthonormal basis of the space.

Remember that all of the energy spaces we introduced above are separable. Hence each of them has a countable orthonormal basis (nonunique, of course). If a Hilbert space is not separable, by Bessel's inequality and Lemma 1.16.5, it follows that for any element x of a nonseparable space the set of nonzero coefficients of Fourier α_k is countable. Repeating the above considerations we can get that (2.8.2) is valid in this case as well.

In conclusion, we consider whether the system (2.8.1) is a basis of the complex space $L^2(0, 2\pi)$. From standard calculus it is known that the system is orthonormal in $L^2(0, 2\pi)$ (the reader, however, can check this). Weierstrass's theorem on the approximation of a function continuous on $[0, 2\pi]$ can be formulated as the statement that the set of trigonometric polynomials, i.e., finite sums of the form $\sum_k \alpha_k e^{ikx}$, is dense in the complex space $C(0, 2\pi)$. But the set of functions $C(0, 2\pi)$ is the base for construction of $L^2(0, 2\pi)$, hence the finite sums $\sum_k \alpha_k e^{ikx}$ are dense in $L^2(0, 2\pi)$. This shows that (2.8.1) is an orthonormal basis of $L^2(0, 2\pi)$.

2.9 Weak Convergence in a Hilbert Space

We know that in $\mathbb{R}^n$, the convergence of a sequence of vectors is equivalent to coordinate-wise convergence.

In a Hilbert space H, the Fourier coefficients (f, g_k) of an element $f \in H$ play the role of the coordinates of f. Suppose $\{g_k\}$ is an orthonormal basis of H. What can we say about convergence of a sequence $\{f_n\}$ if, for every fixed k, the numerical sequence $\{(f_n, g_k)\}$ is convergent?

Let us consider $\{g_n\}$ as a sequence. It is seen that for every k,

$$\lim_{n\to\infty} (g_n, g_k) = 0 ,$$

hence we have coordinate-wise convergence of $\{g_n\}$ to zero. But the sequence $\{g_n\}$ is not convergent, since

$$\|g_n - g_m\| = \sqrt{2} \text{ for } n \neq m .$$

Therefore, coordinate-wise convergence in a Hilbert space is not equivalent to the usual form of convergence in the space. We define a new type of convergence in a Hilbert space.

Definition 2.9.1. Let H be a Hilbert space. A sequence $\{x_k\} \subset H$ is *weakly convergent* to $x_0 \in H$ if for every continuous linear functional F in H,

$$\lim_{k\to\infty} F(x_k) = F(x_0) .$$

If every numerical sequence $\{F(x_k)\}$ is a Cauchy sequence, then $\{x_k\}$ is a *weak Cauchy sequence*.

To distinguish between weak convergence and convergence as defined on p. 29, we shall refer to the latter as *strong convergence*. We retain the notation $x_k \to x$ for strong convergence and adopt $x_k \rightharpoonup x$ for weak convergence.

Definition 2.9.1 is given in a form which (with suitable modifications) is valid in a metric space. But in a Hilbert space any continuous linear functional, by the Riesz representation theorem, takes the form $F(x) = (x, f)$ where f is an element of H. So Definition 2.9.1 may be rewritten as follows:

Definition 2.9.2. Let H be a Hilbert space. A sequence $\{x_n\} \subset H$ is weakly convergent to $x_0 \in H$ if for every element $f \in H$ we have

$$\lim_{n\to\infty} (x_n, f) = (x_0, f) .$$

If every numerical sequence $\{(x_n, f)\}$ is a Cauchy sequence, then $\{x_k\}$ is a weak Cauchy sequence.

We have seen that some weak Cauchy sequences in H are not strong Cauchy sequences. But a strong Cauchy sequence is always a weak Cauchy sequence, by virtue of the continuity of the linear functionals in the definition.

We formulate a simple sufficient condition for strong convergence of a weakly convergent sequence:

Theorem 2.9.1. Suppose that $x_k \rightharpoonup x_0$, where x_k, x_0 belong to a Hilbert space H. Then $x_k \to x_0$ if $\|x_k\| \to \|x_0\|$.

Proof. Consider $\|x_k - x_0\|^2$. We get

$$\|x_k - x_0\|^2 = (x_k - x_0, x_k - x_0) = \|x_k\|^2 - (x_0, x_k) - (x_k, x_0) + \|x_0\|^2 \ .$$

By Definition 2.9.2 we have

$$\lim_{k\to\infty} [(x_0, x_k) + (x_k, x_0)] = 2\,\|x_0\|^2 \ ,$$

hence $\|x_k - x_0\|^2 \to 0$. □

We shall see later that for some numerical methods it is easier to first establish weak convergence of approximate solutions and then strong convergence, than to establish strong convergence directly. The last theorem allows us to justify a method successively, beginning with a simple approximate result and then passing to the needed one. That is why weak convergence is a major preoccupation in this presentation.

Theorem 2.9.2. In a Hilbert space, every weak Cauchy sequence $\{x_n\}$ is bounded.

Proof. We will prove the theorem for a complex Hilbert space; the proof is valid for a real space as well. Suppose to the contrary that there is a weak Cauchy sequence $\{x_n\}$ which is not bounded in H. So let $\|x_n\| \to \infty$ as $n \to \infty$. We will show that this yields a contradiction.

First we consider the set U of all numbers of the form (x_n, y), where y belongs to a closed ball $\overline{B}(y_0, \varepsilon)$ with arbitrary $\varepsilon > 0$ and center $y_0 \in H$, which are momentarily fixed. We claim that U is unbounded from above. Indeed, elements of the form $y_n = y_0 + \varepsilon x_n/(2\,\|x_n\|)$ belong to $\overline{B}(y_0, \varepsilon)$ since

$$\|y_n - y_0\| = \left\| \frac{\varepsilon x_n}{2\,\|x_n\|} \right\| = \frac{\varepsilon}{2} \ .$$

As $\{x_n\}$ is a weak Cauchy sequence, the numerical sequence $\{(x_n, y_0)\}$ converges and therefore is bounded. Since $\|x_n\| \to \infty$ we get

$$|(x_n, y_n)| = \left|(x_n, y_0) + \frac{\varepsilon}{2\,\|x_n\|}(x_n, x_n)\right| = \left|(x_n, y_0) + \frac{\varepsilon}{2}\,\|x_n\|\right| \to \infty$$

as $n \to \infty$. We see that U is unbounded for any fixed y_0.

Now we show that unboundedness of any set U for any y_0 yields a contradiction. Take the ball $\overline{B}(y_0, \varepsilon_1)$ with $\varepsilon_1 = 1$ and $y_0 = 0$. Because U is unbounded from above, we can take any $y_1 \in \overline{B}(y_0, \varepsilon_1)$ and then find x_{n_1} such that

$$|(x_{n_1}, y_1)| > 1 \ . \tag{2.9.1}$$

By continuity of the inner product in both its variables, we can find a closed ball $\overline{B}(y_1, \varepsilon_2)$ such that $\overline{B}(y_1, \varepsilon_2) \subset \overline{B}(y_0, \varepsilon_1)$ and such that (2.9.1) holds not only for y_1 but for all $y \in \overline{B}(y_1, \varepsilon_2)$:

$$|(x_{n_1}, y)| > 1 \ \text{ for all } y \in \overline{B}(y_1, \varepsilon_2)\ .$$

Then, in the ball $\overline{B}(y_1, \varepsilon_2)$, we similarly take an interior point y_2 and find x_{n_2}, with $n_2 > n_1$, such that

$$|(x_{n_2}, y_2)| > 2\ ,$$

and, after this, a closed ball $\overline{B}(y_2, \varepsilon_3)$ such that $\overline{B}(y_2, \varepsilon_3) \subset \overline{B}(y_1, \varepsilon_2)$ and

$$|(x_{n_2}, y)| > 2 \ \text{ for all } y \in \overline{B}(y_2, \varepsilon_3)\ .$$

Repeating this procedure ad infinitum, we produce a sequence of closed balls $\overline{B}(y_k, \varepsilon_{k+1})$ such that $\overline{B}(y_0, \varepsilon_1) \supset \overline{B}(y_1, \varepsilon_2) \supset \overline{B}(y_2, \varepsilon_3) \supset \cdots$, and corresponding terms x_{n_k}, $n_{k+1} > n_k$, of the sequence $\{x_n\}$ such that

$$|(x_{n_k}, y)| > k \ \text{ for all } y \in \overline{B}(y_k, \varepsilon_{k+1})\ .$$

Since H is a Hilbert space there is at least one element y^* which belongs to every $\overline{B}(y_k, \varepsilon_{k+1})$, so

$$|(x_{n_k}, y^*)| > k\ .$$

Thus we find a continuous linear functional $F^*(x) = (x, y^*)$ for which the numerical sequence $\{F^*(x_{n_k})\}$ is not a Cauchy sequence. This contradicts the definition of weak convergence of $\{x_k\}$. □

This proof yields another important result:

Lemma 2.9.1. Assume $\{x_k\}$ is an unbounded sequence in H, i.e., $\|x_k\| \to \infty$. Then there exists $y^* \in H$ and a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, y^*) \to \infty$ as $k \to \infty$.

Proof. Let $z_n = x_n/\|x_n\|$. For any y with unit norm, the numerical sequence (z_n, y) is bounded and thus we can select a convergent subsequence from it. If there exists such a unit element y^* and a subsequence $\{z_{n_k}\}$ for which $(z_{n_k}, y^*) \to a \neq 0$, then the statement of the lemma is valid for the subsequence $\{x_{n_k}\}$ and y^* if $a > 0$; if $a < 0$, then y^* must be changed to $-y^*$. Indeed, if $a > 0$ then $(x_{n_k}, y^*) = (z_{n_k}, y^*)\,\|x_{n_k}\| \to \infty$.

Now we suppose that we cannot find such an element y^* and a subsequence $\{z_{n_k}\}$ for which $(z_{n_k}, y^*) \to a \neq 0$. So $(z_n, y) \to 0$ for any $y \in H$. By the Riesz representation theorem, this means that $\{z_n\}$ converges weakly to zero. We will prove that the statement of Lemma 2.9.1 holds for the latter class of sequences as well. For this we repeat two steps of the proof of Theorem 2.9.2.

First we show that for any center y_0 and radius ε, the numerical set (x_n, y) with y running over $\overline{B}(y_0, \varepsilon)$ is unbounded. Indeed, taking the sequence $y_n = y_0 + \varepsilon/(2\,\|x_n\|)\, x_n$ we get an element from $\overline{B}(y_0, \varepsilon)$. Next,

$$(x_n, y_n) = (x_n, y_0) + \frac{\varepsilon}{2\,\|x_n\|}(x_n, x_n) = \left((z_n, y_0) + \frac{\varepsilon}{2}\right)\|x_n\|\ .$$

Since ε is finite and $(z_n, y_0) \to 0$ as $n \to \infty$, we have $(x_n, y_n) \to \infty$.

Another step of the proof of Theorem 2.9.2, establishing the existence of a subsequence $\{x_{n_k}\}$ and an element y^* such that $(x_{n_k}, y^*) \to \infty$, requires only that $\|x_n\| \to \infty$ and that for any $\varepsilon > 0$ the set (x_n, y) is unbounded when y runs over $\overline{B}(y_0, \varepsilon)$, which was just proved. Thus we immediately state the validity of Lemma 2.9.1 for all the unbounded sequences. □

This is used in proving the *principle of uniform boundedness*, which we have established in a more general form (Theorem 1.23.2).

Theorem 2.9.3. Let $\{F_k(x)\}$ $(k = 1, 2, \ldots)$ be a family of continuous linear functionals defined on a Hilbert space H. If $\sup_k |F_k(x)| < \infty$, then $\sup_k \|F_k\| < \infty$.

Proof. By the Riesz representation theorem, each of the functionals $F_k(x)$ has the form

$$F_k(x) = (x, f_k) , \quad \text{where} \quad f_k \in H , \quad \|f_k\| = \|F_k\| .$$

So the condition of the theorem can be rewritten as

$$\sup_k |(x, f_k)| < \infty . \tag{2.9.2}$$

By Lemma 2.9.1, the assumption that $\sup_k \|f_k\| = \infty$ implies the existence of $x_0 \in H$ and $\{f_{k_n}\}$ such that

$$|(x_0, f_{k_n})| \to \infty \ \text{ as } k \to \infty .$$

This contradicts (2.9.2). □

Corollary 2.9.1. Let $\{F_k(x)\}$ be a sequence of continuous linear functionals given on H, such that for every $x \in H$ the numerical sequence $\{F_k(x)\}$ is a Cauchy sequence. Then there is a continuous linear functional $F(x)$ on H such that

$$F(x) = \lim_{k\to\infty} F_k(x) \ \text{ for all } x \in H \tag{2.9.3}$$

and

$$\|F\| \le \liminf_{k\to\infty} \|F_k\| < \infty . \tag{2.9.4}$$

Proof. The limit on the right-hand side of (2.9.3), existing by the condition, defines a functional $F(x)$ which is clearly linear. Since the condition of Theorem 2.9.3 is met, we have $\sup_k \|F_k\| < \infty$; from

$$|F(x)| = \lim_{k\to\infty} |F_k(x)| \le \sup_k \|F_k\| \, \|x\|$$

it follows that $F(x)$ is continuous. Moreover (recall Problem 1.23.1),

$$|F(x)| = \lim_{k\to\infty} |F_k(x)| \le \liminf_{k\to\infty} \|F_k\| \, \|x\| ,$$

i.e., (2.9.4) is proved also. □

The following theorem gives an equivalent but more convenient definition of weak convergence.

Theorem 2.9.4. A sequence $\{x_n\}$ is weakly Cauchy in a Hilbert space H if and only if the following pair of conditions holds:

(i) $\{x_n\}$ is bounded in H, i.e., there is a constant M such that $\|x_n\| \le M$;

(ii) for any $f_\alpha \in H$ from a system $\{f_\alpha\}$ which is complete in H, the numerical sequence (x_n, f_α) is a Cauchy sequence.

Proof. Necessity of the conditions follows from the definition of weak convergence and Theorem 2.9.2.

Now we prove sufficiency. Suppose the conditions (i) and (ii) hold. Take an arbitrary continuous linear functional defined, by the Riesz representation theorem, by an element $f \in H$ and consider the numerical sequence

$$d_{nm} = (x_n, f) - (x_m, f) .$$

As the system $\{f_\alpha\}$ is complete, there is a linear combination

$$f_\varepsilon = \sum_{k=1}^{N} c_k f_k$$

such that

$$\|f - f_\varepsilon\| < \varepsilon/3M .$$

Then

$$\begin{aligned} |d_{nm}| &= |(x_n - x_m, f)| \\ &= |(x_n - x_m, f_\varepsilon + f - f_\varepsilon)| \\ &\le |(x_n - x_m, f_\varepsilon)| + |(x_n - x_m, f - f_\varepsilon)| \\ &\le \sum_{k=1}^{N} |c_k|\,|(x_n - x_m, f_k)| + (\|x_n\| + \|x_m\|)\,\|f - f_\varepsilon\| . \end{aligned}$$

Since, by (ii), the sequences $\{(x_n, f_k)\}$ $(k = 1, \ldots, N)$ are Cauchy sequences, we can find a number R such that

$$\sum_{k=1}^{N} |c_k|\,|(x_n - x_m, f_k)| < \varepsilon/3 \quad \text{for all } m, n > R$$

hence

$$|d_{nm}| \le \varepsilon/3 + 2M\varepsilon/(3M) = \varepsilon \quad \text{for } m, n > R .$$

This means that $\{(x_n, f)\}$ is a weak Cauchy sequence. □

Problem 2.9.1. Show that a sequence $\{x_n\}$ is weakly convergent to x_0 in H if and only if the following pair of conditions holds:

(i) $\{x_n\}$ is bounded in H;

(ii) for any f_α from a system $\{f_\alpha\}$, $f_\alpha \in H$, which is complete in H, we have $\lim_{n\to\infty}(x_n, f_\alpha) = (x_0, f_\alpha)$. □

Because weak convergence differs from strong convergence, we are led to consider *weak completeness* of a Hilbert space.

Theorem 2.9.5. Any weak Cauchy sequence $\{x_n\}$ in a Hilbert space H converges weakly to an element of this space.

In other words, a Hilbert space H is also weakly complete.

Proof. For any fixed $y \in H$ we define $F(y) = \lim_{n\to\infty}(y, x_n)$. The functional $F(y)$, whose linearity is evident, is defined on the whole of H. From the inequality

$$|(y, x_n)| \leq M\,\|y\|$$

where M is a constant such that $\|x_n\| \leq M$, it follows that

$$|F(y)| \leq M\,\|y\| \quad \text{and} \quad \|F\| \leq M\ .$$

Therefore $F(y)$ is a continuous linear functional which, by the Riesz representation theorem, can be written in the form

$$F(y) = (y, f)\ , \qquad f \in H\ , \quad \|f\| = \|F\| \leq M\ .$$

But this means that f is a weak limit of $\{x_n\}$. □

From this proof also follows

Lemma 2.9.2. If a sequence $\{x_n\} \subset H$ converges weakly to x_0 in H and $\|x_n\| \leq M$ for all n, then $\|x_0\| \leq M$.

Problem 2.9.2. Provide the details. □

This states that a closed ball about zero is weakly closed. Any closed subspace of a Hilbert space is also weakly closed. We also formulate Mazur's theorem that any closed convex set in a Hilbert space is weakly closed. The interested reader can find a proof in Yosida [44].

Theorem 2.9.6 (Mazur). Assume that a sequence $\{x_n\}$ in a Hilbert space H converges weakly to $x_0 \in H$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequence of arithmetic means $\frac{1}{N}\sum_{k=1}^{N} x_{n_k}$ converges strongly to x_0.

Problem 2.9.3. Show that a weakly closed set is closed. □

Let us consider the problem of *weak compactness* of a set in a Hilbert space. We have seen that a ball in an infinite dimensional Hilbert space is not strongly compact. But for weak compactness, an analog of the Bolzano–Weierstrass theorem holds as follows:

Theorem 2.9.7. A bounded sequence $\{x_n\}$ in a separable Hilbert space contains a weak Cauchy subsequence.

In other words, a bounded set in a Hilbert space is *weakly precompact*.

Proof. In a separable Hilbert space there is an orthonormal basis $\{g_n\}$. By Theorem 2.9.4 it suffices to show that there is a subsequence $\{x_{n_k}\}$ such that, for each fixed g_m, the numerical sequence $\{(x_{n_k}, g_m)\}$ is a Cauchy sequence.

The bounded numerical sequence $\{(x_n, g_1)\}$ contains a convergent subsequence $\{(x_{n_1}, g_1)\}$. Considering the numerical sequence $\{(x_{n_1}, g_2)\}$, for the same reason we can choose a convergent sequence $\{(x_{n_2}, g_2)\}$. Continuing this process, on the kth step we obtain a convergent numerical subsequence $\{(x_{n_k}, g_k)\}$.

Choosing now the elements x_{n_n}, we obtain a sequence $\{x_{n_n}\}$ such that for any fixed g_m the numerical sequence $\{(x_{n_n}, g_m)\}$ is a Cauchy sequence. That is, $\{x_{n_n}\}$ is a weak Cauchy sequence. □

This theorem has important applications. In justifying certain numerical methods we can sometimes prove boundedness of the set of approximate solutions in a Hilbert (as a rule, energy) space, and hence obtain a subsequence of approximations that converges weakly to an element; then we can show that this element is a solution.

Let us apply this procedure to the approximation problem, namely, we want to minimize a functional

$$F(x) = \|x - x_0\|^2 \tag{2.9.5}$$

over a real Hilbert space when x_0 is a fixed element of H, $x_0 \notin M$, and x is an arbitrary element of a closed subspace $M \subset H$.

In Sect. 1.19 we established the existence of a minimizer of $F(x)$. We now treat this problem once more, as though this existence were unknown to us.

This very simple problem (at least in theory) exhibits the following typical steps, which are common for the justification of approximate solutions to many boundary value problems:

1. the formulation of an approximation problem and the demonstration of its solvability;
2. a global *a priori* estimate of the approximate solutions that does not depend on the step of approximation;
3. the demonstration of convergence of the approximate solutions to a solution of the initial problem, and a study of the nature of convergence.

Thus we begin to study our problem with *Step 1*, the formulation of the approximation problem.

We try to solve the problem approximately, using the *Ritz method*. Assume $\{g_k\}$ is a complete system in M such that any of its finite subsystems is linearly independent. Consider M_n spanned by $(g_1, \ldots, g_n)$ and find an element which minimizes $F(x)$ on M_n. A solution of this problem, denoted by x_n, is the nth *Ritz approximation* of the solution.

A real-valued function $f(t) = F(x_n + tg_k)$ of the real variable t takes its minimal value at $t = 0$ and, thanks to differentiability of $f(t)$,

$$\left.\frac{df(t)}{dt}\right|_{t=0} = 0 .$$

This yields

$$0 = \frac{d}{dt}\|x_n - x_0 + tg_k\|^2\Big|_{t=0} = \frac{d}{dt}(x_n - x_0 + tg_k, x_n - x_0 + tg_k)\Big|_{t=0} = 2(x_n - x_0, g_k) ,$$

so $x_n - x_0$ is orthogonal to each g_k $(k = 1, \dots, n)$.

Using the representation

$$x_n = \sum_{k=1}^{n} c_{kn} g_k ,$$

we get a linear system of algebraic equations called the Ritz system of nth approximation:

$$\sum_{k=1}^{n} c_{kn}(g_k, g_m) = (x_0, g_m) \qquad (m = 1, \dots, n) . \tag{2.9.6}$$

The determinant of this system is the Gram determinant of a linearly independent system $(g_1, \dots, g_m)$ that is not equal to zero. So the system (2.9.6) has a unique solution $(\hat{c}_{1n}, \dots, \hat{c}_{kn})$.

Step 2. Now we will find a global estimate of the approximate solutions that does not depend on n. Although, in this case, we know that the approximate solution exists, we can get the estimate without this knowledge. Hence it is called an *a priori* estimate.

We begin with the definition of x_n:

$$\|x_n - x_0\|^2 \le \|x - x_0\|^2 \text{ for all } x \in M_n .$$

As $x = 0 \in M_n$, it follows that

$$\|x_n - x_0\|^2 \le \|x_0\|^2 ,$$

from which

$$\|x_n\|^2 \le 2\,\|x_n\|\,\|x_0\| ,$$

hence

$$\|x_n\| \le 2\,\|x_0\| . \tag{2.9.7}$$

This is the required estimate.

Remark 2.9.1. It is possible to get a sharper estimate than (2.9.7); however, for this problem it is only necessary to establish the existence of a bound. □

Step 3. Our last goal is to show that the sequence of approximations converges to a solution of the problem. First we demonstrate that this convergence is weak, and then that it is strong.

By (2.9.7), the sequence $\{x_n\}$ is bounded and, thanks to Theorem 2.9.7, contains a weakly convergent subsequence $\{x_{n_k}\}$ whose weak limit x^* belongs to M (remember that a closed subspace is weakly closed).

For any fixed g_m, we can pass to the limit as $k \to \infty$ in the equality

$$(x_{n_k} - x_0, g_m) = 0$$

and get

$$(x^* - x_0, g_m) = 0$$

because (x, g_m) is a continuous linear functional in $x \in H$.

Now consider $(x^* - x_0, h)$ where $h \in M$ is arbitrary but fixed. By completeness of the system $g_1, g_2, g_3, \ldots$ in M, given $\varepsilon > 0$ we can find a finite linear combination

$$h_\varepsilon = \sum_{k=1}^{N} c_k g_k$$

such that

$$\|h - h_\varepsilon\| \le \varepsilon/(3\,\|x_0\|) .$$

Then

$$\begin{aligned}
|(x^* - x_0, h)| &= |(x^* - x_0, h - h_\varepsilon + h_\varepsilon)| \\
&\le |(x^* - x_0, h - h_\varepsilon)| + |(x^* - x_0, h_\varepsilon)| \\
&= |(x^* - x_0, h - h_\varepsilon)| \\
&\le \|x^* - x_0\|\,\|h - h_\varepsilon\| \\
&\le (\|x^*\| + \|x_0\|)\,\|h - h_\varepsilon\| \\
&\le (2\,\|x_0\| + \|x_0\|)\,\varepsilon/(3\,\|x_0\|) = \varepsilon .
\end{aligned}$$

Therefore, for any $h \in M$ we get

$$(x^* - x_0, h) = 0 . \tag{2.9.8}$$

Finally, considering values of (2.9.5) on elements of the form $x = x^* + h$ when $h \in M$, we obtain, by (2.9.8),

$$\begin{aligned}
F(x^* + h) &= (x^* - x_0 + h, x^* - x_0 + h) \\
&= \|x^* - x_0\|^2 + 2(x^* - x_0, h) + \|h\|^2 \\
&= \|x^* - x_0\|^2 + \|h\|^2 \ge \|x^* - x_0\|^2 = F(x^*) .
\end{aligned}$$

It follows that x^* is a solution of the problem, and existence of solution has been proved.

Now we can show that the approximation sequence converges strongly to a solution of the problem. By Theorem 1.19.3, a minimizer of $F(x)$ is unique; this gives us weak convergence of the sequence $\{x_n\}$ on the whole. Indeed, suppose to the contrary that $\{x_n\}$ does not converge weakly to x^*. Then there is an element $f \in H$ such that

$$(x_n, f) \nrightarrow (x^*, f) . \tag{2.9.9}$$

By boundedness of the numerical set $\{(x_n, f)\}$, the statement (2.9.9) implies that there is a subsequence $\{x_{n_k}\}$ such that there exists

$$\lim_{k\to\infty} (x_{n_k}, f) \neq (x^*, f) . \tag{2.9.10}$$

Problem 2.9.4. Prove (2.9.10). □

But, for the subsequence $\{x_{n_k}\}$, we can repeat the above considerations and find that $\{x_{n_k}\}$ contains a subsequence which converges weakly to a solution of the problem. Since the solution is unique, this contradicts (2.9.10). Finally, multiplying both sides of (2.9.6) by the Ritz coefficient $\hat{c}_{mn}$ and summing over m, we get

$$(x_n, x_n) = (x_0, x_n) .$$

We can pass to the limit as $n \to \infty$, obtaining

$$\lim_{n\to\infty} (x_n, x_n) = \lim_{n\to\infty} (x_0, x_n) = (x_0, x^*) .$$

By (2.9.8) with $h = x^*$ we have

$$(x_0, x^*) = (x^*, x^*) ,$$

so

$$\lim_{n\to\infty} \|x_n\|^2 = \|x^*\|^2 .$$

Therefore, by Theorem 2.9.1, the sequence $\{x_n\}$ converges strongly to x^*.

So we have demonstrated, via the Ritz method, a general way of justifying the solution of a minimal problem and the Ritz method itself. The method is common to a wide variety of problems, some nonlinear. In the latter case, many difficulties center on Steps 2 or 3, depending on the problem. The problem under discussion can also be interpreted another way, and this is of so much importance that we devote a separate section to it.

2.10 The Ritz and Bubnov–Galerkin Methods in Linear Problems

We reconsider the problem of minimizing the quadratic functional (2.4.8) in a Hilbert space, namely,

$$I(x) = \|x\|^2 + 2\Phi(x) \to \min_{x\in H} . \tag{2.10.1}$$

Assuming $\Phi(x)$ is a continuous linear functional, the Riesz representation theorem yields

$$\Phi(x) = (x, -x_0)$$

where $x_0 \in H$ is uniquely defined by $\Phi(x)$. Then

$$I(x) = \|x\|^2 - 2(x, x_0) = \|x - x_0\|^2 - \|x_0\|^2 .$$

Since $\|x_0\|^2$ is fixed, the problem (2.4.1) is equivalent to

$$F(x) = \|x - x_0\|^2 \to \min_{x \in H} .$$

This problem has the unique (and obvious) solution $x = x_0$. Of much interest is the fact that it coincides with the problem of the previous section if $M = H$. So application of the Ritz method in this problem is justified. Let us recall those results in terms of the new problem.

Let $\{g_k\}$ be a complete system in H, every finite subsystem of which is linearly independent, and let the nth Ritz approximation to a minimizer be

$$x_n = \sum_{k=1}^{n} c_{kn} g_k .$$

The system giving the nth approximation of the Ritz method is

$$\sum_{k=1}^{n} c_{kn}(g_k, g_m) = -\Phi(g_m) \qquad (m = 1, \ldots, n) . \tag{2.10.2}$$

Let us collect the results in

Theorem 2.10.1. The following statements hold.

(i) For each $n \geq 1$, the system (2.10.2) of nth approximation of the Ritz method has the unique solution $c_{1n}, \ldots, c_{nn}$.

(ii) The sequence $\{x_n\}$ of Ritz approximations defined by (2.10.2) converges strongly to the minimizer of the quadratic functional $\|x\|^2 + 2\Phi(x)$, where $\Phi(x)$ is a continuous linear functional on H.

It is interesting to note that if $\{g_k\}$ is an orthonormal basis of H, then (2.10.2) gives the Fourier coefficients of the solution in H.

Concerning Bubnov's method, we only mention that it appeared when A.S. Bubnov, reviewing an article by S. Timoshenko, noted that the Ritz equations can be obtained by multiplying by g_m, a function of a complete system, the differential equation of equilibrium in which u was replaced by

$$u_n = \sum_{k=1}^{n} c_{kn} g_k ,$$

integrating the latter over the region, and then integrating by parts. In our terms this is

$$(u_n, g_m) = -\Phi(g_m) \qquad (m = 1, \ldots, n)\,.$$

Since this system indeed coincides with (2.10.2), Theorem 2.10.1 also justifies Bubnov's method.

Galerkin was the first to propose multiplying by f_m, a function of another system, for better approximation of the residual. The corresponding system is, in our notation,

$$(u_n, f_m) = -\Phi(f_m) \qquad (m = 1, \ldots, n)\,.$$

Discussion of this modification of the method can be found in Mikhlin [27].

Finally, we note that the finite element method for solution of mechanics problems is a particular case of the Bubnov–Galerkin method, hence it is also justified for the problems we consider.

2.11 Curvilinear Coordinates, Nonhomogeneous Boundary Conditions

We have considered some problems of mechanics using the Cartesian coordinate system. Almost all of the textbooks present the theory of the same problems in Cartesian frames; the few exceptions are the textbooks on the theory of shells and curvilinear beams, where it is impossible to consider the problems in Cartesian frames. However, in practice other coordinate systems occur frequently. The question arises whether it is necessary to investigate the boundary value problems for other coordinates, or whether it is enough to reformulate the results for Cartesian systems. For the generalized statements of mechanics problems in energy spaces, the answer is simple: it is possible to reformulate the results, and a key tool is a simple change of the coordinates. This change allows us to reformulate the imbedding theorems in energy spaces, to establish the requirements for admitting classes of loads, etc. We note that it is a hard problem to obtain similar results independently, without the use of coordinate transformations, if the coordinate frame has singular points.

Let us consider a simple example of a circular membrane with fixed edge (Dirichlet problem). In Cartesian coordinates we have the Sobolev imbedding theorem

$$\left(\iint_\Omega |u(x)|^p\,dx\,dy\right)^{1/p} \le m\left\{\iint_\Omega \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy\right\}^{1/2} \tag{2.11.1}$$

for $p \ge 1$, which is valid for any $u \in \dot{W}^{1,2}(\Omega) \equiv E_{MC}$ satisfying the boundary condition

$$u\big|_{\partial\Omega} = 0\,. \tag{2.11.2}$$

Taking a function $u \in C^{(1)}(\Omega)$ satisfying (2.11.2), in both integrals of (2.11.1) we pass to the polar coordinate system:

$$\left(\int_0^R \int_0^{2\pi} |u|^p r \, d\phi \, dr \right)^{1/p} \le m \left\{ \int_0^R \int_0^{2\pi} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \phi} \right)^2 \right] r \, d\phi \, dr \right\}^{1/2} \quad (2.11.3)$$

where (r, ϕ) are the polar coordinates in a disk of radius R. Passing to the limit along a Cauchy sequence of E_{MC} in the inequality (2.11.1), which is valid in Cartesian coordinates, shows us that it remains valid in the form (2.11.3) in polar coordinates. Inequality (2.11.3) is an imbedding theorem in the energy space of the circular membrane in terms of polar coordinates. The expression

$$\|u\| = \left\{ \int_0^R \int_0^{2\pi} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \phi} \right)^2 \right] r \, d\phi \, dr \right\}^{1/2} \quad (2.11.4)$$

is the norm in this coordinate system and

$$(u, v) = \int_0^R \int_0^{2\pi} \left(\frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial u}{\partial \phi} \frac{\partial v}{\partial \phi} \right) r \, d\phi \, dr$$

is the corresponding inner product.

The requirement imposed on forces for existence of a generalized solution has the form

$$\int_0^R \int_0^{2\pi} |F|^q r \, d\phi \, dr < \infty \qquad (q > 1) \, .$$

We have a natural form of the norm in the energy space (which is determined by the energy itself) using curvilinear coordinates, as well as a form of the imbedding theorem (i.e., properties of elements of the energy space and natural requirements on forces for the problem to be uniquely solvable).

Then we note that we can replace formally the Cartesian system by any other system of coordinates which is admissible for smooth functions, and also change formally any variables in any expression which makes sense in the energy space considered in Cartesian coordinates.

Finally, note that a norm like (2.11.4) is usually called a *weighted norm* because of the presence of weight factors, here connected with powers of r. There is an abstract theory of such weighted Sobolev spaces, not being so elementary as in the space we have considered.

For more complicated problems such as elasticity problems, we can use the same method of introducing curvilinear coordinates; here we can change not only the independent variables (x_1, x_2, x_3), but also unknown components of vectors of displacements and prescribed forces, to the new coordinate system. We leave it to the reader to write down an equation determining a generalized solution, the forms of norm and scalar product, and restrictions for forces as well as imbedding inequalities, in other curvilinear coordinate systems such as cylindrical and spherical.

Now let us consider two questions connected with nonhomogeneous boundary value problems in mechanics. The first is to identify the whole class of admissible external forces for which an energy solution exists. We know that the condition for existence of a solution is that the functional of external forces

$$\int_\Omega F(\mathbf{x})v(\mathbf{x})\,d\Omega \tag{2.11.5}$$

(say, in the membrane problem) is continuous and linear with respect to $v(\mathbf{x})$ on an energy space. We shall show how this condition can be expressed in terms of so-called spaces with negative norms, a notion due to P.D. Lax [17].

The functional (2.11.5) can be considered as the scalar product of $F(\mathbf{x})$ by $v(\mathbf{x})$ in $L^2(\Omega)$. But $v(\mathbf{x})$ belongs to an energy space E whose norm, for simplicity, is assumed to be such that $\|v\|_E = 0$ implies $v = 0$. We know that $v \in L^2(\Omega)$ if $v \in E$; moreover, E is dense in $L^2(\Omega)$. For any $F(\mathbf{x}) \in L^2(\Omega)$, we can introduce a new norm

$$\|F\|_E = \sup_{\|v\|_E \le 1} \left| \int_\Omega F(\mathbf{x})v(\mathbf{x})\,d\Omega \right| .$$

It is clear that $L^2(\Omega)$ with this norm is not complete (since all $v \in L^p(\Omega)$ for any $p > 2$, $p < \infty$). The completion of $L^2(\Omega)$ in the norm $\|\cdot\|_E$ is called the space with negative norm, denoted E^-. In Lax [17] (and in other books, for example, Yosida [44]) it is shown that the set of all continuous linear functionals on E can be identified with E^- since E is dense in $L^2(\Omega)$.

So the condition $F(\mathbf{x}) \in E^-$ is necessary and sufficient for the work functional (2.11.5) to be continuous with respect to $v(\mathbf{x})$ on E.

In Lax [17], such a construction was introduced for a Sobolev space $\dot{W}^{k,2}(\Omega)$; the corresponding space with negative norm was denoted by $W^{-k,2}(\Omega)$. An equivalent approach to the introduction of $W^{-k,2}(\Omega)$ involves use of the Fourier transformation in Sobolev spaces (cf., Yosida [44]).

The notion of the space with negative norm is useful for studying problems, but it is not too informative when we want to know whether certain forces are of a needed class; here sufficient conditions are more convenient.

Secondly, we discuss how to handle nonhomogeneous boundary conditions (of Dirichlet type). Consider, for example, the problem

$$-\Delta v = F , \tag{2.11.6}$$

$$v\Big|_{\partial\Omega} = \varphi . \tag{2.11.7}$$

We can try the classical approach, finding a function $v_0(\mathbf{x})$ that satisfies (2.11.7), i.e.,

$$v_0\Big|_{\partial\Omega} = \varphi .$$

Now we are seeking $v(\mathbf{x})$ in the form $v = u + v_0$, where $u(\mathbf{x})$ satisfies the homogeneous boundary condition

$$u\Big|_{\partial\Omega} = 0 . \tag{2.11.8}$$

An integro-differential equation of equilibrium of the membrane is

$$\iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \psi}{\partial y}\right) d\Omega + \iint_\Omega \left(\frac{\partial v_0}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial v_0}{\partial y}\frac{\partial \psi}{\partial y}\right) d\Omega = \iint_\Omega F\psi \, d\Omega \quad (2.11.9)$$

wherein virtual displacements must also satisfy (2.11.8):

$$\psi|_{\partial\Omega} = 0 \,.$$

We recognize the term

$$\iint_\Omega \left(\frac{\partial v_0}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial v_0}{\partial y}\frac{\partial \psi}{\partial y}\right) d\Omega$$

as a continuous linear functional on E_{MC} if $\partial v_0/\partial x$ and $\partial v_0/\partial y$ belong to $L^2(\Omega)$. In that case there is a generalized solution to the problem, i.e., $u \in E_{MC}$ satisfying (2.11.9) for any $\psi \in E_{MC}$.

We have supposed that there exists an element of $W^{1,2}(\Omega)$ satisfying (2.11.7). In more detailed textbooks on the theory of partial differential equations, one may find the conditions for a function φ given on the boundary that are sufficient for the existence of v_0. Corresponding theorems for v_0 from Sobolev spaces are called trace theorems. The trace theorems assume the boundary is sufficiently smooth. The case of a piecewise smooth boundary, frequently encountered in practice, has not been completely studied yet. The problem of the traces of functions is beyond the scope of this book.

A final remark is in order. In mathematics we normally deal with dimensionless quantities, and we have followed that practice here. However, variables with dimensional units can be used without difficulty, provided we check carefully for units in all inequalities and equations, and introduce additional factors as needed. In particular, the constants in imbedding theorems normally carry dimensional units, hence these constants change if the units are changed.

2.12 Bramble–Hilbert Lemma and Its Applications

This lemma is widely used to establish the convergence rate for the finite element method (see, for example, Ciarlet [7]). It gives a bound for a functional with special properties in a Sobolev space. We remark that the lemma can be viewed as a simple consequence of the theorem on equivalent norming of $W^{l,p}(\Omega)$ in Sobolev [33].

Recall Poincaré's inequality (2.3.9),

$$\iint_S u^2 \, dS \le m \left\{ \left(\iint_S u \, dS \right)^2 + \iint_S \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dS \right\}, \quad (2.12.1)$$

which was derived when S was the square $[0, a] \times [0, a]$.

The proof of (2.3.9) is easily extended to the case of an n-dimensional cube. We now discuss how to extend it to a compact set Ω that is star-shaped with respect to a square S; that is, any ray starting in S intersects the boundary of Ω exactly once. We shall establish the following estimate, which is also called *Poincaré's inequality*:

$$\iint_\Omega u^2\,d\Omega \le m_1\Big(\iint_S u\,dS\Big)^2 + m_2\iint_\Omega\left[\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial u}{\partial y}\Big)^2\right]d\Omega\,. \tag{2.12.2}$$

Let us rewrite this in a system of polar coordinates (r,ϕ) having origin at the center of S. Let $\partial\Omega$ be given by the equation $r = R(\phi) \ge a/2$, $R(\phi) < R_0$. Then (2.12.2) has the form

$$\int_0^{2\pi}\int_0^{R(\phi)} u^2 r\,dr\,d\phi \le m_1\Big(\iint_S u\,dS\Big)^2 + m_2\int_0^{2\pi}\int_0^{R(\phi)}\left[\Big(\frac{\partial u}{\partial r}\Big)^2+\frac{1}{r^2}\Big(\frac{\partial u}{\partial \phi}\Big)^2\right] r\,dr\,d\phi\,.$$

Because of (2.12.1), it follows that it is sufficient to get the estimate

$$\int_0^{2\pi}\int_{a/2}^{R(\phi)} u^2 r\,dr\,d\phi \le m_3\int_0^{2\pi}\int_{a/4}^{a/2} u^2 r\,dr\,d\phi + m_4\int_0^{2\pi}\int_{a/4}^{R(\phi)} r\Big(\frac{\partial u}{\partial r}\Big)^2 dr\,d\phi \tag{2.12.3}$$

with constants independent of $u \in C^{(1)}(\Omega)$ ($C^{(1)}(\Omega)$ is introduced in Cartesian coordinates!). We now proceed to prove this.

The starting point is the representation

$$u(r_2,\phi) = u(r_1,\phi) + \int_{r_1}^{r_2}\frac{\partial u(r,\phi)}{\partial r}\,dr\,, \qquad \begin{aligned} &a/4 \le r_1 \le a/2\,,\\ &a/4 \le r_2 \le R_0\,,\end{aligned}$$

from which, by squaring both sides and applying elementary transformations, we get

$$\begin{aligned} u^2(r_2,\phi) &\le 2u^2(r_1,\phi) + 2\left[\int_{r_1}^{r_2}\frac{1}{\sqrt{r}}\Big(\sqrt{r}\,\frac{\partial u}{\partial r}\Big)dr\right]^2\\ &\le 2u^2(r_1,\phi) + 2\int_{r_1}^{r_2}\frac{dr}{r}\int_{r_1}^{r_2} r\Big(\frac{\partial u}{\partial r}\Big)^2 dr\\ &\le 2u^2(r_1,\phi) + m_5\int_{a/4}^{R(\phi)} r\Big(\frac{\partial u}{\partial r}\Big)^2 dr\,, \quad m_5 = 2\ln\frac{4R_0}{a}\,.\end{aligned}$$

Multiplying this chain of inequalities by $r_1 r_2$ and then integrating it first with respect to r_2 from $a/2$ to $R(\phi)$ and then with respect to r_1 from $a/4$ to $a/2$, we have

$$\begin{aligned}\int_{a/4}^{a/2} r_1\int_{a/2}^{R(\phi)} u^2(r_2,\phi) r_2\,dr_2\,dr_1 &\le 2\int_{a/4}^{a/2} u^2(r_1,\phi) r_1\,dr_1\int_{a/2}^{R(\phi)} r_2\,dr_2\\ &\quad + m_5\int_{a/4}^{a/2}\int_{a/2}^{R(\phi)} r_1 r_2\,dr_1\,dr_2\int_{a/4}^{R(\phi)} r\Big(\frac{\partial u}{\partial r}\Big)^2 dr\end{aligned}$$

or

$$\tfrac{3}{32}a^2 \int_{a/2}^{R(\phi)} u^2(r,\phi) r\,dr \le R_0^2 \int_{a/4}^{a/2} u^2(r,\phi) r\,dr + \tfrac{3}{64}a^2 R_0^2 m_5 \int_{a/4}^{R(\phi)} r\left(\frac{\partial u}{\partial r}\right)^2 dr\,.$$

Finally, integrating this with respect to ϕ over $[0, 2\pi]$ and multiplying it by $32/(3a^2)$, we establish (2.12.3) and hence (2.12.2).

We can similarly extend Poincaré's inequality to the case of a multiconnected domain Ω which is a union of star-shaped domains, and to the case of an n-dimensional domain Ω with $n > 2$. The latter extension is

$$\int_\Omega u^2\,d\Omega \le m_1\left(\int_C u\,d\Omega\right)^2 + m_2 \sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 d\Omega\,, \tag{2.12.4}$$

where $C \subset \Omega$ is a hypercube in $\mathbb{R}^n$.

We can apply the inequality (2.12.4) to any derivative $D^\alpha u$, $|\alpha| < k$. Combining these estimates successively, we derive the inequality needed to prove the Bramble–Hilbert lemma:

$$\|u\|^2_{W^{k,2}(\Omega)} \le m_3 \sum_{0\le|\alpha|<k} \left(\int_C D^\alpha u\,d\Omega\right)^2 + m_4 \sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^2\,d\Omega\,. \tag{2.12.5}$$

This estimate permits us to introduce another form of equivalent norm in $W^{k,2}(\Omega)$. (Question to the reader: Which one?) Note that the estimate was obtained for functions of $C^{(k)}(\Omega)$, but the now standard procedure of completion provides that it is valid for any $u \in W^{k,2}(\Omega)$.

Lemma 2.12.1 (Bramble–Hilbert [5]). Assume $F(u)$ is a continuous linear functional on $W^{k,2}(\Omega)$ such that for any polynomial $P_r(\mathbf{x})$ of order less than k,

$$F(P_r(\mathbf{x})) = 0\,. \tag{2.12.6}$$

Then there is a constant m^* depending only on Ω such that

$$|F(u)| \le m^*\,\|F\|_{W^{k,2}(\Omega)} \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^2\,d\Omega\right)^{1/2}. \tag{2.12.7}$$

Proof. From (2.12.5) and continuity of $F(u)$ on $W^{k,2}(\Omega)$, it follows that

$$|F(u)| \le m\,\|F\|_{W^{k,2}(\Omega)} \left[\sum_{0\le|\alpha|<k} \left(\int_C D^\alpha u\,d\Omega\right)^2 + \sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^2\,d\Omega\right]^{1/2}. \tag{2.12.8}$$

By (2.12.6),

$$F(u(\mathbf{x}) + P_{k-1}(\mathbf{x})) = F(u(\mathbf{x}))$$

where $P_{k-1}(\mathbf{x})$ is an arbitrary polynomial of order $k-1$. Fixing $u(\mathbf{x}) \in W^{k,2}(\Omega)$, we can always choose a polynomial $P^*_{k-1}(\mathbf{x})$ such that

$$\int_C D^\alpha(u(\mathbf{x}) + P^*_{k-1}(\mathbf{x}))\,d\Omega = 0 \quad \text{for all } 0 \le |\alpha| \le k-1 \,.$$

Substituting $u(\mathbf{x}) + P^*_{k-1}(\mathbf{x})$ into (2.12.8), we get (2.12.7) since

$$D^\alpha P^*_{k-1}(\mathbf{x}) = 0 \quad \text{for } |\alpha| = k \,.$$

This completes the proof. □

Let us consider some simple applications of this lemma. Assume that we find numerically, by Simpson's rule,

$$\int_0^1 u(x)\,dx \quad \text{for } u(x) \in W^{2,2}(0,1) \,.$$

What is a bound on the error? First we find the error in one step of the trapezoidal rule:

$$F_k(u) = \int_{x_k}^{x_k+h} u(x)\,dx - \frac{h}{2}[u(x_k+h) + u(x_k)] \,.$$

It is clear that $F_k(u)$ is a linear and continuous functional in $W^{2,2}(0,1)$. Making the change of variable $x = x_k + hz$ in the integral, we get

$$|F_k(u)| = h\left|\int_0^1 u(x_k+hz)\,dz - \tfrac{1}{2}[u(x_k) + u(x_k+h)]\right| \le 2h \max_{z\in[0,1]} |u(x_k+zh)| \,. \tag{2.12.9}$$

By the elementary inequality (Problem 2.12.1 below)

$$\max_{x\in[0,1]} |f(x)| \le \sqrt{2}\left[\int_0^1 \left(f^2(x) + [f'(x)]^2\right) dx\right]^{1/2} \le \sqrt{2}\,\|f\|_{W^{2,2}(0,1)} \,, \tag{2.12.10}$$

relation (2.12.9) gives

$$|F_k(u)| \le 2\sqrt{2}h\,\|u(x_k+hz)\|_{W^{2,2}(0,1)} \,.$$

Since $F_k(a+bx) = 0$ for any constants a, b we can apply the Bramble–Hilbert lemma and obtain

$$|F_k(u)| \le 2\sqrt{2}hm\left(\int_0^1 [u''(x_k+hz)]^2\,dz\right)^{1/2} = m_1 h^{5/2}\left(\int_{x_k}^{x_k+h} [u''(x)]^2\,dx\right)^{1/2}.$$

This is the needed error bound for one step of integration.

Consider now the bound on total error when $[0,1]$ is subdivided into N equal parts

$$F(u) = \int_0^1 u(x)\,dx - \frac{h}{2}\sum_{k=0}^{N-1}[u(x_k) + u(x_{k+1})] \,, \qquad x_k = kh \,.$$

This is linear and continuous in $W^{2,2}(0,1)$, and

$$f(u) = \sum_{k=0}^{N-1} F_k(u)\,.$$

We get

$$|F(u)| = \left|\sum_{k=0}^{N-1} F_k(u)\right| \le \sum_{k=0}^{N-1} |F_k(u)| \le m_1 h^{5/2} \sum_{k=0}^{N-1} \left(\int_{x_k}^{x_k+h} [u''(x)]^2\,dx \right)^{1/2}$$
$$\le m_1 h^{5/2}\sqrt{N} \left(\sum_{k=0}^{N-1} \int_{x_k}^{x_k+h} [u''(x)]^2\,dx \right)^{1/2}.$$

Thus the needed bound on the error of the trapezoidal rule is

$$|F(u)| \le m_1 h^2 \left(\int_0^1 [u''(x)]^2\,dx \right)^{1/2}.$$

No improvements in the order of the error result if we take functions smoother than those from $W^{2,2}(0,1)$. But if $v \in W^{1,2}(0,1)$, the bound is worse:

$$|F(v)| \le m_2 h \left(\int_0^1 [v'(x)]^2\,dx \right)^{1/2}.$$

Problem 2.12.1. Prove (2.12.10). □

Another example of the application of Lemma 2.12.1 is given by

Problem 2.12.2. Show that the local error of approximation of the first derivatives of a function $u(x_1, x_2) \in W^{3,2}(\Omega)$, $\Omega \subset \mathbb{R}^2$, by symmetric differences, is

$$l(u) = \left| \frac{\partial u(0,0)}{\partial x_1} - \frac{u(h_1,0) - u(-h_1,0)}{2h_1} \right| + \left| \frac{\partial u(0,0)}{\partial x_2} - \frac{u(0,h_2) - u(0,-h_2)}{2h_2} \right|$$
$$\le \frac{M(h_1^2 + h_2^2)}{\sqrt{h_1 h_2}}\, \|u\|_{W^{3,2}(\Omega)}$$

if $0 < c_1 < h_1/h_2 < c_2 < \infty$. □

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