

Chapter 2

Stochastic Taylor Formulas and Riemannian Geometry

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To David Nualart, with admiration and respect.

Received 11/22/2011; Accepted 2/12/2012; Final 2/21/2012

1 Introduction

Let (X_t, P_x) be the standard Brownian motion on a complete Riemannian manifold. We investigate the asymptotic behavior of the moments of the exit time from a geodesic ball when the radius tends to zero. This is combined with a “stochastic Taylor formula” to obtain a new expansion for the mean value of a function on the boundary of a geodesic ball.

Several authors [3–5] have considered mean value formulas on a Riemannian manifold, using the exponential mapping and integration over the unit sphere of the tangent space, at each point. In general the stochastic mean value is not equal to the exponential mean value. If these coincide, the manifold must be Einsteinian (constant Ricci curvature). Our expansions are used to answer some *inverse questions* in stochastic Riemannian geometry. If the mean exit time from every small ball is the same as for a flat manifold then the manifold is flat, in case $d = 2$ or $d = 3$. Our method of proof begins with an expansion of the Laplacian in a system of Riemannian normal coordinates [6]. The successive correction terms in the expansion of the moments is obtained by a perturbation expansion, the rigorous validity of which is established by systematic application of the stochastic Taylor formula.

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We use the summation convention throughout. The notation $|x| < \epsilon \rightarrow 0$ means that the indicated asymptotic estimate holds uniformly in the ball of radius ϵ , when $\epsilon \rightarrow 0$.

2 Stochastic Taylor Formula

For completeness we include the proof of this formula, which was previously treated by several authors (Athreya-Kurtz, Airault-Follmer, and Van der Bei). Let (X_t, P_x) be a diffusion process on a locally compact and separable space V . Let f be a real-valued function such that

$$f \in \mathcal{D}(A^k) \quad k = 1, 2, \dots, N + 1,$$

where A is the infinitesimal generator defined by

$$Af(x) = \lim_{t \rightarrow 0} t^{-1} E_x[f(X_t) - f(x)].$$

Let T be a stopping time with

$$E_x(T^{N+1}) < \infty.$$

Proposition 2.1. *Under the above conditions we have for $N \geq 1$*

$$E_x f(X_T) = f(x) + \sum_{k=1}^{N-1} \frac{(-1)^{k+1}}{k!} E_x (T^k A^k f(X_T)), + (-1)^{N-1} R_{N-1}, \quad (2.1)$$

where

$$R = R_{N-1} = \frac{1}{(N-1)!} E_x \left(\int_0^T u^{N-1} A^N f(X_u) du \right). \quad (2.2)$$

The empty sum is defined to be zero.

Proof. For $N = 1$ we have an empty sum and Dynkin's identity: $E_x f(X_T) = f(x) + R_1$. In general for $N \geq 1$ we have

$$A^k f(X_t) - A^k f(X_0) = \int_0^t A^{k+1} f(X_u) du + M_t, \quad (2.3)$$

where M_t is a local martingale. From the stochastic product rule we have

$$d(t^k A^k f(X_t)) = t^k dM_t + (t^k A^{k+1} f(X_t) + k t^{k-1} A^k f(X_t)) dt. \quad (2.4)$$

Integrating and taking the expectation, we have

$$E_x [T^k A^k f(X_T)] = E_x \int_0^T t^k A^{k+1} f(X_t) dt + k E_x \int_0^T t^{k-1} A^k f(X_t) dt. \quad (2.5)$$

Dividing by $k!$, we have

$$\frac{1}{k!} E_x [T^k A^k f(X_T)] = R_{k+1} + R_k. \quad (2.6)$$

Multiplying by $(-1)^{k+1}$ and summing for $k = 1, \dots, N$ gives the stated result. $\square$

2.1 Some Special Cases

The stochastic Taylor formula allows one to pass freely between classical solutions of equations and moments of various functionals.

Example 2.1a. Let B be the unit ball of R^n centered at the origin and T be the first exit time from B . If $f = 0$ on $S = \partial B$ and solves $Af = -1$ in B , then we have

$$0 = E_x f(X_T) = f(x) + E_x \int_0^T Af(X_u) du = f(x) - E_x(T).$$

Thus the mean exit time can be retrieved from the value at the starting point. Explicit computation gives $f(x) = (1 - |x|^2)/2n$.

The stochastic Taylor formula can be used to give a probabilistic representation of the solution of some higher-order elliptic boundary-value problems.

Example 2.1b. Let B be a bounded region of Euclidean space with boundary hypersurface $S = \partial B$. Let u be the solution of the boundary-value problem

$$u|_S = 0, \dots, A^{N-1}u|_S = 0, A^N u|_B = g. \quad (2.7)$$

We first note that the left side of Eq. (2.1) is $0 - u(x)$ whereas all of the terms on the right are zero, save the last term, which is $1/(N-1)!$ times

$$E_x \left(\int_0^T s^{N-1} A^N f(X_s) ds \right) = E_x \left(\int_0^T s^{N-1} g(X_s) ds \right),$$

so that we have

$$u(x) = \frac{1}{(N-1)!} \left(E_x \int_0^T s^{N-1} g(X_s) ds \right),$$

which is the desired probabilistic representation. In the special case $g = 1$ we can perform the integration to obtain

$$u(x) = \frac{E_x(T^N)}{N!}.$$

Stated otherwise, the higher moments of the exit time are obtained by solving a higher-order boundary-value problem.

One can also make conclusions based on *approximate solutions*.

Example 2.1c. If $f = 0$ on $S = \partial B$ and solves¹ $Af = -1 + \epsilon$, then we have

$$0 = E_x f(X_T) = f(x) + E_x \int_0^t Af(X_u) du = f(x) - E_x(T)(1 \pm \epsilon);$$

hence $E_x(T) = f(x) \times (1 \pm \epsilon)$. Thus the mean exit time can be retrieved with ϵ accuracy from the value at the starting point.

3 Expansion of the Laplacian

Now consider the case of an n -dimensional Riemannian manifold V . Let $O \in V$ and consider the exponential mapping

$$\exp_O V_O \rightarrow V.$$

A choice of an orthonormal basis in V_O gives rise to normal coordinates $(x_1, \dots, x_n)$.

Let Δ be the Laplacian in normal coordinates:

$$\Delta f = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f) \quad g := \det(g_{ij}). \quad (2.8)$$

From Eq. (2.8) it follows that Δ is self-adjoint with respect to the weight function $x \rightarrow \sqrt{g}$.

Let τ_ϵ be the dilation operator, defined by

$$(\tau_\epsilon f)(x) = f\left(\frac{x}{\epsilon}\right). \quad (2.9)$$

Definition. A differential operator A is said to be homogeneous of degree j if and only if for every homogeneous polynomial Q of degree k , AQ is homogeneous of degree $k + j$.

Proposition 3.1. For every integer $N \geq 0$, there exists a finite set of second-order differential operators $\Delta_0, \dots, \Delta_N$ such that Δ_j is homogeneous of degree j and we have the asymptotic expansion

$$\tau_\epsilon^{-1} \circ \Delta \circ \tau_\epsilon f = \epsilon^{-2} \Delta_{-2} f + \sum_{j=0}^N \epsilon^j \Delta_j f + O(\epsilon^{N+1}), \quad (2.10)$$

¹This means that $\sup_{x \in B} |Af(x) + 1| \leq \epsilon$.

where

$$\Delta_{-2}f = \sum_{i=1}^n (\partial_i \partial_i) f$$

for any twice differentiable function f .

Proof. In normal coordinates we have the asymptotic expansion [5]

$$g_{ij} = \delta_{ij} + a_{ijkl} x^k x^l + \text{terms of order 3 and higher} \quad (2.11)$$

with similar expressions for $\sqrt{g}$, g^{ij} . Substitution into Eq. (2.8) and collecting terms give the formula

$$\Delta f - \Delta_{-2}f = P_{ij} \partial_i \partial_j f + Q_i \partial_i f, \quad (2.12)$$

where P_{ij} begins with the quadratic terms and Q_i begins with linear terms. Now replace f by $\tau_\epsilon f$ in Eq. (2.12). Clearly $\tau_\epsilon^{-1} \circ (\Delta_{-2}) \circ \tau_\epsilon f = \epsilon^{-2} \Delta_{-2}f$. On the other hand when we apply the right side of Eq. (2.12) to $\tau_\epsilon f$, we obtain an asymptotic series in ϵ . The coefficient of ϵ^0 is the quadratic term in P_{ij} plus the linear terms in Q_i . Proceeding to the next stage, the coefficient of ϵ consists of the cubic term in P_{ij} and the quadratic terms in $Q_j f$. Continuing this to higher powers of ϵ we can compute the coefficient operators $\Delta_0, \Delta_1, \dots$; hence we have completed the proof of Proposition 3.1. $\square$

3.1 Computation of Δ_0

To compute $\Delta_0 f$ we begin with Cartan's formula

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{ijkl} x^k x^l + O(|x|^3), (x \rightarrow 0), \quad (2.13)$$

$$g^{\alpha\beta} = \delta^{\alpha\beta} - \frac{1}{3} R_{\alpha\beta kl} x^k x^l + O(|x|^3), (x \rightarrow 0). \quad (2.14)$$

The Christoffel coefficients are computed from Eq. (2.13). We use the covariant form of the Laplacian:

$$\Delta f = g^{ij} \left(\partial_i \partial_j f - \Gamma_{ij}^k \partial_k f \right). \quad (2.15)$$

When we substitute these expansions and collect terms, there results

$$\begin{aligned} \Delta f = & \partial_i \partial_i f + \left(-\frac{1}{3} R_{ijkl} x^k x^l + O(|x|^3) \right) \partial_i \partial_j f \\ & + \left(-\frac{2}{3} R_{ikk'i} x^{k'} \partial_i f + O(|x|^2) \right). \end{aligned}$$

Hence

$$\Delta_0 f = -\frac{1}{3} R_{ijkl} x^k x^l \partial_i \partial_j f - \frac{2}{3} R_{ikk'i} x^{k'} \partial_k f. \quad (2.16)$$

Of particular interest is the case $f = \phi(r)$, where $\phi \in C^2[0, \infty)$. This yields

$$\Delta_0(\phi \circ r) = -\frac{1}{3} \rho_{kl} \frac{x^k x^l}{r} \phi'(r),$$

where ρ_{kl} is the *Ricci tensor*, defined by $\rho_{kl} = R_{ikil}$ (sum on i).

3.2 Computation of $\Delta_k, k \geq 1$

To compute the higher corrections $\Delta_1, \Delta_2 \dots$, we introduce polar coordinates in V_O ; letting $x = r\theta$, we have

$$\Delta_k = b_k(\theta) r^{1+k} \frac{\partial}{\partial r} + r^k \tilde{\Delta}_k, \quad (2.17)$$

where $\tilde{\Delta}_k$ are second-order differential operators in $\theta_1, \dots, \theta_n$ and b_k is a homogeneous function of degree zero. By Gauss' lemma, the terms with $\partial^2/\partial r^2$ or $\partial^2/\partial r \partial \theta_i$ are not present. To compute $b_k(\theta)$, we recall the expansion of Gray–Vanhecke [4] for the determinant of the exponential map of a Riemannian manifold:

$$\omega = 1 - \frac{1}{6} \beta_2 r^2 - \frac{1}{12} \beta_3 r^3 + \frac{1}{24} \beta_4 r^4 + O(r^5), \quad (r \rightarrow 0), \quad (2.18)$$

where

$$\beta_2 = \rho_{ij} \theta_i \theta_j, \quad (2.19)$$

$$\beta_3 = \nabla_i \rho_{jk} \theta_i \theta_j \theta_k, \quad (2.20)$$

$$\beta_4 = \left(1 - \frac{3}{5} \nabla_{ij}^2 + \frac{1}{3} \rho_{ij} \rho_{kl} - \frac{2}{15} R_{iajk} R_{kalb} \right) \theta_i \theta_j \theta_k \theta_l. \quad (2.21)$$

Performing the long division we have

$$\frac{\omega'}{\omega} = b_0 r + b_1 r^2 + b_2 r^3 + O(r^4),$$

$$b_0 = -\frac{1}{3} \rho_i \rho_j \theta_i \theta_j,$$

$$b_1 = -\frac{1}{4} \rho_j \rho_k \partial_i \partial_i \partial_j \theta_j \theta_k,$$

$$b_2 = -\frac{1}{9} \nabla_{ij}^2 \rho_{kl} - \frac{1}{45} R_{iajb} R_{kalb} \theta_i \theta_j \theta_k \theta_l.$$

4 Estimate of the Moments $E_x(T_r^k), r \rightarrow 0$

Let V be an N -dimensional complete Riemannian manifold and (X_t, P_x) be the Brownian motion process on V , a diffusion process with infinitesimal generator Δ . The *exit time* from a ball of radius R centered at $O \in V$ is defined by

$$T_R := \inf\{t : t > 0, d(X_t, O) = R\}. \quad (2.22)$$

We emphasize that the computations are made as a function of the unspecified starting point $p = X_0$ where we set $p = O$ at the end. We will prove the following.

Proposition 4.1. *For each $k \geq 1, E_O(T_R^k) \sim c_{nk} R^{2k}, R \rightarrow 0$, where c_{nk} are positive constants. In case $V = R^n$ we have for each $k \geq 1, E_O(T_R^k) = c_{nk} R^{2k}$ for all $R > 0$.*

Proof. Let $B = \{y : d(y, O) \leq R\}$ and let $f_0^k(r)$ be the solution of

$$\Delta_{-2} f_0^k = -f_0^{k-1}, \quad f_0^k|_{\partial B} = 0, \quad f_0^0 \equiv 1 \quad (k \geq 1). \quad (2.23)$$

In case $V = R^n$ $f_0^k(r)$ is a polynomial of degree $2k$. The substitution $r \rightarrow r/\epsilon$ with subsequent normalization allows one to study the case $R = 1$. In detail, we have $f_0^k(r) \rightarrow \epsilon^{2k} f_0^k(r/\epsilon)$.

By explicit computation, we have

$$\begin{aligned} -f_0^1(r) &= \frac{r^2 - 1}{2n}, \quad f_0^1(0) = \frac{1}{2n} \\ -f_0^2(r) &= \frac{r^4}{8n(n+2)} - \frac{r^2}{4n^2} + \frac{n+4}{8n^2(n+2)}, \quad f_0^2(0) = \frac{n+4}{8n^2(n+2)}, \\ -f_0^3(r) &= \frac{r^6}{48n(n+2)(n+4)} - \frac{r^4}{16n^2(n+2)} \\ &\quad + \frac{r^2(n+4)}{16n^3(n+2)} + \frac{n^2 + 12n + 48}{48n^3(n+2)(n+4)}. \end{aligned}$$

In general f_0^k is a polynomial of the form

$$-f_0^k(r) = \frac{r^{2k}}{2^k} k! n(n+2) \cdots (n+2k-2) + \sum_{j=0}^{k-1} (-1)^j c_{knj} r^{2k-2j}.$$

□

In order to confirm the polynomial character of $f_0^k(r)$, one can either use mathematical induction or proceed directly, as follows: given functions $g(x), h(x)$, it is required to solve the differential equation $f''(x) + (g'/g)f'(x) = -h$ on the

interval $0 \leq x < R$, with the boundary conditions that $f(R) = 0$ and f remains bounded when $x \rightarrow 0$. To do this, multiply the differential equation by g to obtain $(gf')' = -gh$, $gf' = C - \int_x^1 gh$. Solving for f' and doing the integral, we obtain the formula

$$f(x) = \int_x^1 \frac{1}{g(t)} \left(\int_0^t h(u)g(u)du \right) dt, \quad (2.24)$$

which can be directly checked as follows:

$$f'' + (g'/g)f' = (1/g)(gf')' = (1/g)(-gh) = -h.$$

The boundary condition $f(1) = 0$ follows from Eq. (2.24). The integral defining $f(0)$ is absolutely convergent since f is displayed in terms of the ratio of two terms each of which tends to zero. l'Hospital's rule applies to the ratio of the derivatives, namely

$$\lim_{t \rightarrow 0} \frac{\int_0^t h(u)g(u)du}{g(t)} = \lim_{t \rightarrow 0} \frac{h(t)g(t)}{g'(t)} = h(0) \lim_{t \rightarrow 0} \frac{g(t)}{g'(t)}.$$

This shows that the integral in Eq. (2.24) is well defined whenever g/g' is integrable; in particular this is the case where $g(x) = x^{n-1}$ and $g/g' = x/(n-1)$.

In particular if $g(x)$ is a monomial— $g(x) = x^n$ and $h(x)$ is a polynomial, then $f(x)$ is also a polynomial.

From the stochastic Taylor formula we have

$$E_x[f(X_T)] = f(x) + \sum_{k=1}^{N-1} \frac{(-1)^{k+1}}{k!} E_x(T^k A^k f(X_T)) + (-1)^{N-1} R_{N-1}, \quad (2.25)$$

where

$$R = R_{N-1} = \frac{1}{(N-1)!} E_x \left(\int_0^T u^{N-1} A^N f(X_u) du \right). \quad (2.26)$$

By construction, f_0^k is identically zero on the sphere S when $k < N$. We can write $0 - f(x) = E_x(T^N)/N!$ which proves that $E_O(T^N) = c_N R^{2N}$. This proves the proposition in case of Euclidean space.

In the general case, we can still use the stochastic Taylor formula with $A = \Delta$, $T = T_B$, $f = f_0^k$. Then

$$f(X_T) = O(\epsilon), \dots, A^{N-1} f(X_T) = O(\epsilon), A^N f(X_T) = -1 + O(\epsilon).$$

From the decomposition of the Laplacian of V we have $\Delta f - \Delta_{-2} f = O(\epsilon)$

$$f_0^N(r) = \frac{1}{N!} E_x(T^N) (1 + O(\epsilon^2)).$$

Taking $r = 0$, $R = \epsilon \rightarrow 0$, we see that

$$\limsup_{R \rightarrow 0} \frac{E_O(T_R)}{R^N} = 1.$$

To compute an explicit form, we specialize to Euclidean space, $V = R^N$, $f(x) = e^{<\alpha, x>} \alpha \in R^N$. The stochastic Taylor formula gives

$$E_0 e^{<\alpha, X_{T_r}>} = \sum_{k=0}^{\infty} \frac{|\alpha|^{2k}}{k!} E_0(T_r^k).$$

But the P_O hitting measure of X_T is uniform on the sphere $|x| = r$. Hence

$$\begin{aligned} E_0 e^{<\alpha, X_{T_r}>} &= \int_{S^{n-1}} \exp(|\alpha| r \cos \theta) d\theta \\ &= \sum_{m=0}^{\infty} \frac{|\alpha|^m}{m!}. \end{aligned}$$

Comparing the two expansions, we have

$$\frac{E(T_r^k)}{r^{2k}} = \frac{k!}{(2k)!} \int_{S^{n-1}} (\cos \theta)^{2k} = \frac{\Gamma(\frac{n}{2})}{\Gamma(k + \frac{n}{2})} 2^{-2k}.$$

Hence $c_{nk} = \frac{\Gamma(\frac{n}{2})}{\Gamma(k + \frac{n}{2})}$.

5 Refined Estimate of $E(T_r)$, $r \downarrow 0$

The mean exit time of Brownian motion is affected by the curvature σ of the manifold. When σ is zero, we have the Euclidean space, which has been well studied. If $\sigma > 0$ a geodesic disk has smaller volume than the Euclidean counterpart and the mean exit time is greater. Finally in the case of negative curvature the volume is strictly smaller and the mean exit time is strictly larger than the corresponding disk in Euclidean space. Here is a precise result.

Proposition 5.1.

$$E_0(T_r) = \frac{r^2}{2n} + \sigma \frac{r^4}{12n^2(n+2)} + O(r^5) \quad (r \downarrow 0). \quad (2.27)$$

Proof. For this purpose, let

$$F = f_0^1 + \epsilon^2 f_2^1 + \epsilon^3 f_3^1 + \epsilon^4 f_4^1, \quad (2.28)$$

where

$$f_0^1 = (1 - r^2)/2n, \quad (2.29)$$

$$\Delta_{-2}f_2^1 + \Delta_0f_0^1 = 0, \quad \text{in } B, \quad f_2|_{\partial B} = 0, \quad (2.30)$$

$$\Delta_{-2}f_3^1 + \Delta_0f_1^1 + \Delta_1f_0^1 = 0 \quad \text{in } B, \quad f_3|_{\partial B} = 0, \quad (2.31)$$

$$\Delta_{-2}f_4^1 + \Delta_0f_2^1 + \Delta_2f_0^1 = 0 \quad \text{in } B, \quad f_4|_{\partial B} = 0. \quad (2.32)$$

Now we set $f = \epsilon^2 \tau_\epsilon F$. From Eq. (2.10) and Proposition 2.1 we have

$$\Delta f = -1 + O(\epsilon^5), \quad \Delta^2 f = O(\epsilon^3), \quad f|_{\partial B_\epsilon} = 0.$$

Therefore, from the stochastic Taylor formula with $N = 2$

$$0 = f(x) + E_x(T_\epsilon)(-1 + O(\epsilon^5)),$$

$$E_x(T_\epsilon) = \epsilon^2 [f_0^1(x) + \epsilon^2 f_2^1(x) + \epsilon^3 f_3(x) + \epsilon^4 f_4(x)] + O(\epsilon^7), \quad |x| < \epsilon \rightarrow 0.$$

In the previous section it was proved that $f_0^1(0) = 1/2n$. To compute $f_2^1(0)$ we first note that

$$\Delta_{-2}f_2^1 = -\rho_{ij}\theta_i\theta_j r^2/2n = \frac{r^2}{2n}b(\theta).$$

To compute $f_2^1(0)$, $f_4^1(0)$ we invoke the following lemma. Integration over $S = S^{n-1}$ is taken with respect to the uniform probability measure $d\theta$. $\square$

Lemma 5.1. *Let $j(\theta), g(\theta)$ be polynomials which satisfy the relation $\Delta_{-2}j = r^k g(\theta)$, $g|_{\partial S} = 0$ where k is an integer. Then*

$$j(0) = \frac{1}{(n+k)(k+2)} \int_S g(\theta).$$

Proof. We can write j in the form $j = r^{k+2}c(\theta) + h$ where c is a homogeneous polynomial and h is a harmonic function, solution of $\Delta_{-2}h = 0$. Then

$$[(k+2)(n+1) + (n-1)(k+2)]c(\theta) \Delta_{-2}c(\theta) = -g(\theta).$$

Integrating over S , we have $(k+2)(n+k) \int_S c(\theta) = - \int g(\theta)$. But $j(0) = h(0) = - \int_S c(\theta) d\theta = 1/(n+k)(k+2) \int_S g(\theta)$.

Hence

$$f_2^1(0) = \frac{1}{4n(n+2)} \int_S b_0(\theta) = \frac{\sigma}{12n^2(n+2)}. \quad (2.33)$$

To compute $f_4^1(0)$, we have $\Delta_{-2}f_4^1 = -\Delta_0f_2^1 - \Delta_2f_0^1$, $f_4|_S = 0$. We treat the two terms separately.

Recalling that $f_0^1(r) = (1-r^2)/2n$, $\Delta_2 = r^3 b_2(\theta) \frac{\partial}{\partial r}$, it follows that $\Delta_2 f_0^1(r) = -r^4 b_2(\theta) d\theta$. By Lemma 5.1, this term contributes to $f_4^1(0)$ in the amount

$$\frac{1}{6(n+4)} \int_S b_2(\theta) d\theta.$$

To handle the second term, we use the fact that $\Delta_0 + \beta_2 \Delta_{-2}$ is self-adjoint with respect to Lebesgue measure on B . Thus

$$\begin{aligned} \int_S \Delta_0 f_2 &= - \int_S \beta_2 \Delta_{-2} f_2 \\ &= \int_S \beta_2 f_0 \\ &= 3 \int_S b_0^2(\theta). \end{aligned}$$

Thus the second term contributes to $f_4^1(0)$ in the amount $1/2(n+4) \int_S b_0(\theta)^2$. Combining this with the previous computations, we have

$$f_4^1(0) = \frac{1}{6(n+4)} \int_S b_2(\theta) + \frac{1}{2(n+4)} \int_S b_0(\theta)^2.$$

Finally we note that $f_3^1(0) = 0$, since $\Delta_{-2} f_3^1 = -\Delta_1 f_0^1 = r^3 b_1(\theta)/n$ which is a cubic polynomial; hence the integral is zero. $\square$

6 Mean Value Formulas for General Manifolds

On Euclidean space one may define the mean value of a function on a sphere. This can be effectively computed in terms of the radius of the sphere and the values of $\Delta f, \Delta^2 f, \dots$ at the center of the sphere. The resulting series expansion is known as Pizetti's theorem in classical differential geometry.

When we pass to manifolds of variable curvature, the corresponding series expansion is much more complicated than for the Euclidean case, leading one to ask for a simpler mean value. In the following sections we will explore the properties of the *stochastic mean value formula*.

6.1 Stochastic Mean Value

The stochastic mean value of a continuous function f on a sphere is defined as the following linear functional on continuous functions:

$$\Phi_O^\epsilon f = E_O[f(X_{T_\epsilon})] = \int_S f(y) dS^\epsilon(y),$$

where the existence of the probability measure dS_ϵ is guaranteed by the Riesz representation theorem. We propose to find a three-term asymptotic expansion of the measure dS^ϵ when $\epsilon \rightarrow 0$. Equivalently we can find a three-term expansion of the linear functional Φ_O^ϵ when $\epsilon \rightarrow 0$. The key to success is through Dynkin's formula, which is transformed into the identity

$$\Phi_O^\epsilon f = E_O[f(X_{T_\epsilon})] = E_O f + E_O \left(\int_0^{T_\epsilon} \Delta f(X_s) ds \right), \quad \epsilon > 0, f \in C(V). \quad (2.34)$$

For any $f \in C^5(V)$, we can write for $|x| < \epsilon \rightarrow 0$

$$\Delta f = d_0 + \sum_{i=1}^n d_i x_i + \frac{1}{2} \sum_{i,j=1}^n d_{ij} x_i x_j + O(|x|^3).$$

Equivalently

$$d_0 = \Delta f(O), \quad d_i = \partial_i \Delta f(O), \quad d_{ij} = \partial_i \partial_j \Delta f(O).$$

Therefore it is sufficient to consider test functions f which are either constant, linear, or purely quadratic. From Dynkin's formula, to study Φ , it is equivalent to study the integral (2.34). In the next section we study the equation

$$E_p \int_0^{T_\epsilon} \Delta f(X_s) ds = d_0 u_0^\epsilon(x) + \sum_{i=1}^n d_i u_i^\epsilon(x) + \sum_{i,j=1}^n d_{ij} u_{ij}^\epsilon(x) + O(\epsilon^5). \quad (2.35)$$

6.2 Solution of Poisson's Equation in a Geodesic Ball

In order to proceed further, we develop the properties of the mean value of three functionals as follows: the result of these computations is that the Euclidean mean value differs from the non-Euclidean mean value to within $O(\epsilon^5)$ when $\epsilon \rightarrow 0$.

Lemma 6.1. *For each $\epsilon > 0$, let $B = B_\epsilon$ denote the geodesic ball of radius ϵ in the tangent space. Functions u_0, u_i , and u_{ij} are defined as the solutions of the following boundary-value problems and respective stochastic representations:*

$$\begin{aligned} \Delta u_0^\epsilon|_B &= -1, \quad u_0^\epsilon|_{\partial B} = 0 & \Rightarrow u_0 &= E_p(T_\epsilon), \\ \Delta u_i^\epsilon|_B &= -x_i, \quad u_i^\epsilon|_{\partial B} = 0, 1 \leq i \leq n, & \Rightarrow u_i(p) &= E_p \left(\int_0^{T_\epsilon} X_i(s) ds \right), \\ \Delta u_{ij}^\epsilon|_B &= -x_i x_j, \quad u_{ij}^\epsilon|_{\partial B} = 0, 1 \leq i, j \leq n & \Rightarrow u_{ij}(p) &= E_p \left(\int_0^{T_\epsilon} X_i(s) X_j(s) ds \right). \end{aligned}$$

To see this, apply Dynkin's formula successively to $f = u_0^\epsilon$, $f = u_i^\epsilon$, $f = u_{ij}^\epsilon$. We have the asymptotic formulas when $\epsilon \rightarrow 0$:

$$u_0^\epsilon(x) = \epsilon^2 U_0(x/\epsilon) + \epsilon^4 V_0(x/\epsilon) + O(\epsilon^5), \quad (2.36)$$

$$u_i^\epsilon(x) = \epsilon^3 U_i(x/\epsilon) + O(\epsilon^5), \quad (2.37)$$

$$u_{ij}^\epsilon(x) = \epsilon^4 U_{ij}(x/\epsilon) + O(\epsilon^5), \quad (2.38)$$

where

$$\begin{aligned} U_0(x) &= \frac{1 - |x|^2}{2n}, \quad U_i(x) = \frac{x_i(1 - |x|^2)}{2n + 4}, \quad U_{ij}(x) = \frac{x_i x_j(1 - |x|^2)}{2n + 8}, \\ (n + 4)V_0(x) &= \frac{(1 - |x|^2)}{6n} \left(\sum_{i=1}^n \lambda_i x_i^2 \right) + \sigma \frac{(1 - |x|^2)}{6n^2} - \sigma \frac{(1 - |x|^4)}{12n(n + 2)}, \end{aligned}$$

where (λ_i) are the eigenvalues of the Ricci tensor.

Proof. The stochastic representations follow immediately from Dynkin's formula, applied successively to u_0 , u_i , and u_{ij} .

In terms of the Euclidean Laplacian Δ_{-2} , we verify that U_0 , U_i , and U_{ij} satisfy the following identities:

$$\begin{aligned} \Delta_{-2} U_0 &= -1, \\ \Delta_{-2}(x_i x_j) &= 2\delta_{ij}, \\ \Delta_{-2}(x_i^2 |x|^2) &= 2|x|^2 + (2n + 8)x_i^2, \\ \Delta_{-2}|x|^4 &= (4n + 8)|x|^2, \\ \Delta_{-2}(x_i x_j |x|^2) &= (2n + 8)x_i x_j, \end{aligned}$$

where we use the identities

$$\Delta_{-2}(|x|^2) = 2n, \text{ etc, etc.}$$

We combine this with Dynkin's formula—the stochastic Taylor formula with $N = 1$

$$E_p f(X_{T_\epsilon}) = f(p) + E_p \left(\int_0^{T_\epsilon} \Delta f(X_s) ds \right), \quad p \in B, f \in C^2(V). \quad (2.39)$$

To prove Eq. (2.36) apply Dynkin's formula with the choice $f = \epsilon^2 U_0(x/\epsilon) + \epsilon^4 V_0(x/\epsilon)$ for which $f = 0$ on S . Then we obtain

$$\Delta f = -1 + O(\epsilon^3) \Rightarrow 0 = f(p) + \int_0^T (-1 + O(\epsilon^3)) ds,$$

which proves Eq. (2.36).

To prove Eq. (2.37), apply Dynkin's formula with $f = \epsilon^2 U_i(x/\epsilon)$. Then $\Delta f = -x_i + O(\epsilon^3)$.

For Eq. (2.38), apply Dynkin to the choice $f = \epsilon^4 U_{ij}$, for which $\Delta f = -x_i x_j + O(\epsilon^4) \in B$, $f = 0$ on S . $\square$

7 Comparison with the Gray–Willmore Expansion

When we combine the results of Sects. 1–6, we find the following asymptotic formula:

$$\begin{aligned} E_O f(X_{T_\epsilon}) - f(O) &= \Delta f(O) [\epsilon^2 U_0(y) + \epsilon^4 V_0(y)] + \epsilon^3 \sum_{i=1}^n \partial_i (\Delta f) + O(\epsilon^5), \\ E_O f(X_{T_r}) &= f(O) + \frac{r^2}{2n} \Delta f(O) + r^4 \left[\frac{\Delta^2 f(O)}{8n(n+2)} - \frac{\sigma \Delta f(O)}{12n^2(n+2)} \right] + O(r^5), \quad r \rightarrow 0. \end{aligned} \quad (2.40)$$

It is interesting to compare this with the Gray–Willmore expansion as defined by integration in the tangent space:

$$\begin{aligned} M(r, f) &= f(O) + \frac{r^2}{2n} \Delta f(O) + \frac{r^4}{24n(n+2)} (3\Delta^2 f(O) - 2 \langle \nabla^2 f, \rho \rangle \\ &\quad - 3 \langle \nabla f, \nabla \sigma \rangle + \frac{4}{n} \sigma \Delta f(O)) + O(r^5), \quad r \rightarrow 0. \end{aligned}$$

Proposition 7.1. *Suppose that for all $f \in C^2(V)$, $M(r, f) = E_O f(X_{T_r})$ for all $r > 0$. Then V is an Einstein manifold.*

Proof. From the above formulas we have

$$\lim_{r \rightarrow 0} \frac{[E_O f(X_{T_r}) - M(r, f)]}{r^4} = \frac{\langle \nabla^2 f, \rho \rangle}{12n(n+2)} + \frac{\langle \nabla f, \nabla \sigma \rangle}{8n(n+2)} - \frac{\langle \sigma, \Delta f \rangle}{12n^2(n+2)}. \quad (2.41)$$

If the right side is zero for all $f \in C^2(V)$, we first take f to be linear in the normal coordinates so that ∇f is perpendicular to $\nabla \sigma$, from which we find $\nabla \sigma = 0$. The remaining term of Eq. (2.41) is

$$\frac{1}{12n(n+2)} \left[\langle \nabla^2 f, \rho \rangle - \frac{\sigma}{n} \Delta f \right].$$

Take normal coordinates so that the Ricci tensor is diagonal at O ; thus $\rho_{ij} = 0$ for $i \neq j$. Now choose f so that $\partial_i \partial_j = 0$ for $i \neq j$ from which we conclude that $\rho_{jj} = \frac{1}{n}(\rho_{11} + \cdots + \rho_{nn})$. Hence the Ricci tensor is a multiple of the identity, i.e., V is an Einstein manifold. $\square$

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<http://www.springer.com/978-1-4614-5905-7>

Malliavin Calculus and Stochastic Analysis
A Festschrift in Honor of David Nualart
Viens, F.; Feng, J.; Hu, Y.; Nualart, E. (Eds.)
2013, XI, 583 p. 13 illus., Hardcover
ISBN: 978-1-4614-5905-7