2

Continuum Mechanics and Linearized Elasticity

We will be concerned with bodies that at the macroscopic level may be regarded as being composed of material that is continuously distributed. By this it is meant, first, that such a body occupies a region of three-dimensional space that may be identified with $\mathbb{R}^3$. The region occupied by the body will of course vary with time as the body deforms. It is convenient, then, for the purpose of keeping track of the evolution of the body's behavior to locate any point in the body by its position vector $\boldsymbol{x}$ with respect to some previously chosen origin $\mathbf{0}$, at a fixed time. For simplicity we will take this to be at the time $t = 0$, and we will assume that the body is undeformed and unstressed in this state, unless stated otherwise. The region occupied by the body at the time $t = 0$ is denoted by Ω, and is called the *reference configuration*. To emphasize the identification between points in the region Ω and points in the undeformed body we will often refer to a point $\boldsymbol{x} \in \Omega$ as a *material point*. If we go one step further and place a set of Cartesian axes with the origin $\mathbf{0}$, then the position vector $\boldsymbol{x}$ has components x_i $(i = 1, 2, 3)$ with respect to the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ associated with this set of axes. The situation is illustrated in Figure 2.1, in which Ω_t is the current configuration, the region occupied by the body at the current time t.

The objective will be to obtain a complete description of the motion and deformations of the body, for given loading conditions, within the framework of continuum mechanics. There is an extensive literature on continuum mechanics; the texts [28, 31, 69] are examples of works that may be consulted for further details.

Second, it is assumed that both the properties and the behavior of such a body can be described in terms of functions of position $\boldsymbol{x}$ in the body and time t. Thus, for example, we may associate with the body a scalar temperature distribution θ that varies within the body and with the passage of time, so that the value of the temperature of a material point $\boldsymbol{x}$ at time t is represented by the function $\theta(\boldsymbol{x}, t)$, or equivalently by $\theta(x_1, x_2, x_3, t)$.

It will be necessary at some stage to stipulate the properties assumed or expected of functions defined on the body. For the time being there is no need to be too specific about this, except to say that functions will be assumed to possess as many derivatives as are required in order for what follows to make sense. Later we will have to be careful about the specification of function spaces to which these functions are required to belong.

The study of the behavior of continuous media conveniently begins with a development of a suitable framework within which to describe the motion of the body. This framework is quite independent of any agencies acting on the body, and it is also independent of the constitution of the body. In other words, we are concerned in the first instance solely with the geometry of motion. This is known as kinematics, and we now proceed to set out a framework that will be adequate for future needs.

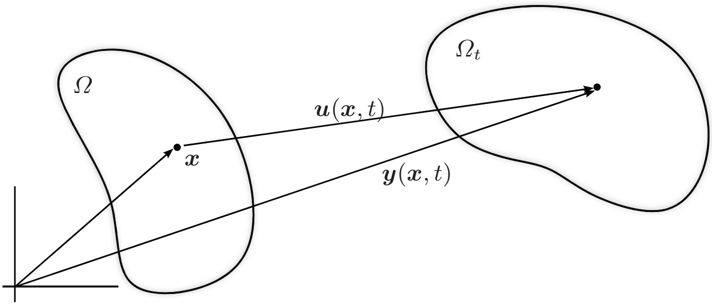

Fig. 2.1. Current and undeformed configurations of an arbitrary material body

2.1 Kinematics

As mentioned above, the position of a body in an undeformed state is identified with a region Ω in $\mathbb{R}^3$. With time the body moves and deforms, as a result of the action of various forces (we are not interested in the details of these forces at this point), so that at time t it occupies a new region Ω_t, called the *current configuration* at time t, as is shown in Figure 2.1. This deformation may be described mathematically by introducing a vector-valued function $\boldsymbol{y}$ of position and time, called the *motion*. Thus a material particle initially located at $\boldsymbol{x}$ will have position $\boldsymbol{y}(\boldsymbol{x},t)$ at time t. Clearly, we must have $\boldsymbol{y}(\boldsymbol{x},0)=\boldsymbol{x}$. For simplicity we denote functions and their values by the same symbol, so that the motion is described by the equation

$$\boldsymbol{y}=\boldsymbol{y}(\boldsymbol{x},t), \tag{2.1}$$

or in component form,

$$y_i = y_i(x_1, x_2, x_3, t), \quad 1 \le i \le 3,$$

for $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$.

The function $\boldsymbol{y}$ will have to satisfy certain conditions if it is to be used to model adequately the motion of the body. First, we must ensure that no two points get mapped to a single point by $\boldsymbol{y}$; in other words, $\boldsymbol{y}$ must be one-to-one, and hence invertible. Second, we must ensure that the motion is orientation-preserving; that is, the *Jacobian* J, defined by

$$J = \det\left(\nabla \boldsymbol{y}\right), \tag{2.2}$$

must be positive. Here,

$$\nabla \boldsymbol{y} = \left(\frac{\partial y_i}{\partial x_j}\right)$$

stands for the Jacobian matrix whose (i, j)th element is $\partial y_i / \partial x_j$. Hence, every element of nonzero volume in Ω is mapped to an element of nonzero volume in Ω_t. We recall a result from calculus: $dy = J\,dx$, where dx and dy denote the volume elements in Ω and Ω_t.

A sufficient condition for the motion $\boldsymbol{y}$ to be invertible is that there exist a constant $c(\Omega) > 0$, depending only on Ω, such that

$$\sup_{\Omega} |\nabla \boldsymbol{y} - \boldsymbol{I}| < c(\Omega).$$

In the case where Ω is convex, we may take $c(\Omega) = 1$. This result, as well as others on the invertibility of the motion, may be found in [31, § 5.5, § 5.6].

Instead of adopting the function $\boldsymbol{y}$ as the primary unknown variable, it is more convenient to introduce the *displacement* vector $\boldsymbol{u}$ by

$$\boldsymbol{u}(\boldsymbol{x}, t) := \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{x}$$

and to replace the motion by the displacement as the primary unknown. Of course, the displacement alone does not give complete information about the deformation of the body. We need to be able to distinguish, for example, between a simple *rigid body motion*, in which the body is translated and rotated to a new position without deformation (Figure 2.2), and a situation in which the body indeed assumes a new shape. The quantity that we use to measure deformation is the *strain tensor*. Let us now see how this quantity arises.

Consider a point $\boldsymbol{x}$ in Ω and two fibers of material particles emanating from $\boldsymbol{x}$. These fibers are described by vectors $\Delta\boldsymbol{x}$ and $\delta\boldsymbol{x}$, as is shown in Figure 2.3. The notion of strain emerges naturally if we consider the changes in lengths of these fibers, and the change in the angle between them, under the motion $\boldsymbol{y}$. The fiber $\Delta\boldsymbol{x}$ is mapped to the fiber $\Delta\boldsymbol{y} := \boldsymbol{y}(\boldsymbol{x} + \Delta\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x})$. Likewise, the fiber $\delta\boldsymbol{x}$ becomes the fiber $\delta\boldsymbol{y} := \boldsymbol{y}(\boldsymbol{x} + \delta\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x})$. Here, for simplicity in writing, we drop the time variable t in the expression for the motion $\boldsymbol{y}$. We are now in a position to measure changes in lengths and angles.

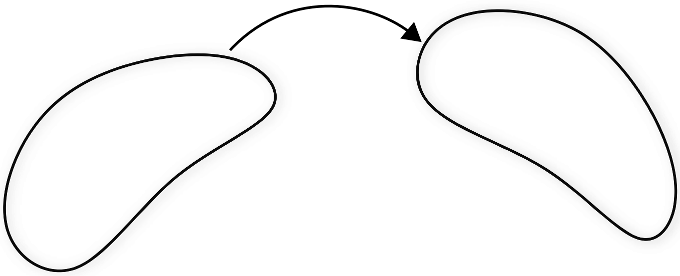

Fig. 2.2. An example of rigid body motion

We assume that the motion is smooth and may be differentiated as many times as required. Then it is possible to expand the term $\boldsymbol{y}(\boldsymbol{x} + \Delta\boldsymbol{x})$ in a Taylor series about $\boldsymbol{x}$ to get

$$\boldsymbol{y}(\boldsymbol{x} + \Delta\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x}) + (\nabla\boldsymbol{y})\Delta\boldsymbol{x} + o(|\Delta\boldsymbol{x}|),$$

with a similar expression for $\boldsymbol{y}(\boldsymbol{x} + \delta\boldsymbol{x})$. Thus

$$\Delta\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x} + \Delta\boldsymbol{x}) - \boldsymbol{y}(\boldsymbol{x}) = (\nabla\boldsymbol{y})\Delta\boldsymbol{x} + o(|\Delta\boldsymbol{x}|).$$

Since $\nabla\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{I} + \nabla\boldsymbol{u}(\boldsymbol{x})$, it follows that

$$\Delta\boldsymbol{y} = \Delta\boldsymbol{x} + (\nabla\boldsymbol{u})\Delta\boldsymbol{x} + o(|\Delta\boldsymbol{x}|).$$

In exactly the same way we arrive at the expression

$$\delta\boldsymbol{y} = \delta\boldsymbol{x} + (\nabla\boldsymbol{u})\,\delta\boldsymbol{x} + o(|\delta\boldsymbol{x}|).$$

We can now consider the expression

$$\begin{aligned}\Delta\boldsymbol{y}\cdot\delta\boldsymbol{y} - \Delta\boldsymbol{x}\cdot\delta\boldsymbol{x} &= (\nabla\boldsymbol{u})\Delta\boldsymbol{x}\cdot\delta\boldsymbol{x} + (\nabla\boldsymbol{u})\delta\boldsymbol{x}\cdot\Delta\boldsymbol{x}\\ &\quad + (\nabla\boldsymbol{u})\Delta\boldsymbol{x}\cdot(\nabla\boldsymbol{u})\delta\boldsymbol{x} + o(|\delta\boldsymbol{x}|^2 + |\Delta\boldsymbol{x}|^2).\end{aligned} \tag{2.3}$$

Though no confusion need arise, it is worth emphasizing that the gradient in (2.3) is with respect to the variable $\boldsymbol{x}$.

The point about the expression (2.3) is that if the body deforms as a rigid body, then obviously we must have $\Delta\boldsymbol{y}\cdot\delta\boldsymbol{y} = \Delta\boldsymbol{x}\cdot\delta\boldsymbol{x}$ for any pair of fibers emanating from any point in the body, since these fibers will not change in length, nor will the angle between them. Thus the right-hand side of (2.3) is identically zero in a rigid body motion. We now go one step further and consider the limit of (2.3) as the lengths of the fibers go to zero. Set $h = \max\{|\Delta\boldsymbol{x}|, |\delta\boldsymbol{x}|\}$, $\boldsymbol{n} = \Delta\boldsymbol{x}/h$, and $\boldsymbol{m} = \delta\boldsymbol{x}/h$; these are assumed to be fixed vectors independent of h. Now divide both sides of (2.3) by h^2, and take the limit as $h \to 0$. This gives

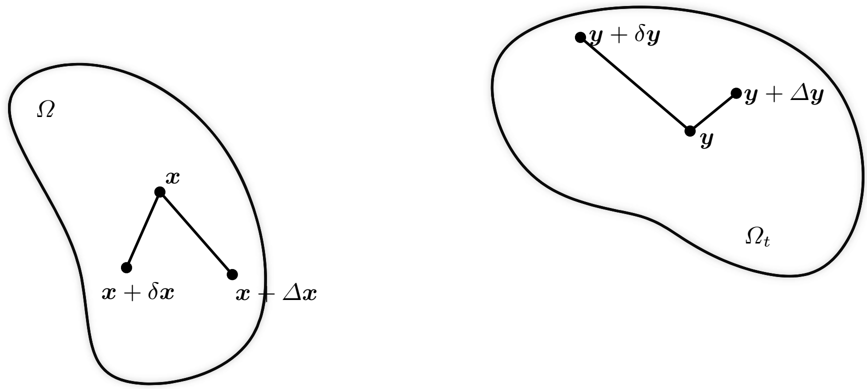

Fig. 2.3. Deformed and undeformed configurations of material line elements

$$\lim_{h\to 0}\frac{\Delta\boldsymbol{y}\cdot\delta\boldsymbol{y}-\Delta\boldsymbol{x}\cdot\delta\boldsymbol{x}}{h^2}=2\,\boldsymbol{n}\cdot\boldsymbol{\eta}(\boldsymbol{u})\boldsymbol{m}. \tag{2.4}$$

where $\boldsymbol{\eta}$ is the *strain tensor* associated with the displacement $\boldsymbol{u}$, defined by

$$\boldsymbol{\eta}(\boldsymbol{u}):=\tfrac{1}{2}\left[\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T+(\nabla\boldsymbol{u})^T\nabla\boldsymbol{u}\right]; \tag{2.5}$$

in component form this expression reads

$$\eta_{ij}(\boldsymbol{u})=\tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}).$$

Though we have been explicit about the fact that the strain is defined for a particular displacement field by writing $\boldsymbol{\eta}(\boldsymbol{u})$, very often we will simply denote the strain by $\boldsymbol{\eta}$ or η_{ij} when there is no danger of confusion.

So we see that the strain tensor is defined in such a way that it is zero if the body undergoes a rigid body motion.

The components of $\boldsymbol{\eta}$ are easily interpreted by referring back to equation (2.4) and by giving the fibers $\Delta\boldsymbol{x}$ and $\delta\boldsymbol{x}$ specific orientations. First, suppose that we identify $\delta\boldsymbol{x}$ with $\Delta\boldsymbol{x}$ at an arbitrary point in the body, and suppose that $\Delta\boldsymbol{x}$ is chosen so that it lies parallel to the x_1-axis. Then (2.4) becomes

$$\lim_{h\to 0}\frac{|\Delta\boldsymbol{y}|^2-|\Delta\boldsymbol{x}|^2}{h^2}=2\,\boldsymbol{e}_1\cdot\boldsymbol{\eta}\boldsymbol{e}_1=2\,\eta_{11},$$

since $\Delta\boldsymbol{x}/h=\boldsymbol{e}_1$ here. Thus we see that in this situation η_{11} equals half the net *change in length* (squared) of a material fiber originally oriented so that it points in the x_1 direction. The other two diagonal components of the strain are interpreted in a similar way.

To see how the off-diagonal components of $\boldsymbol{\eta}$ may be interpreted we return to (2.4) and now choose $\Delta\boldsymbol{x}$ and $\delta\boldsymbol{x}$ at an arbitrary point in the body in such a way that they have equal length h and lie parallel to the x_1 and x_2 axes, respectively. Then (2.4) gives

$$\lim_{h\to 0} \frac{\Delta\boldsymbol{y}\cdot\delta\boldsymbol{y} - \Delta\boldsymbol{x}\cdot\delta\boldsymbol{x}}{h^2} = \lim_{h\to 0}\frac{\Delta\boldsymbol{y}\cdot\delta\boldsymbol{y}}{h^2} = 2\,\boldsymbol{e}_1\cdot\boldsymbol{\eta}\boldsymbol{e}_2 = 2\,\eta_{12}. \tag{2.6}$$

Thus the component η_{12} gives a measure of the *change in angle* between two fibers originally at right angles to each other and oriented so that they were in the x_1 and x_2 directions. The remaining off-diagonal components are interpreted in a similar way.

Because the components of the strain have the interpretations described above, the diagonal components are referred to as *direct strains*, while the off-diagonal components are referred to as *shear strains*.

Earlier we had the result that for a rigid body motion the strain tensor is zero. Now consider a situation in which the strain tensor is zero; then we see from the above interpretation of its components and the observation that the axes may be chosen arbitrarily that no changes in length of fibers take place, nor are there any changes in angles between fibers. Thus the converse is also true: if $\boldsymbol{\eta} = \boldsymbol{0}$, then the body necessarily undergoes a rigid body motion.

Infinitesimal strain. There are many problems of practical interest for which the deformations can be regarded as "small" in some sense, and under such circumstances it is natural to consider whether the formulation of the problem might be simplified by exploiting this feature. Of course, it is necessary first to formalize and to quantify what is meant by "small". For the purposes of this work the following definition suffices: a body is said to undergo *infinitesimal deformation* if the displacement gradient $\nabla\boldsymbol{u}$ is sufficiently small so that the nonlinear term in (2.5) can be neglected. When this is the case, we may replace the strain tensor $\boldsymbol{\eta}$ by the *infinitesimal strain tensor* $\boldsymbol{\epsilon}$, which is defined by

$$\boldsymbol{\epsilon}(\boldsymbol{u}) := \tfrac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T). \tag{2.7}$$

Setting $h = |\nabla\boldsymbol{u}|$, in the case of infinitesimal strains we assume that $h \ll 1$ and that to within an error of $O(h^2)$ as $h \to 0$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ coincide.

Characterization of rigid body motions for infinitesimal strain. We have seen earlier that the strain tensor $\boldsymbol{\eta}$ vanishes if and only if the body undergoes a rigid body motion. Since we will study problems in the context of infinitesimal strains, it is necessary to characterize a rigid body motion for situations in which terms of $O(h^2)$ are neglected. Suppose that the body undergoes an infinitesimal rigid body motion, that is, one for which

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{0}.$$

Then

$$\nabla\boldsymbol{u} = -(\nabla\boldsymbol{u})^T,$$

so that the displacement gradient is skew. Thus the most general representation of the motion in such a situation is given by

$$\boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{y}_0 + \boldsymbol{\omega}(\boldsymbol{x} - \boldsymbol{x}_0),$$

or, equivalently, by

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \boldsymbol{\omega}(\boldsymbol{x} - \boldsymbol{x}_0),$$

where $\boldsymbol{x}_0$ is any point, $\boldsymbol{\omega}$ is a *skew* tensor, and $\boldsymbol{y}_0$ and $\boldsymbol{u}_0$ are either given or arbitrary vectors (for a proof of this result, see [106, Section 3.6]). If the motion is a pure translation, then $\boldsymbol{\omega} = \boldsymbol{0}$, while if on the other hand the motion is a pure rotation, then $\boldsymbol{u}_0 = \boldsymbol{0}$. An infinitesimal rigid body motion may be written alternatively as

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0 + \boldsymbol{w} \wedge (\boldsymbol{x} - \boldsymbol{x}_0),$$

where $\boldsymbol{w}$ is the unique *axial vector* corresponding to $\boldsymbol{\omega}$; that is, $\boldsymbol{\omega a} = \boldsymbol{w} \wedge \boldsymbol{a}$ for any vector $\boldsymbol{a}$.

Changes in volume; incompressibility. We require a simple measure of the local change in volume accompanying a motion. The volume of the reference configuration is

$$V_0 = \int_\Omega dx,$$

while the volume of the current configuration is

$$V_t = \int_{\Omega_t} dy.$$

Thus the change in volume as a result of the deformation $\boldsymbol{y}$ is simply given by

$$\Delta V := V_t - V_0 = \int_{\Omega_t} dy - \int_\Omega dx.$$

Since $\Omega_t = \boldsymbol{y}(\Omega, t)$, we may use the conventional technique for change of variables in an integral to write

$$\int_{\Omega_t} dy = \int_\Omega J\, dx,$$

where the Jacobian J has been defined in (2.2). Thus the change in volume is

$$\Delta V = \int_\Omega (J(\boldsymbol{x}) - 1)\, dx. \tag{2.8}$$

Once again we are interested in determining the expression for the change in volume for situations in which the underlying deformation can be regarded as infinitesimal. For this purpose we set $h = |\nabla \boldsymbol{u}|$ and write the Jacobian in terms of $\boldsymbol{u}$; thus

$$J = \det(\nabla \boldsymbol{y}) = \det(\boldsymbol{I} + \nabla \boldsymbol{u}) = 1 + \operatorname{div} \boldsymbol{u} + O(h^2).$$

The last equality follows directly from the definition of the determinant or from the identity (see, for example, [28, page 48])

$$\det(\boldsymbol{A}+\boldsymbol{B}) = (1+\boldsymbol{B}:\boldsymbol{A}^{-T})\det\boldsymbol{A} + (1+\boldsymbol{A}:\boldsymbol{B}^{-T})\det\boldsymbol{B}$$

for all invertible matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Substitution in (2.8) yields the result that to within an error of $O(h^2)$,

$$\varDelta V = \int_{\varOmega} \operatorname{div}\boldsymbol{u}\, dx.$$

In other words, the quantity $\operatorname{div}\boldsymbol{u}$ represents the change in volume per unit volume in an infinitesimal deformation.

A deformation that experiences no change in volume is called *isochoric*; for such a deformation,

$$J(\boldsymbol{x},t) = 1 \quad \forall\,\boldsymbol{x}\in\varOmega,\ t\in[0,T]. \tag{2.9}$$

When an isochoric deformation is infinitesimal, then to within an error of $O(h^2)$ the displacement field satisfies the condition

$$\operatorname{tr}\boldsymbol{\epsilon}(\boldsymbol{u}(\boldsymbol{x},t)) = \operatorname{div}\boldsymbol{u}(\boldsymbol{x},t) = 0 \quad \forall\,\boldsymbol{x}\in\varOmega,\ t\in[0,T]. \tag{2.10}$$

It may alternatively happen that a material has the property, possibly idealized, that it is unable to experience a change in volume. This idealization is often made in the case of materials for which, for the range of conditions under which they are being analyzed, the volume change observed is negligible. Such materials are referred to as *incompressible*. Note the difference between isochoric deformations and incompressible materials; in the former case a particular *deformation* is accompanied by no change in volume so that (2.9) and (2.10) are consequences of the deformation, while in the latter case it is a property of the *material* that no matter what the deformation, the body is unable to undergo any change in volume. In this case the conditions (2.9) or (2.10) represent constraints on the possible classes of deformations that are admitted.

2.2 Balance of Momentum; Stress

In this section we move away from the purely geometric nature of kinematics and investigate the consequences for material bodies of the fundamental laws of balance of linear and angular momentum. A further development is the introduction in this context of the notion of stress as a tensorial quantity that characterizes the state of internal forces acting in a body. All variables are assumed to have the requisite degree of smoothness consistent with developments in this section.

It is particularly convenient to develop the notions of momentum and stress in the context of the *reference configuration*; that is, we exploit the fact that field variables are functions of reference position $\boldsymbol{x}$ and time t, so that while the momentum and stress at time t are quantities associated with the configuration of the body at time t, these can easily be expressed, via the mapping (2.1), as functions defined over the reference configuration Ω.

The equations corresponding to local balance of linear and angular momentum are obtained by writing down the expressions that correspond to balance of linear and angular momentum for an arbitrary subset of the body. The local forms of these laws then follow from the arbitrariness of the subset and appropriate smoothness assumptions on the variables.

Now let Ω represent the reference configuration of the body, as before, and Ω_t the current configuration. Furthermore, let Ω' be an arbitrary subset of Ω, which is mapped by the motion to an arbitrary subset Ω_t' of Ω_t. Under these circumstances we may express global quantities associated with the current configuration as integrals over the reference configuration.

The *velocity field* $\dot{\boldsymbol{u}}$ and *acceleration field* $\ddot{\boldsymbol{u}}$ corresponding to a displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$ are defined by

$$\dot{\boldsymbol{u}}(\boldsymbol{x}, t) := \frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t},$$

$$\ddot{\boldsymbol{u}}(\boldsymbol{x}, t) := \frac{\partial^2 \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t^2}.$$

Thus, the *linear momentum* of the subset Ω_t' of Ω_t at time t is defined by

$$\int_{\Omega'} \rho \dot{\boldsymbol{u}} \, dx,$$

and its *angular momentum* by

$$\int_{\Omega'} \boldsymbol{x} \wedge \rho \dot{\boldsymbol{u}} \, dx,$$

in which ρ denotes the mass density of the body, that is, the mass per unit reference volume of the body.

The body is subjected to a system of forces, which are of two kinds. There is the *body force* $\boldsymbol{b}(\boldsymbol{x}, t)$, which represents the force per unit reference volume exerted on the material point $\boldsymbol{x}$ at time t by agencies external to the body; gravity is a canonical example, the body force in this case being $\rho g \boldsymbol{e}$, where g is the gravitational acceleration and $\boldsymbol{e}$ is the unit vector pointing in the downward vertical direction. The second kind of force acting on the body is the *surface traction*. To define this force field it is convenient to begin by introducing, for a given unit vector $\boldsymbol{n}$, the *stress vector* $\boldsymbol{s}_n(\boldsymbol{x}, t)$: if γ is a regular surface in $\bar{\Omega}$ passing through $\boldsymbol{x}$ and having unit normal $\boldsymbol{n}$ at $\boldsymbol{x}$, then $\boldsymbol{s}_n(\boldsymbol{x}, t)$ is the current force per unit reference area exerted by the portion of Ω on the side of γ towards which $\boldsymbol{n}$ points, on the portion of Ω that lies on the other side. Let Γ' denote the boundary of Ω'; then the surface traction at time t is defined to be the stress vector $\boldsymbol{s}_n(\boldsymbol{x}, t)$ ($\boldsymbol{x} \in \Gamma'$) acting on Γ', with $\boldsymbol{n}$ defined to be the outward unit normal on Γ' (see Figure 2.4). While

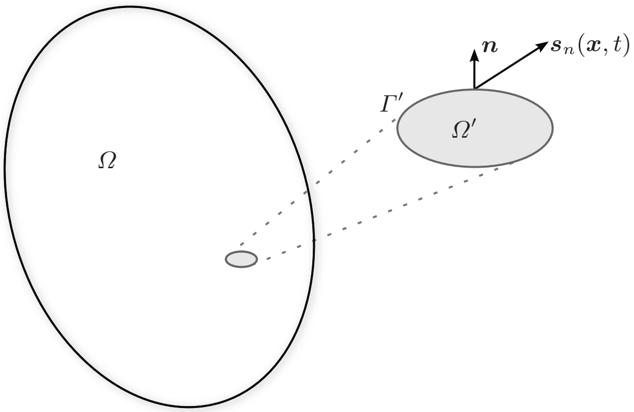

Fig. 2.4. The surface traction vector field

we have chosen to define quantities such as forces and momentum in terms of the reference configuration of the body, there is no difficulty in restating these definitions in terms of the current configuration.

The laws of balance of linear and angular momentum may now be stated.

Balance of linear momentum. The total force acting on Ω'_t is equal to the rate of change of the linear momentum of Ω'_t; expressed in terms of integrals over the reference configuration,

$$\int_{\Omega'} \rho \ddot{\boldsymbol{u}}\, dx = \int_{\Omega'} \boldsymbol{b}\, dx + \int_{\Gamma'} \boldsymbol{s}_n\, ds. \tag{2.11}$$

Note that in this identity we have used the fact that

$$\frac{\partial}{\partial t} \int_{\Omega'} (\cdot)\, dx = \int_{\Omega'} \frac{\partial}{\partial t} (\cdot)\, dx,$$

since Ω' is chosen independent of time.

Balance of angular momentum. The total moment acting on Ω'_t is equal to the rate of change of the angular momentum of Ω'_t; expressed in terms of integrals over the reference configuration,

$$\int_{\Omega'} \boldsymbol{x} \wedge \rho \ddot{\boldsymbol{u}}\, dx = \int_{\Omega'} \boldsymbol{x} \wedge \boldsymbol{b}\, dx + \int_{\Gamma'} \boldsymbol{x} \wedge \boldsymbol{s}_n\, ds. \tag{2.12}$$

We have the following two important results.

CAUCHY'S RECIPROCAL THEOREM. Given any unit vector $\boldsymbol{n}$,

$$\boldsymbol{s}_n = -\boldsymbol{s}_{-n}. \tag{2.13}$$

This result is clearly a generalization to deformable bodies of Newton's third law of action and reaction.

EXISTENCE OF THE STRESS TENSOR. There exists on $\Omega \times [0, T]$ a second-order tensor field $\boldsymbol{\tau}$, called the first Piola–Kirchhoff stress field, with the property that

$$\boldsymbol{\tau}\boldsymbol{n} = \boldsymbol{s}_n \tag{2.14}$$

for each unit vector $\boldsymbol{n}$.

The derivation of the reciprocal theorem of Cauchy and the proof of the existence of the stress tensor are treated in detail in [4, Chapter 12], [65, page 45], and [106, Section 4.1].

We are now in a position to obtain *local* forms of the two balance laws. In the following we assume that all variables have the degree of differentiability consistent with the manipulations that are carried out.

We begin with the law of balance of linear momentum. From the relationship (2.14) between the surface traction and stress tensor we obtain, using a variant of the Green–Gauss theorem,

$$\int_{\Gamma'} \boldsymbol{s}_n \, ds = \int_{\Gamma'} \boldsymbol{\tau}\boldsymbol{n} \, ds = \int_{\Omega'} \operatorname{Div} \boldsymbol{\tau} \, dx,$$

so that (2.11) becomes

$$\int_{\Omega'} (\rho \ddot{\boldsymbol{u}} - \boldsymbol{b} - \operatorname{Div} \boldsymbol{\tau}) \, dx = \boldsymbol{0}. \tag{2.15}$$

Here Div is the divergence operator with respect to the reference configuration and expressed in terms of derivatives with respect to components of $\boldsymbol{x}$:

$$(\operatorname{Div} \boldsymbol{\tau})_i = \frac{\partial \tau_{ij}}{\partial x_j}.$$

Since the domain Ω' is arbitrary, the integrand in (2.15) must vanish. We thus obtain in local form the *equation of motion*

$$\operatorname{Div} \boldsymbol{\tau} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}. \tag{2.16}$$

In component form, the equation of motion reads

$$\frac{\partial \tau_{ij}}{\partial x_j} + b_i = \rho \ddot{u}_i, \quad 1 \leq i \leq 3.$$

For situations in which all the given data are independent of time, the response of the body will also be independent of time. In this case we have

$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x})$, and the equation of motion becomes the *equation of equilibrium*

$$\frac{\partial \tau_{ij}}{\partial x_j} + b_i = 0, \quad 1 \leq i \leq 3. \tag{2.17}$$

We have chosen to present the arguments leading to the equation of motion in the setting of the reference configuration, with $\boldsymbol{x}$ and t as independent variables. Since the motion

$$\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}, t)$$

is invertible, it is also possible to treat $\boldsymbol{y}$ as the independent variable and to carry out the development in the *current* configuration. That is, we have $\boldsymbol{x} = \overline{\boldsymbol{x}}(\boldsymbol{y}, t)$ after carrying out the inversion, and so, for example, the velocity $\dot{\boldsymbol{u}}$ has the alternative representation

$$\frac{\partial}{\partial t}\boldsymbol{y}(\boldsymbol{x}, t) = \dot{\boldsymbol{u}}(\overline{\boldsymbol{x}}(\boldsymbol{y}, t), t) = \boldsymbol{v}(\boldsymbol{y}, t).$$

Similar transformations can be carried out with respect to all variables, and the principles of balance of linear and angular momentum are then expressed in terms of integrals over the current configuration Ω_t. As far as the stress goes, an argument identical to that which leads to the existence of the first Piola–Kirchhoff stress tensor gives the existence of a tensor $\boldsymbol{\sigma}$, called the *Cauchy stress*, that has the property that the force per unit *current* area $\boldsymbol{t}_\nu$ on an elemental surface having unit normal $\boldsymbol{\nu}$ is given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{t}_\nu. \tag{2.18}$$

The Cauchy stress therefore has the same relationship to the current configuration as does the first Piola–Kirchhoff stress to the reference configuration.

The use of the principle of balance of linear momentum, when applied in the current configuration, leads to the equation of motion in the form

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} = \rho_t \boldsymbol{a}$$

in which $\boldsymbol{a}$ is the acceleration and ρ_t is the mass density per unit current volume. Here div is the divergence operator in the current configuration, so that $\operatorname{div} \boldsymbol{\sigma} = (\partial \sigma_{ij}/\partial y_j)\boldsymbol{e}_i$.

It can be shown that the first Piola–Kirchhoff and Cauchy stresses are related according to

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{\tau}\,(\boldsymbol{I} + \nabla \boldsymbol{u})^T. \tag{2.19}$$

We have not as yet examined the consequences of the equation (2.12) for balance of angular momentum; by carrying out manipulations similar to those that lead to (2.16), it is possible to show that this balance law implies that

$$\boldsymbol{\tau}\,(\boldsymbol{I} + \nabla \boldsymbol{u})^T$$

is symmetric. Equivalently, we have the classical result that the Cauchy stress is *symmetric*:

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma} \quad \text{or} \quad \sigma_{ji} = \sigma_{ij}. \tag{2.20}$$

Stress and the balance laws for infinitesimal deformations. For problems in which deformations are assumed infinitesimal, the distinction between the reference and current configurations may be ignored. To begin with, we may neglect the term $\nabla \boldsymbol{u}$ appearing in (2.19); furthermore, since $J = \det(\boldsymbol{I} + \nabla \boldsymbol{u}) = 1 + \operatorname{div} \boldsymbol{u} + O(h^2)$, we may set $J \approx 1$. Likewise, $\rho_t = J^{-1}\rho \approx \rho$, to within an error $O(h)$. Thus the distinction between the first Piola–Kirchhoff and Cauchy stresses disappears. In addition, since

$$\frac{\partial}{\partial x_j} = \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i} = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)\frac{\partial}{\partial y_i},$$

it follows that when $\nabla \boldsymbol{u}$ is small, we may replace derivatives with respect to y_j by derivatives with respect to x_j. In summary, then, the principles of balance of linear and angular momentum are, in local form and for infinitesimal deformations,

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}, \tag{2.21}$$

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}. \tag{2.22}$$

2.3 Linearly Elastic Materials

We are moving towards a situation in which the behavior of a material body is described by a system of partial differential equations. So far, we have the equation of motion (2.16) and the strain–displacement relation (2.5); equivalently, if we assume that the deformation is infinitesimal, we will deal with equations (2.21) and (2.7). In either case these represent, when written out in component form, a total of nine equations: three from the equation of motion and six from the strain–displacement relation (taking into account the symmetry of $\boldsymbol{\epsilon}$). The total number of unknowns is fifteen: three components of displacement, six components of the strain and six components of the stress (again accounting for the symmetry of $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$). Thus it is clear that six additional equations are required if we are to have a problem that is at least in principle solvable.

Physical considerations also dictate that the description of the problem so far is incomplete: The kinematics have been described, and the balance laws are accounted for, but as yet there is no description of the particular material behavior. This information, embodied in the *constitutive equations* of the material, will provide the remaining equations of the problem.

Later on, we will embark on a detailed study of the constitutive equations that describe elastoplastic behavior. An essential precursor to such a study is an understanding of the equations governing elastic behavior. We review in this section the salient ideas, confining attention to linearly elastic materials.

A body is *linearly elastic* if the stress depends linearly on the infinitesimal strain, that is, if the stress and strain are related to each other through an equation of the form

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}, \tag{2.23}$$

where $\boldsymbol{C}$, called the *elasticity tensor*, is a linear map from the space of symmetric matrices or second-order symmetric tensors into itself. Like $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, $\boldsymbol{u}$, and other variables, the elasticity tensor is a function of position in the body. It does not, however, depend on time. If the density ρ and the elasticity tensor $\boldsymbol{C}$ are independent of position, the body is said to be *homogeneous*.

The map $\boldsymbol{C}$ may be represented as a fourth-order tensor as follows: relative to the orthonormal basis $\{\boldsymbol{e}_i\}$ we have

$$\begin{aligned}
\sigma_{ij} &= \boldsymbol{e}_i \cdot \boldsymbol{\sigma}\boldsymbol{e}_j \\
&= \boldsymbol{e}_i \cdot (\boldsymbol{C}\boldsymbol{\epsilon})\boldsymbol{e}_j \\
&= \boldsymbol{e}_i \cdot \left(\boldsymbol{C}(\epsilon_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\right)\boldsymbol{e}_j \\
&= \boldsymbol{e}_i \cdot \left(\boldsymbol{C}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\right)\boldsymbol{e}_j \epsilon_{kl} \\
&= C_{ijkl}\epsilon_{kl},
\end{aligned}$$

where C_{ijkl}, the components of $\boldsymbol{C}$, are defined by

$$C_{ijkl} = \boldsymbol{e}_i \cdot \left(\boldsymbol{C}(\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\right)\boldsymbol{e}_j.$$

It follows that the constitutive equation (2.23) has the component form

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}. \tag{2.24}$$

Properties of the elasticity tensor. Without loss of generality, we may assume the elasticity tensor to have the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \tag{2.25}$$

This is argued as follows. Since $\boldsymbol{\epsilon}$ is symmetric, we have, from (2.24),

$$\sigma_{ij} = C_{ijkl}\epsilon_{lk} = C_{ijlk}\epsilon_{kl}.$$

Hence

$$\sigma_{ij} = \frac{1}{2}\left(C_{ijkl} + C_{ijlk}\right)\epsilon_{kl}.$$

Similarly, using the symmetry of $\boldsymbol{\sigma}$, we have

$$\sigma_{ij} = \frac{1}{2}\left(C_{ijkl} + C_{jikl}\right)\epsilon_{kl}.$$

Therefore, the relation (2.24) can be equivalently expressed as

$$\sigma_{ij} = \frac{1}{4}\left(C_{ijkl} + C_{ijlk} + C_{jikl} + C_{jilk}\right)\epsilon_{kl}.$$

In other words, if necessary, we may redefine the tensor $\boldsymbol{C}$ for the relation (2.24) such that the symmetry properties (2.25) hold.

Later, when we consider elastic constitutive equations that are derived from a strain energy or free energy function, it will be seen that the elasticity tensor possesses the additional symmetry property

$$C_{ijkl} = C_{klij}. \tag{2.26}$$

The elasticity tensor is *positive definite* if

$$\boldsymbol{\epsilon} : \boldsymbol{C}\boldsymbol{\epsilon} > 0 \quad \text{for all nonzero symmetric second-order tensors } \boldsymbol{\epsilon}. \tag{2.27}$$

Furthermore, $\boldsymbol{C}$ is said to be *strongly elliptic* (see [114, 183]) if

$$(\boldsymbol{a} \otimes \boldsymbol{b}) : \boldsymbol{C}(\boldsymbol{a} \otimes \boldsymbol{b}) > 0 \quad \text{for all nonzero vectors } \boldsymbol{a} \text{ and } \boldsymbol{b}. \tag{2.28}$$

In component form, (2.28) reads

$$C_{ijkl}a_i a_k b_j b_l > 0 \quad \text{if } a_i a_i > 0 \text{ and } b_i b_i > 0.$$

Finally, $\boldsymbol{C}$ is said to be *pointwise stable* ([114, page 321]) if there exists a constant $\alpha > 0$ such that

$$\boldsymbol{\epsilon} : \boldsymbol{C}\boldsymbol{\epsilon} \geq \alpha \, |\boldsymbol{\epsilon}|^2 \quad \text{for all symmetric second-order tensors } \boldsymbol{\epsilon}. \tag{2.29}$$

It should be clear from these definitions that pointwise stability implies, but is not implied by, strong ellipticity. It is also clear that pointwise stability is equivalent to pointwise positive definiteness, under the assumption that $\boldsymbol{C}$ is continuous on $\overline{\Omega}$.

Sometimes it is convenient to work not with stress as a function of strain, but the other way around. If the relationship (2.23) is invertible (and this will always be the case for real materials) then we may write

$$\boldsymbol{\epsilon} = \boldsymbol{A}\boldsymbol{\sigma}, \tag{2.30}$$

where the fourth-order tensor $\boldsymbol{A}$ is known as the *compliance tensor*; it is the inverse of $\boldsymbol{C}$ and therefore has the property that

$$\boldsymbol{A}(\boldsymbol{C}\boldsymbol{\epsilon}) = \boldsymbol{\epsilon} \quad \forall\, \boldsymbol{\epsilon},\ \boldsymbol{\epsilon}^T = \boldsymbol{\epsilon},$$

and

$$\boldsymbol{C}(\boldsymbol{A}\boldsymbol{\sigma}) = \boldsymbol{\sigma} \quad \forall\, \boldsymbol{\sigma},\ \boldsymbol{\sigma}^T = \boldsymbol{\sigma}.$$

2.4 Isotropic Elasticity

It is often the case that materials possess preferred directions or symmetries. For example, timber can be regarded as an orthotropic material, in the sense that it possesses particular constitutive properties along the grain and at right angles to the grain of the wood. The greatest degree of symmetry is possessed by a material that has no preferred directions; that is, its response to a force is independent of its orientation. This property is known as isotropy, and a material with such a property is called *isotropic*.

Isotropic linearly elastic materials occur in abundance, and so form an important subclass of materials whose properties we need to model mathematically. The most striking mathematical effect of isotropy is that it reduces the twenty-one independent components C_{ijkl} of $\boldsymbol{C}$ (taking account of the symmetry properties (2.25) and (2.26)) to *two*. Of course, the choice of these two material coefficients is not unique, and a new pair may be generated by combining a given pair in different ways. The most appropriate choice of material coefficients for isotropic elastic materials will depend on the application in mind. We will discuss some of the more common variants.

First, for an isotropic linearly elastic material we have the result that the components of the elasticity tensor are given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.31}$$

where δ_{ij} is the Kronecker delta. In coordinate-free form the elasticity tensor is defined to be the fourth-order tensor $\boldsymbol{C}$ that satisfies

$$(\boldsymbol{a}\otimes\boldsymbol{b}) : \boldsymbol{C}(\boldsymbol{c}\otimes\boldsymbol{d}) = \lambda\,(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{c}\cdot\boldsymbol{d}) + \mu\,[(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) + (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})] \tag{2.32}$$

for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$. The scalars λ and μ are called *Lamé moduli*. The stress-strain relation (2.23) in this case is easily found to be given by

$$\boldsymbol{\sigma} = \lambda\,(\mathrm{tr}\,\boldsymbol{\epsilon})\boldsymbol{I} + 2\,\mu\,\boldsymbol{\epsilon}. \tag{2.33}$$

For the purpose of interpreting the moduli, and of defining alternative pairs of moduli for isotropic elastic materials, it is convenient to carry out an orthogonal decomposition of both the stress and the strain into what are known as spherical and deviatoric components; the first is associated solely with volumetric changes, while the latter is associated with shearing stresses and deformations. To achieve this decomposition we recall that any second-order tensor $\boldsymbol{\tau}$ may be written in the form

$$\boldsymbol{\tau} = \boldsymbol{\tau}^D + \boldsymbol{\tau}^S, \tag{2.34}$$

where the deviatoric and spherical parts $\boldsymbol{\tau}^D$ and $\boldsymbol{\tau}^S$ of $\boldsymbol{\tau}$ are defined, respectively, by

$$\boldsymbol{\tau}^D := \boldsymbol{\tau} - \tfrac{1}{3}(\mathrm{tr}\boldsymbol{\tau})\boldsymbol{I}, \quad \boldsymbol{\tau}^S := \tfrac{1}{3}(\mathrm{tr}\,\boldsymbol{\tau})\boldsymbol{I}. \tag{2.35}$$

The maps $\boldsymbol{\tau}\mapsto\boldsymbol{\tau}^D$ and $\boldsymbol{\tau}\mapsto\boldsymbol{\tau}^S$ can be regarded as orthogonal projections on the space of second-order tensors when this space is equipped with the inner product $\boldsymbol{\tau}:\boldsymbol{\sigma} = \tau_{ij}\sigma_{ij}$. Indeed, we have $(\boldsymbol{\tau}^D)^S = (\boldsymbol{\tau}^S)^D = \boldsymbol{0}$, and

$$\begin{aligned}\boldsymbol{\tau}^D : \boldsymbol{\tau}^S &= (\boldsymbol{\tau} - \boldsymbol{\tau}^S) : \boldsymbol{\tau}^S \\ &= \boldsymbol{\tau} : \boldsymbol{\tau}^S - |\boldsymbol{\tau}^S|^2 \\ &= \tau_{ij} \tfrac{1}{3} \tau_{kk} \delta_{ij} - |\boldsymbol{\tau}^S|^2 \\ &= \tfrac{1}{3} \tau_{ii} \tau_{kk} - |\boldsymbol{\tau}^S|^2 \\ &= 0,\end{aligned}$$

since $|\boldsymbol{\tau}^S|^2 = \frac{1}{3}(\tau_{ii})^2$. The constitutive equation can thus be written in the *uncoupled form* (by applying the operators $(\cdot)^D$ and $(\cdot)^S$ successively to (2.33))

$$\boldsymbol{\sigma}^D = 2\,\mu\,\boldsymbol{\epsilon}^D, \tag{2.36}$$

$$\boldsymbol{\sigma}^S = \lambda\,(\mathrm{tr}\,\boldsymbol{\epsilon})\boldsymbol{I}^S + 2\,\mu\,\boldsymbol{\epsilon}^S = 3\,(\lambda + \tfrac{2}{3}\mu)\,\boldsymbol{\epsilon}^S. \tag{2.37}$$

The scalar μ is also known as the *shear modulus* (for reasons that are evident from (2.36)), while the material coefficient $K \equiv \lambda + \frac{2}{3}\mu$ is known as the *bulk modulus* because it measures the ratio between the spherical stress and volume change. Thus an alternative pair of elastic coefficients to the Lamé moduli is $\{\mu, K\}$. Note that the shear modulus is often denoted by G, especially in the engineering literature.

Yet another important alternative pair of material coefficients arises from direct consideration of the behavior of the length of an elastic rod when it is subjected to a uniaxial stress. Suppose that the Cartesian axes are aligned in such a way that an isotropic elastic rod lies parallel to the x_1-axis (see Figure 2.5) and is subjected to a uniform stress with $\sigma_{11} \neq 0$ and all other components being zero. The effect will be that the rod experiences only direct strains, on account of its isotropy. We are interested here first in the ratio $\sigma_{11}/\epsilon_{11}$ and second in the ratio $\epsilon_{22}/\epsilon_{11}$, or, equivalently, $\epsilon_{33}/\epsilon_{11}$. The associated material coefficients are known, respectively, as *Young's modulus* and *Poisson's ratio*:

$$\begin{aligned}\text{Young's modulus} \quad E &= \frac{\sigma_{11}}{\epsilon_{11}}, \\ \text{Poisson's ratio} \quad \nu &= -\frac{\epsilon_{22}}{\epsilon_{11}}.\end{aligned}$$

Thus Young's modulus measures the slope of the stress–strain curve and is analogous to the stiffness of a spring, while Poisson's ratio measures lateral contraction. Since we expect a tensile stress to be accompanied by an extension of the material and since we also know from experience that most common materials would respond to an extension in one direction with a contraction in the transverse direction (think of what happens when a rubber band is extended), it follows that one expects both E and ν to be positive quantities. We will see later that further restrictions are placed on the ranges of E and ν by thermodynamic or mathematical considerations.

From (2.31) it is a straightforward task to obtain a relationship between the pairs $\{\lambda, \mu\}$ and $\{E, \nu\}$. Since for the case of pure tension we have

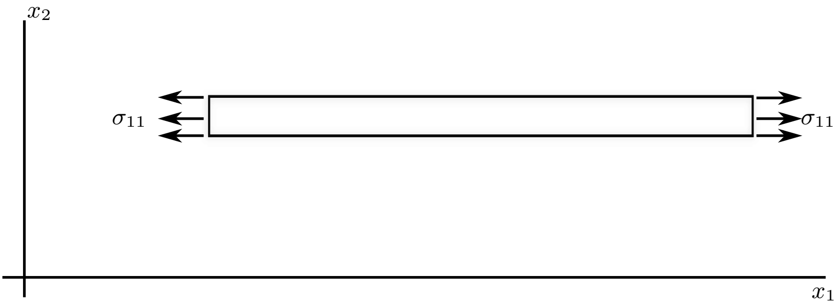

Fig. 2.5. A rod in a state of uniaxial stress

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{22} \end{pmatrix},$$

it follows that

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \tag{2.38}$$

and

$$\nu = \frac{\lambda}{2(\mu + \lambda)}\,. \tag{2.39}$$

The constitutive relation (2.33) can be put in an alternative useful form involving E and ν by inverting it and making use of (2.38) and (2.39); this gives

$$\boldsymbol{\epsilon} = E^{-1}[(1+\nu)\boldsymbol{\sigma} - \nu(\operatorname{tr}\boldsymbol{\sigma})\boldsymbol{I}]. \tag{2.40}$$

The conditions of pointwise stability and strong ellipticity introduced earlier both lead to constraints on admissible ranges for the material constants. Indeed, it is possible to show ([114, page 241]) that an isotropic linearly elastic material is

(a) pointwise stable if and only if $\mu > 0$ and $3\lambda + 2\mu > 0$ (or, in terms of Young's modulus and Poisson's ratio, if and only if $E > 0$ and $-1 < \nu < \frac{1}{2}$);
(b) strongly elliptic if and only if $\mu > 0$ and $\lambda + 2\mu > 0$ (or if and only if $E > 0$, and $\nu < \frac{1}{2}$ or $\nu > 1$).

2.5 A Thermodynamic Framework for Elasticity

The developments in the preceding sections were described in a purely mechanical framework, without bringing into play any thermodynamic considerations. Since it is our intention in this monograph to deal only with processes that take place under isothermal conditions, it would appear that there is indeed no need to take account of thermodynamics. This is, however, not quite the case. Since the primary goal is to present a theory of elastoplasticity and since plasticity as a constitutive theory can be conveniently developed within a thermodynamic framework, it will be necessary to bring thermodynamics into play, albeit in the context of isothermal processes. Plasticity is most conveniently described in the framework of thermodynamics with *internal variables*. We postpone discussion of internal variable theories to Section 2.7, while in this section we sketch the basic thermodynamic theory within which linearized elasticity can be described.

Suppose that a material body is subjected to a body force $\boldsymbol{b}$ in its interior and a surface traction $\boldsymbol{s}$ on the boundary. Suppose also, for now, that the body is subjected to thermal equivalents of these mechanical sources: in its interior the *heat source* r per unit volume, and across its boundary the *heat flux* $\boldsymbol{q}$ per unit area.

We begin with the *first law of thermodynamics*, which is essentially a statement of balance of energy. This law states that for any part Ω' of the body Ω, the rate of change of total internal energy plus kinetic energy is equal to the rate of work done on that part of the body by the mechanical forces, plus that by the heat supply. Mathematically the law may be expressed in the form

$$\frac{d}{dt}\int_{\Omega'}(e+\tfrac{1}{2}\,\rho|\dot{\boldsymbol{u}}|^2)\,dx=\int_{\Omega'}\boldsymbol{b}\cdot\dot{\boldsymbol{u}}\,dx+\int_{\Gamma'}\boldsymbol{s}\cdot\dot{\boldsymbol{u}}\,ds+\int_{\Omega'}r\,dx-\int_{\Gamma'}\boldsymbol{q}\cdot\boldsymbol{n}\,ds. \quad (2.41)$$

Here e represents the internal energy per unit volume, $\dot{\boldsymbol{u}}$ is the velocity vector, and $\Gamma'=\partial\Omega'$ is the boundary of Ω'. The minus sign in front of the term involving the heat flux appears because $\boldsymbol{n}$ is the outward unit normal vector to the surface, while $\boldsymbol{q}$ is the heat flux per unit area in the direction of $\boldsymbol{n}$, so that $-\int_{\Gamma'}\boldsymbol{q}\cdot\boldsymbol{n}\,ds$ is the total flow of heat across Γ' *into* the body. This law may be simplified by the use of the divergence theorem: indeed, observe that

$$\begin{aligned}\int_{\Gamma'}\boldsymbol{s}\cdot\dot{\boldsymbol{u}}\,ds&=\int_{\Gamma'}\boldsymbol{\sigma n}\cdot\dot{\boldsymbol{u}}\,ds\\&=\int_{\Omega'}\boldsymbol{\sigma}:\nabla\dot{\boldsymbol{u}}\,dx+\int_{\Omega'}\operatorname{div}\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}}\,dx\\&=\int_{\Omega'}\boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}}\,dx+\int_{\Omega'}\operatorname{div}\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}}\,dx,\end{aligned}$$

where in the last step we invoked the symmetry of $\boldsymbol{\sigma}$. Here and below we use the notation $\dot{\boldsymbol{\epsilon}}=\boldsymbol{\epsilon}(\dot{\boldsymbol{u}})$. Substituting this result in (2.41) and making use of equation (2.21) of balance of momentum, we obtain the first law in the form

$$\frac{d}{dt}\int_{\Omega'}e\,dx=\int_{\Omega'}\boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}}\,dx+\int_{\Omega'}r\,dx-\int_{\Gamma'}\boldsymbol{q}\cdot\boldsymbol{n}\,ds.$$

The *local* form of this law may be obtained by assuming first that all variables in the above relation are sufficiently smooth, and then by converting the surface integral involving the heat flux to a volume integral with the use of the divergence theorem. This gives

$$\int_{\Omega'} \left(\dot{e} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - r + \operatorname{div} \boldsymbol{q}\right) dx = 0,$$

which in turn leads to the local form

$$\dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + r - \operatorname{div} \boldsymbol{q}. \tag{2.42}$$

The second essential postulate of thermodynamics is the *second law*. For this we require first the notion of the *entropy* η per unit volume, and the *absolute temperature* $\theta > 0$. The entropy flux across the bounding surface Γ' into the body Ω' is given by $-\int_{\Gamma'} \theta^{-1} \boldsymbol{q} \cdot \boldsymbol{n}\, ds$, while the entropy supplied by the exterior is given by $\int_{\Omega'} \theta^{-1} r\, dx$. The second law states that the rate of increase in entropy in the body is not less than the total entropy supplied to the body by the heat sources. That is,

$$\frac{d}{dt} \int_{\Omega'} \eta\, dx \geq \int_{\Omega'} \theta^{-1} r\, dx - \int_{\Gamma'} \theta^{-1} \boldsymbol{q} \cdot \boldsymbol{n}\, ds. \tag{2.43}$$

By the same process used to obtain the local form (2.42) of the first law from (2.41) we may obtain the local form of the second law, which reads

$$\dot{\eta} \geq -\operatorname{div} (\theta^{-1} \boldsymbol{q}) + \theta^{-1} r. \tag{2.44}$$

The inequalities (2.43) and (2.44) are known as the *Clausius–Duhem form* of the second law of thermodynamics.

It is customary in elasticity and elastoplasticity to work with the *Helmholtz free energy* ψ, defined by

$$\psi = e - \eta\, \theta,$$

rather than with the internal energy. With this substitution and the use of (2.42), the local form of the second law becomes

$$\dot{\psi} + \eta\, \dot{\theta} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \theta^{-1} \boldsymbol{q} \cdot \nabla \theta \leq 0. \tag{2.45}$$

The inequality (2.45) is known as the *local dissipation inequality*.

Now we specialize to the situation in which subsequent developments will take place, namely, that of isothermal processes. Thus the temperature distribution in a body is assumed to be uniform and equal to the ambient temperature. Furthermore, it is assumed that there is no flow of heat, and also that there is no heat supply from the exterior. Under these circumstances the local dissipation inequality takes the simpler form

$$\dot{\psi} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} \leq 0. \tag{2.46}$$

Henceforth we will at all times make the assumptions just described, so that temperature will not appear as a variable. Furthermore, both the heat flux vector and heat supply will be assumed zero in what follows.

Elastic constitutive equations. We are now in a position to obtain the equations describing elastic material behavior. We define an elastic material to be one for which the constitutive equations take the form

$$\psi = \psi(\boldsymbol{\epsilon}), \tag{2.47}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}). \tag{2.48}$$

That is, the free energy and stress depend only on the current strain; there is no dependence on the history of behavior, for example. It should be remarked that the more general point of departure is to take the free energy and stress to be functions of the *displacement gradient* $\nabla \boldsymbol{u}$ rather than the strain. That these variables in fact depend on $\nabla \boldsymbol{u}$ through its symmetric part, the strain $\boldsymbol{\epsilon}$, is a consequence of the principle of material frame indifference (see [114]). We circumvent these considerations by assuming from the outset a dependence on $\boldsymbol{\epsilon}$ rather than on $\nabla \boldsymbol{u}$.

The functions appearing in (2.47) and (2.48) are assumed to be sufficiently smooth with respect of their arguments that as many derivatives as required may be taken.

It is an immediate consequence of the local dissipation inequality that the stress is determined by ψ through the relation

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\epsilon}}. \tag{2.49}$$

To see this, we substitute (2.47) in the local dissipation inequality (2.46) to obtain

$$\left(\frac{\partial \psi}{\partial \boldsymbol{\epsilon}} - \boldsymbol{\sigma} \right) : \dot{\boldsymbol{\epsilon}} \leq 0. \tag{2.50}$$

Then (2.49) follows from the fact that (2.50) holds for all $\dot{\boldsymbol{\epsilon}}$. The *linearly elastic material* is recovered from (2.49) by assuming that the free energy is a quadratic function of the strain; that is,

$$\psi(\boldsymbol{\epsilon}) = \tfrac{1}{2} \boldsymbol{\epsilon} : \boldsymbol{C} \boldsymbol{\epsilon}, \tag{2.51}$$

or

$$\psi(\boldsymbol{\epsilon}) = \tfrac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}.$$

Then the constitutive equation (2.23) is immediately recovered from (2.49) by substitution of (2.51). The thermodynamic framework is not entirely equivalent to the mechanical framework adopted earlier, though. One distinction lies in the symmetries of $\boldsymbol{C}$. From (2.49) and (2.51) we find that

$$\boldsymbol{\sigma} = \tfrac{1}{2} \left(\boldsymbol{C} + \boldsymbol{C}^T \right) \boldsymbol{\epsilon}.$$

Here $\frac{1}{2}\left(\boldsymbol{C} + \boldsymbol{C}^T \right)$ is the symmetric part of $\boldsymbol{C}$. Replacing $\boldsymbol{C}$ by $\frac{1}{2}\left(\boldsymbol{C} + \boldsymbol{C}^T \right)$ in (2.51) does not change the value of $\psi(\boldsymbol{\epsilon})$. Hence in the definition (2.51) we will replace $\boldsymbol{C}$ by its symmetric part, though for convenience we continue to denote this symmetrized tensor by $\boldsymbol{C}$. We then have

$$\boldsymbol{C} = \frac{\partial^2 \psi}{\partial \boldsymbol{\epsilon} \partial \boldsymbol{\epsilon}}, \tag{2.52}$$

and in addition to the symmetries given in (2.25) (these still hold, in view of the symmetry of the stress and strain), we must have the additional symmetry

$$C_{ijkl} = C_{klij}. \tag{2.53}$$

We will henceforth take as a basis for the description of linearly elastic material behavior the thermodynamic framework, so that in particular, the symmetry (2.53) will be assumed valid. Note that this symmetry is satisfied with the coefficients (2.31) for *isotropic* elastic materials.

2.6 Initial–Boundary and Boundary Value Problems for Linearized Elasticity

It is now possible to give a complete formulation of the problems to be solved in order to obtain a description of the deformation of, and stresses in, a linearly elastic body. Suppose that such a body initially occupies a domain $\Omega \subset \mathbb{R}^3$ and that the body has boundary Γ, which comprises nonoverlapping parts Γ_u and Γ_t with $\Gamma = \overline{\Gamma_u} \cup \overline{\Gamma_t}$. Suppose that the body force $\boldsymbol{b}(\boldsymbol{x}, t)$ is given in Ω, the displacement $\bar{\boldsymbol{u}}(\boldsymbol{x}, t)$ is given on the part Γ_u of the boundary, and the surface traction $\bar{\boldsymbol{s}}(\boldsymbol{x}, t)$ is given on the remainder Γ_t of the boundary, for $t \in [0, T]$. The initial values of the displacement and velocity are given by $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$ and $\dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$. Then the *initial–boundary value problem of linearized elasticity* is the following: find the displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$ that satisfies, for $\boldsymbol{x} \in \Omega$ and for $t \in [0, T]$,

the *equation of motion*

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}, \tag{2.54}$$

the *strain–displacement relation*

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \tfrac{1}{2} \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right), \tag{2.55}$$

the *elastic constitutive relation*

$$\boldsymbol{\sigma} = \boldsymbol{C} \boldsymbol{\epsilon}, \tag{2.56}$$

the *boundary conditions*

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \Gamma_u \quad \text{and} \quad \boldsymbol{\sigma n} = \bar{\boldsymbol{s}} \quad \text{on } \Gamma_t, \tag{2.57}$$

and the *initial conditions*

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{and} \quad \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{2.58}$$

We may take the displacement vector field as the primary unknown, and eliminate the stress and strain from the governing equations by substitution; this gives the equation of motion in the form

$$\operatorname{div}(\boldsymbol{C}\boldsymbol{\epsilon}(\boldsymbol{u})) + \boldsymbol{b} = \rho\ddot{\boldsymbol{u}}. \tag{2.59}$$

Similarly, the second boundary condition in (2.57) becomes

$$(\boldsymbol{C}\boldsymbol{\epsilon}(\boldsymbol{u}))\,\boldsymbol{n} = \bar{\boldsymbol{s}} \quad \text{on } \Gamma_t. \tag{2.60}$$

When the data are independent of the time, or when the data can be reasonably approximated as being time-independent, the initial–boundary value problem reduces to a *boundary value problem.* In this case the body force $\boldsymbol{b}(\boldsymbol{x})$ is given in Ω, the displacement $\bar{\boldsymbol{u}}(\boldsymbol{x})$ is given on Γ_u and the surface traction $\bar{\boldsymbol{s}}$ is given on Γ_t. The problem is now to find the displacement field $\boldsymbol{u}(\boldsymbol{x})$ that satisfies the *equation of equilibrium*

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \quad \text{in } \Omega \tag{2.61}$$

together with (2.55)–(2.57). As before, the stress can be eliminated from this problem to give (2.59) with the right-hand side equal to zero.

The variational formulation of the boundary value problem for linearized elasticity, (2.61) and (2.55)–(2.57), will be discussed in Chapter 6, as well as the question of well-posedness of this problem.

2.7 Thermodynamics with Internal Variables

The thermodynamic theory presented in Section 2.5 is not entirely adequate for modeling the behavior of a wide range of phenomena. There are situations involving chemically reacting continuous media, for example, in which it is necessary to account for the individual reactions taking place. This may be accomplished by adding to the conventional variables (temperature, strain, and so on) a number of *internal variables* that represent the degree of advancement of the various reactions.

A similar situation holds in the case of elastoplastic media, the focus of attention of this monograph. Whereas the theory of continuum thermodynamics in its standard form, as presented in Section 2.5, is quite adequate as a framework for the discussion of elasticity, and even of thermoelasticity, it is essential that hidden or internal variables be introduced in order that the theory may serve as a basis for the mathematical description of elastoplastic material behavior. The characteristic features of plasticity will be discussed at length in Chapter 3 and subsequent chapters. In this concluding section of Chapter 2 we extend the thermodynamic theory of Section 2.5 by presenting the theory of thermodynamics with internal variables in a form that will suffice as a basis for the theory of plasticity later. The fundamental references here are those of Coleman and Gurtin [34] and Halphen and Nguyen [71]; in addition, the survey article of Gurtin [64] is a good source for further details, as is the text by Lemaitre and Chaboche [106].

The first and second laws of thermodynamics remain valid in their earlier forms (2.42) and (2.45); here we are concerned with a constitutive theory that will be an extension of that for elastic materials presented earlier. As in that situation we specialize from the outset to isothermal processes in which the temperature is constant and there is no heat flux.

Then we consider materials for which the Helmholtz free energy and stress are given as functions of the strain *and* a set of m internal variables $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$, ..., $\boldsymbol{\xi}_m$. Some of these may be scalars and some tensors, depending on the application.

The constitutive equations are thus of the form

$$\psi = \psi(\boldsymbol{\epsilon}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m), \tag{2.62}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m). \tag{2.63}$$

Unlike the case of elasticity, in which historical effects are irrelevant, the above representations do not suffice for the case in which internal variables are present, and it is necessary to add to this pair of equations an *evolution equation* in which the rate of change of each of the $\boldsymbol{\xi}_i$ is given by an equation of the form

$$\dot{\boldsymbol{\xi}}_i = \boldsymbol{\beta}_i(\boldsymbol{\epsilon}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m), \quad 1 \le i \le m. \tag{2.64}$$

Later we will adopt a specialized form of (2.64), but for now it is important merely to note that such an equation is necessary to complete the description of constitutive behavior.

As in Section 2.5 we assume that all functions appearing in (2.62)–(2.64) are sufficiently smooth with respect to their arguments that as many derivatives as required may be taken.

By introducing (2.62) and (2.64) in the reduced dissipation inequality (2.45) we find that

$$\left(\frac{\partial \psi}{\partial \boldsymbol{\epsilon}} - \boldsymbol{\sigma} \right) : \dot{\boldsymbol{\epsilon}} + \frac{\partial \psi}{\partial \boldsymbol{\xi}_i} : \dot{\boldsymbol{\xi}}_i \le 0. \tag{2.65}$$

In view of the arbitrariness of the rate of change $\dot{\boldsymbol{\epsilon}}$ appearing in (2.65) we conclude that

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\epsilon}}. \tag{2.66}$$

We now introduce the *thermodynamic forces* $\boldsymbol{\chi}_i$ conjugate to $\boldsymbol{\xi}_i$; these are defined by

$$\boldsymbol{\chi}_i = -\frac{\partial \psi}{\partial \boldsymbol{\xi}_i}, \quad 1 \le i \le m. \tag{2.67}$$

Then, taking account of (2.66) we see that

$$\boldsymbol{\chi}_i : \dot{\boldsymbol{\xi}}_i \geq 0. \tag{2.68}$$

The inequality (2.68) will play a major role later in the construction of a constitutive theory for plastic materials. The left-hand side may be interpreted as a rate of dissipation due to those internal agencies modeled by the internal variables; indeed, we have here a quantity that is a scalar product of force-like variables ($\boldsymbol{\chi}_i$) with the rate of change of strain-like variables ($\boldsymbol{\xi}_i$). Under these circumstances (2.68) declares that the dissipation rate due to internal agencies is nonnegative.

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