Chapter 2

Analysis of Metric Spaces

Metric spaces were introduced and studied by the French mathematician, Maurice René Fréchet (in his doctoral dissertation published in 1906), and developed later by the German Felix Hausdorff (in his 1914 book *Grundzüge der Mengenlehre*). It was apparent that to the end of the nineteenth century the mathematical world (partly inspired by Cantor's fundamental work in set theory) was eager to structure more general sets than conventional $\mathbb{R}^n$. On the other hand, the needs of complex analysis and the rash development of differential equations accelerated this process. Typical examples are uniform convergence in function spaces, approximation of continuous functions by polynomials and the Riemann mapping theorem. After 1920, the theory of metric spaces, especially fundamental work on normed spaces and their applications to functional analysis, was further developed by the Pole Stefan Banach and his school. Paying tribute to their achievements and of their fellow countrymen followers, an important subclass of metric spaces was named “Polish.” A series of studies of metric spaces was further undertaken in the late 1920s by the Russian school of analysis. At this time, metric spaces have become generalized to topological spaces.

In this chapter we introduce the main principles of metric spaces and their special case: normed vector spaces. This part of analysis traditionally precedes the more general theory of topology and functional analysis.

1. DEFINITIONS AND NOTATIONS

The concept of "metric" (measuring distances in space) is at the root of mathematical (geometric) thinking. Starting with that concept we show how the concepts of limits of sequences and continuity of functions can be extended by metrization of spaces more general than Euclidean spaces introduced in calculus. Recall that a point x is a *limit of a sequence* $\{x_n\}$ if all terms of the sequence numbered with $k, k+1, \ldots$ for some k are sufficiently "close" to x. The closeness of these points to x was defined in terms of the Euclidean distance $|x - x_k|$, which determined the specific structure employed on $\mathbb{R}$. In many applications, an underlying carrier is often more general than $\mathbb{R}$ or even $\mathbb{R}^n$. So, the question arises, "how do we build the analysis in the general space?" Since the distance was crucial in the formation of analysis on the real line, we therefore introduce this notion also for the general space, emphasizing the main properties of the distance we have had experience with. Once a distance (or *metric*) between any two points of a given set is defined, the set becomes "well-structured" or *metrized*, and then is ranked as a *space*, more precisely, a *metric space*.

1.1 Definitions.

(i) Let X be a nonempty set. A *metric* d (or *distance*) on X is any nonnegative function $d: X^2 \to \mathbb{R}_+$ such that:

(a) $\forall x, y \in X,\ d(x,y) = 0 \Leftrightarrow x = y.$
(b) $\forall x, y \in X,\ d(x,y) = d(y,x).$
(c) $\forall x, y, z \in X, d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

The pair (X, d) is called a *metric space.* As usual, we refer to set X as a *carrier*. Sometimes, for brevity, carrier X itself is called the *metric space*.

(ii) If for $x, y \in X,\ x = y$ implies $d(x,y) = 0$, but the converse does not hold [i.e., $d(x,y) = 0$ does not yield $x = y$], and (b) and (c) hold, then d is called a *pseudo-metric*. Correspondingly, the pair (X, d) is called a *pseudo-metric space.*

Any pseudo-metric can be "made" a metric by introducing the equivalence classes generated by d in such a way that x and y will belong to one and the same class whenever $d(x,y)=0$. The corresponding metric space will then turn to $(X|\mathrm{E}_d, \boldsymbol{d})$, where $X|\mathrm{E}_d$ is the quotient set of X modulo E_d and $\boldsymbol{d}$ is the reducer or d (as per Theorem 4.4, Chapter 1). □

1.2 Remark. By the triangle inequality we have

$$d(x,y) - d(z,y) \le d(x,z), \tag{1.2a}$$

which holds for all $x, y, z \in X$. Then, interchanging x and z in the last inequality we arrive at

$$d(x,y) - d(z,y) \ge \; - d(x,z). \tag{1.2b}$$

Inequalities (1.2a) and (1.2b) yield

$$|d(x,y) - d(y,z)| \le \; d(x,z), \; \forall x, y, z \in X. \tag{1.2c}$$

□

Let $Y \subseteq X$. Then the pair (Y,d) is also a metric space, called a *subspace of* (X,d).

1.3 Examples (of metric spaces).

(i) The *discrete metric* is defined on a nonempty set X as

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y\,. \end{cases}$$

The triangle inequality does not hold if and only if $d(x,y)=1$ and $d(x,z)=d(z,y)=0$. However, the latter can only be possible when $x=z=y$. Hence, $d(x,y)$ cannot equal 1.

(ii) Let $X=(0,\infty)$ and $d(x,y)=\left|\frac{1}{x} - \frac{1}{y}\right|$. The triangle inequality follows from

$$d(x,y) = \left|\tfrac{1}{x} - \tfrac{1}{y}\right| \; = \; \left|\tfrac{1}{x} - \tfrac{1}{z} + \tfrac{1}{z} - \tfrac{1}{y}\right|$$

$$\leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| = d(x,z) + d(z,y)\,.$$

(iii) Let X consist of all sequences $\{x_n\} \subset \mathbb{R}$. Such a carrier X is denoted by $\mathbb{R}^{\mathbb{N}}$. Recall that a subset of $\mathbb{R}^{\mathbb{N}}$ is the l^1 *space* if it contains only absolutely convergent sequences, i.e., those with

$$\sum_{n=1}^{\infty} \mid x_n \mid \, < \infty.$$

Let us define the function d on l^1 as $d(x,y) = \sum_{n=1}^{\infty} \mid x_n - y_n \mid$. Then

$$d(x,y) = \sum_{n=1}^{\infty} \mid x_n - z_n + z_n - y_n \mid$$

$$\leq \sum_{n=1}^{\infty} \mid x_n - z_n \mid \, + \sum_{n=1}^{\infty} \mid z_n - y_n \mid$$

$$= d(x,z) + d(z,y).$$

Thus, d is a metric on l^1, because the other properties of d as a metric are obvious.

(iv) Let $\mathcal{C}_{[a,b]}$ denote the set of all continuous functions on interval $[a,b] \subseteq \mathbb{R}$. Let us define

$$d_\infty(x,y) = \sup\{|x(t) - y(t)| : t \in [a,b]\},$$

called the *supremum metric*. Because any continuous function on a closed and bounded interval assumes maximum and minimum values, the definition of d makes sense. The inequalities

$$|x(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)|$$

$$\leq \sup|x(t) - z(t)| + \sup|z(t) - y(t)|$$

hold for all $t \in [a,b]$. Therefore we have

$$\sup|x(t) - y(t)| \leq \sup|x(t) - z(t)| + \sup|z(t) - y(t)|,$$

which is exactly the triangle inequality. Property (a) of a metric obviously holds. Hence d is a metric on $\mathcal{C}_{[a,b]}$.

(v) Now, define another metric on $\mathcal{C}_{[a,b]}$:

$$d(x,y) = \int_a^b |x(t) - y(t)| dt.$$

It is easy to see that $d(x,y) = 0$ if and only if $x(t) = y(t)$ for all $t \in [a,b]$. (See Problem 1.11). The triangle inequality is obvious. $\square$

PROBLEMS

1.1 Let $X = \mathbb{R}$ and $d(x,y) = \sin^2(x-y)$. Is (X,d) a metric space?

1.2 Let $X = \mathbb{R}$ and $d(x,y) = \sqrt{|x-y|}$. Is (X,d) a metric space?

1.3 Let $X = \mathbb{R}^n$. Define on X, $d_\infty(x,y) = \max\{|x_k - y_k| : k = 1,\ldots,n\}$ $\forall x = (x_1,\ldots,x_n),\ y = (y_1,\ldots,y_n)$. Show that (X, d_∞) is a metric space.

1.4 Let d be a metric on X. Define

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}\ .$$

Show that ρ is a metric on X.

1.5 Two real numbers $p > 1$ and $q > 1$ are called *conjugate exponents,* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Show that for all $x, y \in \mathbb{R}_+$ and for conjugate exponents p and q, the following inequality holds.

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Hint: Work with the function $f(z) = \frac{z}{p} + \frac{1}{q} - z^{1/p}$ and then substitute $z = \frac{x^p}{y^q}$.

1.6 (a) Prove Hölder's inequality for finite sums: for conjugate exponents $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $a_1, \ldots, a_n \geq 0$, and $b_1, \ldots, b_n \geq 0$,

$$\sum_{i=1}^{n} a_i b_i \leq \left[\sum_{i=1}^{n} a_i^p\right]^{1/p} \left[\sum_{i=1}^{n} b_i^q\right]^{1/q}.$$

Hint : Apply Problem 1.5 to $x = a_i/A$ and $y = b_i/B$, where

$$A = \left[\sum_{i=1}^{n} a_i^p\right]^{1/p} \text{ and } B = \left[\sum_{i=1}^{n} b_i^q\right]^{1/q}.$$

(b) Generalize Hölder's inequality for infinite sums.

1.7 $a)$ Prove Minkowski's inequality (for finite sums): for $p \geq 1$, $a_1, \ldots, a_n \geq 0$, and $b_1, \ldots, b_n \geq 0$, it holds true that

$$\left[\sum_{i=1}^{n} (a_i + b_i)^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} a_i^p\right]^{1/p} + \left[\sum_{i=1}^{n} b_i^p\right]^{1/p}.$$

(*Hint:* Make use of $(a+b)^p = a(a+b)^{p-1} + b(a+b)^{p-1}$ and then apply Hölder's inequality.)

$b)$ Generalize Minkowski's inequality for infinite sums.

1.8 The *Euclidean metric* or *Euclidean distance* is defined in $\mathbb{R}^n$ by

$$d_e(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}, \tag{P1.8}$$

$$x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n). \tag{P1.8a}$$

[Specifically, if $n = 1$, we have $d(x, y) = \sqrt{(x - y)^2} = |x - y|$.] Show that d_e is indeed a metric (*Hint*: Apply Minkowski's inequality.)

1.9 In Problem 1.8 we defined the Euclidean metric on $\mathbb{R}^n$ by equation (P1.8). This metric can be regarded as

$$d_P(x, y) = \sqrt{\sum_{k=1}^{n} d_k(x_k, y_k)^2}\,, \tag{P1.9}$$

$$x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n), \tag{P1.9a}$$

where $d_k(x_k, y_k)$ is the one-dimensional Euclidean metric on the kth coordinate axis (kth factor space). We can extend this notion and define a metric on the n-times Cartesian product set $Y = Y_1 \times \cdots \times Y_n$ by formulas (P1.9 - P1.9a). This problem asserts that such d_p is indeed a metric on Y. We call this metric the *product metric* and the corresponding metric space (Y, d_P) the product space. In notation, $\times\,\{(Y_k,\, d_k) : k = 1, \ldots, n\}$.

Prove the statement: *Let* $(Y_k,\, d_k),\ k = 1, \ldots,\, n,$ *be a collection of metric spaces and let* Y *be the Cartesian product of* $Y_1, \ldots, Y_n$. *Then the function* d_P *on* $Y \times Y$ *defined by* (P1.9-P1.9a) *is a metric on* Y.

1.10 Show that the function $\rho(x, y) = \sum_{k=1}^{n} d_k(x_k, y_k)$ is also a metric on $Y = Y_1 \times Y_2 \times \cdots \times Y_n$.

1.11 In Example 1.3 (v), why is $d(x, y) = 0$ if and only if $x = y$?

NEW TERMS:

2. THE STRUCTURE OF METRIC SPACES

The structural properties of metric spaces stem from the notion of the open ball, with the aid of which we are be able to introduce open and closed sets, interior, closure, and accumulation points. Open balls, due to a particular metric, generate convergence and continuity, the principles of any analysis, which we explore in this chapter and Chapter 3.

2.1 Definition. Let (X, d) be a metric space and let $x \in X$ and $r > 0$. The subset of X,

$$B(x, r) = \{y \in X : d(x, y) < r\},$$

is called the *open ball centered at* x *with radius* r $($*with respect to metric* $d)$. [Whenever we need to emphasize that the ball is with respect to metric d, we write it as $B_d(x, r)$. This notation makes sense if X is endowed with more than one metric.] □

2.2 Examples.

(i) The open ball $B(x, r)$ in Euclidean space $(\mathbb{R}, d_e)$ is the open interval $(x - r,\, x + r)$.

(ii) The open ball $B(x, r)$ in Euclidean space $(\mathbb{R}^2, d_e)$ is the open disc centered at x with radius r in the usual sense.

(iii) Different metric choices on a given carrier give rise to different spaces and, as the result, to different shapes of open balls. Indeed, in metric spaces other than Euclidean, the shape of open balls may be quite surprising to our usual perception of them. Consider, for instance, an open ball $B(x, r)$ in $(\mathbb{R}^2, d)$, where d is the supremum metric defined as in Problem 1.3, for $n = 2$:

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It is easy to see that the open ball $B_\infty(x, r)$ is square shaped and that the Euclidean ball $B_e(x, r)$ is inscribed in this square (see Figure 2.1 below).

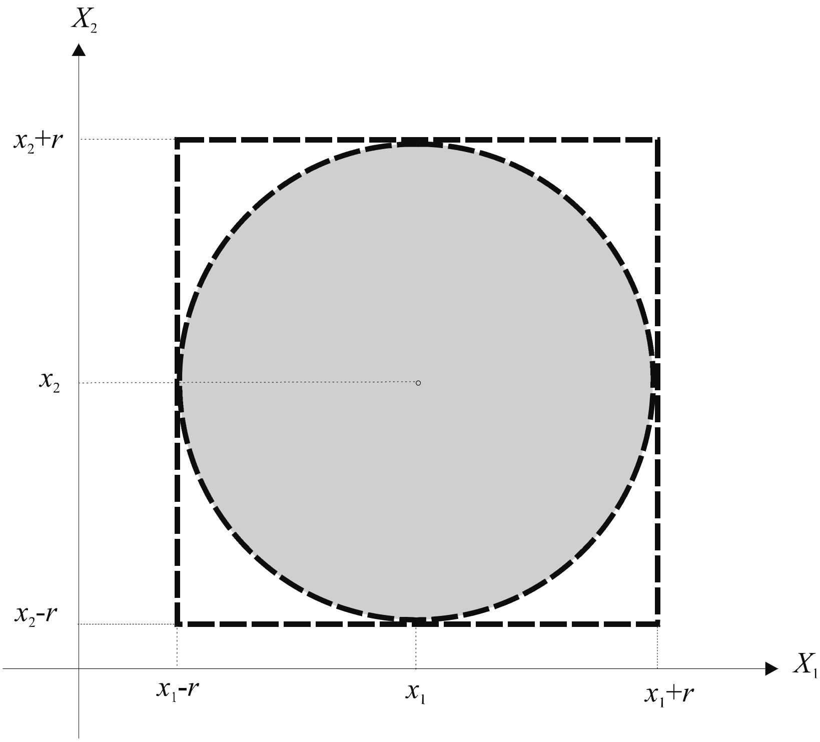

Figure 2.1

(iv) Let (X, d) be a discrete metric space with the metric defined in Example 1.3(i). Then, for any $x \in X$, an open ball centered at x is

$$B(x,r) = \begin{cases} \{x\}, & r \leq 1 \\ X, & r > 1. \end{cases}$$

Indeed, because $d(x, y)$ is either 0 or 1, for $r \leq 1$, $B(x,r) = \{y \in X : 0 = d(x,y) < 1\}$, whereas for $r > 1$, $B(x,r) = \{y \in X : d(x,y) = 1\}$.

(v) Let (X, d) be the metric space defined in Example 1.3(iv), where $X = \mathcal{C}_{[a,b]}$, and

$$d_\infty(x,y) = \sup\{|x(t) - y(t)|\colon t \in [a,b]\}.$$

Then the open ball $B_\infty(x, r)$ has a shape as depicted in Figure 2.2 below.

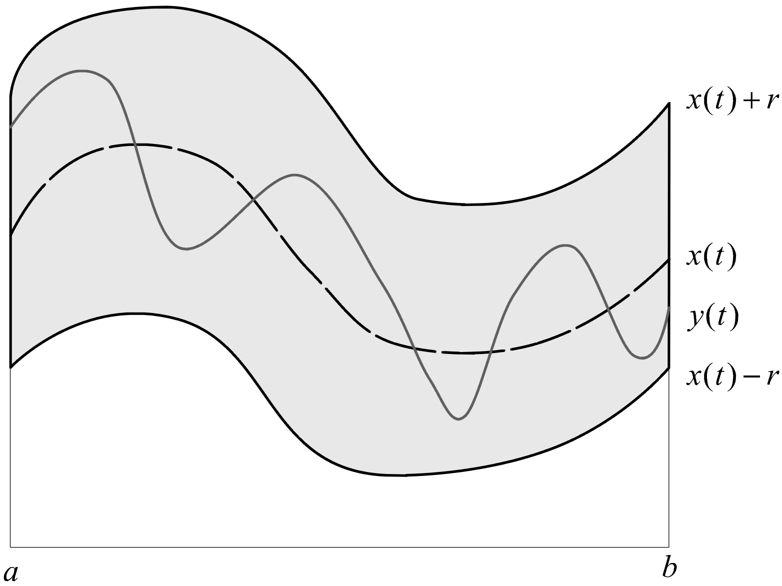

Figure 2.2

$\square$

2.3 Definition. Let (X, d) be a metric space. A subset A of X is called a d-*open set* (or just *open set*) if every point x of A can serve as the center of an open ball inscribed in A, that is, there is an $r > 0$ such that $B(x, r) \subseteq A$. $\square$

2.4 Examples.

(i) Every open ball is an open set itself. Indeed, if $x_1 \in B(x, r)$ then $r - d(x, x_1) > 0$. Take $r_1 = r - d(x, x_1)$ and show that $B(x_1, r_1) \subseteq B(x, r)$. For every $z \in B(x_1, r_1)$, by the triangle inequality,

$$d(x, z) \leq d(x, x_1) + d(x_1, z) < d(x, x_1) + r_1 = r.$$

Thus $z \in B(x, r)$ (see Figure 2.3 below).

(ii) The set $[a, b)$, for $a < b$, in $(\mathbb{R}, d_e)$ is not open, because there is no open ball $B(a, r) \subseteq [a, b)$.

(iii) Any carrier X is obviously open.

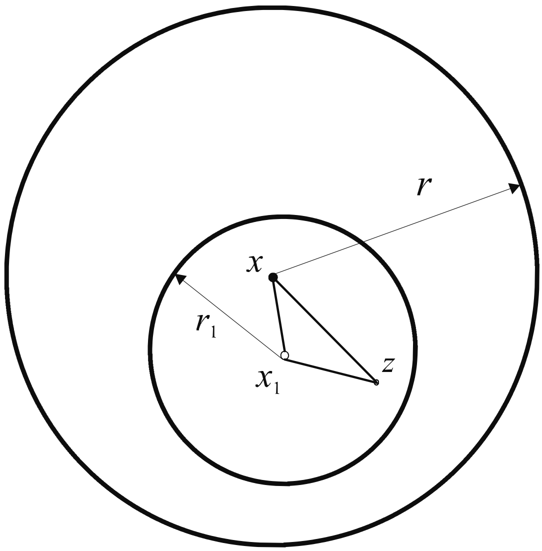

Figure 2.3

(iv) A set A is not open if there is at least one point $x \in A$ such that there is no ball $B(x, r)$ that can be inscribed in A. The empty set does not have any point. Thus it is reasonable to assign it to the class of open sets. In other words, we define Ø as an open set.

(v) In the Euclidean space $(\mathbb{R}, d_e)$, $\mathbb{R}$ is an open set but not an open ball (why?). $\square$

2.5 Theorem. *For every metric space* (X, d), *the following statements hold true.*

(i) *Arbitrary unions of open sets are open sets.*

(ii) *Finite intersections of open sets are open sets.*

Proof.

(i) Let $\{A_k : k \in I\}$ be an indexed family of open sets in X and let $A = \bigcup_{k \in I} A_k$. If $x \in A$, then there is an i such that $x \in A_i$. Because A_i is open, there is an $r > 0$ such that

$$B(x,r) \subseteq A_i \subseteq \bigcup_{k \in I} A_k .$$

Therefore, A is open.

(ii) Let $A_1, \ldots, A_n$ be open subsets of X and let $A = \bigcap_{k=1}^{n} A_k$. If $x \in A$ then $x \in A_k,\ k = 1, \ldots, n$. It follows that there are $r_1, \ldots, r_n$ such that $B(x, r_k) \subseteq A_k,\ k = 1, \ldots, n$. Let $r = \min\{r_1, \ldots, r_n\}$. Then, obviously, $B(x,r) \neq \emptyset$ and $B(x,r) \subseteq A_k,\ k = 1, \ldots, n$. Thus, $B(x,r) \subseteq A$ and A is open. ☐

2.6 Remark. The intersection of more than finitely many open sets need not be open. The reason is that because $r = \min\{r_k : k \in I\}$ can be zero. For example, let

$$A_n = (1 - \tfrac{1}{n}, 1 + \tfrac{1}{n}) \text{ in } (\mathbb{R}, d_e).$$

Then $1 \in A_n,\ n = 1, 2, \ldots,$ which implies that $1 \in \bigcap_{n=1}^{\infty} A_n$ and hence $\{1\} = \bigcap_{n=1}^{\infty} A_n$. However, the set $\{1\}$ is not open in $(\mathbb{R}, d_e)$. ☐

2.7 Example. Let (X,d) be a discrete metric space. Then the power set $\mathcal{P}(X)$ coincides with the set of all open sets. Indeed, in Example 2.2 (iv), we showed that in any discrete metric space, every singleton $\{x\}$ and the carrier X are open balls. In addition, Ø is an open set. Any subset A of X can be represented as the union of all points of A. Thus by Theorem 2.5(i), it follows that A is also open. Specifically, in $\mathbb{R}$ endowed with the discrete metric, all singletons are open, whereas in Euclidean space $(\mathbb{R}, d_e)$ they are not. ☐

2.8 Definitions.

(i) A point $x \in A \subseteq X$ is called an *interior point* of A if there exists an open ball $B(x,r) \subseteq A$. The set of all interior points of set A is denoted by $\mathring{A}$ or *Int*(A) and called the *interior* of A.

(Clearly, $\overset{\circ}{A}$ is the largest open subset of A, which yields that A is open if and only if $A = \overset{\circ}{A}$. Indeed, let $C \subseteq A$ be an open set, larger than $\overset{\circ}{A}$. Then, there is an $x \in C$ such that $x \notin \overset{\circ}{A}$. But this is a contradiction, because x must be an interior point of A.)

(ii) A subset A of X is called *closed* if its complement A^c is open. [Specifically, the carrier X and the empty set Ø are both closed. Any singleton is obviously closed.] □

2.9 Proposition. *Arbitrary intersections or finite unions of closed sets are closed sets.*

Proof. The statements follow from Theorem 2.5 by applying De Morgan's laws. □

2.10 Example. Because the set of all open subsets of a discrete metric space (X, d) coincides with its power set, the set of all closed subsets is also the power set. Particularly, in a discrete metric space all subsets are simultaneously open and closed. □

2.11 Definitions.

(i) A point $x \in X$ is called a *closure point* of $A \subseteq X$ if every open ball centered at x contains at least one element of A. We also say, "if every open ball centered at x *meets* A." The set of all closure points of A is denoted by $\overline{A}$ or by *Cl*(A) and called the *closure* of A. [From Definition 2.8 (i) it follows that $A \subseteq \overline{A}$.]

(ii) There are two types of closures points. Type 1: x is an *isolated closure point of* A if there is a ball $B(x, r)$ that does not contain any points of A, other than x. In other words, $B(x, r) \cap (A \backslash \{x\}) = Ø$. Clearly, an isolated closure point must belong to A. Type 2: x is an *accumulation point*, if it is not an isolated point. That means, for each $r > 0$, $B(x, r) \cap (A \backslash \{x\}) \neq Ø$. Unlike an isolated closure point, an accumulation point need not belong to A.

The set of all accumulation points of A is called a *derived set*, in notation A'. From the above definition it follows that $A' \subseteq \overline{A}$.

(iii) A point $x \in X$ is called a *boundary point* of A if every open ball at x contains points from A and from A^c. The set of all boundary points of A is called the *boundary* of A and is denoted by ∂A. □

2.12 Examples.

(i) In $(\mathbb{R}, d_e)$, let $A = (0, 2) \cup \{5\}$. $\{5\}$ is an isolated closure point, because $B(5, r)$ contains only $\{5\}$ for all $r \leq 3$. Thus, $\overline{A} = [0, 2] \cup \{5\}$, and $A' = [0, 2]$.

(ii) Note that $\partial A = \partial A^c = \overline{A} \cap \overline{A^c}$. (See Problem 2.1.) Therefore, ∂A (as follows from Proposition 2.13 below) is a closed set.

(iii) In the next section we learn that the closure $\overline{\mathbb{Q}}$ of all rational numbers in $(\mathbb{R}, d_e)$ is entire $\mathbb{R}$. So is the closure $\overline{\mathbb{Q}^c}$ of irrational numbers. Consequently, by (ii), $\partial \mathbb{Q} = \mathbb{R}$, which is quite surprising. [Note that $\partial \mathbb{Q} = \mathbb{R}$ also directly follows from Definition 2.11(iii).]

(iv) The boundaries of X and Ø are the empty sets.

(v) Let $A = [0, 1) \cup \{2\}$. Then, $\mathring{A} = (0, 1)$, $\overline{A} = [0, 1] \cup \{2\}$, $A' = [0, 1]$, $\partial A = \{0, 1, 2\}$ (because $A^c = (-\infty, 0) \cup [1, 2) \cup (2, \infty)$, $\overline{A^c} = (-\infty, 0] \cup [1, \infty)$, and $\overline{A} \cap \overline{A^c} = \{0, 1, 2\}$).

(vi) Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \subseteq (\mathbb{R}, d_e)$. Because 0 is the limit of the sequence $\{\frac{1}{n}\}$ (in terms of Euclidean distance), it is also an accumulation point of A. Any open ball at 0 contains at least one point of A. This is the only accumulation point of A. By the way, A is not closed, for 0 is a closure point of A. So we have $A' = \{0\}$, and $\overline{A} = A \cup \{0\}$. □

2.13 Proposition. *For any subset A of X, $\overline{A}$ is a closed set and furthermore, $\overline{A}$ is the smallest closed superset of A.*

Proof.

(i) We show first that $\overline{A}$ is a closed set, that is, $(Cl(A))^c$ is open. Let $x \in (Cl(A))^c$. Then there exists an open ball $B(x, r)$ such that $B(x, r) \cap A = $ Ø (because, otherwise, x would belong to $\overline{A}$ by the definition). However, we have not proved yet that $B(x, r) \cap \overline{A} = $ Ø, which would immediately imply that $(Cl(A))^c$ is open. Now we show that no point of $B(x, r)$ is a closure point of A. Take an arbitrary point $t \in B(x, r)$. Because $B(x, r)$ is an open set, there is an open ball $B(t, r_t) \subseteq B(x, r)$ also disjoint from A. By the definition of a closure point, this means that $t \notin \overline{A}$. In as much as t was an arbitrary point of $B(x,r)$, $B(x,r) \subseteq (Cl(A))^c$.

(ii) Now we show that the closure of A is the smallest closed set containing A. Let B be an arbitrary closed set such that $A \subseteq B$. We prove that $B^c \subseteq (\overline{A})^c$. Because B^c is open, for each $x \in B^c$, there is an open ball $B(x,r) \subseteq B^c$. This implies that $B(x,r) \cap B = \emptyset$ and that $B(x,r) \cap A = \emptyset$.

Thus $x \notin \overline{A}$ (by the definition of a closure point), which is equivalent to $x \in (Cl(A))^c$. Therefore, we have proved that $x \in B^c$ yields that $x \in (Cl(A))^c$, that is, $B^c \subseteq (Cl(A))^c$. The latter is obviously equivalent to $\overline{A} \subseteq B$. □

2.14 Corollary. *A set A is closed if and only if $A = \overline{A}$.*

(See Problem 2.15.)

2.15 Remark. Consider the set $C = C(x,r) = \{y \in X : d(x,y) \leq r\}$. It can be easily shown that C is a closed set. (See Problem 2.4.) Such a C is called a *closed ball centered at x with radius r*. Evidently, $B(x,r) \subseteq C(x,r)$ implies that the closure of an open ball is a subset of C [i.e., $\overline{B(x,r)} \subseteq C(x,r)$], because $\overline{B}$ is the smallest closed set containing B. However, $C(x,r)$ need not coincide with the closure of the corresponding open ball $B(x,r)$. For instance, let (X,d) be a discrete metric space, where any open ball is both a closed and open set: $B(x,r) = \overline{B(x,r)}$. Because

$$C(x,r) = \begin{cases} \{x\}, & r < 1 \\ X, & r \geq 1, \end{cases}$$

we have $B(x,r) = C(x,r) = X$ for $r > 1$ or $B(x,r) = C(x,r) = \{x\}$ for $r < 1$. For $r = 1$, $\overline{B(x,r)} = \{x\} \subset C(x,r) = X$, unless X is a singleton. □

2.16 Example. The set of all rational numbers $\mathbb{Q}$ is neither open nor closed. Indeed, it is known that each irrational point x is a limit of a sequence of rational points $\{x_n\}$. Therefore, if x is irrational, there is no open ball $B(x,r)$ that does not contain rational points. This implies that $\mathbb{Q}^c$ is not open either, or equivalently, $\mathbb{Q}$ is not closed. On the other hand, $\mathbb{Q}$ cannot be open, because otherwise, every rational point q could be the center of an open ball (interval) containing just rational numbers. This is absurd, in as much as any interval is a continuum. Therefore, the set of all rational numbers is neither open nor closed. It also follows that the set of all irrational numbers is neither open nor closed. $\square$

In Problem 1.9, we introduced a product metric. We wonder how open sets can look like in a product metric space. A remarkable property of such a metric is given by the following theorem to be proved in Chapter 3 for the general topological product spaces.

2.17 Theorem. *Let* $\{(Y_k, d_k) : k = 1, \ldots, n\}$ *be a finite family of metric spaces and let* $(Y, d) = \times\{(Y_k, d_k) : k = 1, \ldots, n\}$ *be the product space. Then* $O \subseteq (Y, d)$ *is open if and only if* O *is the union of sets of the form* $\prod_{i=1}^{n} O_i$, where each O_i is open in (Y_i, d_i).

PROBLEMS

2.1 Show that $\partial A = \partial A^c = \overline{A} \cap \overline{A^c}$.

2.2 Is it true that $A \subseteq B$ yields

(a) $\overline{A} \subseteq B$?

(b) $\overline{A} \subseteq \overline{B}$?

2.3 Show that $[\overline{A^c}]^c \subseteq \overline{A}$.

2.4 Prove that a closed ball $C(x, r)$ is a closed set.

2.5 If $x \in \partial A$, must x be an accumulation point?

2.6 Show that $\overline{A} = A \cup A'$.

2.7 Let $A \subseteq (X, d)$, where X is an infinite set. Show that if x is an accumulation point of A, then every open set containing x contains infinitely many points of A.

2.8 Give an example of a continuum closed set that does not have any accumulation point.

2.9 Describe the shape of open balls in the metric space (X, d) introduced in Example 1.3(ii).

2.10 Show that set $[1, \infty)$ is closed in the metric space of Example 1.3(ii).

2.11 Prove that $\overline{A} = \mathring{A} + \partial A$.

2.12 Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2.13 Show that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

2.14 Show that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

2.15 Prove Corollary 2.14: *A set A is closed if and only if $A = \overline{A}$.*

2.16 Show that in any discrete metric space, the boundary of any subset is empty.

2.17 Define $S(x, r) = \{y \in X : d(x, y) = r\}$ and call it a *sphere centered at x with radius* $r\,(r > 0)$. In Euclidean space $(\mathbb{R}^n, d_e)$, obviously, $S(x, r) = \partial B(x, r)$. Is this true for metric spaces? If it is not, then is there at least a relationship between $S(x, r)$ and $\partial B(x, r)$ common for all metric spaces? Are there any exceptional cases? Explain.

2.18 Show that set $(0, 1]$ is closed in the metric space of Example 1.3 (ii).

2.19 Let O be an open set in $(\mathbb{R}^n, d_e)$. Show that for each $x \in O$, there is an open ball $B(q, r) \ni x$, such that $q \in \mathbb{Q}^n$ (the set of all vectors in $\mathbb{R}^n$ with rational coordinates) and $r \in \mathbb{Q}$.

NEW TERMS:

open ball $B(x, r)$, 101
radius of an open ball, 101
supremum metric,101, 102
open ball with respect to the Euclidean metric, 102
open ball with respect to the supremum metric, 102, 103
open (d-open) set, 103
interior point, 105
interior $\mathring{A}$ or *Int*(A) of set A, 105
closed set, 106
closure point, 106
closure $\overline{A}$ or $Cl(A)$ of set A, 106
isolated closure point, 106
accumulation point, 106
derived set (A'), 107
boundary point, 107
boundary ∂A of set A, 107
closed ball $C(x, r)$, 108
sphere $S(x, r)$, 110

3. CONVERGENCE IN METRIC SPACES

This section introduces the reader to one of the central notions in the analysis of metric spaces: convergence. Among different things, we discuss the relation between limit and closure points.

3.1 Definitions.

(i) A function $[\mathbb{N}, X, f]$ is called a *sequence*; its most commonly used notation is $\{x_n\} = f$, with $x_n = f(n)$. Let $\{x_n\} \subseteq (X, d)$ be a sequence and let $\varepsilon > 0$ and N be a positive integer. A subsequence $T_N = \{x_N, x_{N+1}, \dots\}$ is called an $N(\varepsilon)$-*tail* of $\{x_n\}$ if for any $x_m, x_n \in T_N$, $d(x_m, x_n) < \varepsilon$. The sequence $\{x_n\}$ is called a *Cauchy sequence*, in notation

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0,$$

if for every $\varepsilon > 0,$ there is an $N(\varepsilon)$-tail of $\{x_n\}$.

(ii) Let $x \in X$. Any $N(\varepsilon)$-tail, T_N, is called an $N(x, \varepsilon)$-*tail of* $\{x_n\}$ if $T_N \subseteq B(x, \varepsilon)$. The sequence $\{x_n\}$ is said to *converge* to a point $x \in X$ if for every $\varepsilon > 0,$ there is an $N(x, \varepsilon)$-tail. In notation,

$$\lim_{n\to\infty} d(x_n, x) = 0$$

(also $d\text{-}\lim_{n\to\infty} x_n = x$ or just $x_n \to x$). x is called a *limit point* of the sequence $\{x_n\}$. A sequence is *convergent* if it is convergent to at least one limit point that belongs to X.

(iii) A point x is said to be a *sequential limit point of a set* A if there is a sequence $\{x_n\} \subseteq A$ convergent to x.

(iv) A set A in a metric space (X, d) is called *complete* if every Cauchy sequence in A is convergent in A. Consequently, the metric space (X, d) is complete if so is X.

(v) A sequence $\{x_n\}$ is called *bounded* if for every n, $d(x_1, x_n) \leq M,$ where M is a positive real number. $\square$

3.2 Remark. A sequence in a metric space can have at most one limit point. Indeed, let x, y be limits of a sequence $\{x_n\} \subseteq (X, d)$

and let $\varepsilon > 0$ be arbitrary. Then, there are $N_1(x, \varepsilon/2)$ and $N_2(y, \varepsilon/2)$ tails such that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

for all $n \geq N = \max\{N_1, N_2\}$ [i.e., $d(x, y)$ can be made arbitrarily small]. Thus, $x = y$. $\square$

3.3 Theorem. *Let $A \subseteq (X, d)$. Then a point x is a closure point of a set A if and only if x is a sequential limit point of A (i.e., there is a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$).*

Proof.

(i) Let x be a closure point of A. If $x \in A$ then the proof becomes trivial (if we set $x_n = x,\ n = 1, 2, \dots$). Let $x \in \overline{A} \backslash A$. By the definition of a closure point, every open ball $B(x, r)$ meets A. Thus for every n, there is a point, $x_n \in A \cap B(x, \frac{1}{n})$, so that $d(x, x_n) < \frac{1}{n}$. Therefore, $\{x_n\}$ is a pledged sequence convergent to x.

(ii) Let $\{x_n\} \subseteq A$ such that $\lim_{n \to \infty} x_n = x$. We prove that $x \in \overline{A}$. The convergence implies that for every $\varepsilon > 0$, there is a $T_N = \{x_N, x_{N+1}, \dots\}$ such that $T_N \subseteq B(x, \varepsilon)$ and $T_N \subseteq A$. Thus, $\forall \varepsilon > 0,\ B(x, \varepsilon) \cap A \neq \emptyset$, which yields that $x \in \overline{A}$.

Part (i) implies that if A is an infinite set and $x \in A' \backslash A \neq \emptyset$, i.e., x is an accumulation point of A, then there is a sequence $\{x_n\}$ with all distinct terms such that $x_n \to x$.) $\square$

3.4 Corollary. *A subset A of a metric space (X, d) is closed if and only if it contains all of its sequential limit points.*

Proof.

(i) Let A be closed and let $\{x_n\} \subseteq A$ be a sequence convergent to a point x. Then, by Theorem 3.3, x is a sequential limit point and a closure point of A at the same time:

$$\lim_{n \to \infty} x_n = x \in \overline{A}.$$

Since A is closed, $A = \overline{A}$ and $x \in A$. Thus, A contains all of its sequential limit points.

(ii) Suppose A contains all of its sequential limit points and let $x \in \overline{A}$. Then, by Theorem 3.3, there is a sequence $\{x_n\} \subseteq A$ such that $\lim_{n\to\infty} x_n = x$. By our assumption, x also belongs to A or, equivalently, $\overline{A} \subseteq A$ implying that $A = \overline{A}$ and hence A is closed. □

3.5 Definitions.

(i) A subset $A \subseteq (X, d)$ is called *dense* in X, if $\overline{A} = X$. (By Theorem 3.3, A is dense in X if and only if the set of all limit points of A coincides with X, or, in other words, if and only if for every $x \in X$, there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$.)

(ii) A set $A \subseteq (X, d)$ is called *nowhere dense* if its closure has the empty set as its interior, that is, if *Int*(*Cl*$(A)) = \varnothing$. □

3.6 Examples.

(i) Because each irrational number can be represented as the limit of a sequence of rational numbers, $\mathbb{Q}$ is dense in $\mathbb{R}$ (in terms of the Euclidean metric).

(ii) Let $A = \{1, 5, 10\} \subseteq (\mathbb{R}, d_e)$. Then A is nowhere dense.

(iii) $\{\frac{1}{n} : n = 1, 2, \dots\}$ is nowhere dense in $(\mathbb{R}, d_e)$.

PROBLEMS

3.1 Show that every convergent sequence is a Cauchy sequence. Give an example when the converse is not true.

3.2 Prove that a set $A \subseteq (X, d)$ is nowhere dense in X if and only if the complement of its closure is dense in X.

3.3 Assuming that $(\mathbb{R}, d_e)$ is complete (a known fact from calculus), prove that $(\mathbb{R}^n, d_e)$ is also complete.

3.4 Show that any Cauchy sequence is bounded.

3.5 Show that in a discrete metric space any convergent sequence has at most finitely many distinct terms.

3.6 Show that any discrete metric space is complete.

3.7 Show that if $\{x_n\} \subseteq (X, d)$ is a Cauchy sequence and $\{x_{n_k}\}$ is a subsequence convergent to a point $a \in X$, then $x_n \to a$.

3.8 Show that in $(\mathbb{R}^n, d_e)$, $\overline{B(x, r)} = C(x, r)$, where $C(x, r)$ is a closed ball of Remark 2.15.

NEW TERMS:

sequence, 112
$N(x,\varepsilon)$-tail, 112
Cauchy sequence, 112
convergent sequence, 112
limit point of a sequence, 112
sequential limit point of a set, 112
complete metric space, 112
bounded sequence, 112
dense set, 114
nowhere dense set, 114

4. CONTINUOUS MAPPINGS IN METRIC SPACES

4.1 Definition. Let (X, d) and (Y, ρ) be two metric spaces. A function $f : (X, d) \to (Y, \rho)$ is called *continuous at a point* $x_0 \in X$ if for each $\varepsilon > 0,$ there is $\delta > 0$ such that $\rho(f(x), f(x_0)) < \varepsilon$ for all x with $d(x, x_0) < \delta$. Function f is called *continuous on* X or simply *continuous* if f is continuous at every point of X. □

4.2 Remark. (The anatomy of continuity.) Since for any function, $x_0 \in f^*\Big(\{f(x_0)\}\Big)$, we always have $x_0 \in f^*(B_\rho(f(x_0), \varepsilon))$ [the inverse image of an open ball in Y centered at $f(x_0)$], the key question is whether x_0 is an interior point of $f^*(B_\rho(f(x_0), \varepsilon))$ for any $\varepsilon > 0$. We show that f is continuous at x_0 if and only if the inverse image under f^* of any open ball centered at $f(x_0)$ contains x_0 as an interior point. (See Figure 4.1.)

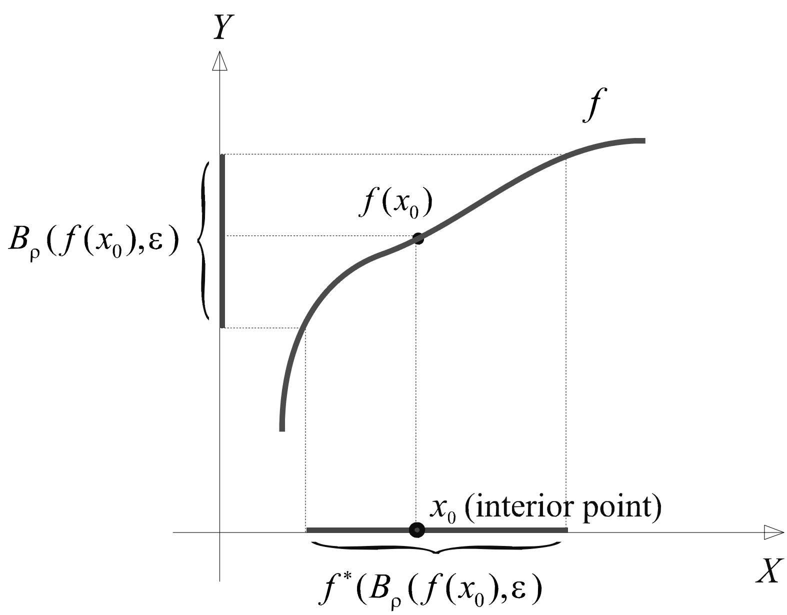

Figure 4.1

If x_0 is an interior point of $f^*(B_\rho(f(x_0),\varepsilon))$, as conjectured, there must be an open ball $B_d(x_0,\delta) \subseteq f^*(B_\rho(f(x_0),\varepsilon))$. It implies that: 1) such a positive δ exists, and 2) the image of $B_d(x_0,\delta)$ under f_* is a subset of $B_\rho(f(x_0),\varepsilon)$ (see arguments in Theorem 4.3 below), which guarantees that $\rho(f(x),f(x_0)) < \varepsilon$ for all x with $d(x,x_0) < \delta$.

Now, if f is not continuous at x_0, as depicted in Figure 4.2, x_0 need not be an interior point of $f^*(B_\rho(f(x_0),\varepsilon))$. If this is the case, no ball $B_d(x_0,\delta)$ can be inscribed in $f^*(B_\rho(f(x_0),\varepsilon))$ and consequently, no positive δ exists to warrant that $\rho(f(x),f(x_0))$ is less than ε for all x with $d(x,x_0) < \delta$.

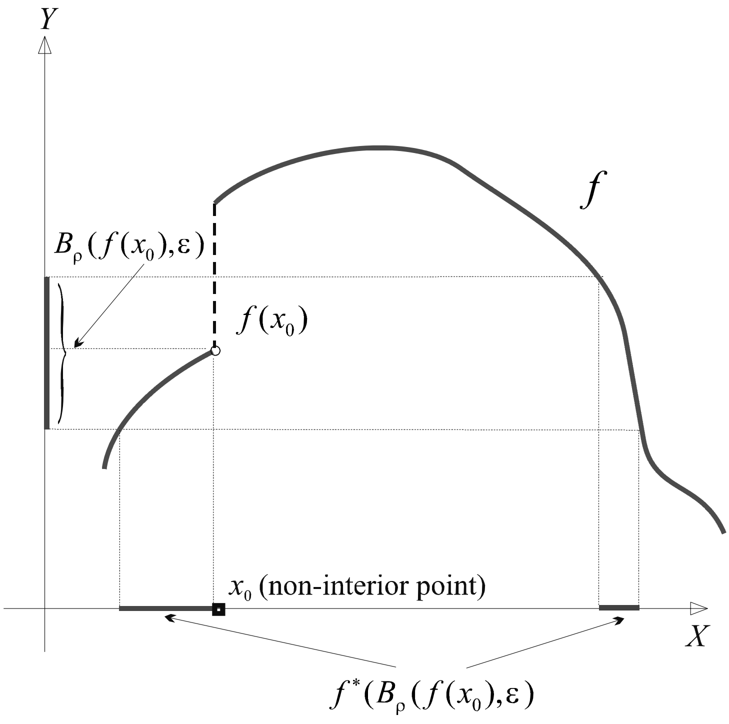

Figure 4.2

On the other hand, the reader should not jump to a quick conclusion that Figure 4.2 (which serves for a mere illustration) necessarily represents a discontinuous function. For instance, if X is endowed with the discrete metric, then x_0 is an interior point of the set $f^*(B_\rho(f(x_0),\varepsilon))$, because $\{x_0\}$ is an open ball with any radius $r \le 1$. Furthermore, $f^*(B_\rho(f(x_0),\varepsilon))$ is an open set in X for all $\varepsilon > 0$. □

The following theorem elaborates the above discussion and also generalizes the principle of continuity.

4.3 Theorem. *A function* $f:(X,d) \to (Y,\rho)$ *is continuous if and only if the inverse image of any open set in* (Y,ρ) *under* f *is open in* (X,d).

Proof.

1) According to the blueprint in Remark 4.2, we begin the proof by showing the validity of the following assertion.

> f *is continuous at* x_0 *if and only if* x_0 *is an interior point of the inverse image under* f^* *of any open ball* $B_\rho(f(x_0),\varepsilon)$.

Let x_0 be an interior point of $f^*(B_\rho(f(x_0),\varepsilon))$. Then, there is an open ball

$$B_d(x_0,\delta) \subseteq f^*(B_\rho(f(x_0),\varepsilon)),$$

and because inclusions are preserved under f_*, by Problems 3.6(a) and 2.6 of Chapter 1,

$$f_*(B_d(x_0,\delta)) \subseteq f_*(f^*(B_\rho(f(x_0),\varepsilon))) \subseteq B_\rho(f(x_0),\varepsilon),$$

which yields continuity of f at x_0, as per Remark 4.2.

Now, let f be continuous at x_0. Then, the inclusion $f_*(B_d(x_0,\delta)) \subseteq B_\rho(f(x_0),\varepsilon)$ holds, which, along with Problem 2.5, Chapter 1, leads to the following sequence of inclusions.

$$B_d(x_0,\delta) \subseteq f^*(f_*(B_d(x_0,\delta))) \subseteq f^*(B_\rho(f(x_0),\varepsilon)).$$

Because x_0 is the center of $B_d(x_0,\delta)$, it is an interior point of this ball and, due to the last inclusion, an interior point of the set $f^*(B_\rho(f(x_0),\varepsilon))$.

2) Suppose f is continuous on X. We show that for each open set $O \subseteq Y,$ $f^*(O)$ is open in (X, d). Pick a point $x_0 \in f^*(O)$. Then, $f(x_0) \in f_*(f^*(O)) \subseteq O$ and, because O is open, $f(x_0)$ is its interior point. Thus, O is a superset of the open ball $B_\rho(f(x_0), \varepsilon),$ for some $\varepsilon,$ and consequently,

$$f^*(B_\rho(f(x_0), \varepsilon)) \subseteq f^*(O). \tag{4.3}$$

Because f is continuous at $x_0,$ by assertion 1), x_0 must be an interior point of $f^*(B_\rho(f(x_0), \varepsilon)),$ and, by (4.3), an interior point of $f^*(O)$. Thus, $f^*(O)$ is open.

3) Let $f^*(O)$ be open in (X, d) for every open subset O of Y. Take $x_0 \in X$ and construct an open ball $B_\rho(f(x_0), \varepsilon)$. By our assumption, the set $f^*(B_\rho(f(x_0), \varepsilon))$ is open in (X, d). Because $f(x_0) \in B_\rho(f(x_0), \varepsilon),$ we have that

$$x_0 \in f^*(\{f(x_0)\}) \subseteq f^*(B_\rho(f(x_0), \varepsilon))$$

and, therefore, $x_0 \in f^*(B_\rho(f(x_0), \varepsilon))$ and it is an interior point of $f^*(B_\rho(f(x_0), \varepsilon))$. By 1), f must then be continuous at x_0. □

There is yet another useful criterion of continuity, known as Heine's criterion (after the German mathematician Heinrich Eduard Heine, 1821 – 1861).

4.4 Theorem (H.E. Heine). *A function $f\colon (X, d) \to (Y, \rho)$ is continuous at $x \in X$ if and only if for every sequence $\{x_n\},$ d-convergent to $x,$ its image sequence $\{f(x_n)\}$ is ρ-convergent to $f(x)$.*

We prove a version of this theorem for a more general case in Chapter 3 (Theorems 4.9 and 4.10).

4.5 Definition. Let (X, d) be a metric space and $\tau(d)$ be the collection of all open subsets of X with respect to metric d. Then $\tau(d)$ (or just τ) is said to be the *topology on X generated by d.* □

Theorem 4.3 can now be rephrased as follows.

4.6 Theorem. *Let $f\colon (X, d) \to (Y, \rho)$ be a function and let $\tau(d)$ and $\tau(\rho)$ be the topologies generated by metrics d and $\rho,$*

respectively. Then f is continuous on X if and only if $f^{**}(\tau(\rho)) \subseteq \tau(d)$ [i.e., $\forall O \in \tau(\rho),\ f^*(O) \in \tau(d)$]. □

4.7 Example. Let $f\colon\ (\mathbb{R}, d) \to (\mathbb{R}, d_e)$ be the *Dirichlet function* defined as $f = \mathbf{1}_{\mathbb{Q}}$, where $\mathbb{Q}$ is the set of rational numbers. If $d = d_e$ is the Euclidean metric then f is discontinuous at every point. If d is the discrete metric, by Theorem 4.3, f is continuous on $\mathbb{R}$, because the inverse image of any open set in $(\mathbb{R}, d_e)$ under f is clearly an element of the power set coinciding with the *discrete topology* generated by the discrete metric (see Example 2.7). □

We are further interested in the conditions under which two different metrics on X generate one and the same topology. This property of metrics satisfies an equivalence relation on the set of all topologies on X and hence is regarded as equivalence of metrics.

4.8 Definition. Two metrics d_1 and d_2 on X are called *equivalent* if $\tau(d_1) = \tau(d_2)$ (in notation $d_1 \approx d_2$). □

According to Definition 4.8, all metrics sharing one and the same topology τ on X form one equivalence class, say $[d]_\tau$. Therefore, if $\mathfrak{D}_X$ is the set of all metrics on X, then $\mathfrak{D}_X|\mathrm{E}_\tau$ is the quotient set of $\mathfrak{D}_X$ modulo E_τ.

4.9 Remark. Let (X, d_1) and (X, d_2) be two metric spaces and let $f\colon\ (X, d_1) \to (X, d_2)$ be the identity map $(f(x) = x,\ x \in X)$. If d_1 and d_2 are equivalent [i.e., $\tau(d_1) = \tau(d_2)$], then for every open set O in (X, d_2), $f^*(O) = O \in \tau(d_1)$. Thus f is continuous on X. According to Heine's Theorem 4.4, this is equivalent to the statement that

$$\lim_{n \to \infty} d_1(x_n, x) = 0$$

implying that

$$\lim_{n \to \infty} d_2(f(x_n), f(x)) = \lim_{n \to \infty} d_2(x_n, x) = 0.$$

In a nutshell, assuming that

(i) $\tau(d_1) = \tau(d_2)$

we showed that

$$(ii)\ \ \lim_{n\to\infty} d_1(x_n, x) = 0 \ \Leftrightarrow\ \lim_{n\to\infty} d_2(x_n, x) = 0.$$

By Heine's theorem, it follows that the converse is also true: statement (ii) implies statement (i). Hence, we may call two metrics $\tau(d_1)$ and $\tau(d_2)$ on X equivalent if (i) or (ii) holds.

In short, two metrics d_1 and d_2 on X are equivalent if and only if any sequence $\{x_n\} \subseteq X$ convergent in d_1 is convergent in d_2 and vice versa. □

From Theorem 4.3, it also follows that the identity map in Remark 4.9 is continuous "under equivalent metrics." However, an identity map need not be continuous under d_1 and d_2 if they are not equivalent.

4.10 Definitions.

(i) Let A be a subset in a metric space (X, d). The real number or infinity

$$d(A) = \sup\{d(x, y) \colon x, y \in A\}$$

is called the *diameter* of A. The set A is called *d-bounded* or just *bounded* if $d(A) < \infty$. Particularly, the metric space (X, d) (or just metric d) is called *bounded* if so is X. Set A is said to be *unbounded* if $d(A) = \infty$.

(ii) A subset A in a metric space (X, d) is called *totally bounded* if for every $\varepsilon > 0$, A can be covered by finitely many ε-balls (i.e., balls with common radius ε). □

4.11 Example. According to Problem 1.4, the function

$$\rho(x, y) = \frac{d(x,y)}{1 + d(x,y)}$$

defined on a metric space (X, d) is a metric on X. Obviously

$$\lim_{n\to\infty} d(x_n, x) = 0$$

if and only if $\lim_{n\to\infty} \rho(x_n, x) = 0$ (due to $d = \frac{\rho}{1-\rho}$). Therefore, d and ρ are equivalent. Observe that ρ is clearly bounded whereas d is arbitrary. □

We close this section with a short discussion on uniform continuity. This concept is developed further in Section 6 and Chapter 3. Uniform continuity is an important notion in analysis. Real-valued uniformly continuous functions can be approximated by step functions and this is a core issue when proving the Riemann integrability of continuous functions. The generalization of uniform continuity for metric spaces is useful in more abstract settings and is relatively straightforward.

4.12 Definition. A function f: $(X, d) \to (Y, \rho)$ is called *uniformly continuous* on X if for every $\varepsilon > 0$, there is a positive real number δ such that $d(x, y) < \delta$ implies that $\rho(f(x), f(y)) < \varepsilon$, for every $x, y \in X$. □

In the case of usual continuity, a delta depends upon a particular point $x \in X$, where the continuity holds, so that a common delta, good for all points $x \in X$, need not exist. Unlike continuity, uniform continuity guarantees the existence of such positive δ (for every fixed ε) and for all points of X simultaneously. Clearly, uniform continuity implies continuity.

Uniform continuity can also be defined on some subset A of X, so that in Definition 4.12, X will be replaced with A.

4.13 Examples.

(i) Consider $f : (\mathbb{R}, d_e) \to (\mathbb{R}, d_e)$ such that $f(x) = x^2$. Then

$$|x_0 - x| < \delta$$

implies that

$$|x + x_0| = |x - x_0 + 2x_0|$$

$$\leq |x - x_0| + 2|x_0| < \delta + 2|x_0|$$

and

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot |x + x_0|$$
$$< \delta \cdot (\delta + 2|x_0|).$$

Take $\delta \cdot (\delta + 2|x_0|)$ as ε. Then δ can be found explicitly as a function of ε such that

$$\delta = \sqrt{x_0^2 + \varepsilon} - |x_0|.$$

Therefore, function $x \mapsto x^2$ is d_e-continuous at every point $x_0 \in \mathbb{R}$. However, x^2 is not uniformly continuous on $\mathbb{R}$, because δ depends upon x_0 as well. Specifically, $\delta \to 0$ when $x_0 \to \infty$. Consequently, we cannot find a $\delta > 0$ good for all x_0.

(ii) Let $f(x) = x^2$ be given as

$$f : ([0, 3], d_e) \to ([0, 9], d_e).$$

From the last inequality above we derive

$$|\ f(x) - f(x_0)\ | < \delta(\delta + 2|x_0|) \leq \delta(\delta + 6).$$

The latter inequality implies that $\delta = \sqrt{9 + \varepsilon} - 3$, where $\varepsilon = \delta(\delta + 6)$. Thus $d_e(f(x), f(x_0)) < \varepsilon$ whenever $d_e(x, x_0) < \delta = \sqrt{9 + \varepsilon} - 3$. Because δ is independent of x_0, $f(x)$ is uniformly continuous. Observe that f has been given on a closed and bounded interval which provides the uniform continuity. However, in this case, f would also be uniformly continuous even if f were defined on any bounded but not necessarily closed interval, for instance $(0, 3)$ (why?).

(iii) A continuous function can be uniformly continuous over unbounded sets as, for example, functions $f(x) = \frac{1}{x}$, $x \in [1, \infty)$, and $f(x) = \sin x$, $x \in \mathbb{R}$.

(iv) If (X, d) and (Y, ρ) are two metric spaces, an *isometry* from X to Y is any bijective map $[X, Y, f]$ such that

$$d(x_1, x_2) = \rho(f(x_1), f(x_2)), \text{ for all } x_1, x_2 \in X.$$

Obviously, such an f is uniformly continuous on X.

(v) A function f from (X, d) to (Y, ρ) is said to *satisfy the Lipschitz condition on* X if there is a nonnegative constant C such that for each $x, y \in X, \rho(f(x), f(y)) \leq Cd(x, y)$. The smallest such C is called the *Lipschitz constant of* f. The function f is uniformly continuous on X if f satisfies the Lipschitz condition. (See Problem 4.17.) □

There is an analytical result, known as the Heine-Borel theorem, stating that any continuous function defined on a closed and bounded set in any Euclidean metric space is also uniformly continuous. The general form of this result will be discussed in Section 6 (Theorem 6.13).

4.14 Definition. A function f from (X, d) to (Y, ρ) is said to be *Cauchy continuous* if the image under f_* of every d-Cauchy sequence $\{x_n\}$ [i.e., $f_*(\{x_n\})$] is a ρ-Cauchy sequence. □

4.15 Example. Notice that continuous functions do not map Cauchy sequences to Cauchy sequences, that is, continuous functions need not be Cauchy continuous. Let us introduce the function $[(-\frac{\pi}{2}, \frac{\pi}{2}), \mathbb{R}, \text{rtan}]$, where rtan $:= \text{Res}_{(-\frac{\pi}{2},\frac{\pi}{2})}\tan$, and set arctan $:= \text{rtan}^{-1}$. Consider the sequence

$$x_n := \arctan(n),\ n = 1, 2, \ldots\ .$$

Then, although $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}, d_e)$, $\{\text{rtan}(x_n)\}$ is not. (See Problem 4.14.) □

Notice that continuous functions do not map Cauchy sequences to Cauchy sequences as we learned it from Example 4.15. However, uniformly continuous functions do it.

The statement below is an easy exercise. (See Problem 4.15.)

4.16 Proposition. *If f from (X, d) to (Y, ρ) is uniformly continuous, then it is also Cauchy continuous.* □

4.17 Remark. Suppose $f : (X, d) \to (Y, \rho)$ is a map. What can go wrong with f if it is not uniformly continuous? Suppose f is continuous, but not uniformly continuous. Thus, given an $\varepsilon > 0$ and an x_1, there is a $\delta_1 := \delta(x_1, \varepsilon)$ such that $f_*(B_d(x_1, \delta_1)) \subseteq B_\rho(f(x_1), \varepsilon)$. Using the same ε, we can find for another point x_2 a $\delta_2 := \delta(x_2, \varepsilon)$ such that $f_*(B_d(x_2, \delta_2)) \subseteq B_\rho(f(x_2), \varepsilon)$. If we continue with this process point after point, it may happen that along some sequential path $\{x_n\}$, it is getting increasingly harder to "squeeze" $f_*(B_d(x_n, \delta_n))$ into $B_\rho(f(x_n), \varepsilon)$ while δ_n becomes smaller and smaller, as we can see in Figure 4.3 below.

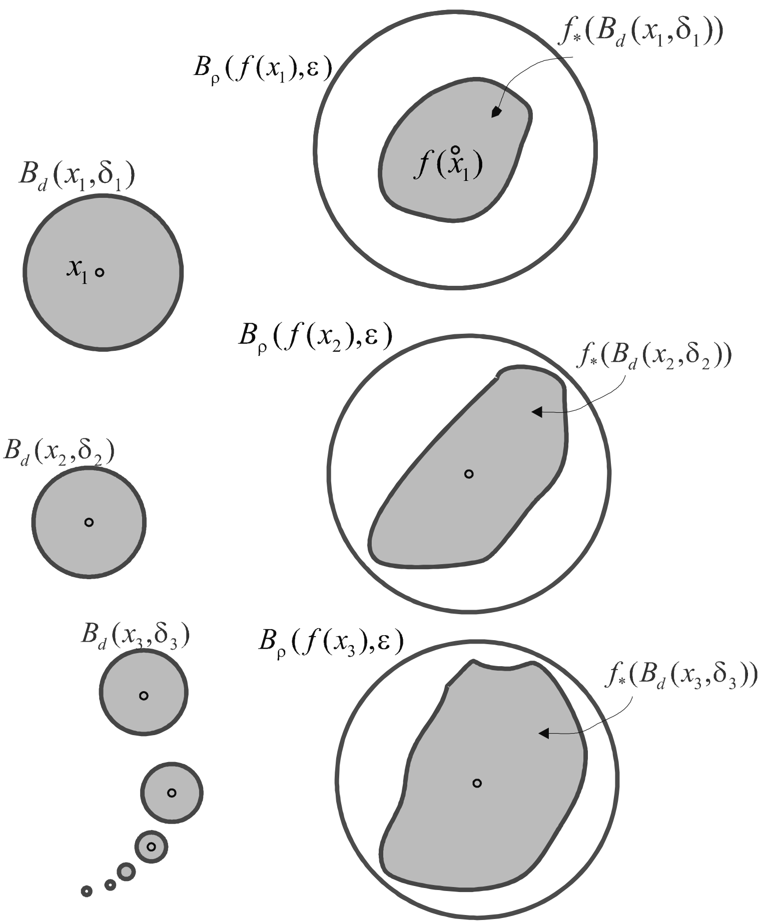

Figure 4.3

So, along the path $\{x_n\}$, $\delta_n \downarrow 0$. The sequence $\{x_n\}$ was brought for an illustration. In general, if f is not uniformly continuous and X is an infinite set, then there is no positive $\delta(\varepsilon)$ good for all $x \in X$ such that $f_*(B_d(x, \delta(\varepsilon))) \subseteq B_\rho(f(x), \varepsilon)$. However, if such a positive $\delta(\varepsilon)$ does exist, f is uniformly continuous.

There is another way to say this. Suppose f is continuous on X and let $\varepsilon > 0$ be chosen. Then for each $x \in X$, there is a $\delta_x = \delta_x(\varepsilon) > 0$ such that $f_*(B_d(x, \delta_x)) \subseteq B_\rho(f(x), \varepsilon)$. Let $\Delta := \{\delta_x(\varepsilon) : x \in X\}$. If f is not uniformly continuous, then for at least one ε_0, $\inf\Delta = 0$ and thus it holds for all $\varepsilon < \varepsilon_0$. $\square$

4.18 Remark. It is known from calculus that the space of all real-valued continuous functions defined on $\mathbb{R}^n$ is closed under the formation of main algebraic operations. What if the functions were defined on an arbitrary space (X, d)? We give here some informal discussion on this matter. Let $\mathbb{R}^X$ be the space of all real-valued functions defined on a set X and let $f, g \in \mathbb{R}^X$. Define the following.

(i) $f \pm g$ is the function such that for each point $x \in X$, $(f \pm g)(x) = f(x) \pm g(x)$.

(ii) fg is the function such that for each $x \in X$, $(fg)(x) = f(x) \cdot g(x)$.

(iii) $+\infty$ and $-\infty$ are not real numbers. Consequently, f/g is the function such that for all $x \in X$, $(f/g)(x) = f(x)/g(x)$, excluding $x \in X$ for which $g(x) = 0$. At all those values, the function f/g is either undefined or should be specified.

(iv) As a special case, any real-valued function multiplied by a real number is a real-valued function too.

(v) The associative (relative to multiplication) and distributive laws of functions relative to the addition and multiplication defined in (i) and (ii) are the corresponding consequences of these laws for real numbers.

Bearing in mind these observations, we conclude that the space $\mathbb{R}^X$ with the above operations is a commutative algebra over the field $\mathbb{R}$ with unity, and a vector lattice (which was also mentioned in

Example 7.7(ix), Chapter 1). A subset $\mathcal{C}((X,d);(\mathbb{R},\rho))$ (of $\mathbb{R}^X$) of all continuous functions is a subalgebra characterized by the following properties.

$$(a) \quad f,g\in\mathcal{C} \;\Rightarrow\; af+bg\in\mathcal{C}\,,\ \forall a,b\in\mathbb{R}.$$
$$(b) \quad f,g\in\mathcal{C} \;\Rightarrow\; fg\in\mathcal{C}. \qquad \square$$

PROBLEMS

4.1 Show that if A is totally bounded then A is bounded. Give an example, where a bounded set is not totally bounded.

4.2 Prove that $\mathcal{C}$ in Remark 4.14 is indeed a subalgebra with properties (a) and (b).

4.3 Show that a continuous bounded function on a bounded interval need not be uniformly continuous.

In Problems 4.4 - 4.8 it is assumed that f and g are functions from $(\mathbb{R},d_e)$ to $(\mathbb{R},d_e)$.

4.4 Let $f:((-\infty,0),d_e)\to((-\infty,0),d_e)$ be a function given by $f(x)=\frac{1}{x}$. Show that f is continuous. Explain why $f(x)$ is not uniformly continuous.

4.5 Let $f:A\to\mathbb{R}$ be a differentiable function such that its derivative f' is bounded over A, where A is an arbitrary (bounded or unbounded) interval. Show that f is uniformly continuous on A.

4.6 Show that if f and g are uniformly continuous on $\mathbb{R}$ and bounded then fg is uniformly continuous on $\mathbb{R}$ too. However, if f and g are uniformly continuous then fg need not be uniformly continuous.

4.7 Which of the following functions are uniformly continuous?

$a)$ $f(x) = \sin^2 x \ \ (x \in \mathbb{R})$.
$b)$ $f(x) = x^3 \cos x \ (x \in \mathbb{R})$.
$c)$ $f(x) = x \sin x \ (x \in \mathbb{R})$.
$d)$ $f(x) = \ln x \ (x \in [1, \infty)$.
$e)$ $f(x) = x^2 \ln x \ (x \in (1, 100))$.
$f)$ $f(x) = \sqrt{x}, \ x \in \mathbb{R}_+$.
$g)$ $f(x) = \sin x^2$.

4.8 Let f be a continuous function and g a uniformly continuous function on a set A such that $|f| \leq |g|$. Is f then uniformly continuous?

4.9 Show that in $(\mathbb{R}^n, d_e)$, any bounded set is also totally bounded.

4.10 Show that in $\mathbb{R}^n$, Euclidean and supremum metrics are equivalent.

4.11 Let A be a nonempty subset in a metric space (X, d). Define the function $f(x) = d(A, x) = \inf\{d(x, y) : y \in A\}$ as the distance from $x \in X$ to set A. Show that f is uniformly continuous.

4.12 Let (X, d) be a metric space and define the function $\rho : (X^2, X^2) \to \mathbb{R}_+$ as $\rho(\mathbf{x}, \mathbf{y}) = d(x_1, y_1) + d(x_2, y_2)$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. (a) Show that ρ is a metric on X^2. (b) Define the function $f(\mathbf{x}) = d(x_1, x_2)$. Show that $f : X^2 \to \mathbb{R}_+$ is uniformly continuous on (X^2, ρ).

4.13 Let (X, d) and (Y, ρ) be two metric spaces. Let function $[X, Y, f]$ be a map. Is the following argument true? "If f is not uniformly continuous, then there is an $\varepsilon > 0$ such that for each $\delta > 0$, there is a set $S_\delta \subseteq X$ with $d : S_\delta \times S_\delta \to (0, \delta)$, and

$$\rho : f_*(S_\delta) \times f_*(S_\delta) \to [\varepsilon, \infty).$$

If S_δ is finite, then we choose $\widetilde{\delta} = \min\{d(x, y) : x, y \in S_\delta\}$. It shows that δ is reducible to $\widetilde{\delta}$. Herewith we empty S_δ and thus for all $x, y \in X, d(x,y) < \widetilde{\delta}$ and $\rho(f(x), f(y)) < \varepsilon$. Thus S_δ is infinite."

4.14 Give an example of a continuous function $[X, Y, f]$ from (X, d) to (Y, ρ) such that f is not Cauchy continuous.

4.15 Prove Proposition 4.16. *Let* (X, d) *and* (Y, ρ) *be two metric spaces and let function* $[X, Y, f]$ *be uniformly continuous on* X. *Then* f *is also Cauchy continuous.*

4.16 Prove the following statement. *Let* (X, d) *and* (Y, ρ) *be two metric spaces and let* $[X, Y, f]$ *and* $[X, Y, g]$ *be two continuous mappings. The subset* $A = \{x \in X : f(x) = g(x)\}$ *is closed.*

4.17 Recall [Example 4.13 (v)] that a function f from (X, d) to (Y, ρ) is said to *satisfy the Lipschitz condition on* X if there is a nonnegative constant C such that for each $x, y \in X$, $\rho(f(x), f(y)) \leq Cd(x, y)$. The smallest such C is called the *Lipschitz constant of* f. Show that f is uniformly continuous on X if f satisfies the Lipschitz condition.

4.18 Let (X, d) be a metric space such that d is bounded, that means $d(X) < \infty$. Is the function ρ defined on $\mathcal{P}(X) \times \mathcal{P}(X)$ as

$$\rho(A, B) = \sup\{d(x, y) : x \in A, y \in B\}$$

a metric on $\mathcal{P}(X)$? If it is not, show which properties of a metric hold and which do not.

NEW TERMS:

5. COMPLETE METRIC SPACES

In this section we are concerned with completeness of metric spaces previously introduced in Definition 3.1(iv). According to Problems 5.10 and 5.11, completeness is a stronger property of a subset than closeness. However, these two notions of a set coincide if the metric space is complete, as we learn it from the following theorem.

5.1 Theorem. *Let* (X,d) *be a complete metric space. Then, a subspace* (A,d) *is complete if and only if A is closed.*

Proof. Let A be closed and let $\{x_n\} \subseteq A$ be any Cauchy sequence. Because (X,d) is complete, there is a point $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Then, by Corollary 3.4, $x \in A$. Thus, (A,d) is complete. Now, let (A,d) be complete and $\{x_n\}$ be any convergent sequence in A. Then this sequence is also a Cauchy sequence and hence A contains its limit. Therefore, A is closed, again, by Corollary 3.4. $\square$

The reader should be aware of the differences between the notions of completeness and closeness of a subspace. (See Problem 5.4.)

5.2 Theorem. *A metric space* (X,d) *is complete if and only if every nested sequence* $\{C(x_n,r_n)\}$ *of closed balls, with* $r_n \downarrow 0$ *as* $n \to \infty$, *has a nonempty intersection.*

Proof. Because $r_n \downarrow 0$, for any $\varepsilon > 0$, there is an integer ν such that $r_\nu < \frac{1}{2}\varepsilon$. Given that $k > n \geq \nu$,

$$C(x_k,r_k) \subset C(x_n,r_n) \subseteq C(x_\nu,r_\nu)$$

and, consequently,

$$d(x_k,x_n) \leq 2r_\nu < \varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence.

Now, assume that (X,d) is complete. Then $\{x_n\}$ converges to a point, say $x \in X$. In as much as each ball $C(x_n,r_n)$ contains the tail

$$\{x_n, x_{n+1}, \dots\}$$

of the sequence $\{x_n\}$ and because it is closed, it must contain x. Thus, $\bigcap\limits_{n=1}^{\infty} C(x_n, r_n)$ contains x and hence it is not empty.

Now, let any nested sequence of closed balls have a nonempty intersection and let $\{x_k\}$ be a Cauchy sequence in X. By Definition 3.1 (iii), it implies the existence of an increasing subsequence $\{\nu_1, \nu_2, \dots\}$ of indices of $\{x_k\}$ such that for each n,

$$d(x_s, x_{\nu_n}) < \tfrac{1}{2^{n+1}}, \text{ for } s > \nu_n.$$

We show that the sequence $\left\{C_n = C\left(x_{\nu_n}, \tfrac{1}{2^n}\right)\right\}$ is nested. Indeed, let $y \in C_{n+1}$. Then

$$d(y, x_{\nu_{n+1}}) \leq \tfrac{1}{2^{n+1}} \quad \text{and} \quad d(x_{\nu_n}, x_{\nu_{n+1}}) < \tfrac{1}{2^{n+1}}.$$

Therefore,

$$d(y, x_{\nu_n}) < \tfrac{1}{2^n},$$

which yields that y is an interior point of C_n and thus $C_n \supset C_{n+1}$. Because by our assumption, the intersection $\bigcap\limits_{n=1}^{\infty} C_n \neq \varnothing$, there is at least one point, say x that belongs to all balls. Furthermore, because the sequence $\{r_n\}$ of their radii is convergent to zero, the subsequence $\{x_{\nu_n}\}$ of their centers must converge to $x \in X$ and thus, by Problem 3.9, $\{x_k\}$ also converges to x. □

5.3 Remark. In the final phrase of the last theorem, x is a unique point of the intersection $\bigcap\limits_{n=1}^{\infty} C_n$. Theorem 5.4 is a useful refinement of this statement due to Georg Cantor. Because of its similarity to Theorem 5.2, its proof is suggested as an exercise (Problem 5.4). □

5.4 Theorem (Cantor). *Let (X, d) be a complete metric space and let $\{A_n\}\downarrow \subseteq X$ be a sequence of nonempty closed subsets with*

$$\lim_{n\to\infty} d(A_n) = 0.$$

Then $\bigcap_{n=1}^{\infty} A_n$ *consists of exactly one element.* □

5.5 Definition. A function $[X, (Y, \rho), f]$ is called *ρ-bounded* if Y is a vector space and there is a nonnegative real number M such that $\rho(f(x), \mathbf{0}(x)) \leq M, \ \forall x \in X,$ where $\mathbf{0}$ is the function identically equal to $\theta \in Y$ (the origin of Y). □

5.6 Examples.

(i) Let X be a nonempty set, (Y, ρ) a metric vector space, and let $\mathcal{F}_* = \mathcal{F}_*(X; (Y, \rho))$ be the set of all ρ-bounded functions from X to Y. For all $f, g \in \mathcal{F}_*$ define

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\}.$$

It can be shown (Problem 5.5) that d_∞ is a metric on $\mathcal{F}_*,$ called a *uniform* (or *supremum*) *metric*. Consequently, the convergence in $(\mathcal{F}_*, d_\infty)$ is called the *uniform convergence*. A subset of functions $\mathcal{F} \subseteq \mathcal{F}_*$ is said to be *uniformly bounded on* X if $\mathcal{F}$ is d_∞-bounded; that is, $\operatorname{diam} \mathcal{F} = \sup\{d_\infty(f, g) : f, g \in \mathcal{F}\} \leq M$ (a positive real number).

We show that any Cauchy sequence in $(\mathcal{F}_*, \rho)$ is uniformly bounded. We will make use of Problem 5.5. Let $\{f_n\}$ be a Cauchy sequence in $(\mathcal{F}_*, d_\infty)$. Therefore, for $\varepsilon = 1$, there is an $N = N(1)$ such that $d_\infty(f_n , f_k) < 1, \ n, k \geq N$. Let $k = N(1)$. Then,

$$d_\infty(f_n, \mathbf{0}) \leq d_\infty(f_n , f_N) + d_\infty(f_N, \mathbf{0}) < 1 + M(f_N),$$

where $M(f_N)$ is a "ρ-bound" of function f_N. If $M(f_i)$ is a bound of $f_i,$ then $M,$ defined as

$$M = \max\{M(f_1), \ldots, M(f_{N-1}), 1 + M(f_N)\},$$

d_∞-dominates the whole sequence $\{f_n\}$. By Problem 5.5, we have that $\{f_n\}$ is d_∞-bounded.

(ii) Assume that (Y, ρ) is a complete metric vector space. Let us show then that $(\mathcal{F}_*, d_\infty)$ is also complete. Consider a Cauchy sequence $\{f_n\} \subseteq (\mathcal{F}_*, d_\infty)$. It is obvious that for each fixed $x \in X,$

the sequence $\{f_n(x)\}$ is also Cauchy in (Y, ρ). Because (Y, ρ) is by our assumption complete, the "pointwise limit" of $\{f_n\}$ exists. Denote it by f. In other words,

$$\lim_{n\to\infty} \rho(f_n(x), f(x)) = 0,\ \forall x \in X.$$

We need to show that $f \in (\mathcal{F}_*, d_\infty)$. Because $\{f_n\}$ is a Cauchy sequence, according to (i) it is uniformly bounded by a real number M. Thus we have

$$\begin{aligned}\rho(f(x), \mathbf{0}(x)) &\le \rho(f(x), f_n(x)) + \rho(f_n(x), \mathbf{0}(x)) \\ &\le \rho(f(x), f_n(x)) + d_\infty(f_n, \mathbf{0}) \le M,\end{aligned}$$

that is,

$$\rho(f(x), 0(x)) \le \rho(f(x), f_n(x)) + M.$$

The last inequality holds for every $x \in X$ if $n \to \infty$, which yields

$$\rho(f(x), \mathbf{0}(x)) \le M, \text{ for all } x \in X.$$

Consequently, $d_\infty(f, \mathbf{0}) \le M$ and hence, $f \in (\mathcal{F}_*, d_\infty)$.

We only showed that

$$f_n(x) \xrightarrow{\rho} f(x),$$

for each $x \in X$, and that $f \in \mathcal{F}_*$. The assertion $f_n \xrightarrow{d_\infty} f$ is subject to Problem 5.7. □

PROBLEMS

5.1 Using similar arguments as in Example 5.6, show that the limit of any uniformly convergent sequence of continuous bounded functions from (X, d_0) to (Y, ρ) is a bounded and continuous function.

5.2 Prove Cantor's Theorem 5.4.

5.3 Let $\{C_n\}$ be a sequence of closed balls in $(\mathbb{R}^n, d_e)$ such that each of the balls C_n is centered at a point $x_0 \in \mathbb{R}^n$ and has radius $\frac{1}{n}$, $n = 1, 2, \ldots$. Find the intersection $\bigcap_{n=1}^{\infty} C_n$.

5.4 Show that if a metric space (X, d) is not complete then a closed subspace (A, d) need not be complete either. $($*Hint*: Consider the metric space in Problems 2.9 and 2.10.$)$

5.5 Show that d_∞, defined in Example 5.6(i), is a metric on $\mathcal{F}_*$.

5.6 Let $\mathcal{F} \subseteq \mathcal{F}_*(X; (Y, \rho))$, where Y is a vector space. Prove that $\mathcal{F}$ is d_∞-bounded if and only if there is a positive constant M such that for all $f \in \mathcal{F}$, $d_\infty(f, \mathbf{0}) \leq M$.

5.7 Show that in Example 5.6 (ii) $f_n \overset{d_\infty}{\to} f$.

5.8 We can make use of the fact that the Euclidean and uniform (supremum) metrics are equivalent (see Problem 4.10) to show completeness of $(\mathbb{R}^n, d_e)$. For $n = 1$, it is well known from calculus. Prove completeness of $(\mathbb{R}^n, d_e)$ for an arbitrary n. (See Problem 3.5.)

5.9 Let (X, d) be a metric space. A subset $A \subseteq X$ is said to be of the *first category* if it can be represented as a countable union of nowhere dense sets. Otherwise, A is of the second category. Prove Baire's category theorem: *a complete metric space is of the second category.*

5.10 Let (X, d) be a metric space and $A \subseteq X$ be a complete subset. Show that A is closed.

5.11 Show that a closed set in a metric space need not be complete.

NEW TERMS:

completeness criteria, 132
Cantor's Theorem on intersection of closed sets, 133
ρ-bounded function, 134
bounded function, 134
uniform (supremum) metric, 134
supremum (uniform) metric, 134
uniform convergence continuous bounded functions, 135
uniformly bounded set of functions, 136
d_∞-bound of a function, 136
Baire's category theorem, 136

6. COMPACTNESS

Compactness is one of the key concepts in real analysis. We develop it in the present section for metric spaces and then revisit it in Chapter 3 for general topological spaces. It stems from the fact known in $\mathbb{R}$ that every bounded sequence has a convergent subsequence, which implies that any sequence in a closed bounded interval has a subsequence convergent to a point in this interval. In a general metric space, a subset A, in which every sequence has a subsequence convergent to a point in A, is called *sequentially compact* or just *compact*. Although compactness and sequential compactness are generally distinct notions in topological spaces (and they are defined in a different way), they are equivalent in metric spaces as Theorem 6.3 states.

Continuous functions defined on compact sets are uniformly continuous; continuous images of compact sets are compact (hence, closed and bounded) and this means that in normed vector spaces (introduced in the next section), continuous functions on compact sets reach their maximum values. Further applications lead to the celebrated Ascoli and Ascoli - Arzela theorems [Dsh2].

6.1 Definitions.

(i) A family of sets $\{A_i : i \in I\} \subseteq (X, d)$ is called a *cover* of a set $A \subseteq X$ if

$$A \subseteq \bigcup_{i \in I} A_i.$$

Any subfamily of $\{A_i : i \in I\}$, that covers A is called a *subcover* of A. If $\{A_i : i \in I\}$ is a family of open sets, then the corresponding cover (or subcover) is called an *open cover* (or an *open subcover*).

(ii) A set $A \subseteq (X, d)$ is called *compact* if any open cover of A has within itself a finite subcover of A, or we also say that "any open cover of A can be reduced to a finite subcover of A." Correspondingly, (X, d) is a *compact metric space* if so is X. Notice that any finite subset is compact. Consequently, to avoid triviality, in all theorems below we assume that underlying sets of spaces are infinite.

(iii) A set $A \subseteq (X,d)$ is called a *Lindelöf set* if any open cover of A contains a countable subcover of A (or can be reduced to a countable subcover). (X,d) is called a *Lindelöf space* if X is a Lindelöf set. A Lindelöf space was named after the Finnish topologist Ernst Leonard Lindelöf (1870-1946). □

A noteworthy property of Euclidean spaces is given in the following classical result.

6.2 Theorem (Lindelöf). *Any subset of* $\mathbb{R}^n$ *is Lindelöf set in* $(\mathbb{R}^n, d_e)$. *In particular,* $(\mathbb{R}^n, d_e)$ *is a Lindelöf space.* □

(See Problem 6.7.)

6.3 Theorem. *For a subset* $A \subseteq (X,d)$, *the following statements are equivalent.*

(i) *A is compact.*

(ii) *Every infinite subset of A has an accumulation point in A* (in this case A is called *Bolzano-Weierstrass compact*).

(iii) *Every sequence in A has a subsequence that converges in A* (A is called *sequentially compact*). □

Sequential compactness of a subspace implies its completeness. (See Problem 6.6.)

The proofs of the above statements are left for the reader (Problem 6.8).

Definition 6.4. A metric space is called *separable* if it has a dense countable subset. □

Example 6.5. The Euclidean metric space $(\mathbb{R}, d_e)$ is separable. A relevant dense countable subset of $\mathbb{R}$ would be $\mathbb{Q}$, the set of rational numbers. Another example is the n-dimensional Euclidean metric space with the countable, dense subset $\mathbb{Q}^n$. □

Theorem 6.6. *Any compact metric space is separable.*

Proof. Let X be compact. It is easy to see that for each $n \in \mathbb{N}$, X can be covered by the family of open balls centered at every $x \in X$ with radius $\frac{1}{n}$. Because (X,d) is compact, this open cover can be reduced to a finite subcover, such that $\bigcup_{x\in F_n} B(x, \frac{1}{n})$ contains X, where $F_n = \{x_1^n, \dots, x_{k_n}^n\}$. Denote $F = \bigcup_{n=1}^{\infty} F_n$, which obviously is a countable subset of X. We show that F is dense in X, that is, $\overline{F} = X$. It is sufficient to prove that, for each $y \in X$ and $r > 0$, the open ball $B(y, r)$ contains at least one point of the set F, that is, y is a closure point of F. Taking some y and r we choose any n such that $\frac{1}{n} < r$. Then if

$$y \in X \subseteq \bigcup_{x\in F_n} B(x, \tfrac{1}{n}),$$

there is a point $x_{j_n}^n \in F_n$ such that $y \in B(x_{j_n}^n, r)$. This implies that $d(x_{j_n}^n, y) < r$ and, therefore, $x_{j_n}^n \in B(y, r)$. Consequently, $B(y, r) \cap F_n \neq \emptyset$ and $B(y, r) \cap F \neq \emptyset$. The proof of the statement is complete. □

The following two theorems belong to central results in analysis.

Theorem 6.7. *Let $A \subseteq (X, d)$ be compact. Then A is closed and bounded.*

Proof.

(i) We show that A is bounded. Obviously A is covered by the family of open balls $\{B(x, 1) : x \in A\}$. Since A is compact, this cover can be reduced to a finite subcover, that is, $A \subseteq \bigcup_{k=1}^{h} B(x_k, 1)$ for some integer h. Let $M = \max\{d(x_i, x_j) : i, j = 1, \dots, h\}$. Then M is finite. For any $x, y \in A$, there are x_i and x_j such that $x \in B(x_i, 1)$ and $y \in B(x_j, 1)$. The following holds due to the triangle inequality.

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < 1 + M + 1 < \infty.$$

Therefore, A is bounded.

(ii) We show that A is closed, that is, that $A = \overline{A}$. Let $x \in \overline{A}$. By Theorem 3.3, there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$. By Theorem 6.3, if A is compact, every sequence $\{x_n\} \subseteq A$ has a subsequence that converges in A. By Problem 3.7, such a subsequence must have the same limit as $\{x_n\}$, that is, $x \in A$. Therefore, A is closed. □

The history of what today is called the *Heine–Borel theorem* (one of the most cited theorems in analysis) begins in the second half of nineteenth century in the work of the German mathematician Peter Gustav Dirichlet. The key issue was the concept of uniform continuity and the theorem stating that every continuous function on a closed interval is uniformly continuous. In his proof, Dirichlet implicitly used the existence of a finite subcover of a given open cover of a closed interval. He presented this proof in his lectures taught in 1862, which were published only in 1904. The first known proof of the theorem was rendered in 1872 by Émile Borel and not by Heinrich Eduard Heine (who, by the way, was a student of Dirichlet). However, Borel's formulation was restricted to countable covers and he used techniques similar to those that Heine used to prove that continuous functions on closed intervals are uniformly continuous. Consequently, Heine's name was attached due to the similarity in Heine's and Borel's approaches. Henri Lebesgue in 1898, among a few others, generalized the Heine - Borel theorem to arbitrary covers.

Theorem 6.8 (Heine - Borel). *A set $A \subseteq (\mathbb{R}^n, d_e)$ is compact if and only if A is closed and bounded.*

Proof.

(i) If A is compact, it is closed and bounded as a special case of Theorem 6.7.

(ii) If $A \subseteq (\mathbb{R}^n, d_e)$ is closed and bounded, $d_e(x,y) \leq M < \infty$, $\forall\ x, y \in A$. Let y and $\mathbf{a} = (a_1, \ldots, a_n)$ be two elements of A. Then we have:

$$|a_i| = \sqrt{a_i^2} \leq \sqrt{\sum_{k=1}^{n} (a_k - 0)^2} = d_e(\mathbf{a}, \theta)$$

$$\leq d_e(\mathbf{a}, y) + d_e(y, \theta) \leq M + d_e(y, \theta),\ i = 1, \ldots, n,$$

where θ denotes the origin of $\mathbb{R}^n$. Because each y has a finite distance to the origin, every other point of A, like $\mathbf{a}$, has also a finite distance to the origin bounded by $M + d_e(y, \theta)$. Note that even though $d_e(\mathbf{a}, \theta) < \infty$, $d_e(\mathbf{a}, \theta)$ in general, need not have a uniform bound, unless A were bounded, which is assumed. Now we show that any d_e-bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

The steps below represent an appropriate selection procedure. Let $\{x_k\} \subseteq A$. Then $\{x_k^i\} \subseteq \mathbb{R}$ is a bounded sequence of i-coordinates (the ith-component sequence), $i = 1, \ldots, n$. A bounded sequence need not to converge but does have a convergent subsequence.

For $i = 1$, let such a subsequence be $\{x_{r_1}^1, x_{r_2}^1, \ldots\}$ with the limit point $x^{(1)}$. Select from the second-component sequence, the subsequence with the same indices $\{x_{r_1}^2, x_{r_2}^2, \ldots\}$. This subsequence is also bounded and hence contains a convergent subsequence, say $\{x_{k_1}^2, x_{k_2}^2, \ldots\}$, with a limit point $x^{(2)}$, so that the set of indices $\{k_1, k_2, \ldots\} \subseteq \{r_1, r_2, \ldots\}$. If we return to the subsequence $\{x_{r_1}^1, x_{r_2}^1, \ldots\}$ and select single out the subsequence $\{x_{k_1}^1, x_{k_2}^1, \ldots\}$, then this subsequence is also convergent and has the same limit $x^{(1)}$.

We can continue this process by taking the third-component sequence, selecting the subsequence $\{x_{k_1}^3, x_{k_2}^3, \ldots\}$ and from this sequence a convergent subsequence with a limit point $x^{(3)}$. Then the above first- and second-component subsequences will be reduced to the those with the indices from the third selection and so on. Let $x = (x^{(1)}, \ldots, x^{(n)})$ be the limit of the selected subsequence of $\{x_k\}$. In as much as A is closed, x must belong to A.

Therefore, we proved that an arbitrary sequence in A has a convergent subsequence in A, implying that A is sequentially compact. By Theorem 6.3, A is then compact. $\square$

6.9 Remark. The second part of the Heine-Borel theorem does not hold for general metric spaces. That is, if A is closed and bounded, it need not be compact. For example, let X be an infinite set and let d be the discrete metric (which is finite) on X. Then X is closed and bounded.

Now consider

$$X \subseteq \bigcup_{x \in X} B(x, 1).$$

Because each of the balls covers just one point, the open cover $\{B(x,1)\colon\ x\in X\}$ cannot be reduced to a finite subcover. Therefore, X is not compact. □

The following theorem is very important in many forthcoming applications.

6.10 Theorem. *Let $f:(X,d)\to(Y,\rho)$ be a continuous function and let K be a compact subset of X. Then the image $f_*(K)$ is compact. In short, the image of a compact set under a continuous function is compact.*

Proof. Take any open cover $\{O_i'\colon i\in I\}$ of $f_*(K)$ to have $f_*(K)\subseteq\bigcup_{i\in I}O_i'$. Then,

$$K\subseteq f^*(f_*(K))\subseteq\bigcup_{i\in I}f^*(O_i').$$

Because f is continuous, $f^*(O_i')$ is open, and because K is compact, there is a finite subcover of K by sets $f^*(O_i')$, without loss of generality indexed by $1,\ldots,n,$ that is,

$$K\subseteq\bigcup_{k=1}^{n}f^*(O_k').$$

Because by Problem 3.6(a, b), Chapter 1, maps preserve inclusions and unions, we have

$$f_*(K)\subseteq\bigcup_{k=1}^{n}f_*(f^*(O_k'))\subseteq\bigcup_{k=1}^{n}O_k'.$$

The last inclusion is due to Problem 2.6, Chapter 1. □

6.11 Remark. Let $f:(X,d)\to(\mathbb{R},d_e)$ be a continuous map and let $A\subseteq(X,d)$ be compact. Then, by Theorem 6.10, $f(A)$ is compact in $(\mathbb{R},d_e)$. By Theorem 6.7, $f(A)$ is then closed and bounded, which means that the diameter of $f(A)$ equals some $M<\infty$. As mentioned in part (ii) in the proof of the Heine-Borel Theorem, this implies that all points of $f(A)$ have a finite distance (i.e., are bounded by some M_0) to the origin, or equivalently, $|f(x)|\leq M_0$, for all

$x \in A$. We have therefore shown that a continuous real-valued map on a compact set assumes a minimum and a maximum value. $\square$

6.12 Examples.

(i) In $(\mathbb{R}, d_e)$, $\mathbb{R}$ is closed but not bounded. Therefore, by the Heine-Borel theorem, $\mathbb{R}$ is not compact.

(ii) Take as $A \subseteq (\mathbb{R}, d_e)$ the set $(0, 1]$ which is bounded but not closed and therefore is not compact. We use a different argument to prove noncompactness of $(0, 1]$. Consider the open cover of A given by the family of sets $\{(\frac{1}{n},\, 2) : n = 1, 2, \ldots\}$. Obviously,

$$\bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2) = (0, 2) \supseteq A.$$

It is not possible to select any finite subcover of A, for no finite subcover would include the point 0. Yet another argument that A is not compact (due to Theorem 6.3(ii)) is that the sequence $\{\frac{1}{n}\}$ does not converge in A. $\square$

A continuous function need not be uniformly continuous, unless it is defined on a compact set, which is a widely referred to result known for Euclidean spaces.

6.13 Theorem. *Let* $f\colon (X, d) \to (Y, \rho)$ *be a continuous function and let* (X, d) *be compact. Then* f *is uniformly continuous on* X.

Proof. Let f be continuous at x. Then, for each $\varepsilon > 0$, there is a $\delta_x > 0$, such that

$$\rho(f(x), f(y)) < \tfrac{\varepsilon}{2}$$

for all y with $d(x, y) < \delta_x$. Because X is compact, after reduction, there is an n-tuple of open balls such that

$$X \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{x_i}/2).$$

Let $\delta = \frac{1}{2} \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$ and let x, y be such that $d(x, y) < \delta$. Then $x \in B(x_i, \delta_{x_i}/2)$ implies that

$$d(x, x_i) < \tfrac{1}{2}\, \delta_{x_i}$$

and

$$d(x_i, y) \le d(x, y) + d(x, x_i) < \delta + \tfrac{1}{2}\,\delta_{x_i} \le \delta_{x_i}.$$

Thus, y belongs to the ball $B(x_i, \delta_{x_i})$. Because y and x_i are within the distance of δ_{x_i}, due to continuity of f at x_i, given ε,

$$\rho(f(x_i), f(y)) < \tfrac{\varepsilon}{2}.$$

Obviously, $d(x, x_i) < \delta_{x_i}$ yields $\rho(f(x_i), f(x)) < \frac{\varepsilon}{2}$ and, therefore,

$$\rho(f(x), f(y)) \le \rho(f(x_i), f(x)) + \rho(f(x_i), f(y)) < \varepsilon. \qquad \square$$

Here is a very useful criterion of compactness.

6.14 Theorem. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof.

(i) Let (X, d) be compact. Then by Problem 6.6, it is complete. Because $X \subseteq \bigcup\limits_{x \in X} B(x, \varepsilon)$ for some $\varepsilon > 0$, by compactness, the cover $\{B(x, \varepsilon)\colon x \in X\}$ can be reduced to a finite subcover, which implies total boundedness.

(ii) Let (X, d) be complete and totally bounded. We show that (X, d) is sequentially compact, which, by Theorem 6.3, would imply compactness. Let $\{x_n\}$ be a sequence in X. We construct a Cauchy subsequence.

X is totally bounded. Therefore, it can be covered by finitely many open balls of radius 1. Then, at least one of the balls, for instance B_1, contains infinitely many terms, say $\{x_k^1\}$, of this sequence. Furthermore, we cover X by balls of radius $\frac{1}{2}$ and again an infinite subsequence $\{x_k^2\} \subseteq \{x_k^1\}$ (because B_1 will also be covered) is contained in one of the balls, which we label B_2, and so on.

The desired Cauchy sequence is formed by the selection of the first term from each subsequence. Indeed, by the construction, x_1^1 and x_1^2 belong to ball B_1. Thus, $d(x_1^1, x_1^2) < 1$. Then, x_1^3 and x_1^2 belong to ball B_2, which implies that $d(x_1^2, x_1^3) < \frac{1}{2}$, and so on. Because (X, d) is assumed to be complete, this Cauchy sequence must be convergent, thereby yielding sequential compactness of (X, d). $\square$

PROBLEMS

6.1 Show that if $\{x_k\} \subseteq (\mathbb{R}^n, d_e)$ with $d(x_k, 0) \leq 3$, then $\{x_k\}$ has a convergent subsequence.

6.2 Define for each pair $A, B \in (X, d), d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Let A be compact. Show that $\forall B \subseteq X$, $\exists$ an $x \in A$ such that $d(x, B) = d(A, B)$.

6.3 Let $A, B \subseteq (X, d)$ such that A is compact and B is closed. If $A \cap B = \emptyset$, show that $d(A, B) > 0$.

6.4 Let $A \subseteq (X, d)$. Show that if A is totally bounded then $\overline{A}$ is also totally bounded.

6.5 Generalize Theorem 6.6: *Any Lindelöf metric space is separable.*

6.6 Show that sequential compactness of a subspace implies its completeness.

6.7 Prove Theorem 6.2. *Any subset of* $\mathbb{R}^n$ *is Lindelöf set in* $(\mathbb{R}^n, d_e)$. *In particular,* $(\mathbb{R}^n, d_e)$ *is a Lindelöf space.*

6.8 Prove Theorem 6.3: *For a subset* $A \subseteq (X, d)$, *the following statements are equivalent.*

(i) *A is compact.*

(ii) *Every infinite subset of A has an accumulation point in* A (in this case A is called *Bolzano-Weierstrass compact*).

(iii) *Every sequence in A has a subsequence that converges in* A (A is called *sequentially compact*).

6.9 Show that a mapping $f : (X, d) \to (Y, \rho)$ is continuous if and only if, for any compact subset K of X, $\mathrm{Res}_K f$ is continuous.

6.10 Show that a closed subset of a compact metric space is compact.

6.11 Let $\mathcal{K}$ be the family of all compact subsets of a metric space (X, d). List those set operations which are contained within $\mathcal{K}$. Give counterexamples for all set operations, which are not valid in $\mathcal{K}$.

NEW TERMS:

cover, 138
subcover, 138
open cover, 138
open subcover, 138
compact set, 138
compact metric space, 138
Lindelöf set, 139
Lindelöf space, 139
Lindelöf, Ernst Leonard, 139
Lindelöf theorem, 139
compactness criteria (metric space), 139
Bolzano - Weierstrass compactness (metric space), 139
sequential compactness (metric space), 139
separable metric space, 139
separability criterion (metric space), 139
compactness criterion (metric space), 140
Dirichlet, Peter Gustav, 141
Borel, Émile, 141
Heine, Heinrich Eduard, 141
Lebesgue, Henri, 141
Heine - Borel Theorem, 141
compact set under a continuous function (metric space), 143
uniform continuity criterion in compact space, 144
compactness criterion (metric space), 145
separability of a Lindelöf metric space, 146

7. NORMED VECTOR SPACES

We have already mentioned that the Euclidean metric defines the length of a vector in n-dimensional Euclidean vector space. The following generalizes the notion of vector length in a vector space and reconciles it with the notion of a special metric defined on a vector space (initially discussed in Section 5).

It is hard to say when exactly normed vector spaces were born. In 1912, the Austrian mathematician Eduard Helly published an important paper about bounded functionals on the Banach space $\mathcal{C}_{[a,b]}$ followed by his habilitation work (submitted in 1918) and his seminal paper of 1921, "Über Systeme linearer Gleichungen mit unendlich vielen Unbekannten", *Monatshefte für Mathematik und Physik*, **31**, 60 - 91. Here he considers subspaces of $\mathbb{C}^{\mathbb{N}}$ and defines a *norm* on one such subspace X, although he neither calls it the "norm" nor does he use the common notation $\|\cdot\|$. However, he uses instead the word *Abstandfunktion* (distance function) $D : X \to \mathbb{R}_+$ such that, with $x \in X$, also $\alpha x \in X$ ($\alpha \in \mathbb{F}$), with $D(\alpha x) = |\alpha| D(x)$, with $x, y \in X$, also $x + y \in X$ and $D(x + y) \leq D(x) + D(y)$, and finally, $D(x) = 0$ implies that $x = \theta$. Although norms in various forms had been introduced earlier, Helly was the first one to utilize the relationship between the norm and convexity. Shortly thereafter, in 1922, Stefan Banach defended his thesis (submitted in 1920) in which a well structured concept of the norm in abstract vector spaces has been laid out. And not only that: Banach also considered complete normed vector spaces, which Maurice Fréchet in 1928 suggested be named *Banach spaces*.

7.1 Definition. Let (X, d) be a metric space such that X is a vector space over a field $\mathbb{F}$ (which is $\mathbb{R}$ or $\mathbb{C}$). The metric d is said to be:

a) translation invariant if for all $a,\ x,\ y \in X,\ d(x + a, y + a) = d(x, y)$.

b) homothetic if for all $\alpha \in \mathbb{F}$ and $x, y \in X$, $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.

If a metric d is translation invariant and homothetic we abbreviate it by TIH. □

If d is a metric on a vector space X, then we are able to measure the length of vectors, thus comparing them by setting the distance from any point $x \in X$ to one fixed point of X, the origin. If, in addition, d is TIH then we can use the properties of X as a vector space, and in some particular cases, employ even geometry, thereby emulating the Euclidean space and preserving the generality needed in applications.

7.2 Definition. Let d be a TIH metric on a vector space X, with the origin θ, over $\mathbb{F}$ (assuming that $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Then for all $x \in X$, we call the distance $d(x,\theta)$ the *norm of vector* x and denote it by $\|x\|$. We also call $\|\cdot\|$ the *norm on X induced by the TIH metric d.* The pair $(X, \|\cdot\|)$ will be referred to as a *normed vector space* (*NVS*). □

7.3 Theorem. *Let* $\|\cdot\|$ *be a norm on X in Definition 7.2. Then, the following properties of* $\|\cdot\|$ *hold true.*

(i) $\|x\| = 0 \quad \Leftrightarrow \quad x = \theta.$

(ii) $\|\alpha x\| = |\alpha|\,\|x\|, \forall \alpha \in \mathbb{F},\ \forall\, x \in X.$

(iii) $\|x + y\| \leq \|x\| + \|y\|,\ \forall\, x, y \in X.$

(iv) *The norm* $\|\cdot\|$ *is a continuous mapping from* $(X, \|\cdot\|)$ *to* $(\mathbb{R}_+, |\cdot|)$.

Proof. Property (i) is obvious.

(ii)
$$\|\alpha x\| = d(\alpha x, \theta) = d(\alpha x, \alpha\theta)$$
$$= |\alpha| d(x,\theta) = |\alpha|\|x\|.$$

(iii)
$$\|x + y\| = d(x + y, \theta) = d(x, -y)$$
$$\leq d(x,\theta) + d(\theta, -y) = \|x\|$$

$$+|-1|\|y\| = \|x\| + \|y\|.$$

(iv) From inequality (1.2c) we have

$$|d(x,\theta) - d(z,\theta)| = |\|x\| - \|z\|| \leq d(x,z) = \|x - z\|. \quad (7.3)$$

Thus, if $f(x) :=\|x\|,$ we have from Definition 4.1 that $f : (X, \|\cdot\|) \to (\mathbb{R}, |\cdot|)$ is continuous at x if for each $\varepsilon > 0,$ there is a $\delta > 0$ such that $|f(x) - f(z)| < \varepsilon$ for all $z \in X$ with the property that $\|x - z\| < \delta$. Now, letting $\varepsilon = \delta$ we have the latter hold due to inequality (7.3). □

7.4 Remark. Conversely, if $\|\cdot\|$ is a real-valued nonnegative function defined on a vector space X and has properties $(i$ - $iii)$ of Theorem 7.3, then $\|\cdot\|$ generates a TIH metric on X by setting $d(x,y) = \| x - y \|$ (show it, see Problem 7.6). □

7.5 Definition. If d in Definition 7.2 is a TIH pseudometric, then the function $\|\cdot\|$ is called a *seminorm* and correspondingly, the pair $(X, \|\cdot\|)$ is called a *seminormed vector space* (*SNVS*). □

It is easy to show that the Euclidean metric d_e on $\mathbb{R}^n$ is TIH. The associated norm induced by d_e is called the *Euclidean norm* and it is denoted $\|\cdot\|_e$.

A very important class of NVS's is introduced below.

7.6 Definition. An NVS is called a *Banach space* if it is complete with respect to the metric induced by the norm (or the norm induced by a TIH metric, as per Remark 7.4). □

7.7 Examples.

(i) The NVS $(\mathbb{R}^n, \|\cdot\|_e)$ over the field $\mathbb{R}$ with $\|x\| = \sqrt{\sum\limits_{i=1}^{n} x_i^2}$ is a Banach space with the Euclidean norm (see Problem 7.1).

(ii) The NVS l^p $(p \geq 1)$ over the field $\mathbb{C}$ with the norm $\|x\|_p = [\sum_{n=1}^{\infty} |x_n|^p]^{1/p}$ is a Banach space. Observe that $\|\cdot\|_p$ indeed defines a norm (called the l^p *norm*). (See Problem 7.5.) Now let $\{x^{(n)}\}$ be a Cauchy sequence. Then as a Cauchy sequence it is bounded (see Problem 3.4), say, by some $M \in \mathbb{R}_+$. Let $x = (x_1, x_2, \dots)$ be the pointwise limit of the sequence $\{x^{(n)}\}$. This limit exists, because each x_i is the limit of the ith-component sequence in $(\mathbb{C}, d_e)$ which is complete. First we need to show that x is an element of l^p (that is, $\|x\|_p < \infty$) and secondly, that

$$x^{(n)} \overset{l^p}{\longrightarrow} x$$

(i.e., $\{x^{(n)}\}$ converges to x in l^p norm).

We have

$$\left[\sum_{k=1}^{r} |x_k|^p\right]^{1/p} = \left[\sum_{k=1}^{r} \left|x_k - x_k^{(n)} + x_k^{(n)}\right|^p\right]^{1/p}$$

(by Minkowski's inequality, Problem 1.7 (a),

$$\left[\sum_{i=1}^{n} (a_i + b_i)^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} a_i^p\right]^{1/p} + \left[\sum_{i=1}^{n} b_i^p\right]^{1/p}$$

with $a_k = x_k - x_k^{(n)}$ and $b_k = x_k^{(n)}$)

$$\leq \left[\sum_{k=1}^{r} \left|x_k - x_k^{(n)}\right|^p\right]^{1/p} + \left[\sum_{k=1}^{r} \left|x_k^{(n)}\right|^p\right]^{1/p}$$

$$\leq \left[\sum_{k=1}^{r} \mid x_k - x_k^{(n)} \mid ^p\right]^{1/p} + \left\|x^{(n)}\right\|_p$$

$$\leq \left[\sum_{k=1}^{r} \left|x_k - x_k^{(n)}\right|^p\right]^{1/p} + M.$$

Now, letting $n \to \infty$, we have

$$\left[\sum_{k=1}^{r} |x_k|^p\right]^{1/p} \leq M,$$

which holds for all $r = 1, 2, \dots$. Hence, with $\|x\|_p \leq M$, we showed that $x \in l^p$.

The proof that $x^{(n)} \to x$ in l^p norm is left for an exercise. (See Problem 7.7.) So, l^p is complete and therefore it is a Banach space.

(iii) Let $\mathcal{F}_*(X)$ be the space of all bounded functions on X valued in $(\mathbb{R}, d_e)$ or $(\mathbb{C}, d_e)$. Obviously, $\mathcal{F}_*$ is a vector space. The norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ is called the *supremum norm.* The space $(\mathcal{F}_*(X), \|\cdot\|_\infty)$ is then an NVS. The convergence in such a space (i.e., in supremum norm) is referred to as the *uniform convergence*. Now, $(\mathcal{F}_*(X), \|\cdot\|_\infty)$ is a Banach space (see Problem 7.4).

(iv) Consider $\mathcal{C}^n_{[a,b]}$ as the space of all n-times differentiable real-valued functions on a compact interval $[a, b]$. It is easily seen that $\mathcal{C}^n_{[a,b]}$ is a vector space. We introduce the following norm in $\mathcal{C}^n_{[a,b]}$.

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty + \cdots + \left\|f^{(n)}\right\|_\infty$$

Clearly, $\|\cdot\|_\Sigma$ is a norm in $\mathcal{C}^n_{[a,b]}$. We show that $\mathcal{C}^n_{[a,b]}$ is a Banach space under this norm. Let $\{f_k\}$ be a $\|\cdot\|_\Sigma$-Cauchy sequence. Then, for every $\varepsilon > 0$, there is a positive integer N such that $\forall k, j \geq N$,

$$\| f_j - f_k \|_\infty + \| f'_j - f'_k \|_\infty + \cdots + \| f^{(n)}_j - f^{(n)}_k \|_\infty < \varepsilon,$$

which implies

$$\left\|f_j^{(i)} - f_k^{(i)}\right\|_\infty < \varepsilon,\ i = 0, 1, \dots, n.$$

Therefore, by the well-known theorem from calculus (cf. Theorem 4.2, p. 508, in [Fish]), there exists a function $g_i : [a, b] \to \mathbb{R}$ to which the sequence $\{f_j^{(i)} : j = 1, 2, \dots\}$ converges uniformly and g_i is continuous, $i = 0, 1, \dots, n$. On the other hand, it holds that

$$f_k^{(i-1)}(x) - f_k^{(i-1)}(a) = \int_{[a,x]} f_k^{(i)}(u)du,$$

$$i = 1, \ldots, n, \; k = 1, 2, \ldots \; .$$

Let $k \to \infty$ in the above equation in the sense of the pointwise convergence. Because $f_k^{(i)}$ and g_i are bounded functions, by the Lebesgue dominated convergence theorem (see forthcoming Chapter 6 for a rigorous proof of this theorem), we may interchange the limit and the integral and have

$$g_{i-1}(x) - g_{i-1}(a) = \int_{[a,x]} g_i(u)du, \; i = 1, \ldots, n.$$

Consequently, we conclude that g_{i-1} is differentiable on $[a, b]$ and $g'_{i-1}(x) = g_i(x)$. Thus $g_0 \in \mathcal{C}^n_{[a,b]}$ implying that $\|f_k - g_0\|_\Sigma \to 0$ and $\mathcal{C}^n_{[a,b]}$ is a Banach space.

(v) The set $\mathbb{Q}$ of all rational numbers with the Euclidean norm $\|\cdot\|_e$ is an NVS, but it obviously fails to be Banach.

(vi) In Example 1.3(iv) we introduced the vector space $\mathcal{C}_{[a,b]}$ of all continuous functions on interval $[a, b] \subseteq \mathbb{R}$ and defined the supremum metric

$$d(x, y) = \sup\{|x(t) - y(t)| \colon t \in [a, b]\},$$

which induces the supremum norm

$$\|x\|_\infty = \sup\{|x(t)| \colon t \in [a, b]\}.$$

Now, the space $\left(\mathcal{C}_{[a,b]}, \|\cdot\|_\infty\right)$ is Banach. Indeed, let $\{x_n\}$ be a Cauchy sequence. Then, for each $\varepsilon > 0$, there is an $N(\varepsilon)$-tail of $\{x_n\}$. Thus, for each $t \in [a, b]$, $|x_m(t) - x_n(t)| < \varepsilon$, meaning that for each $t \in [a, b]$, $\{x_n(t)\}$ is Cauchy in $(\mathbb{R}, \|\cdot\|_e)$. Because the latter is complete, we conclude that $x_n(t)$ converges pointwise to some function $x(t)$. Consequently, $|x_m(t) - x_n(t)| < \varepsilon$ for all $m, n \geq N(\varepsilon)$ (with $m \to \infty$ for each $t \in [a, b]$) implies that

$$|x(t) - x_n(t)| \leq \varepsilon, \text{ holding } \forall\, t \in [a, b]$$

and thus

$$\|x_n - x\|_\infty \leq \varepsilon.$$

The latter means that $\{x_n\}$ converges to x uniformly on $[a, b]$ and therefore, from calculus we know that x is continuous. Hence, $\left(\mathcal{C}_{[a,b]}, \|\cdot\|_\infty\right)$ is Banach. [The result also follows from (iv)].

(vii) Let $R^1_{[a,b]}$ denote the space of all real-valued Riemann integrable functions on interval $[a, b]$. Obviously, $R^1_{[a,b]}$ is a real vector space. Introduce for an element $x \in R^1_{[a,b]}$, the function

$$\|x\| = \int_a^b |x(t)|dt, \tag{7.7}$$

which obeys all properties of a norm except that $\|x\| = 0$ does not imply that $x(t) = 0$ for all $t \in [a, b]$. This is because there are integrable functions that can differ from x on infinite sets, even uncountable sets. (For the reader familiar with analysis, a function y can differ from zero function $\mathbf{0}$ on a continuum set of Lebesgue measure zero and it must be almost everywhere continuous. We discuss these and other aspects of integration in Euclidean spaces in Chapter 6, Section 3.) Thus the function defined in (7.7) is a seminorm on $R^1_{[a,b]}$.

As we have done for metrics, we can convert $\|\cdot\|$ to a norm and $\left(R^1_{[a,b]}, \|\cdot\|\right)$ to an NVS by turning to the quotient NVS space $\left(R^1_{[a,b]}|\mathrm{E}_{\|\cdot\|}, \boldsymbol{\|\cdot\|}\right)$, where $\boldsymbol{\|\cdot\|}$ (notice the bold face of $\|\cdot\|$) is the restrictor of $\|\cdot\|$. Obviously, an element of $R^1_{[a,b]}|\mathrm{E}_{\|\cdot\|}$ will be an equivalence class $[x]$ of all integrable functions that are equal "almost everywhere on $[a, b]$." □

The following property of NVS's asserts that the summation of their elements and multiplication by a scalar are two continuous functions with respect to the norm. This property makes NLS's an important subclass of "topological vector spaces."

7.8 Proposition. *Let X be an NVS. Then, the functions*

(i) $\varphi : X \times X \to X$ *defined as* $\varphi(x, y) = x + y$

(ii) $\psi : \mathbb{F} \times X \to X$ *defined as* $\psi(\alpha, x) = \alpha x$

are continuous.

Proof. (i) By Theorem 4.4, φ is continuous if and only if for every convergent sequence $(x_n, y_n) \to (x, y)$, the sequence $\varphi(x_n, y_n) \to \varphi(x, y)$, and this obviously is the case, because

$$\|(x_n + y_n) - (x + y)\|$$
$$\leq \|x_n - x\| + \|y_n - y\| \to 0 \text{ as } n \to \infty.$$

The proof of (ii) is similar. □

We close this section with the theorem playing a major role in the next topic on finite-dimensional NVS's.

7.9 Theorem. *Given a linearly independent set* $\{x_1, \ldots, x_n\} \subseteq (X, \|\cdot\|)$, *there is a positive constant* C *such that, for any* n*-tuple of scalars* $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)*, the following inequality holds.*

$$\|\textstyle\sum_{i=1}^n \alpha_i x_i\| \geq C \sum_{i=1}^n |\alpha_i|. \tag{7.9}$$

Proof. Note that (7.7) holds trivially if $\sum_{i=1}^n |\alpha_i| = 0$. We thus assume that at least one $\alpha_i \neq 0$. Then, with $\beta_i := \alpha_i / \sum_{i=1}^n |\alpha_i|$, we have

$$\|\textstyle\sum_{i=1}^n \beta_i x_i\| \geq C, \text{ such that } \sum_{i=1}^n |\beta_i| = 1, \tag{7.9a}$$

which is equivalent to (7.9) if at least one of β_i's is not zero. That is, we need to show that given a linearly independent set of vectors $x_1, \ldots, x_n$, there is a positive constant C such that for any an n-tuple of scalars $\beta_i, \ldots, \beta_n$ with $\sum_{i=1}^n |\beta_i| = 1$ the inequality

$$\|\textstyle\sum_{i=1}^n \beta_i x_i\| \geq C$$

holds. Assume the opposite, that is, for any n-tuple of linearly independent vectors $x_1, \ldots, x_n$, any n-tuple of scalars $\beta_i, \ldots, \beta_n$ with $\sum_{i=1}^n |\beta_i| = 1$, and any positive constant C,

$$\|\textstyle\sum_{i=1}^n \beta_i x_i\| < C.$$

Therefore, for any $k = 1, 2, \dots,$ there are n-tuples of linearly independent vectors $x_1, \dots, x_n$ and an n-tuple of scalars $\beta_1^{(k)}, \dots, \beta_n^{(k)}$ with $\sum_{i=1}^{n} \left|\beta_i^{(k)}\right| = 1$ such that

$$\left\|\sum_{i=1}^{n} \beta_i^{(k)} x_i\right\| < \tfrac{1}{k}.$$

Then,

$$\left\|\sum_{i=1}^{n} \beta_i^{(k)} x_i\right\| \to 0 \text{ as } k \to \infty.$$

Now, the sequence $\left\{\beta_1^{(k)}, \dots, \beta_n^{(k)} : k \in \mathbb{N}\right\}$ is obviously bounded. Thus, by Bolzano-Weierstrass theorem, the sequence $\left\{\beta_1^{(k)} : k \in \mathbb{N}\right\}$ has a convergent subsequence $\left\{\beta_1^{(k_{s1})} : s^1 \in \mathbb{N}\right\}$ whose limit we denote by δ_1. Then, from the subsequence $\left\{\beta_2^{(k_{s1})} : s^1 \in \mathbb{N}\right\}$ there is a convergent subsequence $\left\{\beta_2^{(k_{s2})} : s^2 \in \mathbb{N}\right\}$ whose limit is δ_2. Continuing sifting every next subsequence we finally arrive at the subsequences

$$\left\{\beta_i^{(k_{sn})}, i = 1, \dots, n : s^n \in \mathbb{N}\right\} \to \{\delta_1, \dots, \delta_n\},$$

such that

$$\left\|\sum_{i=1}^{n} \beta_i^{(k_{sn})} x_i\right\| \to 0 \text{ as } s^n \to \infty$$

and

$$\sum_{i=1}^{n} \left|\beta_i^{(k_{sn})}\right| = 1$$

and thus by continuity of the Euclidean norm,

$$\sum_{i=1}^{n} |\delta_i| = 1. \tag{7.9b}$$

On the other hand, by continuity of norm $\|\cdot\|$ (Theorem 7.3(iv)),

$$\left\|\sum_{i=1}^{n} \beta_i^{(k_{sn})} x_i\right\| \to \left\|\sum_{i=1}^{n} \delta_i x_i\right\| = 0.$$

Thus $\sum_{i=1}^{n} \delta_i x_i = \theta$. Because by (7.9b) not all δ_i's are zeros, $x_1, \dots,$ x_n must be linearly dependent, which is a contradiction. □

PROBLEMS

7.1 Show that $(\mathbb{R}^n, \|\cdot\|_e)$ defined in Example 7.7 (i) is an NVS and then show that it is a Banach space.

7.2 Define the space l^∞ as the set of all bounded sequences $x = \{x_1, x_2, \dots\} \subseteq \mathbb{C}$. Show that l^∞ is an NVS with the norm defined as $\|x\| = \sup\{|x_i|\colon i = 1, 2, \dots\}$.

7.3 Define the space c $\subseteq l^\infty$ as the subset of all convergent subsequences and let $c_0 \subseteq$ c be the set of all sequences convergent to zero. Show that c and c_0 are normed vector subspaces of l^∞ with the same norm as that in Problem 7.2.

7.4 Let $\mathcal{F}_*(\Omega)$ be the space of all bounded real-valued functions on Ω. Show that $\mathcal{F}_*$ is a vector space. Let $\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}$ be the supremum norm defined in Example 7.7(iii). Show that the supremum norm in $\mathcal{F}_*$ is indeed a norm and show that $\mathcal{F}_*$ is a Banach space with respect to this norm.

7.5 Show that $\|\cdot\|_p$ in Example 7.7(ii) is a norm.

7.6 Let $\|\cdot\|$ be a real-valued nonnegative function defined on a vector space X over a field $\mathbb{F}$ (which is $\mathbb{R}$ or $\mathbb{C}$) and let it have properties $(i$ - $iii)$ of Theorem 7.3. Show that $\|\cdot\|$ generates a TIH metric on X by $d(x, y) = \|x - y\|$.

7.7 Show that the pointwise limit x of the sequence $\{x^{(n)}\}$ in Example 7.7(ii) is also an l^p-limit.

NEW TERMS:

8. FINITE-DIMENSIONAL NORMED VECTOR SPACES

The notion of a finite-dimensional normed linear space was introduced by the German mathematician Hermann Minkowski (1864 - 1909) in his monograph *Geometrie der Zahlen* (Teubner, Leipzig, 1896). To honor his contribution, such spaces are often called *Minkowski spaces*.

We begin this section with the following significant property of finite-dimensional NVS's:

8.1 Theorem. *Any finite-dimensional NVS is a Banach space.*

Proof. Let n be the dimension of an NVS X and let $\{b_1, \dots, b_n\}$ be a Hamel basis for X. Let $\{x_k\} \subseteq X$ be a Cauchy sequence. Then, every element of the sequence can uniquely be expressed as

$$x_k = \sum_{i=1}^{n} \alpha_i^{(k)} b_i \tag{8.1}$$

Because $\{x_k\}$ is Cauchy, for every $\varepsilon > 0$, there is an $N(\varepsilon)$-tail $\{x_N, x_{N+1}, \dots\}$ of $\{x_k\}$ such that $\|x_k - x_m\| < \varepsilon$ for all elements of the $N(\varepsilon)$-tail. Since $b_1, \dots, b_n$ are linearly independent, due to Theorem 7.7, there is a universal constant $C > 0$ such that

$$\|x_k - x_m\| = \left\|\sum_{i=1}^{n} \left[\alpha_i^{(k)} - \alpha_i^{(m)}\right] b_i\right\| \geq C \sum_{i=1}^{n} \left|\alpha_i^{(k)} - \alpha_i^{(m)}\right|$$

good for all $k, m \geq N(\varepsilon)$.

Therefore, for all k and m greater than or equal to $N(\varepsilon)$,

$$\left|\alpha_i^{(k)} - \alpha_i^{(m)}\right| \leq \sum_{i=1}^{n} \left|\alpha_i^{(k)} - \alpha_i^{(m)}\right| < \varepsilon / C.$$

Consequently, $\left\{\alpha_i^{(k)}\right\}$ is a Cauchy sequence in $(\mathbb{F}, |\cdot|)$ and the latter is complete (whether $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). If α_i is the limit of $\alpha_i^{(k)}, i = 1, \dots, n,$ then the element $x := \sum_{i=1}^{n} \alpha_i b_i$ belongs to X and we have by the triangle inequality:

$$\|x - x_k\| = \left\|\sum_{i=1}^{n}\left[\alpha_i - \alpha_i^{(k)}\right]b_i\right\| \le \sum_{i=1}^{n}\left|\alpha_i - \alpha_i^{(k)}\right|\|b_i\|.$$

For each i, $\alpha_i^{(k)}$ is convergent to α_i. Therefore, for each choice of $\frac{\varepsilon}{\|b_i\|n} > 0$, there is an $N_i\left(\frac{\varepsilon}{\|b_i\|n}\right)$-tail of $\left\{\alpha_i^{(k)}\right\}$ such that for all terms of the $N_i\left(\frac{\varepsilon}{\|b_i\|n}\right)$-tail, the inequality $\left|\alpha_i - \alpha_i^{(k)}\right| < \frac{\varepsilon}{\|b_i\|n}$ holds. If $N = \max\{N_1, \ldots, N_n\}$, then

$$\|x - x_k\| \le \sum_{i=1}^{n}\left|\alpha_i - \alpha_i^{(k)}\right|\|b_i\| < \varepsilon \tag{8.1a}$$

for all $k \ge N$, that is, for each ε, there is an $N(\varepsilon)$-tail of $\{x_k\}$ with property (8.1a). Thus, the sequence $\{x_k\}$ is convergent. □

Combining Theorems 8.1 and 5.1 we arrive at the following.

8.2 Corollary. *Any finite-dimensional subspace of a normed vector subspace is a closed subspace of this space.* □

According to Theorem 6.7, every compact subset of a metric space is closed and bounded. The converse of this theorem was part two of Theorem 6.8 (Heine - Borel) holding for the Euclidean space. Beyond the Heine-Borel theorem, we learned that this converse did not hold in discrete metric spaces. It stands to reason to investigate in which spaces the converse of Theorem 6.7 is valid.

8.3 Definition. We say that an NVS (or metric space) has *the Heine - Borel property* if every closed and bounded subset of the space is compact. □

8.4 Theorem. *Every finite-dimensional NVS space has the Heine - Borel property.*

Proof. Suppose A is a closed and bounded subset of an n-dimensional NVS X. Let $\{x_k\} \subseteq A$ be a sequence and $\{b_1, \ldots, b_n\}$ be a Hamel basis for X. Then,

$$x_k = \sum_{i=1}^{n}\alpha_i^{(k)}b_i,\ k = 1, 2, \ldots \tag{8.3}$$

Because A is bounded, the sequence is also bounded, that is, there is a constant M such that $\|x_k\| \leq M$ for each k. Due to Theorem 7.7, there is a constant $C > 0$ such that

$$\|x_k\| = \left\|\sum_{i=1}^n \alpha_i^{(k)} b_i\right\| \geq C \sum_{i=1}^n \left|\alpha_i^{(k)}\right|. \tag{8.3a}$$

Inequality (8.3a) yields $\left|\alpha_i^{(k)}\right| \leq M$, $i = 1, \ldots, n$. By the Bolzano - Weierstrass theorem, $\left\{\alpha_i^{(k)}\right\}$ has a convergent subsequence for each k. Repeating the same process as in the proof of Theorem 7.7, we can extract from the sequence $\left\{\alpha_1^{(k)}, \ldots, \alpha_n^{(k)} : k \in \mathbb{N}\right\}$ a convergent subsequence $\left\{\alpha_1^{(k_s)}, \ldots, \alpha_n^{(k_s)}\right\}$ whose limit we denote by $\{\delta_1, \ldots, \delta_n\}$. Consequently, by (8.3), the sequence $\{x_k\}$ has a subsequence $\{x_{k_s}\}$ that can easily be shown to converge to point

$$x : = \sum_{i=1}^n \delta_i b_i \in A,$$

in as much as A is closed. In other words, A is sequentially compact. Therefore, by Theorem 6.3, A is compact. ☐

The following theorem by Frigyes Riesz sheds more light on the Heine - Borel property for NVS's, where we establish a necessary and sufficient condition for it.

8.5 Theorem (F. Riesz). *An NVS* $(X, \|\cdot\|)$ *has the Heine - Borel property if and only if it is finite-dimensional.*

To prove this theorem we need the following.

8.6 Lemma. *Suppose* $(X, \|\cdot\|)$ *is an infinite-dimensional NVS. Then, there is a linearly independent sequence* $\{x_n\} \subseteq S(\theta, 1)$ (*where* $S(\theta, 1)$ *is the unit sphere centered at zero*) *such that for each pair* x_j, x_k *of distinct elements,* $\|x_j - x_k\| \geq 1$.

Proof. We use the induction principle to prove this statement. Assume that for $n \geq 1$, there is an n-tuple $\{x_1, \ldots, x_n\}$ of linearly independent elements all belonging to the unit sphere $\partial B(\theta, 1)$ such that $\|x_j - x_k\| \geq 1$. Denote $A = \text{span}\{x_1, \ldots, x_n\}$. Let $z \in A^c$ and let $\delta = d(z, A^c)$. Because, by Corollary 8.2, A is closed, A^c is open and thus there is a ball $B(z, r) \subseteq A^c$ implying that $\delta > 0$.

Now, since A is a subspace, it follows that $\frac{1}{\delta}A = \left\{\frac{1}{\delta}x : x \in A\right\}$ $= A$. Denote $z_0 = z/\delta$. Because the metric induced by the norm is homothetic, we have

$$1 = \tfrac{1}{\delta}d(z, A) = d(z_0, A).$$

From the latter it follows that there is a sequence $\{y_n\} \subseteq A$ such that

$$\|y_n - z_0\| \to 1.$$

(Indeed, $1 = d(z_0, A) = \inf\{\|z_0 - x\| : x \in A\}$ means that for each $\varepsilon > 0$, there is an $x \in A$ such that $1 \le \|z_0 - x\| < 1 + \varepsilon$.)

Now, we have

$$\|y_n\| \le \|y_n - z_0\| + \|z_0\| < R + \|z_0\| = : M.$$

Therefore, $\{y_n\} \subseteq C(\theta, M) \cap A$, which is a closed and bounded subset of A. Because A has the Heine-Borel property, $C(\theta, M) \cap A$ is compact and thus by Theorem 6.3, $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$ that has a limit point, say y, in A. Because $\|y_n - z_0\|$ $\to 1$, we must also have $\|y_{n_k} - z_0\| \to 1$ and thus

$$1 = d(z_0, A) \le \|y - z_0\|$$

$$\le \|y - y_{n_k}\| + \|y_{n_k} - z_0\| \to 0 + 1$$

implying that $\|y - z_0\| = 1$.

Define $x_{n+1} : = y - z_0$. We show that x_{n+1} fits the pattern of the elements from $\{x_1, \dots, x_n\}$.

(i) Obviously, $x_{n+1} \in S(\theta, 1)$.

(ii) If x_{n+1} is linearly dependent with $\{x_1, \dots, x_n\}$, then $\sum_{i=1}^{n}\alpha_i x_i = z_0 - y$. Because $y \in A$, clearly $y = \sum_{i=1}^{n}\beta_i x_i$ and hence $z_0 = \sum_{i=1}^{n}(\alpha_i + \beta_i)x_i$ and thus must belong to A. So, it implies the linear independence of the tuple $\{x_1, \dots, x_{n+1}\}$.

(iii) Finally, we show that $\|x_j - x_{n+1}\| \geq 1$. In as much as A is a subspace, $A = A - y$ if $y \in A$ and thus

$$1 = d(z_0, A) = d(z_0 - y, A - y) = d(x_{n+1}, A)$$

$$\leq d(x_{n+1}, x) = \|x_{n+1} - x\|, \text{ for each } x \in A.$$

In particular, it holds true for $x_j \in \{x_1, \ldots, x_n\} \subseteq A$.

Thus, if we continue this process, we indeed collect a sequence $\{x_n\}$ as the lemma claims. □

Now we are back to the proof of the theorem.

Proof of Theorem 8.5. Suppose that X is infinitely dimensional and let $S(\theta, 1)$ be the unit sphere in X. Obviously, $S(\theta, 1)$ is bounded. Furthermore, $S(\theta, 1) = C(\theta, 1) \backslash B(\theta, 1)$, where $C(\theta, 1)$ is the unit closed ball, and thus $S(\theta, 1) = C(\theta, 1) \cap B^c(\theta, 1)$ is the intersection of two closed subsets. By Lemma 8.6, there is a sequence $\{x_n\} \subseteq S(\theta, 1)$, which due to the property $\|x_m - x_n\| \geq 1$, is not convergent, nor can it have a convergent subsequence. Thus, by Theorem 6.3, $S(\theta, 1)$ is not compact. Therefore, X fails to have the Heine - Borel property. □

8.7 Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are called *equivalent* or *similar* (in notation, $\|\cdot\|_1 \approx \|\cdot\|_2$) if there are two positive constants $K \leq M$ such that for any $x \in X$,

$$K\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1. \qquad \square$$

We show that on a finite-dimensional space, all norms are equivalent.

8.8 Proposition. *The similarity in Definition 8.7 defines an equivalence relation on the set of all norms on X.* □

(See Problem 8.1.)

8.9 Proposition. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on a vector space X and if $\{x_n\}$ is a sequence in X, then*

$$(i) \quad x_n \overset{\|\cdot\|_1}{\to} x \;\Leftrightarrow\; x_n \overset{\|\cdot\|_2}{\to} x.$$

(ii) $\{x_n\}$ is Cauchy in $\|\cdot\|_1$ if and only if $\{x_n\}$ is Cauchy in $\|\cdot\|_2$.

Proof. Because $\|\cdot\|_1 \approx \|\cdot\|_2$, there is a positive constant M such that

$$\|x\|_2 \leq M\|x\|_1. \tag{8.9}$$

(i) Let $x_n \xrightarrow{\|\cdot\|_1} x$. Then, for any $\varepsilon > 0$, there is an $N(\varepsilon)$ tail $\{x_N, x_{N+1}, \ldots\}$ of $\{x_n\}$ such that $\|x - x_n\| < \varepsilon/M$ for all $x_n \in \{x_N, x_{N+1}, \ldots\}$. From (8.9) we have

$$\|x - x_n\|_2 \leq M\|x - x_n\|_1 < M\varepsilon/M = \varepsilon.$$

Obviously, the roles of the norms can be interchanged.

(ii) If $\{x_n\}$ is $\|\cdot\|_1$-Cauchy, then for each $\varepsilon > 0$ there is an $N(\varepsilon)$ tail $\{x_N, x_{N+1}, \ldots\}$ of $\{x_n\}$ such that $\|x_m - x_n\| < \varepsilon/M$ for all $x_m, x_n \in \{x_N, x_{N+1}, \ldots\}$. From (8.9) we have

$$\|x_m - x_n\|_2 \leq M\|x_m - x_n\|_1 < M\varepsilon/M = \varepsilon.$$

Obviously, the roles of the norms can be interchanged. □

8.10 Corollary. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on a vector space X, then $(X, \|\cdot\|_1)$ is Banach if and only if $(X, \|\cdot\|_2)$ is Banach.* □

8.11 Example. In n-dimensional Euclidean space the norms

$$\|x\|_e = \sqrt{x_1^2 + \cdots + x_n^2}$$

and

$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\},$$

where $x = (x_1, \ldots, x_n)'$, are equivalent. Indeed, from

$$n\|x\|_\infty^2 \geq x_1^2 + \cdots + x_n^2$$

we have

$$\|\cdot\|_\infty \geq \tfrac{1}{\sqrt{n}}\|\cdot\|_e.$$

On the other hand, obviously, $\|\cdot\|_\infty \leq \|\cdot\|_e$. Thus, with the choice of $K = \frac{1}{\sqrt{n}}$ and $M = 1$ we have

$$K\|\cdot\|_e \leq \|\cdot\|_\infty \leq M\|\cdot\|_e. \qquad \square$$

Example 8.11 can be generalized in

8.12 Theorem. *In a finite-dimensional vector space all norms are equivalent.*

Proof. Let $\{b_1, \ldots, b_n\}$ be a Hamel basis for X and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be any two norms on X. Then, if $x \in X$ and thus is representable as $x = \sum_{i=1}^n \alpha_i b_i,$ by Theorem 7.9, there is a positive constant C_1 such that

$$\|x\|_1 = \|\textstyle\sum_{i=1}^n \alpha_i b_i\|_1 \geq C_1 \sum_{i=1}^n |\alpha_i|.$$

Let $m :\ = \max\{\|b_i\|_2 : i = 1, \ldots, n\}$. Then,

$$\|x\|_2 \leq \textstyle\sum_{i=1}^n |\alpha_i| \|b_i\|_2 \leq m \sum_{i=1}^n |\alpha_i| \leq M\|x\|_1,$$

where $M = \frac{m}{C_1}$. Analogously,

$$\|x\|_1 \leq \tfrac{1}{K}\|x\|_2. \qquad \square$$

8.13 Definition. Let X be an n-dimensional vector space over $\mathbb{F}$ (which is $\mathbb{R}$ or $\mathbb{C}$) with a Hamel basis $\{b_1, \ldots, b_n\}$. For any $x \in X$ such that $x = \sum_{i=1}^n \alpha_i b_i,\ \alpha_i \in \mathbb{F},$ define

$$\|x\|_* = \sqrt{\sum_{i=1}^{n} |\alpha_i|^2}. \tag{8.13}$$

According to Problem 8.2, this is a norm on $(X, \mathbb{C})$, called the *standard norm*. □

Notice that the standard norm is also a norm on $\mathbb{F}^n$.

8.14 Remarks.

(i) Proposition 8.9(i) reconciles the notions of equivalence between two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and two metrics d_1 and d_2 induced by $\|\cdot\|_1$ and $\|\cdot\|_2$ (see Remark 7.4) in any NVS. Indeed, according to Remark 4.9, any two equivalent norms generate the same topology in an NVS, i.e. $\tau(\|\cdot\|_1) = \tau(\|\cdot\|_2)$. If in addition, an NVS is finite-dimensional, then by Theorem 8.12, any two norms induce the same topology. Furthermore, any norm in a finite-dimensional NVS induces the same topology as that of the standard norm.

(ii) The standard norm in (8.13) clearly resembles the n-dimensional Euclidean norm, which according to (i) induces a unique topology on $(\mathbb{R}, \|\cdot\|)$, with respect to any norm $\|\cdot\|$, called the *usual topology*, in notation $(\mathbb{R}, \tau_e)$. □

PROBLEMS

8.1 Prove Proposition 8.8. *The similarity in Definition* 8.7 *defines an equivalence relation on the set of all norms on X.*

8.2 Prove that the function $\|\cdot\|_*$ in Definition 8.13 is a norm.

8.3 Define the function $f : \mathbb{F}^n \to \mathbb{R}_+$ as

$$f(\alpha_1, \ldots, \alpha_n) = \|\sum_{i=1}^{n} \alpha_i b_i\|.$$

Prove that f is continuous with respect to $\|\cdot\|_e$.

Problems 8.4 - 8.6 offer an alternative proof to Theorem 8.12 by showing that any norm $\|\cdot\|$ on a finite-dimensional NVS X (which is assumed throughout) is equivalent to the standard norm $\|\cdot\|_*$.

8.4 Let $(X, \|\cdot\|)$ be a finite-dimensional NVS and $\|\cdot\|_*$ be the standard norm defined in (8.13). Show that there is a positive constant M such that for each $x \in X$,

$$\|x\| \leq M\|x\|_*.$$

8.5 Let $(X, \|\cdot\|)$ be a finite-dimensional NVS. Let $S(0,1)$ denote the unit sphere in $\mathbb{F}^n$ centered at zero and f be the function on $\mathbb{F}^n$ defined in Problem 8.3. Show that f reaches its maximum and minimum values in $\mathbb{R}_+$ on $S(0,1)$.

8.6 Let $(X, \|\cdot\|)$ be a finite-dimensional NLS. Show that there is a positive K constant such that

$$K\|\cdot\|_* \leq \|\cdot\|.$$

8.7 Let $\|\cdot\|_1, \ldots, \|\cdot\|_k$ be norms in $\mathbb{R}^n$ and let $\|\cdot\| = \sum_{i=1}^k \|\cdot\|_i$. Show that $\|\cdot\|$ is a norm in $\mathbb{R}^n$ and that $\|\cdot\|$ induces a topology, identical to the usual topology.

8.8 Let Y be a finite-dimensional subspace of an NVS $(X, \|\cdot\|)$. Suppose there is a bounded sequence $\{y_n\} \subseteq Y$. Show that this sequence has a subsequence convergent in Y. (This generalizes a known result from calculus that any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.)

NEW TERMS:

Minkowski, Herman, 160
Minkowski spaces, 160
Heine -Borel property, 161
Riesz, Frigyes, 162
Riesz criterion of Heine - Borel property, 162
equivalent (similar) norms, 164
similar (equivalent) norms, 164
$\|x\|_*$, standard norm, 167
$(\mathbb{R}, \tau_e)$, usual topology, 167

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