Chapter II
Analysis tools

The goal of this chapter is to describe the analysis tools that we use in later chapters. We have gathered together fundamental concepts required to study many linear or nonlinear evolution partial differential equations coming from many areas of physics and biology, for instance.

We start in Section 2 by presenting, without proof, some classic results of functional analysis such as the open mapping theorem, the Banach–Steinhaus theorem and the Lax–Milgram theorem, proofs of which can easily be found in the literature (see, e.g., [27, 104, 105]). We also give definitions of weak and weak-$\star$ convergence, which are frequently used in the analysis of partial differential equations. We pay particular attention to the expression of these results into some fundamental spaces, namely the Lebesgue spaces. The section is completed by a short introduction to distribution theory and by the description of some basic properties of Lipschitz continuous functions.

Section 3 aims at describing some tools around the notion of compactness which is fundamental when one deals with nonlinear terms in partial differential equations. We recall in particular the Schauder fixed-point theorem.

In the analysis of evolution problems, one of the usual ways for establishing existence theorems is first to obtain energy estimates. In general, these are deduced from elementary differential inequalities involving real functions of a single real variable (the time variable t, in the problems which concern us). In Section 4 of this chapter therefore, we describe the links between the concepts of weak differentiation and standard differentiation as applied to numerical functions of one single real variable. Finally, we prove the various Gronwall type inequalities, which allows us to obtain the desired estimates in most cases.

Section 5 is dedicated to the introduction and study of the spaces of functions integrable on an interval of $\mathbb{R}$ with values in a Banach space. This is also known as the Bochner integral theory. In particular, we prove the Aubin–Lions–Simon compactness theorem [14, 84, 109], a fundamental result for the study of nonlinear problems. The main ingredient of this proof is the Ascoli

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theorem, reviewed at the start of this chapter. We also present the definition and main properties of the Fourier transform for this class of functions.

We conclude the chapter, in Section 6, with a very short introduction to the spectral theory of self-adjoint unbounded operators with compact resolvent.

1 Main notation

Throughout this book, the space dimension is denoted by $d \in \mathbb{N}^*$ (typically $d = 2$ or $d = 3$ in the case of fluid mechanics applications). The Euclidean norm on $\mathbb{R}^d$ is denoted by $x \mapsto |x|$ and the associated inner product by $(x, y) \mapsto x \cdot y$.

For all multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we denote its length by $|\alpha| = \alpha_1 + \dots + \alpha_d$. For any function f we define $\partial^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} f$ as soon as this partial derivative exists (in a classic or in a weak sense).

For any open set $\Omega \subset \mathbb{R}^d$ we use the following standard functional spaces.

- The set $\mathcal{C}^k(\Omega)$, $k \geq 0$, of functions with continuous partial derivatives up to order k.
- The subset $\mathcal{C}_b^k(\Omega) \subset \mathcal{C}^k(\Omega)$ of functions such that all partial derivatives up to order k are bounded.
- The set $\mathcal{C}^{0,\alpha}(\Omega)$, $\alpha \in]0, 1]$ of α-Hölder continuous functions. In the case $\alpha = 1$, $\mathcal{C}^{0,1}(\Omega)$ is the set of Lipschitz continuous functions. The Lipschitz seminorm of such a function is defined by

$$\mathrm{Lip}(f) = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|} < +\infty.$$

- The set $\mathcal{C}^{k,\alpha}(\Omega)$, $k \geq 0$, $\alpha \in]0, 1]$ of functions in $\mathcal{C}^k(\Omega)$ whose partial derivatives of order k are α-Hölder continuous.
- The set $\mathcal{C}_c^\infty(\Omega)$ of functions in $\mathcal{C}^\infty(\Omega)$ which are compactly supported in Ω. Another usual notation for this space, in particular in the theory of distributions, is $\mathcal{D}(\Omega)$.
- The set $\mathcal{C}_c^\infty(\overline{\Omega})$ of the restrictions to Ω of functions in $\mathcal{C}_c^\infty(\mathbb{R}^d)$.

Moreover, for any function u defined on Ω we denote as $\bar{u}$ its extension by 0 on the whole space defined by

$$\bar{u}(x) = \begin{cases} u(x), & \text{for } x \in \Omega, \\ 0, & \text{for } x \notin \Omega. \end{cases}$$

For any $x \in \Omega$, we define $\delta(x)$ to be the signed distance from x to the boundary, which is defined by

$$\delta(x) = \begin{cases} d(x, \partial\Omega) & \text{for } x \in \overline{\Omega}, \\ -d(x, \partial\Omega) & \text{for } x \notin \Omega. \end{cases} \tag{II.1}$$

By using the triangle inequality, it is obvious to check that δ is Lipschitz continuous on $\mathbb{R}^d$ and that $\mathrm{Lip}(\delta) \le 1$.

2 Fundamental results from functional analysis

2.1 Banach spaces

In this section we recall essential results of functional analysis. We do not provide proofs; the reader can find these in the classic monographs on the subject such as [27], [104], and [105].

For any normed vector space E, we denote its topological dual as E', that is, the space of continuous linear functionals on E. For $f \in E'$ and $x \in E$, we introduce the duality bracket

$$\langle f, x\rangle_{E',E} = f(x).$$

We reserve the notation $(\cdot,\cdot)_H$ for a scalar product in a Hilbert space, H.

Let E and F be two normed vector spaces and $S : E \mapsto F$ be a continuous linear function. We define the *adjoint* or *transposed* function, denoted ${}^tS : F' \mapsto E'$, by

$$\langle {}^tSf, x\rangle_{E',E} = \langle f, Sx\rangle_{F',F}, \forall f \in F', \forall x \in E.$$

From the Hahn–Banach theorem, we can express the norm of an element from a normed vector space E by duality as follows.

Proposition II.2.1. *Let E be a normed vector space. Then for all $x \in E$, we have*

$$\|x\|_E = \sup_{f\in E', f\neq 0} \frac{|\langle f, x\rangle_{E',E}|}{\|f\|_{E'}} = \sup_{\|f\|_{E'}\le 1} |\langle f, x\rangle_{E',E}|.$$

Another consequence of the Hahn–Banach theorem is the following useful density criterion for a subspace of a given normed space.

Proposition II.2.2. *Let E be a normed vector space and F be a vector subspace of E. We assume that any continuous linear functional on E which vanishes on F is identically zero. Then, F is a dense subspace of E.*

The following result (due to Banach), gives a characterisation of the isomorphisms between Banach spaces.

Theorem II.2.3 (Open mapping). *Let E and F be two Banach spaces. If u is a surjective, continuous linear function from E into F, then u is an open* *map, which means that the image under u of all open sets of E is an open set of F. In particular, if u is bijective, its reciprocal function is continuous and, consequently, spaces E and F are algebraically and topologically isomorphic.*

Finally, the last result that we recall in this section, which is sometimes called the "uniform boundedness principle" shows that if a family of continuous linear functions defined on a Banach space is pointwise bounded, then it is uniformly bounded.

Theorem II.2.4 (Banach–Steinhaus). *Let $(u_i)_{i\in I}$ be a family of continuous linear functions of a Banach space E within a normed vector space F, indexed by a set I. We assume that for all $x \in E$, the family*

$$(u_i(x))_{i\in I},$$

is bounded in F. Then, the family $(u_i)_{i\in I}$ is uniformly bounded in the sense of the norm of the operators; that is,

$$\sup_{i\in I} \|u_i\|_{L(E,F)} < +\infty;$$

or equivalently,

$$\exists C > 0, \textit{such that } \|u_i(x)\|_F \leq C\|x\|_E, \forall i \in I, \forall x \in E.$$

We conclude this section by introducing the Lax–Milgram theorem, which is an important tool in the study of linear partial differential problems in variational formulation.

Theorem II.2.5 (Lax–Milgram). *Let V be a Hilbert space, $a : V \times V \to \mathbb{R}$ a bilinear form, and $L : V \to \mathbb{R}$ a linear form.*

Assume that a and L are continuous and that a is coercive, that is,

$$\exists \alpha > 0,\ a(v, v) \geq \alpha\|v\|_V^2,\ \forall v \in V;$$

then there exists a unique solution $v \in V$ to the problem

$$a(v, w) = L(w), \forall w \in V. \tag{II.2}$$

Moreover, this solution satisfies

$$\|v\|_V \leq \frac{\|L\|_{V'}}{\alpha}. \tag{II.3}$$

2.2 Weak and weak-$\star$ convergences

We do not go into the details here of the general theory of weak and weak-$\star$ topologies (see [27] for a more complete study). Rather, we simply recall the sequential properties of these topologies, which are essential later. In this book, these notions are mainly used in the framework of Lebesgue and Sobolev spaces (see Section 2.3.4).

Definition II.2.6. *Let E be a Banach space and E' its dual space.*

- *We say that a sequence $(u_n)_n$ of elements of E weakly converges towards $u \in E$, if for any $f \in E'$ we have*

$$f(u_n) = \langle f, u_n \rangle_{E',E} \xrightarrow[n\to\infty]{} \langle f, u \rangle_{E',E} = f(u).$$

- *We say that a sequence $(f_n)_n$ of elements of E' weakly-$\star$ converges towards $f \in E'$, if for any $u \in E$, we have*

$$f_n(u) = \langle f_n, u \rangle_{E',E} \xrightarrow[n\to\infty]{} \langle f, u \rangle_{E',E} = f(u).$$

Of course, as soon as the space on which we are working is infinite-dimensional (functional spaces of type $L^p(\Omega)$, for example), the closed bounded subsets of this space are not necessarily compact for the topology of the norm on E. Nevertheless, the following result establishes the property of weak compactness of closed bounded sets.

Theorem II.2.7 (Weak and Weak-$\star$ compactness).

- *Let E be a reflexive Banach space (i.e., E is isomorphic with E'' via the natural embedding). Then, from any* bounded *sequence of elements of E, we can extract a subsequence which weakly converges in E.*
- *Let E be a separable Banach space (i.e., one which contains a dense countable subset). Then, from any* bounded *sequence of elements of E', we can extract a subsequence which weakly-$\star$ converges in E'.*

One of the important consequences of the Banach–Steinhaus theorem (Theorem II.2.4) is the property of lower semicontinuity of the norm for weak and weak-$\star$ topologies on a Banach space.

Corollary II.2.8. *Let E be a Banach space, and $(u_n)_n$ be a sequence of elements of E (or E', respectively) which weakly converges (or weakly-$\star$, respectively) towards $u \in E$ (or $u \in E'$, respectively). Then the sequence $(u_n)_n$ is bounded in E (or in E', respectively) and we have*

$$\|u\|_E \leq \liminf_{n\to\infty} \|u_n\|_E, \text{ (resp., } \|u\|_{E'} \leq \liminf_{n\to\infty} \|u_n\|_{E'}\text{)}.$$

The following proposition is often used to prove the weak (resp., weak-$\star$) convergence of a whole sequence.

Proposition II.2.9. *Let E be a reflexive Banach space (resp., the dual of a separable Banach space) and $(x_n)_n$ a bounded sequence in E.*

We assume that there exists $x \in E$ such that every weakly convergent (resp., weakly-$\star$ convergent) subsequence of $(x_n)_n$ has a limit equal to x; then the whole sequence $(x_n)_n$ weakly converges (resp., weakly-$\star$ converges) to x.

Proof.

We only give the proof in the reflexive case, the other case being similar. Assume that $(x_n)_n$ does not weakly converge to x. This means that there exists $f \in E'$ such that $(\langle f, x_n \rangle_{E',E})_n$ does not converge to $\langle f, x \rangle_{E',E}$. Hence, there exists $\varepsilon > 0$ and a subsequence $(x_{\varphi(n)})_n$ such that

$$|\langle f, x_{\varphi(n)} - x \rangle_{E',E}| \geq \varepsilon, \forall n \geq 0. \tag{II.4}$$

Since $(x_{\varphi(n)})_n$ is bounded in E which is reflexive, Theorem II.2.7 shows that there exists a new subsequence $(x_{\varphi(\psi(n))})_n$ that weakly converges in E. By assumption its weak limit is necessarily equal to x which implies that

$$\langle f, x_{\varphi(\psi(n))} - x \rangle_{E',E} \xrightarrow[n\to\infty]{} 0.$$

This is a contradiction with (II.4) and the claim is proved.

□

A consequence of this result is that a bound in a "small" space and a weak convergence in a "large" space implies the weak convergence in the "small" space. The precise statement is the following.

Proposition II.2.10. *Let E, F, G be three Banach spaces such that $E \subset G$, $F \subset G$ with continuous embeddings. We assume that F is reflexive.*

Let $(x_n)_n$ be a sequence in $E \cap F$ such that there exists $x \in E$ satisfying

$$(x_n)_n \text{ is bounded in } F,$$

$$(x_n)_n \text{ weakly converges towards } x \text{ in } E.$$

Then,

$$(x_n)_n \text{ weakly converges towards } x \text{ in } F.$$

Proof.

From Proposition II.2.9, the claim will be proved if we show that x is the unique possible weak limit in F of subsequences of $(x_n)_n$.

Let $(x_{\varphi(n)})_n$ a subsequence which weakly converges in F towards some limit $y \in F$. The embedding $F \subset G$ is continuous, therefore we know that $(x_{\varphi(n)})_n$ weakly converges to y in G.

On the other hand, we know by assumption that $(x_{\varphi(n)})_n$ weakly converges towards x in E. The embedding $E \subset G$ is continuous, therefore we deduce that $(x_{\varphi(n)})_n$ weakly converges to x in G. It follows that $y = x$ and the claim is proved.

□

Remark II.2.1. This result can be easily adapted to the case of the weak-$\star$ convergence.

We can now give a useful criterion of strong convergence for weakly convergent sequences in a Hilbert space.

Proposition II.2.11. *Let H be a Hilbert space and $(u_n)_n$ be a sequence of elements of H which weakly converges towards u in H. Let us assume that*

$$\limsup_{n\to\infty} \|u_n\|_H \leq \|u\|_H,$$

then the sequence $(u_n)_n$ strongly converges towards u in H.

Proof.
It is sufficient to write

$$\|u - u_n\|_H^2 = \|u\|_H^2 + \|u_n\|_H^2 - 2(u, u_n)_H.$$

Since the weak convergence gives $(u_n, u)_H \xrightarrow[n\to\infty]{} \|u\|_H^2$, we have

$$\limsup_{n\to\infty} \|u - u_n\|_H^2 = \|u\|_H^2 + \limsup_{n\to\infty} \|u_n\|_H^2 - 2\|u\|_H^2 \leq 0,$$

by using the assumption.

□

We later show (Proposition II.2.32) that this result is also valid in some Banach spaces (e.g. in the spaces $L^p(\Omega)$ with $1 < p < +\infty$).

Unfortunately, the concept of weak convergence, although easier to use, does not generally allow passing to the limit in nonlinear terms. As an example (see Section 2.3 for the main properties of Lebesgue spaces), let the sequence of functions $(u_n)_n$ be defined on $]0,1[$ by $u_n(x) = \sin(nx)$. Then $(u_n)_n$ weakly converges towards 0 in $L^2(]0,1[)$ (Riemann–Lebesgue lemma) and $\int_0^1 u_n^2\,dx$ converges towards $1/2$. Hence the sequence $(u_n^2)_n$ does not weakly converge towards 0 in $L^2(]0,1[)$. However, we note that the sequence $(u_n^2)_n$ does weakly converge in $L^2(]0,1[)$ but its limit is the constant function equal to $1/2$ and not 0.

Nevertheless, we prove in the following result that the product of a strongly converging sequence with a weakly converging one is a sequence which weakly converges towards the product of the limits.

Proposition II.2.12. *Let E, F, and G be three Banach spaces and let B be a continuous bilinear function of $E \times F$ in G. If $(u_n)_n$ is a sequence of elements of E which strongly converges towards u and $(v_n)_n$ is a sequence of elements of F which weakly converges towards v, then the sequence $(B(u_n, v_n))_n$ weakly converges towards $B(u,v)$ in G.*

Proof.
Let $\varphi \in G'$; we need to show that

$$\langle \varphi, B(u_n, v_n)\rangle_{G',G} \xrightarrow[n\to\infty]{} \langle \varphi, B(u,v)\rangle_{G',G}.$$

By using the bilinearity of B, we have

$$\begin{aligned} &|\langle \varphi, B(u_n, v_n) - B(u,v)\rangle_{G',G}| \\ \leq &\|\varphi\|_{G'} \|B(u-u_n, v_n)\|_G + |\langle \varphi, B(u, v_n - v)\rangle_{G',G}| \,. \end{aligned}$$

From Corollary II.2.8, the sequence $(v_n)_n$ is bounded. Hence, since the function B is continuous, the first term is estimated by

$$\|\varphi\|_{G'} \|B(u-u_n, v_n)\|_G \leq \|\varphi\|_{G'} \|B\| \|u-u_n\|_E \|v_n\|_F \leq C\|u-u_n\|_E \xrightarrow[n\to\infty]{} 0.$$

The function $x \in F \mapsto \langle \varphi, B(u,x)\rangle_{G',G}$ is a continuous linear functional on F because u is fixed in E and B is continuous. Hence from the definition of weak convergence, the second term also tends towards 0 when n tends towards infinity.

□

Remark II.2.2. If the space G is reflexive and if $(u_n)_n$ and $(v_n)_n$ converge only weakly towards u (or, respectively, v), then the sequence $(B(u_n, v_n))_n$ is bounded in G (because B is continuous and $(u_n)_n$ and $(v_n)_n$ are bounded). Hence, Theorem II.2.7 shows us that we can extract a subsequence which weakly converges in G towards a certain g.

The problem is that without the property of strong convergence, we cannot in general conclude that g is equal to the expected limit, which would be $B(u,v)$ as shown in the example given above.

As shown in later chapters, we need to establish strong convergence properties of the sequence studied in larger spaces in order to identify the limit g to be the product $B(u,v)$.

Therefore, in order to deal with nonlinearities, it is necessary to obtain strong convergence in one way or another. One way to do this is to use the compactness properties described in Section 3.

2.3 Lebesgue spaces

2.3.1 Definitions and main properties

Definition II.2.13 (Conjugate exponent). *For all $1 \leq p \leq +\infty$, we define* the conjugate exponent *p' of p by*

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

with the obvious conventions for $p = 1$ and $p = +\infty$.

This notation is used systematically throughout this text. We also note that for all p, we have $(p')' = p$.

- For $1 \leq p < +\infty$, the space $L^p(\Omega)$ is the set of Lebesgue-measurable functions on any open set Ω with real values, for which the pth power of the absolute value is integrable for the Lebesgue measure. For each $f \in L^p(\Omega)$ we set $\|f\|_{L^p} = \left(\int_\Omega |f|^p \, dx\right)^{1/p}$.
- The space $L^\infty(\Omega)$ is the set of Lebesgue-measurable functions which are essentially bounded on Ω. For each $f \in L^\infty(\Omega)$, we set $\|f\|_{L^\infty} = \mathrm{esssup}_\Omega |f|$.

In fact, the elements of these spaces have to be considered as the classes of functions which coincide except over null Lebesgue measure sets.

It can be shown (see Proposition II.2.21 and Remark II.2.3) that $\|.\|_{L^p}$ is a norm on $L^p(\Omega)$. Moreover, these spaces are Banach spaces.

- For $1 < p < +\infty$, the space $L^p(\Omega)$ is separable and reflexive. Moreover its dual is isomorphic with $L^{p'}(\Omega)$ where p' is the conjugate exponent of p.
- The space $L^1(\Omega)$ is separable but not reflexive, its dual being isomorphic with $L^\infty(\Omega)$.
- By contrast, the space $L^\infty(\Omega)$ is neither separable nor reflexive and its dual is strictly larger than $L^1(\Omega)$.

We conclude this introduction by recalling the following version of the change of variable theorem.

Definition II.2.14. *Let Ω, $\tilde{\Omega}$ be two open sets in $\mathbb{R}^d$. A map $T : \tilde{\Omega} \to \Omega$ is said to be a* Lipschitz diffeomorphism *if and only if*

- *T is a bijection.*
- *T and T^{-1} are Lipschitz-continuous.*

Notice that such a map is not in general a diffeomorphism in the usual sense because, in particular, it is not necessarily differentiable everywhere.

Proposition II.2.15. *Let Ω, $\tilde{\Omega}$ be two open sets in $\mathbb{R}^d$ and $T : \tilde{\Omega} \to \Omega$ a Lipschitz diffeomorphism.*

For any measurable function $u : \Omega \to \mathbb{R}$ and $1 \leq p \leq \infty$, we have

$$u \in L^p(\Omega) \Longleftrightarrow u \circ T \in L^p(\tilde{\Omega}).$$

Moreover, we have

$$C_1 \|u\|_{L^p(\Omega)} \leq \|u \circ T\|_{L^p(\tilde{\Omega})} \leq C_2 \|u\|_{L^p(\Omega)},$$

for some $C_1, C_2 > 0$ depending only on T.

2.3.2 Elementary inequalities

We give here rather general versions of Young's and Hölder's inequalities, without proof. We use these repeatedly in the following sections, without necessarily explicitly referencing them.

Proposition II.2.16 (Young's inequality). *Let $n \geq 2$, and $x_1, \dots, x_n$ be non-negative real numbers. Also, let $p_1, \dots, p_n$ be positive real numbers such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1.$$

We then have:

$$x_1 \cdots x_n \leq \frac{x_1^{p_1}}{p_1} + \cdots + \frac{x_n^{p_n}}{p_n}.$$

The proof of this inequality is a simple application of the concavity of the logarithm function. We can directly deduce an useful version of this inequality.

Corollary II.2.17. *Let $p_1, \dots, p_n$ be real numbers satisfying the hypothesis of the preceding proposition. For all positive $\varepsilon_1, \dots, \varepsilon_{n-1}$, there exists a $C(\varepsilon_1, \dots, \varepsilon_{n-1}) > 0$, such that for all positive $x_1, \dots, x_n$, we have*

$$x_1 \cdots x_n \leq \varepsilon_1 x_1^{p_1} + \cdots + \varepsilon_{n-1} x_{n-1}^{p_{n-1}} + C(\varepsilon_1, \dots, \varepsilon_{n-1}) x_n^{p_n}.$$

In other words, in Young's inequality, all the coefficients can be fixed except for one. Of course, the coefficient $C(\varepsilon_1, \dots, \varepsilon_{n-1})$ blows up when one of the ε_i tends towards 0.

From Young's inequality we can deduce Hölder's inequality which is stated in the following way.

Proposition II.2.18 (Hölder's inequality). *Let Ω be an open set of $\mathbb{R}^d$ and let $p_1, \dots, p_n$ be positive real numbers (possibly infinite). Let $r \in [1, +\infty]$ such that*

$$\frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}.$$

For all functions $f_1, \dots, f_n$, with $f_i \in L^{p_i}(\Omega)$, the product $f_1 \cdots f_n$ belongs to $L^r(\Omega)$ and we have

$$\|f_1 \cdots f_n\|_{L^r} \leq \|f_1\|_{L^{p_1}} \cdots \|f_n\|_{L^{p_n}}.$$

We also need the following generalisation of Fubini's theorem.

Proposition II.2.19. *Let $d \geq 2$ and $f_1, \dots, f_d : \mathbb{R}^{d-1} \to \mathbb{R}$ be d functions belonging to $L^{d-1}(\mathbb{R}^{d-1})$. We define the following product*

$$f(x) = f_1(x_2, \dots, x_d) f_2(x_1, x_3, \dots, x_d) \cdots f_d(x_1, \dots, x_{d-1}), \forall x \in \mathbb{R}^d,$$

where the term with f_i depends on all the variables except x_i.

Then, f belongs to $L^1(\mathbb{R}^d)$ and we have

$$\|f\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^{d} \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

Proof.

In the case $d = 2$, we have $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ and by assumption $f_1, f_2 \in L^1(\mathbb{R})$. Therefore, Fubini's theorem implies that $f \in L^1(\mathbb{R}^2)$ and that $\|f\|_{L^1(\mathbb{R}^2)} = \|f_1\|_{L^1(\mathbb{R})}\|f_2\|_{L^1(\mathbb{R})}$. This proves the result (in this particular case the claimed inequality is an equality).

Let us only prove the result for $d = 3$ because the general case follows by a simple induction using Hölder's inequality (see [27], for instance, for a complete proof). Let us integrate the definition of $|f|$ with respect to the variable x_3 and apply the Cauchy–Schwarz inequality

$$\begin{aligned}
&\int_{\mathbb{R}} |f|(x_1, x_2, x_3)\, dx_3 \\
=&|f_3|(x_1, x_2) \left(\int_{\mathbb{R}} |f_1|(x_2, x_3)|f_2|(x_1, x_3)\, dx_3 \right) \\
\leq&|f_3|(x_1, x_2) \underbrace{\left(\int_{\mathbb{R}} |f_1|^2(x_2, x_3)\, dx_3 \right)^{1/2}}_{=(g_1(x_1))^{1/2}} \underbrace{\left(\int_{\mathbb{R}^3} |f_2|^2(x_1, x_3)\, dx_3 \right)^{1/2}}_{=(g_2(x_2))^{1/2}}.
\end{aligned}$$

We apply once more the Cauchy–Schwarz inequality to get

$$\int_{R^3} |f|\, dx \leq \left(\int_{\mathbb{R}^2} |f_3|^2(x_1, x_2)\, dx_1\, dx_2 \right)^{1/2} \left(\int_{\mathbb{R}^2} g_1(x_1)g_2(x_2)\, dx_1\, dx_2 \right)^{1/2}.$$

The last term is estimated by using the induction assumption (i.e., Fubini's theorem) to get the claim

$$\|f\|_{L^1(\mathbb{R}^3)} \leq \|f_3\|_{L^2(\mathbb{R}^2)}\|g_1\|^{1/2}_{L^1(\mathbb{R})}\|g_2\|^{1/2}_{L^1(\mathbb{R})} = \|f_3\|_{L^2(\mathbb{R}^2)}\|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}.$$

□

We also need the following version of Jensen's inequality.

Proposition II.2.20 (Jensen's inequality). *Let Ω be an open set of $\mathbb{R}^d$ and $\eta \in L^1(\Omega)$ a nonnegative function. For any function f such that $|f|^p\eta \in L^1(\Omega)$, for some $1 \leq p < +\infty$, we have $f\eta \in L^1(\Omega)$ and*

$$\left| \int_{\Omega} f\eta\, dx \right|^p \leq \|\eta\|^{p-1}_{L^1} \int_{\Omega} |f|^p \eta\, dx.$$

Proof.

We write $f\eta = (f\eta^{1/p})\eta^{1/p'}$ and we use the Hölder inequality with exponents p and $p' = p/(p-1)$.

□

Let us now establish a general version of the classic Minkowski inequality that we need below.

Proposition II.2.21 (Minkowski's inequality). *Let (X_1, μ_1) and (X_2, μ_2) be two σ-finite measure spaces. Then for any nonnegative measurable function f defined on $X_1 \times X_2$ and any $r \geq 1$ we have*

$$\left(\int_{X_1}\left(\int_{X_2} f(x_1,x_2)\,d\mu_2\right)^r d\mu_1\right)^{1/r} \leq \int_{X_2}\left(\int_{X_1} f(x_1,x_2)^r d\mu_1\right)^{1/r} d\mu_2.$$

Remark II.2.3. If one takes $X_2 = \{0,1\}$ and μ_2 the counting measure, the above inequality can be written as

$$\|f+g\|_{L^r} \leq \|f\|_{L^r} + \|g\|_{L^r},$$

for any nonnegative f, g and any $r \geq 1$.

Proof.

For all $x_1 \in X_1$ we denote $J(x_1) = \displaystyle\int_{X_2} f(x_1,x_2)\,d\mu_2$. Then, by using the Hölder inequality and the Fubini theorem, we have

$$\begin{aligned}
\int_{X_1} J(x_1)^r\,d\mu_1 &= \int_{X_1} J(x_1)^{r-1}\left(\int_{X_2} f(x_1,x_2)\,d\mu_2\right) d\mu_1\\
&= \int_{X_1}\int_{X_2} J(x_1)^{r-1} f(x_1,x_2)\,d\mu_2\,d\mu_1\\
&= \int_{X_2}\int_{X_1} J(x_1)^{r-1} f(x_1,x_2)\,d\mu_1\,d\mu_2\\
&\leq \int_{X_2}\left(\int_{X_1} J(x_1)^r\,d\mu_1\right)^{(r-1)/r}\left(\int_{X_1} f(x_1,x_2)^r\,d\mu_1\right)^{1/r} d\mu_2\\
&= \left(\int_{X_1} J(x_1)^r\,d\mu_1\right)^{(r-1)/r}\int_{X_2}\left(\int_{X_1} f(x_1,x_2)^r\,d\mu_1\right)^{1/r} d\mu_2.
\end{aligned}$$

From where we deduce the claim.

□

Let us also mention the following reverse Minkowski inequality that we state in a simple framework sufficient for our purposes.

Proposition II.2.22. *Let $0 < q < 1$ and Ω an open set of $\mathbb{R}^d$. For any nonnegative measurable functions $f, g : \Omega \to \mathbb{R}$, we have*

$$\|f+g\|_{L^q} \geq \|f\|_{L^q} + \|g\|_{L^q}.$$

Proof.

For any $x \in \Omega$ we write

$$\frac{f(x)+g(x)}{\|f\|_{L^q}+\|g\|_{L^q}} = \frac{\|f\|_{L^q}}{\|f\|_{L^q}+\|g\|_{L^q}}\frac{f(x)}{\|f\|_{L^q}} + \frac{\|g\|_{L^q}}{\|f\|_{L^q}+\|g\|_{L^q}}\frac{g(x)}{\|g\|_{L^q}},$$

that is to say,

$$\frac{f(x)+g(x)}{\|f\|_{L^q}+\|g\|_{L^q}} = \theta\frac{f(x)}{\|f\|_{L^q}} + (1-\theta)\frac{g(x)}{\|g\|_{L^q}},$$

with $\theta \in [0,1]$. From the assumption on q we know that the map $s \mapsto s^q$ is concave on $\mathbb{R}^+$. Therefore we get

$$\left(\frac{f(x)+g(x)}{\|f\|_{L^q}+\|g\|_{L^q}}\right)^q \geq \theta\left(\frac{f(x)}{\|f\|_{L^q}}\right)^q + (1-\theta)\left(\frac{g(x)}{\|g\|_{L^q}}\right)^q.$$

By integrating this inequality on Ω, we observe that we obtain 1 in the right-hand side. The claim follows immediately.

□

2.3.3 Mollifying kernels. Density result

Mollifying is a central procedure in functional analysis. It in particular allows us to prove density results in suitable functional spaces related to mathematical fluid mechanics. It is also crucial in the renormalized solutions theory for the transport equation that we describe in detail in Chapter VI.

Definition II.2.23. *A map $\eta : \mathbb{R}^d \to \mathbb{R}$ is called a mollifying kernel if*

- $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, *with* $\operatorname{Supp}\eta \subset B$, *the unit ball of* $\mathbb{R}^d$.
- $\eta \geq 0$ *and* $\int_{\mathbb{R}^d}\eta\,dx = \int_B \eta\,dx = 1$.
- $\eta(x)$ *only depends on* $|x|$.

Note first that the last condition is not necessary but it sometimes allows simplifications in some computations. It is also worth noticing that such a function actually exists.

For any $\varepsilon > 0$, we can now define

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right) \text{ and } (\nabla\eta)_\varepsilon(x) = \frac{1}{\varepsilon^d}(\nabla\eta)\left(\frac{x}{\varepsilon}\right),$$

in such a way that $\nabla\eta_\varepsilon = (1/\varepsilon)(\nabla\eta)_\varepsilon$.

Definition II.2.24. *For any $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ and any $\varepsilon > 0$ we define the convolution*

$$
\begin{aligned}
(f \star \eta_\varepsilon)(x) &= \int_{\mathbb{R}^d} f(y)\eta_\varepsilon(x-y)\,dy \\
&= \int_{\mathbb{R}^d} f(x-y)\eta_\varepsilon(y)\,dy = \int_B f(x-\varepsilon z)\eta(z)\,dz.
\end{aligned}
\tag{II.5}
$$

Notice that, inasmuch as η is bounded and compactly supported, all the integrals in the definition above make sense.

Proposition II.2.25. *For any $\varepsilon > 0$, we have*

$$
\begin{aligned}
f \star \eta_\varepsilon &\in \mathcal{C}^\infty(\mathbb{R}^d), \\
\|f \star \eta_\varepsilon\|_{L^\infty} &\le \frac{C}{\varepsilon^{d/p}}\|f\|_{L^p}, \\
\|\nabla(f \star \eta_\varepsilon)\|_{L^\infty} &\le \frac{C}{\varepsilon^{1+d/p}}\|f\|_{L^p}, \\
\|f \star \eta_\varepsilon\|_{L^p} &\le C\|f\|_{L^p},
\end{aligned}
\tag{II.6}
$$

for some $C > 0$ depending only on η and p. Finally, if $p < +\infty$ we have

$$
f \star \eta_\varepsilon \xrightarrow[\varepsilon \to 0]{} f, \quad \text{in } L^p(\mathbb{R}^d).
$$

Proof.

The regularity of $f \star \eta_\varepsilon$ follows from the regularity of the kernel η and usual results of differentiation under the integral sign. The L^∞ estimates simply follow from Hölder's inequality and the fact that $\|\eta_\varepsilon\|_{L^{p'}} = \|\eta\|_{L^{p'}}/\varepsilon^{d/p}$.

To prove the L^p estimate (II.6) for $p < +\infty$, we first use the Jensen inequality (Proposition II.2.20) to get, for any $x \in \mathbb{R}^d$,

$$
|(f \star \eta_\varepsilon)(x)|^p \le \int_B |f(x-\varepsilon z)|^p \eta(z)\,dz.
$$

By integrating with respect to x and using Fubini's theorem, we get

$$
\begin{aligned}
\|f \star \eta_\varepsilon\|_{L^p}^p &\le \int_{\mathbb{R}^d}\int_B |f(x-\varepsilon z)|^p \eta(z)\,dz\,dx \\
&= \int_B \eta(z)\left(\int_{\mathbb{R}^d} |f(x-\varepsilon z)|^p\,dx\right)dz = \|f\|_{L^p}^p.
\end{aligned}
$$

Let us now show the convergence property. Since $p < +\infty$, we can use the density of $\mathcal{C}_c^0(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ (this property comes from the regularity of the Lebesgue measure). Therefore, there exists a sequence $(f_n)_n$ of functions in $\mathcal{C}_c^0(\mathbb{R}^d)$ which converges towards f in $L^p(\mathbb{R}^d)$. Each function f_n is uniformly continuous and we denote by ω_n its modulus of continuity.

Using the properties of the kernel η, we observe that for each n we have

$$
|f_n \star \eta_\varepsilon(x) - f_n(x)| \le \int_B |f_n(x-\varepsilon z) - f_n(x)|\eta(z)\,dz \le \omega_n(\varepsilon),
$$

and therefore, since f_n is compactly supported we have

$$\|f_n \star \eta_\varepsilon - f_n\|_{L^p} \leq C_n \omega_n(\varepsilon).$$

Using the triangle inequality and (II.6) we get

$$\begin{aligned}\|f \star \eta_\varepsilon - f\|_{L^p} &\leq \|(f - f_n) \star \eta_\varepsilon\|_{L^p} + \|f_n \star \eta_\varepsilon - f_n\|_{L^p} + \|f_n - f\|_{L^p} \\ &\leq (1 + C)\|f - f_n\|_{L^p} + C_n \omega_n(\varepsilon).\end{aligned}$$

Taking the superior limit as $\varepsilon \to 0$, we obtain for any n that

$$\limsup_{\varepsilon \to 0} \|f \star \eta_\varepsilon - f\|_{L^p} \leq (1 + C)\|f - f_n\|_{L^p}.$$

Taking now the limit $n \to \infty$, we finally get $\limsup_{\varepsilon \to 0} \|f \star \eta_\varepsilon - f\|_{L^p} = 0$ and the claim is proved.

□

Theorem II.2.26. *For any open set Ω in $\mathbb{R}^d$, the set $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for any $1 \leq p < +\infty$.*

Proof.

Let $f \in L^p(\Omega)$. For any $n \geq 1$ we define the open set $\Omega_n = \{x \in \Omega, d(x, \partial\Omega) > 1/n\}$. By the dominated convergence theorem we see that $f_n = f 1_{\overline{\Omega_n}}$ converges to f in $L^p(\Omega)$. We consider now the function $f_{n,\varepsilon} = f_n \star \eta_\varepsilon$. By the previous proposition, we know that $f_{n,\varepsilon} \in \mathcal{C}^\infty(\mathbb{R}^d)$; moreover, for any $\varepsilon < 1/n$, we observe that the support of $f_{n,\varepsilon}$ is contained in Ω, therefore $f_{n,\varepsilon} \in \mathcal{D}(\Omega)$. The result follows because we have $\lim_{\varepsilon \to \infty} (\lim_{n \to \infty} f_{n,\varepsilon}) = f$ in $L^p(\Omega)$.

□

2.3.4 Weak and weak-$\star$ convergences in Lebesgue spaces

From the recap at the beginning of this section, and in particular from the characterisation of the dual space of $L^p(\Omega)$, we can write the L^p-version of Theorem II.2.7.

Proposition II.2.27. *Let $(u_n)_n$ be a bounded sequence of $L^p(\Omega)$, $1 < p < +\infty$; then we can extract a weakly converging subsequence from the sequence $(u_n)_n$; that is*

$$\exists (u_{n_k})_k, \exists u \in L^p(\Omega), \lim_{k \to \infty} \int_\Omega u_{n_k} \varphi \, dx = \int_\Omega u \varphi \, dx, \forall \varphi \in L^{p'}(\Omega).$$

This result does not hold in $L^1(\Omega)$ because that space is not reflexive. Nevertheless, we have a similar result in $L^\infty(\Omega)$ provided that we consider the weak-$\star$ topology on this space, because it is the dual of the separable space $L^1(\Omega)$.

Proposition II.2.28. *Let $(u_n)_n$ be a bounded sequence of $L^\infty(\Omega)$; then, from the sequence $(u_n)_n$, we can extract a subsequence which is weakly-$\star$ convergent; that is*

$$\exists (u_{n_k})_k, \exists u \in L^\infty(\Omega), \lim_{k\to\infty} \int_\Omega u_{n_k}\varphi\, dx = \int_\Omega u\varphi\, dx, \forall \varphi \in L^1(\Omega).$$

With this characterisation of the weak-$\star$ convergence in $L^\infty(\Omega)$, we can extend the density result given in Theorem II.2.26.

Theorem II.2.29. *For any open set Ω of $\mathbb{R}^d$, the set $\mathcal{D}(\Omega)$ is dense in $L^\infty(\Omega)$ for the weak-$\star$ topology.*

Proof.

Let $f \in L^\infty(\Omega)$. We set $\psi_n = 1_{B(0,n)}$ so that $f\psi_n \in L^1(\Omega) \cap L^\infty(\Omega)$. By using Theorem II.2.26, we know that for each n there exists a function $f_n \in \mathcal{D}(\Omega)$ such that $\|f_n - f\psi_n\|_{L^1} \leq 1/n$. Observe in the proof of this theorem that we have the additional property $\|f_n\|_{L^\infty} \leq \|f\psi_n\|_{L^\infty} \leq \|f\|_{L^\infty}$.

Let now $\varphi \in \mathcal{D}(\Omega)$. We have

$$\left|\int_\Omega f_n\varphi\, dx - \int_\Omega f\varphi\, dx\right| \leq \underbrace{\int_\Omega |f_n - f\psi_n||\varphi|\, dx}_{\|f_n - f\psi_n\|_{L^1}\|\varphi\|_{L^\infty}} + \int_\Omega |\psi_n - 1||f||\varphi|\, dx.$$

The first term in the right-hand side tends to 0 by construction of $(f_n)_n$ and the second one also tends to 0 thanks to the Lebesgue dominated convergence theorem.

Finally, since $(f_n)_n$ is bounded in $L^\infty(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $L^1(\Omega)$, we deduce that

$$\left|\int_\Omega f_n g\, dx - \int_\Omega fg\, dx\right| \xrightarrow[n\to\infty]{} 0, \forall g \in L^1(\Omega),$$

which proves the theorem.

□

By applying Proposition II.2.12 within the framework of L^p-spaces, and by using Hölder's inequality, we obtain the following useful result.

Proposition II.2.30. *Let p, q, and r be three real numbers in $[1, +\infty[$ such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

If $(u_n)_n$ is a sequence of $L^p(\Omega)$ which strongly converges towards u in $L^p(\Omega)$ and if $(v_n)_n$ is a sequence of $L^q(\Omega)$ which weakly converges towards v in $L^q(\Omega)$, then the product sequence $(u_nv_n)_n$ weakly converges towards uv in $L^r(\Omega)$.

We now state the classic inequalities in L^p spaces which prove that these spaces are uniformly convex ([27], [69]) except for $p = 1$ and $p = +\infty$. We can view these inequalities as generalisations of the parallelogram law in Hilbert space.

Lemma II.2.31 (Clarkson's inequalities). *Let $1 < p < +\infty$, and let f, g be in $L^p(\Omega)$.*

- *If $p \geq 2$, we have*

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \leq \frac{1}{2}\|f\|_{L^p}^p + \frac{1}{2}\|g\|_{L^p}^p.$$

- *If $p < 2$, we have*

$$\left\| \frac{f+g}{2} \right\|_{L^p}^{p'} + \left\| \frac{f-g}{2} \right\|_{L^p}^{p'} \leq \left(\frac{1}{2}\|f\|_{L^p}^p + \frac{1}{2}\|g\|_{L^p}^p \right)^{1/(p-1)}.$$

We can now prove the strong convergence criterion for a weakly converging sequence in L^p spaces.

Proposition II.2.32. *Let $1 < p < +\infty$, and let $(u_n)_n$ be a sequence of functions of $L^p(\Omega)$ which weakly converges towards u in $L^p(\Omega)$. If we assume*

$$\limsup_{n \to \infty} \|u_n\|_{L^p} \leq \|u\|_{L^p},$$

then the sequence $(u_n)_n$ strongly converges towards u.

According to Corollary II.2.8, this hypothesis is equivalent to saying that the sequence of norms $(\|u_n\|_{L^p})_n$ converges towards $\|u\|_{L^p}$.

For L^p spaces, this result generalises Proposition II.2.11 which dealt with the Hilbertian case (i.e., $p = 2$).

Proof.

The Clarkson inequalities given by the previous lemma can be written in the general form

$$\left\| \frac{f+g}{2} \right\|_{L^p}^{\alpha p} + \left\| \frac{f-g}{2} \right\|_{L^p}^{\alpha p} \leq \left(\frac{1}{2}\|f\|_{L^p}^p + \frac{1}{2}\|g\|_{L^p}^p \right)^{\alpha},$$

where $\alpha = 1$ if $p \geq 2$ and $\alpha = 1/(p-1)$ if $p < 2$. Let us apply this inequality to $f = u_n$ and $g = u$. We obtain

$$\left\| \frac{u_n+u}{2} \right\|_{L^p}^{\alpha p} + \left\| \frac{u_n-u}{2} \right\|_{L^p}^{\alpha p} \leq \left(\frac{1}{2}\|u_n\|_{L^p}^p + \frac{1}{2}\|u\|_{L^p}^p \right)^{\alpha},$$

and we denote the left-hand side of this inequality as a_n. If we pass to the upper limit in this inequality, then by using the hypothesis, we find that:

$$\limsup_{n\to\infty} a_n \leq \|u\|_{L^p}^{p\alpha}. \tag{II.7}$$

However, we have:

$$\left\|\frac{u_n - u}{2}\right\|_{L^p}^{\alpha p} = a_n - \left\|\frac{u_n + u}{2}\right\|_{L^p}^{\alpha p},$$

which gives

$$\limsup_{n\to\infty}\left\|\frac{u_n - u}{2}\right\|_{L^p}^{\alpha p} \leq \limsup_{n\to\infty} a_n - \liminf_{n\to\infty}\left\|\frac{u_n + u}{2}\right\|_{L^p}^{\alpha p}. \tag{II.8}$$

Moreover, the sequence $(u_n + u)/2$ also weakly converges towards u, so that the Corollary II.2.8 shows us that

$$\|u\|_{L^p}^{p\alpha} \leq \liminf_{n\to\infty}\left\|\frac{u_n + u}{2}\right\|_{L^p}^{\alpha p}. \tag{II.9}$$

By combining (II.7), (II.8), and (II.9), we finally obtain:

$$\limsup_{n\to\infty}\left\|\frac{u_n - u}{2}\right\|_{L^p}^{\alpha p} \leq 0,$$

which concludes the proof.

□

2.3.5 Interpolation between L^p spaces

We now establish an interpolation inequality which is nothing but a convexity property.

Lemma II.2.33. *Let Ω be any open set of $\mathbb{R}^d$ and let $u \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p, q \leq +\infty$. Then for all r such that*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, 0 \leq \theta \leq 1,$$

we have $u \in L^r(\Omega)$ and

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\theta}\|u\|_{L^q}^{1-\theta}.$$

Proof.

We note that

$$1 = \frac{\theta r}{p} + \frac{(1-\theta)r}{q},$$

and we can therefore apply the Hölder inequality in the following way:

$$\begin{aligned}\int_\Omega |u|^r dx &= \int_\Omega |u|^{r\theta}|u|^{r(1-\theta)}dx\\ &\le \left(\int_\Omega |u|^p dx\right)^{r\theta/p}\left(\int_\Omega |u|^q dx\right)^{r(1-\theta)/q}\\ &\le \|u\|_{L^p}^{r\theta}\|u\|_{L^q}^{r(1-\theta)}.\end{aligned}$$

□

The preceding inequality allows us to obtain convergence properties in the intermediate spaces from the convergences in suitable L^p spaces.

Corollary II.2.34. *Let Ω be any open set of $\mathbb{R}^d$. Let $p_1, p_2 \in [1, +\infty]$ and let $(u_n)_n$ be a sequence of functions which strongly converges towards u in $L^{p_1}(\Omega)$ and which weakly converges (weakly-$\star$ if $p_2 = +\infty$) in $L^{p_2}(\Omega)$. Then, for all p included strictly between p_1 and p_2, the sequence $(u_n)_n$ strongly converges towards u in $L^p(\Omega)$.*

Proof.

Since p is strictly included between p_1 and p_2, there exists a $\theta \in]0,1[$ such that

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

From the interpolation inequality given by the preceding lemma, we have

$$\|u - u_n\|_{L^p} \le \|u - u_n\|_{L^{p_1}}^{\theta}\|u - u_n\|_{L^{p_2}}^{1-\theta}.$$

However, the weak (or weak-$\star$) convergence of $(u_n)_n$ in $L^{p_2}(\Omega)$, shows that the sequence $(u-u_n)_n$ is bounded in this space, as well as that the strong convergence in $L^{p_1}(\Omega)$ ensures convergence towards 0 of the first term, because θ is not zero.

□

2.3.6 Local Lebesgue spaces

Definition II.2.35. *For all open sets Ω of $\mathbb{R}^d$ and for all $p \in [1, +\infty[$, we denote as $L^p_{loc}(\Omega)$ the set of measurable functions for which the p-th power of the absolute value is locally integrable, that is, its integral over all compact subsets included in Ω is finite. Similarly, we denote as $L^\infty_{loc}(\Omega)$ the set of measurable functions essentially bounded over all compact sets included in Ω.*

We can say that a sequence $(u_n)_n$ converges towards u in $L^p_{loc}(\Omega)$, if $(u_n)_n$ converges towards u in $L^p(\omega)$ for any bounded open set ω such that $\overline{\omega} \subset \Omega$.

It is clear that $L^p(\Omega) \subset L^p_{loc}(\Omega)$, the inverse inclusion being certainly false. A frequently useful property of sequences of functions in $L^p_{loc}(\Omega)$ is the following.

Proposition II.2.36. *Let Ω be a bounded open set of $\mathbb{R}^d$, $q > 1$ and let $(u_n)_n$ be a sequence of bounded functions in $L^q(\Omega)$. We assume that $(u_n)_n$ converges towards u in $L^p_{loc}(\Omega)$ with $1 \leq p < q$; then we have*

$$u \in L^q(\Omega),$$

and

$$u_n \xrightarrow[n\to\infty]{} u, \quad \text{in } L^p(\Omega).$$

Proof.

The sequence $(u_n)_n$ being bounded in $L^q(\Omega)$, we know from Propositions II.2.27 and II.2.28 that we can extract a subsequence $(u_{n_k})_k$ which weakly converges (weakly-$\star$ if $q = +\infty$) towards a function $v \in L^q(\Omega)$. In particular, we deduce that for any $\omega \subset \overline{\omega} \subset \Omega$, $(u_{n_k})_k$ converges weakly (or weakly-$\star$) towards v in $L^q(\omega) \subset L^p(\omega)$. The convergence in $L^p_{loc}(\Omega)$ implies strong convergence in $L^p(\omega)$, thus we deduce that $u = v \in L^q(\Omega)$.

For any $k \geq 1$, we set $\omega_k = \{x \in \Omega, d(x, \partial\Omega) > 1/k\}$. We have $\overline{\omega_k} \subset \Omega$ so that, by assumption,

$$\|u_n - u\|_{L^p(\omega_k)} \xrightarrow[n\to\infty]{} 0.$$

Moreover, by using the Hölder inequality we get

$$\|u_n - u\|_{L^p(\Omega\backslash\omega_k)} \leq \|u_n - u\|_{L^q(\Omega)}|\Omega\backslash\omega_k|^{(q-p)/q)} \leq 2C|\Omega\backslash\omega_k|^{(q-p)/q},$$

where C is a bound of the sequence $(u_n)_n$ and of the function u in $L^q(\Omega)$. We then set $\varepsilon > 0$ and choose k to be sufficiently large so that $2C|\Omega\backslash\omega_k|^{(q-p)/q} < \varepsilon$. We then choose n_0 sufficiently large that

$$\|u_n - u\|_{L^p(\Omega_k)} \leq \varepsilon, \forall n \geq n_0,$$

and, therefore

$$\|u_n - u\|_{L^p(\Omega)} \leq 2\varepsilon, \forall n \geq n_0.$$

$\square$

2.4 Partitions of unity

Let us start with a useful lemma when studying the local properties of functions.

Lemma II.2.37. *Let Ω be a nonempty open set of $\mathbb{R}^d$ and let ω be a bounded open set of $\mathbb{R}^d$ satisfying $\overline{\omega} \subset \Omega$. Then there exists a function $\varphi \in \mathcal{D}(\Omega)$, such that*

$$0 \leq \varphi \leq 1,$$
$$\varphi(x) = 1, \forall x \in \overline{\omega}.$$

Proof.

By hypothesis $\overline{\omega}$ is compact and disjoint from the closed set $\mathbb{R}^d \backslash \Omega$, hence

$$\delta = d(\overline{\omega}, \mathbb{R}^d \backslash \Omega) > 0.$$

We then introduce the open set $\mathcal{U} = \{x \in \Omega,\ d(x, \overline{\omega}) < \delta/2\}$. It is clear that

$$\overline{\omega} \subset \mathcal{U} \text{ and } \overline{\mathcal{U}} \subset \Omega.$$

The reader can easily convince her- or himself that the function

$$\varphi = 1_{\mathcal{U}} \star \eta_{\frac{\delta}{4}},$$

obtained by convolution with a mollifying kernel of the characteristic function of $\mathcal{U}$ satisfies the stated result.

□

We can now show the essential result of this section.

Lemma II.2.38 (Partition of unity). *Let A be a nonempty set of $\mathbb{R}^d$. We suppose given a covering of A by any family of open sets,*

$$A \subset \bigcup_{i \in I} \omega_i.$$

There exists a family $(\psi_i)_{i \in I}$ of nonnegative functions of $\mathcal{C}^\infty(\mathbb{R}^d)$, indexed on I such that

$$\operatorname{Supp} \psi_i \subset \omega_i, \forall i \in I,$$
$$\sum_{i \in I} \psi_i(x) = 1, \forall x \in A,$$

this sum being locally finite. Moreover, the ψ_i are identically zero except for a countable number of indices $i \in I$.

One such family of functions is called a *partition of unity* associated with the covering $(\omega_i)_{i \in I}$.

Proof.

- Let us consider the set S of points of A with rational coordinates. We then consider the family $(B_j)_{j \in J}$ of spheres centred on S, for which the radius is rational and which are contained in one of the $(\omega_i)_i$. This family is, of course, countable and we therefore index it with the integers $n \in \mathbb{N}$, and by the density of $\mathbb{Q}$ in $\mathbb{R}$ we clearly have

$$A \subset \bigcup_{n=0}^{+\infty} B_n.$$

- For all $n \geq 0$, we let V_n denote the ball with its centre at the same point as B_n and for which the radius is half that of B_n.
  According to Lemma II.2.37, there exists a positive regular function φ_n with compact support included in B_n and which is identically equal to 1 on V_n.
  We then define $\alpha_0 = \varphi_0$ and for all $n \geq 1$,
  $$\alpha_n = (1 - \varphi_0) \cdots (1 - \varphi_{n-1})\varphi_n.$$
  Is is clear that α_n is smooth and nonnegative. Furthermore, by definition, α_n has its support contained in B_n which is itself contained in one ω_i for some $i \in I$.
  Moreover, a straightforward computation implies that
  $$\sum_{n=0}^{N} \alpha_n(x) = 1 - (1 - \varphi_0) \cdots (1 - \varphi_N).$$
  This shows firstly, since $0 \leq \varphi_i \leq 1$, that for all N we have
  $$\sum_{n=0}^{N} \alpha_n(x) \leq 1.$$
  Furthermore, since $\varphi_i = 1$ on V_i, we see that
  $$\sum_{n=0}^{N} \alpha_n(x) = 1, \quad \forall x \in V_1 \cup \cdots \cup V_N.$$
  Inasmuch as the α_i are nonnegative, this implies that for all $n \geq N + 1$, α_n is zero on $V_1 \cup \cdots V_N$, which indeed proves that the sum $\sum_{n \geq 0} \alpha_n$ is locally finite and that
  $$\sum_{n \in \mathbb{N}} \alpha_n(x) = 1, \forall x \in A. \tag{II.10}$$
- For any n, we denote as $i(n) \in I$ an index such that $\operatorname{Supp} \alpha_n \subset \omega_{i(n)}$. We then note that
  $$A \subset \bigcup_{n=0}^{+\infty} \omega_{i(n)}.$$
  Indeed, Equation (II.10), shows that any point of A belongs to the support of at least one function α_n and therefore lies in $\omega_{i(n)}$.
  We now set $\psi_i = 0$ for any $i \in I \backslash \{i(n), n \in \mathbb{N}\}$. It remains to define the functions $\psi_{i(n)}$. To do this, we define

$$\psi_{i(0)}(x) = \sum_{\substack{k\in\mathbb{N},\text{s.t.}\\ \text{Supp}\,(\alpha_k)\subset\omega_{i(0)}}} \alpha_k(x).$$

This sum is perfectly defined because the sum of the family $(\alpha_n)_n$ is locally finite in A. Furthermore, it is clear that $\beta_{i(0)}$ is nonnegative and that its support is contained in $\omega_{i(0)}$. We then define for $n \geq 1$,

$$\psi_{i(n)}(x) = \sum_{\substack{k\in\mathbb{N},\text{s.t. } \text{Supp}\,(\alpha_k)\subset\omega_{i(n)}\\ \forall p\leq n-1,\text{Supp}\,(\alpha_k)\not\subset\omega_{i(p)}}} \alpha_k(x).$$

It is then obvious to check that those $(\psi_i)_{i\in I}$ solve the problem.

$\square$

2.5 A short introduction to distribution theory

Let us first describe the sequential topology of $\mathcal{D}(\Omega)$. A sequence $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ is said to be convergent towards some $\varphi \in \mathcal{D}(\Omega)$ if there is a compact $K \subset \mathbb{R}^d$ which contains the support of φ and of all the functions φ_n and if for any multi-index $\alpha \in \mathbb{N}^d$, the sequence $(\partial^\alpha \varphi_n)_n$ uniformly converges towards $\partial^\alpha \varphi$.

Definition II.2.39 (Distributions). *A linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is called a distribution if it is continuous in the sense that $T(\varphi_n) \xrightarrow[n\to\infty]{} T(\varphi)$ for any sequence $(\varphi_n)_n$ converging towards φ in $\mathcal{D}(\Omega)$.*

The set of distributions is denoted by $\mathcal{D}'(\Omega)$.

Even though $\mathcal{D}(\Omega)$ is not a Banach space, by similarity with the usual duality theory, we also adopt the notation

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = T(\varphi).$$

Definition II.2.40 (Convergence of distributions). *A sequence of distributions $(T_n)_n \subset \mathcal{D}'(\Omega)$ is said to converge towards a distribution $T \in \mathcal{D}'(\Omega)$ if for any $\varphi \in \mathcal{D}(\Omega)$ we have*

$$\langle T_n, \varphi\rangle_{\mathcal{D}',\mathcal{D}} \xrightarrow[n\to\infty]{} \langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}}.$$

Notice that the limit of a sequence of distributions $(T_n)_n$, if it exists, is necessarily unique.

Definition II.2.41 (Derivatives of distributions). *For any distribution $T \in \mathcal{D}'(\Omega)$ and any multi-index $\alpha \in \mathbb{N}^d$, the derivative of T in the distribution sense is the distribution $\partial^\alpha T$ defined by*

$$\langle \partial^\alpha T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = (-1)^{|\alpha|}\langle T, \partial^\alpha \varphi\rangle_{\mathcal{D}',\mathcal{D}}, \forall \varphi \in \mathcal{D}(\Omega).$$

We can associate the distribution $T_f \in \mathcal{D}'(\Omega)$ defined by,

$$\langle T_f, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega f\varphi\, dx$$

with any $f \in L^1_{loc}(\Omega)$. The following property is fundamental.

Proposition II.2.42. *The map*

$$\mathcal{T} : f \in L^1_{loc}(\Omega) \mapsto T_f \in \mathcal{D}'(\Omega)$$

is injective and sequentially continuous.

Proof.

Let $f, g \in L^1_{loc}(\Omega)$ such that $T_f = T_g$. Let us show that $f = g$ almost everywhere.

Let ω be any bounded open subset of Ω. We set $h = \mathrm{sgn}(f-g) \in L^\infty(\omega)$. By using Theorem II.2.29 we can find a sequence $\varphi_n \in \mathcal{D}(\omega)$ such that $(\varphi_n)_n$ converges to h in $L^\infty(\omega)$ weak-$\star$. By extending φ_n by zero on Ω, we see that $\varphi_n \in \mathcal{D}(\Omega)$ and therefore by assumption we have $\langle T_f, \varphi_n\rangle_{\mathcal{D}',\mathcal{D}} = \langle T_g, \varphi_n\rangle_{\mathcal{D}',\mathcal{D}}$; that is,

$$0 = \int_\Omega (f-g)\varphi_n\, dx = \int_\omega (f-g)\varphi_n\, dx.$$

Since $f - g \in L^1(\omega)$ and $(\varphi_n)_n$ converges in $L^\infty(\omega)$ weak-$\star$, we can pass to the limit in this formula and finally obtain

$$0 = \int_\omega (f-g)\,\mathrm{sgn}(f-g)\, dx = \int_\omega |f-g|\, dx.$$

It follows that $f = g$ almost everywhere in ω. This is true for any such ω, thus we have $f = g$ in Ω.

Let $(f_n)_n \subset L^1_{loc}(\Omega)$ which converges towards some $f \in L^1_{loc}(\Omega)$. For any $\varphi \in \mathcal{D}(\Omega)$, the sequence $(\varphi f_n)_n$ converges to φf in $L^1(\Omega)$ because φ is compactly supported. This implies that $\langle T_{f_n}, \varphi\rangle_{\mathcal{D}',\mathcal{D}} \xrightarrow[n\to\infty]{} \langle T_f, \varphi\rangle_{\mathcal{D}',\mathcal{D}}$, and the claim is proved.

□

Thanks to the previous result we see that the map $\mathcal{T}$ let us identify $L^1_{loc}(\Omega)$ to a subspace of $\mathcal{D}'(\Omega)$. By abuse of notation we say that $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ and we systematically identify f and the distribution T_f. Reciprocally, if a distribution $T \in \mathcal{D}'(\Omega)$ is such that $T = T_f$ for some $f \in L^1_{loc}(\Omega)$ we say that $T \in L^1_{loc}(\Omega)$.

Note also that, as soon as f is a smooth enough function, a simple integration by parts shows that we have

$$\partial^\alpha(T_f) = T_{\partial^\alpha f}, \text{in } \mathcal{D}'(\Omega),$$

so that the derivative in the distribution sense coincides with the derivative in the usual sense.

It is also fundamental to recall that the convergence of functions in the distribution sense is weaker than all the weak and weak-$\star$ convergences that we defined above. The precise result, whose proof is straightforward, is the following.

Proposition II.2.43. • *Let $1 \leq p < +\infty$, and $(f_n)_n$ be a sequence in $L^p(\Omega)$ which converges weakly towards $f \in L^p(\Omega)$. Then we have*

$$f_n \xrightarrow[n\to\infty]{} f, \text{ in } \mathcal{D}'(\Omega).$$

- *Let $(f_n)_n$ be a sequence in $L^\infty(\Omega)$ which converges weakly-$\star$ towards $f \in L^\infty(\Omega)$. Then we have*

$$f_n \xrightarrow[n\to\infty]{} f, \text{ in } \mathcal{D}'(\Omega).$$

We conclude the presentation of the distribution theory with the following useful lemma. Despite its very simple statement, the proof is not so straightforward.

Lemma II.2.44. *Let Ω be a connected open set of $\mathbb{R}^d$ and let $T \in \mathcal{D}'(\Omega)$ be a distribution such that $\nabla T = 0$ (in other words $\partial T/\partial x_i = 0$ in $\mathcal{D}'(\Omega)$ for all i). Then, T is constant; that is, there exists some $\alpha \in \mathbb{R}$ such that*

$$T = \alpha.$$

Proof.

- Let us start with the case where Ω is the cube $]0,1[^d$. We fix a function $\theta \in \mathcal{D}(]0,1[)$ to be nonnegative with integral equal to 1. Now let $\varphi \in \mathcal{D}(\Omega)$. We then denote

$$m_i(\varphi)(x_{i+1},\dots,x_d) = \int_0^1 \dots \int_0^1 \varphi(u_1,\dots,u_i,x_{i+1},\dots,x_d)\,du_1\dots du_i.$$

We set

$$\Phi_1(x_1,\dots,x_d) = \int_0^{x_1} \varphi(t,x_2,\dots,x_d)\,dt - m_1(\varphi)(x_2,\dots,x_d)\int_0^{x_1}\theta(t)\,dt.$$

It is clear that Φ_1 is regular and, by choice of θ, this function has compact support in Ω. By hypothesis we have

$$\begin{aligned} 0 = \left\langle \frac{\partial T}{\partial x_1}, \Phi_1 \right\rangle_{\mathcal{D}',\mathcal{D}} &= -\left\langle T, \frac{\partial \Phi_1}{\partial x_1} \right\rangle_{\mathcal{D}',\mathcal{D}} \\ &= -\langle T, \varphi - m_1(\varphi)(x_2,\dots,x_d)\theta(x_1)\rangle_{\mathcal{D}',\mathcal{D}}. \end{aligned}$$

We have therefore shown that for any $\varphi \in \mathcal{D}(\Omega)$, we have

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, m_1(\varphi)\theta(x_1)\rangle_{\mathcal{D}',\mathcal{D}}.$$

We now set

$$\begin{aligned}\Phi_2(x_1,\ldots,x_d) = \theta(x_1)&\int_0^{x_2} m_1(\varphi)(t,x_3,\ldots,x_d)\,dt\\ &- \theta(x_1)m_2(\varphi)(x_3,\ldots,x_d)\int_0^{x_2}\theta(t)\,dt.\end{aligned}$$

By using the fact that $\partial T/\partial x_2 = 0$ for this test function (which belongs indeed to $\mathcal{D}(\Omega)$), we find

$$\begin{aligned}\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} &= \langle T, \theta(x_1)m_1(\varphi)(x_2,\ldots,x_d)\rangle_{\mathcal{D}',\mathcal{D}}\\ &= \langle T, \theta(x_1)\theta(x_2)m_2(\varphi)(x_3,\ldots,x_d)\rangle_{\mathcal{D}',\mathcal{D}}.\end{aligned}$$

Hence, by induction we obtain that

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \theta(x_1)\cdots\theta(x_d)m_d(\varphi)\rangle_{\mathcal{D}',\mathcal{D}}.$$

However, $m_d(\varphi)$ is a constant which is simply the integral of φ on Ω. If we define

$$\alpha = \langle T, \theta(x_1)\cdots\theta(x_d)\rangle_{\mathcal{D}',\mathcal{D}},$$

then we obtain

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \alpha m_d(\varphi) = \int_\Omega \alpha\varphi\,dx_1\cdots dx_n,$$

which proves the result in the case of the unique cube. It is clear that by translations and homothety this proves the result for all the cubes.

- The case of any connected open set:
  We start by covering Ω with a locally finite family $(\omega_i)_i$ of open cubes. For all i, the distribution T restricted to ω_i has zero gradient in $\mathcal{D}'(\omega_i)$ and is therefore constant on ω_i. In other words, there exists some α_i such that for all φ with support in ω_i we have

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \int_{\omega_i} \alpha_i\varphi(x)\,dx = \alpha_i\int_\Omega \varphi(x)\,dx.$$

We now consider a locally finite $\mathcal{C}^\infty$ partition of unity $(\psi_i)_i$ (see Lemma II.2.38) associated with the covering of Ω under consideration. Let $\varphi \in \mathcal{D}(\Omega)$, then since the support of φ is compact, it is included in a finite union of the open sets of the family $(\omega_i)_i$. We therefore obtain the following equality

$$\varphi = \sum_i \varphi\psi_i,$$

the summation being, in fact, finite. Hence, we have

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \sum_i \langle T, \varphi\psi_i\rangle_{\mathcal{D}',\mathcal{D}}.$$

However, the functions $\varphi\psi_i$ have support in the cube ω_i, such that we obtain

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \sum_i \alpha_i \left(\int_\Omega \varphi(x)\psi_i(x)\,dx\right).$$

The summation on i is in reality finite, therefore we have

$$\langle T, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega \varphi(x)\left(\sum_i \alpha_i\psi_i(x)\right)\,dx.$$

The fact that this is valid for all φ shows that the distribution T coincides with the function of class $\mathcal{C}^\infty$ defined by

$$T(x) = \sum_i \alpha_i\psi_i(x).$$

However, by hypothesis, this function has a gradient (in the classic sense) which is 0 on Ω. Since Ω is connected, this shows that the function T is indeed constant on Ω.

□

2.6 Lipschitz continuous functions

This class of function is important in the sequel because we mainly study the equations of fluid mechanics in a domain whose boundary has a Lipschitz regularity (including, in particular, polygonal/polyhedral domains). That is why we need to state here some basic results concerning those functions.

We first give a very simple extension theorem in this class.

Proposition II.2.45 (McShane–Whitney extension). *Let $A \subset \mathbb{R}^d$ be any nonempty set and $f : A \to \mathbb{R}$ be a Lipschitz continuous function on A. There exists a Lipschitz continuous function $F : \mathbb{R}^d \to \mathbb{R}$ such that $F_{|A} = f$ and $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.*

Proof.

If we set $L = \mathrm{Lip}(f)$, it is a simple exercise to check that

$$F(x) = \inf_{y\in A} \left(f(y) + L|x-y|\right),$$

satisfies the required property.

□

In the sequel of this book (in particular in Chapter III), we often use the fact that Lipschitz continuous functions are almost-differentiable functions. The precise result, whose proof is given in [68], for instance, is the following.

Theorem II.2.46 (Rademacher). *Any locally Lipschitz continuous function f defined on an open set of $\mathbb{R}^d$ is differentiable (in the classic sense) almost everywhere.*

We need to analyse carefully the action of mollifying operators on Lipschitz continuous maps. We suppose given a mollifying kernel η as in Definition II.2.23 and we recall that, for any $\varepsilon > 0$, $f \star \eta_\varepsilon$ is defined in (II.5).

Proposition II.2.47. *Assume that f is Lipschitz continuous on $\mathbb{R}^d$, then*

1. *For any $\varepsilon > 0$, we have $\mathrm{Lip}(f \star \eta_\varepsilon) \leq \mathrm{Lip}(f)$.*
2. *$f \star \eta_\varepsilon$ uniformly converges in $\mathbb{R}^d$ towards f as $\varepsilon \to 0$.*
3. *For any $x \in \mathbb{R}^d$ such that f is differentiable at x, we have*

$$\nabla(f \star \eta_\varepsilon)(x) \xrightarrow[\varepsilon \to 0]{} \nabla f(x).$$

Proof.

1. The regularity of $f \star \eta_\varepsilon$ comes from that of kernel η. The estimate of the Lipschitz seminorm is given by the following simple computation

$$\begin{aligned} |f \star \eta_\varepsilon(x) - f \star \eta_\varepsilon(y)| &\leq \int_B |f(x - \varepsilon z) - f(y - \varepsilon z)|\eta(z)\, dz \\ &\leq \mathrm{Lip}(f)|x - y|, \quad \forall x, y \in \mathbb{R}^d. \end{aligned}$$

2. Since $\int_{\mathbb{R}^d} \eta(z)\, dz = 1$, we have

$$\begin{aligned} |f \star \eta_\varepsilon(x) - f(x)| &= \left| \int_B (f(x - \varepsilon z) - f(x))\eta(z)\, dz \right| \\ &\leq \int_B |f(x - \varepsilon z) - f(x)|\eta(z)\, dz \leq \varepsilon\, \mathrm{Lip}(f) \int_B |z|\eta(z)\, dz, \end{aligned}$$

   and the claim is proved.
3. Let x be a point such that f is differentiable at x. Then, there exists $\tau : \mathbb{R}^d \to \mathbb{R}$ such that $\lim_{h \to 0} \tau(h) = 0$ and

$$|f(x + h) - f(x) - \nabla f(x) \cdot h| \leq |h|\tau(h), \forall h \in \mathbb{R}^d.$$

   Let $i \in \{1, \ldots, d\}$. From (II.5), we see that

$$\partial_{x_i}(f \star \eta_\varepsilon)(x) = \frac{1}{\varepsilon} \int_B f(x - \varepsilon z)\partial_{z_i}\eta(z)\, dz.$$

Moreover, by integration by parts, we observe that

$$\int_B \partial_{z_i}\eta(z)\,dz = 0, \text{ and } \int_B z\partial_{z_i}\eta(z)\,dz = -e_i,$$

so that we can write

$$\begin{aligned}&\partial_{x_i}(f \star \eta_\varepsilon)(x) - \partial_{x_i} f(x)\\ =&\frac{1}{\varepsilon}\left(\int_B \Big[f(x-\varepsilon z) - f(x) + \nabla f(x).(\varepsilon z)\Big]\partial_{z_i}\eta(z)\,dz\right),\end{aligned}$$

and then we can conclude, by using the dominated convergence theorem, that

$$|\partial_{x_i}(f \star \eta_\varepsilon)(x) - \partial_{x_i} f(x)| \le \int_B |z|\tau(\varepsilon z)\partial_{z_i}\eta(z)\,dz \xrightarrow[\varepsilon\to 0]{} 0.$$

□

3 Basic compactness results

As we show below, highlighting of the compactness properties of certain sets, or of certain maps, is often a crucial step in proving the existence of solutions to certain nonlinear partial differential equations. In this section we summarise essential definitions and results which are used later.

3.1 Compact sets in function spaces

Ascoli's theorem is one of the fundamental tools of nonlinear analysis. It allows relatively compact sets in $C^0(E, F)$ to be simply characterised, where E is a compact space. This result is central because it underpins the majority of the compactness results used later in this text. A very classic proof of this theorem can be found, for example, in [99].

Theorem II.3.1 (Ascoli). *Let E be a compact metric space, and let F be a metric space. Let $\mathcal{C}^0(E, F)$ be the metric space formed from the continuous functions of E in F equipped with the uniform distance:*

$$d(f, g) = \sup_{x\in E} d(f(x), g(x)).$$

Let $\mathscr{K}$ be a subset of $\mathcal{C}^0(E, F)$. We assume that:

1. For any $x \in E$, the subset of F, $\mathscr{K}(x)$ defined by:

$$\mathscr{K}(x) = \{f(x), f \in \mathscr{K}\}$$

is relatively compact in F.

2. *The set $\mathscr{K}$ is equicontinuous; that is for all $x \in E$ and all $\varepsilon > 0$, there exists an $\eta > 0$ such that*

$$d(f(x), f(y)) < \varepsilon, \ \forall y \in E, \ \text{such that } d(x,y) < \eta, \ \forall f \in \mathscr{K}.$$

Then, $\mathscr{K}$ is relatively compact (i.e., has compact adherence) in $\mathcal{C}^0(E, F)$.

Unfortunately, it is very rare in the analysis of partial differential equations that we work in the set of continuous functions on a compact space. The following theorem, which follows from Ascoli's theorem, gives us a compactness criterion similar to that of Ascoli for the bounded subsets in $L^p(\Omega)$ spaces.

Theorem II.3.2 (Kolmogorov). *Let Ω be any open set of $\mathbb{R}^d$, and let $\mathcal{F}$ be a bounded subset of $L^p(\Omega)$, with $1 \le p < +\infty$. We assume that*

1. *For all $\varepsilon > 0$, and for all bounded open sets ω such that $\overline{\omega} \subset \Omega$, there exists an $\alpha > 0$, with $\alpha < d(\omega, \mathbb{R}^d \backslash \Omega)$ such that*

$$\|\tau_h f - f\|_{L^p(\omega)} \le \varepsilon, \ \forall f \in \mathcal{F}, \forall h \in \mathbb{R}^d, |h| \le \alpha. \tag{II.11}$$

2. *For all $\varepsilon > 0$, there exists a bounded open set ω, such that $\overline{\omega} \subset \Omega$ and such that*

$$\|f\|_{L^p(\Omega\backslash\omega)} \le \varepsilon, \forall f \in \mathcal{F}. \tag{II.12}$$

Then, $\mathcal{F}$ is relatively compact in $L^p(\Omega)$.

In this theorem $\tau_h f$ denotes the translated function defined by

$$\tau_h f(x) = f(x+h).$$

The first condition looks like the equicontinuity condition of Ascoli's theorem; the second tells us that the functions of $\mathcal{F}$ must be "uniformly small" in the L^p norm near the boundary of Ω and near infinity.

Proof.

- We set $\varepsilon > 0$ and choose an open set ω satisfying (II.12). Let $\alpha > 0$ with $\alpha < d(\omega, \mathbb{R}^d \setminus \Omega)$ satisfying (II.11). We observe that, for any $f \in \mathcal{F}$ and any $x \in \omega$ we have

$$|\bar{f} \star \eta_\alpha(x) - f(x)| \le \int_B |\bar{f}(x - \alpha z) - f(x)|\eta(z)\, dz.$$

Using the Jensen inequality, integrating on ω and using Fubini's theorem leads to

$$\|\bar{f} \star \eta_\alpha - f\|_{L^p(\omega)} \le \int_B \eta(z) \|\tau_{-\alpha z} f - f\|_{L^p(\omega)}\, dz.$$

Note that we have used here the fact that, by assumption on α, $x - \alpha z$ belongs to Ω as soon as $x \in \omega$ and $z \in B$ so that $\bar{f}(x - \alpha z) = f(x - \alpha z)$. For any $z \in B$ we have $|x - \alpha z| \leq \alpha$, therefore we deduce from (II.11) that

$$\|\bar{f} \star \eta_\alpha - f\|_{L^p(\omega)} \leq \varepsilon, \ \forall f \in \mathcal{F}. \tag{II.13}$$

- Let us define $\mathcal{F}_\alpha = \{\bar{f} \star \eta_\alpha, \ f \in \mathcal{F}\}$ and $F_\omega = \{f_{|\omega}, \ \ f \in \mathcal{F}\}$. Each function in $\mathcal{F}_\alpha$ is continuous on the compact $\overline{\omega}$ and satisfies

$$\|\nabla(\bar{f} \star \eta_\alpha)\|_{L^\infty} \leq \frac{C}{\alpha^{1+d/p}}\|f\|_{L^p} \leq \frac{C'}{\alpha^{1+d/p}},$$

because $\mathcal{F}$ is a bounded set of $L^p(\Omega)$.
The number $\alpha > 0$ being fixed, we have shown that $\mathcal{F}_\alpha$ satisfies the assumptions of the Ascoli theorem. It follows that $\mathcal{F}_\alpha$ is relatively compact in $\mathcal{C}^0(\overline{\omega})$ and thus in $L^p(\omega)$ by continuity of the embedding $\mathcal{C}^0(\overline{\omega}) \subset L^p(\omega)$.
- As a consequence, there exist a finite number of balls in $L^p(\omega)$ with radius ε which cover $\mathcal{F}_\alpha$ and with (II.13) we deduce that there exist a finite number of balls in $L^p(\omega)$ with radius 2ε which cover $\mathcal{F}_{|\omega}$ in $L^p(\omega)$. We denote such a covering as $(B_{L^p(\omega)}(g_i, 2\varepsilon))_{1 \leq i \leq N}$.
- We finally prove that the balls $(B_{L^p(\Omega)}(\overline{g_i}, 3\varepsilon))_{1 \leq i \leq N}$ actually cover $\mathcal{F}$. Indeed, for any $f \in \mathcal{F}$ there is a $1 \leq i \leq N$ such that $f_{|\omega} \in B_{L^p(\omega)}(g_i, 2\varepsilon)$ and thus

$$\|f - \overline{g_i}\|^p_{L^p(\Omega)} = \|f\|^p_{L^p(\Omega \setminus \omega)} + \|f - g_i\|^p_{L^p(\omega)} \leq \varepsilon^p + (2\varepsilon)^p \leq (3\varepsilon)^p,$$

by using (II.12).
The claim is proved because, for any $\varepsilon > 0$, we have built a finite covering of $\mathcal{F}$ in $L^p(\Omega)$ made of balls of radius 3ε.

□

3.2 Compact maps

Definition II.3.3. *Let E and F be two Banach spaces. We say that a map S from E into F is compact if the image of any bounded subset of E by S is a relatively compact set of F; that is, it is a set having compact closure in F.*

Of course, any compact linear function is continuous because compactness implies that it is bounded in the neighborhood of 0. We generally use the compactness properties of maps in the following form.

Let $(u_n)_n$ be a bounded sequence of points in E then, if S is compact, there exists a subsequence $(u_{n_k})_k$ such that $(Su_{n_k})_k$ converges in F.

In particular, we have the following result.

Proposition II.3.4. *Let $(u_n)_n$ be a sequence of points in E which weakly converges towards u in E, and let $S : E \to F$ be a compact linear map, then the sequence $(Su_n)_n$ strongly converges towards Su in F.*

Proof.

We note first that the sequence $(Su_n)_n$ weakly converges towards Su. Indeed, for $f \in F'$,

$$\langle f, Su_n\rangle_{F',F} = \langle {}^tSf, u_n\rangle_{E',E} \xrightarrow[n\to\infty]{} \langle {}^tSf, u\rangle_{E',E} = \langle f, Su\rangle_{F',F}.$$

Moreover, since $(u_n)_n$ is weakly convergent, it is a bounded sequence (Corollary II.2.8). Hence $(u_n)_n$ belongs to a bounded subset B of E. Inasmuch as S is compact, $\overline{S(B)}$ is compact, and therefore the sequence $(Su_n)_n$ lies in a compact space of F.

However, in a compact metric space, a sequence converges if and only if it has a unique accumulation point. Therefore let $v = \lim_{k\to\infty} Su_{n_k}$ be an accumulation point of $(Su_n)_n$ in F. As we have seen above, the sequence $(Su_{n_k})_k$ weakly converges towards Su. From the uniqueness of the weak limit, this means $v = Su$. Hence, Su is the unique accumulation point of the sequence $(Su_n)_n$, which is therefore convergent.

□

In particular, this makes it possible to recover strong convergence from weak convergence, but in a larger space than the initial space. Indeed, if a space E is embedded into a space F with a compact embedding (we say that E is embedded in a compact way into F), then any sequence of elements of E which is weakly convergent in E, is strongly convergent in F.

On the other hand, the compactness of linear functions is a stable concept by composition and by passing to the adjoint. More precisely, we have the following results.

Lemma II.3.5. *Let E, F, G be three Banach spaces, let S be a continuous linear map from E to F, and let T be a continuous linear map from F to G. If S is compact* or *if T is compact, then $T \circ S$ is compact.*

Proof.

This is essentially a consequence of the fact that the image of a bounded (resp., compact) set by a continuous linear map is another bounded (resp., compact) set.

□

Lemma II.3.6. *Let E and F be two Banach spaces and let S be a compact linear map from E into F. Then the adjoint map ${}^tS$ from F' into E' is compact.*

Proof.

Let $(f_n)_n$ be a bounded sequence of F'. From the definition of the adjoint of S, we have for all $u \in B_E(0,1)$,

$$\langle {}^t S f_n, u\rangle_{E',E} = \langle f_n, Su\rangle_{F',F}.$$

Since S is compact, the set

$$\mathcal{K} = \overline{S\,(B(0,1)_E)},$$

is a compact subset of F. The sequence $(f_n)_n$, being bounded in F', is therefore also bounded in $\mathcal{C}^0(\mathcal{K},\mathbb{R})$. Moreover, for all points $v_1, v_2 \in \mathcal{K}$ we have

$$|\langle f_n, v_1 - v_2\rangle_{F',F}| \leq \|f_n\|_{F'}\|v_1 - v_2\|_F,$$

which, since the sequence $(f_n)_n$ is bounded in F', proves that $(f_n)_n$ is an equicontinuous family on $\mathcal{K}$.

Hence, from Ascoli's theorem (Theorem II.3.1), there exists an extracted sequence $(f_{n_k})_k$ which converges uniformly on $\mathcal{K}$ towards a continuous function f from $\mathcal{K}$ to $\mathbb{R}$.

Therefore, by transposition, for all $u \in B_E(0,1)$ we have

$$\langle {}^t S f_{n_k}, u\rangle_{E',E} = \langle f_{n_k}, Su\rangle_{F',F} \xrightarrow[k\to\infty]{} f(Su),$$

and, moreover, the convergence is uniform in u on $B_E(0,1)$. By homogeneity (i.e., because of the linearity of the functions ${}^t S f_{n_k}$) we deduce that for all $u \in E$, the sequence $(\langle {}^t S f_{n_k}, u\rangle_{E',E})_k$ converges and, furthermore, the convergence is uniform on all the bounded sets of E. The functions ${}^t S f_{n_k}$ are linear and continuous, therefore the limit obtained is necessarily linear and continuous. All this demonstrates that convergence does indeed occur in E' (for which the strong topology is simply the uniform convergence on bounded sets).

□

Let us apply this result in the case where there is a continuous embedding of one Banach space into another.

Proposition II.3.7. *Let E and F be two Banach spaces. We assume that E is continuously embedded into F and that the range of E is dense in F (we say incorrectly that E is dense in F); then the map*

$$T : f \in F' \mapsto T_f \in E',$$

defined by

$$\langle T_f, u\rangle_{E',E} = \langle f, u\rangle_{F',F},\ \ \forall u \in E,$$

is an embedding (said to be canonical *with respect to the considered embedding from E into F) from F' into E'.*

Moreover, if the embedding of E into F is compact, then T is a compact embedding. Finally, if E is reflexive then the range of T is dense in E'.

Proof.

Since the embedding of E into F is continuous, there exists a constant $C > 0$ such that for any $u \in E$ we have $\|u\|_F \leq C\|u\|_E$. Hence, for all $f \in F'$, the function T_f is indeed continuous and linear on E, that is, an element of E'.

Let us prove the injectivity of the function $f \mapsto T_f$. Let $f \in F'$ such that $T_f = 0$. We therefore have $\langle f, u\rangle_{F',F} = 0$ for all u in E, but, since E is dense in F, we can deduce that $f = 0$.

If the embedding from E into F is compact, then the compactness of T results directly from Lemma II.3.6.

Let us now assume that E is reflexive. We need to show that $T(F')$ is dense in E'. To do this, we use Proposition II.2.2. Any continuous linear functional on E' is of the form $f \mapsto \langle f, u\rangle_{E',E}$ for a certain u in E. Let us suppose that one such functional cancels on $T(F')$ and let us show that it cancels on all of E'. To say that this linear functional cancels on $T(F')$ means that

$$\langle f, u\rangle_{F',F} = 0, \ \forall f \in F'.$$

Proposition II.2.1 then shows that $\|u\|_F = 0$ and hence $u = 0$, which proves the result.

□

The Riesz theorem allows this result to be specified in the Hilbertian case.

Corollary II.3.8. *Let V and H be two Hilbert spaces such that V embeds densely into H. According to the Riesz theorem we can identify H and its dual via its scalar product. We then have a double dense embedding*

$$V \subset H \subset V',$$

the second embedding being defined by

$$f \in H \mapsto T_f \in V', \ \textit{with } \langle T_f, v\rangle_{V',V} = (f, v)_H, \ \forall v \in V.$$

If the embedding of V into H is compact, then the embedding of H into V' is also compact.

For obvious reasons, in the situation described by the corollary, the space H is called the *pivot space*. Furthermore, since T is injective, we systematically identify $f \in H$ with its image $T_f \in V'$ so that the duality (V', V) can be expressed, using the scalar product of H, by

$$\langle f, v\rangle_{V',V} = (f, v)_H, \ \forall f \in H, \ \forall v \in V.$$

3.3 The Schauder fixed-point theorem

For solving nonlinear partial differential equations, one can often use a rather classic fixed-point technique. In this book, we follow this strategy for studying the steady Navier–Stokes equations (Section 3 of Chapter V) and for studying the unsteady Navier–Stokes equations for a nonhomogeneous flow in Chapter VI.

The key of this technique lies in the following theorem for which the rather tricky proof can be found, for example, in [114]. It relies on the concept of *topological degree* which is beyond the scope and objectives of this book.

Theorem II.3.9 (Schauder fixed-point theorem). *Let E be a Banach space and let C be a convex compact set in E. If T is a continuous (nonlinear) function from C into C, then it has at least one fixed-point in C.*

We note that this theorem says nothing about the uniqueness of the fixed-point and that in general uniqueness does not hold (consider the identity function). Moreover, the fact that the function T maps the set C into itself is, of course, a crucial fact.

In the particular case where $E = \mathbb{R}$, we recover an elementary result which says that a continuous function from $\mathbb{R}$ into $\mathbb{R}$ which maps a compact interval $[a, b]$ onto itself contains a fixed-point in this interval.

This result also exists in a slightly different form which is given below.

Theorem II.3.10. *Let E be a Banach space and let C be a convex, closed and bounded region of E. If T is a compact and continuous (nonlinear) function from C into C, then it has at least one fixed-point in C.*

In the finite-dimensional framework, this theorem is known as the Brouwer theorem and is equivalent to the following result.

Proposition II.3.11. *Let P be a continuous function from $\mathbb{R}^N$ to $\mathbb{R}^N$, such that there exists a $\rho > 0$ satisfying*

$$\xi \cdot P(\xi) \geq 0, \forall \xi \in \mathbb{R}^N, |\xi| = \rho.$$

Then, there exists $\overline{\xi} \in \mathbb{R}^N$, $|\overline{\xi}| \leq \rho$ such that $P(\overline{\xi}) = 0$.

Proof.

Suppose, by contradiction, that for all $\xi \in \overline{B(0, \rho)}$, $P(\xi) \neq 0$; then the continuous map $Q : \xi \in \mathbb{R}^N \mapsto -(\rho/|P(\xi)|)P(\xi)$ maps the ball $\overline{B(0, \rho)}$ which is compact and convex into itself. Then by application of the Brouwer/Schauder fixed-point theorem, there exists $\xi^\star \in \overline{B(0, \rho)}$ such that

$$\xi^\star = Q(\xi^\star). \tag{II.14}$$

Necessarily, we have $|\xi^\star| = \rho$. By taking the scalar product of each term of (II.14) by $\xi^\star$, one obtains:

$$\rho^2 = -\frac{\rho}{|P(\xi^\star)|}(\xi^\star \cdot P(\xi^\star)).$$

Therefore $\xi^\star \cdot P(\xi^\star) < 0$, which is in contradiction with the hypothesis.

□

4 Functions of one real variable

In this section, we primarily review the links that exist between the concepts of weak derivatives and derivatives in the usual sense. We do this in a limited but sufficient way, for the case that concerns us, for functions of one real variable.

We conclude the section by reviewing Gronwall-type inequalities which are a useful tool for obtaining a priori estimates for solutions of evolution partial differential equations.

4.1 Differentiation and antiderivatives

Let $[a, b]$ be a compact interval of $\mathbb{R}$. We recall that $W^{1,1}(]a, b[)$ is the set of functions of $L^1(]a, b[)$ for which the derivative in the sense of distributions is a function of $L^1(]a, b[)$ (see Chapter III for a more complete study of Sobolev spaces). A fundamental question that we can ask for such a function is if it can be differentiated in the usual sense and if we can write the fundamental theorem of calculus

$$f(y) = f(x) + \int_x^y f'(t)\,dt,\ \forall x, y \in [a, b].$$

Here, we recall some results that concern this question. This material is useful in the sequel of the book in order to justify the validity of the time evolution of the kinetic energy for weak solutions of the Navier–Stokes equations (see in particular Section 1.4 of Chapter V).

Lemma II.4.1. *Let $g \in L^1(]a, b[)$ and $C \in \mathbb{R}$. We consider the function f defined by*

$$f(t) = C + \int_a^t g(s)\,ds.$$

Then, f is continuous on $[a, b]$. Moreover, $f \in W^{1,1}(]a, b[)$ and its derivative in the sense of distributions is g.

Proof.

For all $t_0 \in [a, b[$ and $h > 0$ but sufficiently small, we have

$$f(t_0+h)-f(t_0)=\int_{t_0}^{t_0+h} g(s)\,ds=\int_a^b g(s)1_{[t_0,t_0+h]}(s)\,ds \xrightarrow[h\to 0]{} 0,$$

from Lebesgue's dominated convergence theorem. A similar argument with $h<0$ shows the continuity of f.

We now need to verify that the derivative of f in the sense of distributions is the function g. Let $\varphi\in\mathcal{D}(]a,b[)$; we have

$$-\int_a^b f(t)\varphi'(t)\,dt=-C\int_a^b \varphi'(t)\,dt-\int_a^b\left(\int_a^t g(s)\varphi'(t)\,ds\right)dt.$$

The first term is zero because $\varphi(a)=\varphi(b)=0$ and we apply Fubini's theorem to the second term (the function $(t,s)\mapsto\varphi'(t)g(s)$ is integrable with respect to the two variables). It follows that

$$\begin{aligned}
-\int_a^b f(t)\varphi'(t)\,dt &= -\int_a^b\left(\int_a^b 1_{[a\le s\le t]}g(s)\varphi'(t)\,ds\right)\,dt\\
&= -\int_a^b\left(\int_a^b 1_{[a\le s\le t]}g(s)\varphi'(t)\,dt\right)\,ds\\
&= -\int_a^b\left(g(s)\int_s^b \varphi'(t)\,dt\right)ds=-\int_a^b \left(g(s)(\varphi(b)-\varphi(s))\right)\,ds\\
&= \int_a^b g(s)\varphi(s)\,ds.
\end{aligned}$$

This indeed proves that $g=f'$ in the sense of distributions and therefore $f\in W^{1,1}(]a,b[)$.

$\square$

Corollary II.4.2. *Any function f of $W^{1,1}(]a,b[)$ is equal almost everywhere to a continuous function $\tilde f$ on $[a,b]$ and we have for all $x,y\in[a,b]$,*

$$\tilde f(y)=\tilde f(x)+\int_x^y f'(s)\,ds;$$

in other words, we have for almost every $x,y\in[a,b]$,

$$f(y)=f(x)+\int_x^y f'(s)\,ds.$$

Proof.

Let $f\in W^{1,1}(]a,b[)$. We introduce

$$g(t)=\int_a^t f'(s)\,ds.$$

- According to Lemma II.4.1, the function g is continuous in $W^{1,1}(]a,b[)$ and its derivative in the sense of distributions is f'. Hence, $f-g$ is a function for which the distribution derivative is zero. We then know that there exists a real number C such that $f-g=C$ almost everywhere (see Lemma II.2.44 for a more general case of this result). If we define $\tilde{f}=C+g$, we have shown that f coincides with the continuous function $\tilde{f}$, almost everywhere.
- From the definition of $\tilde{f}$, it is clear that for all $x,y\in[a,b]$, we have

$$\tilde{f}(y)-\tilde{f}(x)=\left(C+\int_a^y f'(s)\,ds\right)-\left(C+\int_a^x f'(s)\,ds\right)=\int_x^y f'(s)\,ds.$$

□

For any point $t_0\in[a,b]$, we denote as $\mathcal{V}_\eta(t_0)$ the set of open neighborhoods ω of t_0 in $[a,b]$ whose Lebesgue measure $|\omega|$ is less than η.

Definition II.4.3 (Lebesgue points). *Let f be a function of $L^1(]a,b[)$ and $t_0\in]a,b[$. We say that t_0 is a Lebesgue point of f if*

$$\sup_{\omega\in\mathcal{V}_\eta(t_0)}\frac{1}{|\omega|}\int_\omega|f(t)-f(t_0)|\,dt\xrightarrow[\eta\to 0]{}0.$$

With this definition at hand, we have the following result.

Proposition II.4.4. *Let f be a function of $L^1(]a,b[)$ and $t_0\in]a,b[$. If t_0 is a Lebesgue point of f, then any antiderivative of f defined by*

$$F(t)=C+\int_a^t f(s)\,ds,$$

can be differentiated in the classic sense at t_0 and moreover we have

$$F'(t_0)=f(t_0).$$

Proof.

It is sufficient to write

$$\begin{aligned}\left|\frac{F(t_0+h)-F(t_0)}{h}-f(x_0)\right|&=\left|\frac{1}{h}\int_{t_0}^{t_0+h}(f(s)-f(t_0))\,ds\right|\\&\leq\frac{1}{h}\left|\int_{t_0}^{t_0+h}|f(s)-f(t_0)|\,ds\right|;\end{aligned}$$

this last quantity tends towards 0 when h tends towards 0, by definition of a Lebesgue point.

□

The fundamental theorem in this section is a rather difficult result from measure theory, which we do not prove. However, a proof can be found, for example, in [69] or in [100].

Theorem II.4.5. *If $f \in L^1(]a,b[)$, then almost every point of $]a,b[$ is a Lebesgue point of f.*

The immediate consequence of the preceding two results is that the antiderivative F of any function f in $L^1(]a,b[)$ is differentiable almost everywhere and satisfies $F' = f$ and the fundamental theorem of calculus

$$F(y) = F(x) + \int_x^y F'(t)\,dt = F(x) + \int_x^y f(t)\,dt, \quad \forall x, y \in [a,b].$$

Finally, the following elementary result is useful.

Proposition II.4.6. *Let $f \in L^1(]a,b[)$. Any point of continuity of f is a Lebesgue point of f and, in particular, any antiderivative of f is differentiable at any point t_0 where f is continuous and its derivative is $f(t_0)$.*

Remark II.4.1. We can prove [69] that the functions of $W^{1,1}(]a,b[)$ are none other than the absolutely continuous functions on $]a,b[$.

We conclude this section by the following result and its corollary.

Lemma II.4.7 (Hardy's inequality). *For any $1 < p < +\infty$ and any nonnegative $f \in L^p(]0,+\infty[)$ we have*

$$\int_0^M \left(\frac{1}{x}\int_0^x f(s)\,ds\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^M f^p(s)\,ds, \ \forall M \in [0,+\infty].$$

Note that a similar inequality does not hold for $p = 1$.

Proof.

We prove the inequality for $M = +\infty$. The general case follows by taking $f_M(s) = 1_{[0,M]}(s) f(s)$.

By density, it is enough to prove this result for functions $f \in \mathcal{C}_c^\infty(]0,+\infty[)$. For such an f, we set $F(x) = (1/x)\int_0^x f(s)\,ds$ and we note that $F = 0$ in the neighborhood of 0 and that $F(x) = C/x$ for some $C \in \mathbb{R}$ and x large enough. In particular, we have $F \in L^p(]0,+\infty[)$. We remark that $d(xF(x))/dx = f(x)$ so that we can integrate by parts as follows

$$\begin{aligned}
\int_0^{+\infty} F(x)^p\,dx &= \int_0^{+\infty} (xF(x))^p \frac{1}{x^p}\,dx \\
&= \frac{p}{p-1}\int_0^{+\infty} (xF(x))^{p-1} f(x).\frac{1}{x^{p-1}}\,dx \\
&= \frac{p}{p-1}\int_0^{+\infty} F(x)^{p-1} f(x)\,dx.
\end{aligned}$$

We conclude the proof by using the Hölder inequality

$$\int_0^{+\infty} F(x)^p\,dx \le \frac{p}{p-1}\|F\|_{L^p}^{p-1}\|f\|_{L^p}.$$

□

We immediately deduce from this inequality the following result which is important in the sequel (see in particular the multidimensional version in Proposition III.2.40).

Corollary II.4.8. *Let $f \in W^{1,p}(]a,b[) \subset W^{1,1}(]a,b[)$ with $1 < p < +\infty$ and such that $f(a) = 0$. Then, the function $g : x \mapsto f(x)/(x-a)$ belongs to $L^p(]a,b[)$ and satisfies*

$$\|g\|_{L^p} \le C\|f'\|_{L^p}.$$

4.2 Differential inequalities and Gronwall's lemma

The following lemmas, concerning ordinary differential equations and inequalities, are very useful ingredients for studying time-dependent partial differential equations, in particular for proving energy estimates.

Lemma II.4.9. *Let there be two real numbers $\alpha > 0$ and $\beta \ge 0$ and let y be a function in $\mathcal{C}^1([0,+\infty[,\mathbb{R})$ satisfying the differential inequality:*

$$y'(t) + \alpha y(t) \le \beta,\ \forall t \ge 0.$$

Then we have

$$y(t) \le y(0)e^{-\alpha t} + \frac{\beta}{\alpha},\ \forall t \ge 0.$$

Proof.

We multiply the two sides of the differential inequality by $e^{\alpha t}$ and then integrate.

□

The following lemma, known as Gronwall's inequality (even though its proof in the present form is due to Bellman [15]), is central in proving a priori estimates on solutions of (partial) differential equations.

Lemma II.4.10. *Let us consider a function $y \in L^\infty(]0,T[)$, a nonnegative function $g \in L^1(]0,T[)$ and $y_0 \in \mathbb{R}$, such that*

$$y(t) \le y_0 + \int_0^t g(s)y(s)\,ds,\ \textit{for almost all } t \in]0,T[,$$

we then have

$$y(t) \le y_0 \exp\left(\int_0^t g(s)\,ds\right),\ \textit{for almost all } t \in]0,T[.$$

Proof.

We set

$$h(t) = y_0 + \int_0^t g(s)y(s)\,ds,$$

that is, the second term of the inequality from the hypothesis. Since y belongs to $L^\infty(]0,T[)$ and g to $L^1(]0,T[)$, the function h lies in $W^{1,1}(]0,T[)$ and is therefore differentiable almost everywhere and its derivative is gy (see Section 4.1). Furthermore, for almost all t, we have

$$h'(t) = g(t)y(t) \leq g(t)h(t),$$

from the hypothesis and because g is nonnegative. Then, if we set $z(t) = h(t)e^{-\int_0^t g}$, we immediately see that z belongs to $W^{1,1}(]0,T[)$ and that

$$z'(t) = \big(h'(t) - g(t)h(t)\big)\exp\left(-\int_0^t g(s)\,ds\right).$$

Therefore, for almost all t, we have $z'(t) \leq 0$. From Corollary II.4.2, this implies that the function z is nonincreasing and therefore we have

$$z(t) \leq z(0) = h(0) = y_0, \forall t \in [0,T],$$

which can be written as

$$h(t) \leq y_0 \exp\left(\int_0^t g(s)\,ds\right),\ \forall t \in [0,T].$$

This proves the claim since, by hypothesis, we have $y \leq h$ almost everywhere.

□

Lemma II.4.11 (Uniform Gronwall lemma [121]). *Let g_1 and g_2 be two nonnegative functions of $L^1_{loc}(\mathbb{R}^+)$ satisfying:*

$$\exists k_1,\ \int_t^{t+1} g_1(s)\,ds \leq k_1, \forall t \in \mathbb{R}^+,$$
$$\exists k_2,\ \int_t^{t+1} g_2(s)\,ds \leq k_2, \forall t \in \mathbb{R}^+.$$

Let y be a function of $\mathcal{C}^1([0,+\infty[,\mathbb{R}^+)$ satisfying

$$y'(t) \leq g_1(t) + g_2(t)y(t), \text{ for almost all } t \geq 0, \tag{II.15}$$

$$y(0) \leq k_3 \text{ and } \int_t^{t+1} y(s)\,ds \leq k_3,\ \forall t \geq 0. \tag{II.16}$$

Then y is bounded on $\mathbb{R}^+$ and we have the following upper bound,

$$y(t) \leq (k_1 + k_3)e^{k_2}, \forall t \geq 0.$$

Proof.
We integrate (II.15) between s and t (with $0 \leq s \leq t$) and we get

$$y(t) \leq y(s) + \int_s^t g_1(\tau)\,d\tau + \int_s^t g_2(\tau)y(\tau)\,d\tau.$$

From Lemma II.4.10, we deduce

$$y(t) \leq \left(y(s) + \int_s^t g_1(\tau)\,d\tau\right)\exp\left(\int_s^t g_2(\tau)\,d\tau\right).$$

- For $t \leq 1$, we take $s = 0$ and we directly obtain the result.
- For $t \geq 1$, we take $s \in [t-1, t]$, and we apply (II.16) to obtain

  $$y(t) \leq (y(s) + k_1)e^{k_2}.$$

  We integrate this last inequality with respect to s between $t-1$ and t (with t fixed), which gives

  $$y(t) \leq (k_3 + k_1)e^{k_2}.$$

□

We conclude this section by giving a result of the same type which is useful in the study of some nonlinear equations. This result is similar to the usual comparison theorems between differential inequalities, except that the hypothesis is formulated in an integral form, which is weaker. This explains the necessity of some monotonicity assumption for the nonlinear term.

Lemma II.4.12 (Bihari's inequality [16]). *Let $f : [0, +\infty[\mapsto [0, +\infty[$ be a nondecreasing continuous function such that $f > 0$ on $]0, +\infty[$ and $\int_1^{+\infty} 1/f(x)\,dx < +\infty$. We denote the antiderivative of $-1/f$ which cancels at $+\infty$ as F.*

Let y be a continuous function which is nonnegative on $[0, +\infty[$ and let g be a nonnegative function in $L^1_{loc}([0, +\infty[)$. We assume that there exists a $y_0 > 0$ such that for all $t \geq 0$ we have the inequality

$$y(t) \leq y_0 + \int_0^t g(s)\,ds + \int_0^t f(y(s))\,ds.$$

Then, there exists a unique T^* which satisfies the equation

$$T^* = F\left(y_0 + \int_0^{T^*} g(s)\,ds\right), \tag{II.17}$$

and, for any $T < T^*$ we have

$$\sup_{t\leq T} y(t) \leq F^{-1}\left(F\left(y_0+\int_0^T g(s)\,ds\right)-T\right).$$

Proof.

The existence and uniqueness of T^* satisfying (II.17) arises from the fact that the function F is nonincreasing and tends towards 0 at $+\infty$, and that g is nonnegative. Let us set T such that $0 < T < T^*$. For all $t \leq T$, since g is nonnegative, we have

$$y(t) \leq y_0 + \int_0^T g(s)\,ds + \int_0^t f(y(s))\,ds. \tag{II.18}$$

Let us denote the right-hand side of this inequality as $z_T(t)$. Since f and y are continuous, the function z_T is of class $\mathcal{C}^1$ and we have $z_T(0) = y_0 + \int_0^T g(s)\,ds$ and for all $t < T$

$$z_T'(t) = f(y(t)) \leq f(z_T(t)),$$

because f is nondecreasing. We note that z_T is an nondecreasing function and since $y_0 > 0$, the function z_T does not cancel. Hence, we have

$$\frac{z_T'(t)}{f(z_T(t))} \leq 1, \forall t < T,$$

which, after integration between times 0 and T, gives

$$F(z_T(T)) - F(z(0)) \geq -T.$$

F is nonincreasing, thus it follows that

$$z_T(T) \leq F^{-1}(F(z_T(0)) - T) = F^{-1}\left(F\left(y_0+\int_0^T g(s)\,ds\right)-T\right). \tag{II.19}$$

We note that this makes sense because the definition of T^* and the condition $T < T^*$ imply that $F(z_T(0)) - T$ belongs to the range of F. From (II.18), and since z_T is nondecreasing, we have

$$y(t) \leq z_T(t) \leq z_T(T), \forall t < T.$$

Inequality (II.19) therefore provides the claim.

□

5 Spaces of Banach-valued functions

5.1 Definitions and main properties

Definition II.5.1. *Let X be a Banach space and let I be an interval of $\mathbb{R}$; we say that a function f from I in X is Lebesgue measurable, if*

- *The inverse image under f of all open sets of X is a Borel set of I.*
- *We can change f on a subset of zero Lebesgue measure of I, so that f takes its values into a separable subspace of X.*

In the case where X is separable, this definition is identical to the traditional definition of measurability. In the case where X is not separable, this definition ensures that one such function is indeed the limit almost everywhere of a sequence of simple functions with values in X, which makes it possible to define clearly the integral of f when it exists. This theory is known as the Bochner integral.

Proposition II.5.2. *Let X be a Banach space and let I be an interval of $\mathbb{R}$. For all $p \in [1, +\infty[$, we denote as $L^p(]0, T[, X)$, the set of Lebesgue measurable functions defined on I and with values in X, such that $t \mapsto \|f(t)\|_X^p$ is integrable on I. This is a Banach space for the norm*

$$\|f\|_{L^p(I,X)} = \left(\int_I \|f(t)\|_X^p \, dt \right)^{1/p}.$$

In the same way, we define, for $p = +\infty$, a Banach space $L^\infty(I, X)$ provided by the norm

$$\|f\|_{L^\infty(I,X)} = \text{esssup}_{t \in I} \|f(t)\|_X.$$

Proposition II.5.3. *If $p < +\infty$, the set of continuous functions on I with values in X is dense in $L^p(I, X)$.*

For all $f \in L^p(I, X)$, we denote as $\tilde{f}$ the extension by 0 of f to the whole time interval $\mathbb{R}$; then, for all $h \in \mathbb{R}$, we denote as $\tau_h f$ the translated function of $\tilde{f}$ defined by

$$\tau_h f(\cdot) = \tilde{f}(\cdot + h). \tag{II.20}$$

The restriction of $\tau_h f$ to the interval I is of course in $L^p(I, X)$ and we have the following result (the proof being identical to the classic case where $X = \mathbb{R}$; see [69] for example).

Corollary II.5.4 (Continuity of the translation operator). *If $p < +\infty$, then for all $f \in L^p(I, X)$ we have*

$$\tau_h f \xrightarrow[h \to 0]{} f, \quad \text{in } L^p(I, X).$$

These results are of course false if $p = +\infty$. In the case of $p < +\infty$, the proposition above shows that we can also define $L^p(I, X)$ as the completion of $\mathcal{C}^0(I, X)$ for the norm $\|\cdot\|_{L^p(I,X)}$.

For any function $f \in L^1(I, X)$ (and hence also if $f \in L^p(I, X)$ and if I is bounded), we can define the integral

$$\int_I f(t)\, dt \in X,$$

in a similar way to the Lebesgue integrals of real-valued functions, that is, by constructing the integral over simple measurable functions (i.e., taking a finite number of values) and by passing to the simple limit. We assume this result, as well as all the usual properties of the integral: Chasles' linearity theorem, and so on. Moreover, for all linear forms $\varphi \in X'$, we have

$$\int_I \langle \varphi, f(t)\rangle_{X',X}\, dt = \left\langle \varphi, \int_I f(t)\, dt \right\rangle_{X',X}.$$

The first examples for such spaces that are very useful in the sequel are given for $p, q \in [1, +\infty]$, by $L^p(I, L^q(\Omega))$.

The properties of L^p spaces, which we gave at the start of this section, naturally transpose into these spaces and we use these later without giving more details. In particular, if $p < +\infty$ and $q < +\infty$, then we have

$$(L^p(I, L^q(\Omega)))' \equiv L^{p'}(I, L^{q'}(\Omega)),$$

the identification of the two spaces being achieved via the natural inner product of the Hilbert space $L^2(I, L^2(\Omega)) \approx L^2(I \times \Omega)$. Hence, Propositions II.2.27, II.2.28, II.2.30, and II.2.32, can be immediately transposed to these spaces.

Among the particularly useful results to keep in mind, we give the following interpolation result as well as its corollary which gives the convergence properties in some intermediate spaces.

Theorem II.5.5. *Let I be an interval of $\mathbb{R}$, let Ω be an open set of $\mathbb{R}^d$, and let p_1,q_1,p_2,q_2 be four real numbers in $[1, +\infty]$. If $f \in L^{p_1}(I, L^{q_1}(\Omega)) \cap L^{p_2}(I, L^{q_2}(\Omega))$ then for all $\theta \in]0, 1[$, the function f belongs to $L^p(I, L^q(\Omega))$ for p and q defined by*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \text{ and } \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \tag{II.21}$$

and we have

$$\|f\|_{L^p(I,L^q(\Omega))} \leq \|f\|^{\theta}_{L^{p_1}(I,L^{q_1}(\Omega))} \|f\|^{1-\theta}_{L^{p_2}(I,L^{q_2}(\Omega))}.$$

Proof.

From Lemma II.2.33, for almost all $t \in I$ we have:

$$\|f(t)\|_{L^q}^p \leq \|f(t)\|_{L^{q_1}}^{p\theta} \|f(t)\|_{L^{q_2}}^{p(1-\theta)}.$$

If we assume that p_1 and p_2 are finite, then the Hölder inequality applied with the conjugate exponents $p_1/(p\theta)$ and $p_2/(p(1-\theta))$ shows that

$$\int_I \|f(t)\|_{L^q}^p \, dt \leq \left(\int_I \|f(t)\|_{L^{q_1}}^{p_1} \, dt \right)^{p\theta/p_1} \left(\int_I \|f(t)\|_{L^{q_2}}^{q_1} \, dt \right)^{p(1-\theta)/p_2},$$

which gives the desired result. The case where p_1 and/or p_2 are infinite is straightforward.

□

Corollary II.5.6. *We consider the same notation as in the previous theorem and we assume further that p_1 and q_1 are finite and that p_2 and q_2 are strictly larger than* 1*.*

If $(u_n)_n$ is a sequence of functions which strongly converges towards u in $L^{p_1}(I, L^{q_1}(\Omega))$ and weakly (or weakly-$\star$ if p_2 and/or q_2 are infinite) in $L^{p_2}(I, L^{q_2}(\Omega))$, then for all θ such that $0 < \theta \leq 1$ the sequence $(u_n)_n$ strongly converges towards u in $L^p(I, L^q(\Omega))$, where p and q are given by (II.21)*.*

Proof.

From the preceding theorem we have for all n,

$$\|u - u_n\|_{L^p(I,L^q(\Omega))} \leq \|u - u_n\|_{L^{p_1}(I,L^{q_1}(\Omega))}^{\theta} \|u - u_n\|_{L^{p_2}(I,L^{q_2}(\Omega))}^{1-\theta}.$$

The weak convergence in $L^{p_2}(I, L^{q_2}(\Omega))$ shows that the sequence $(u - u_n)_n$ is bounded in this space and the strong convergence in $L^{p_1}(I, L^{q_1}(\Omega))$ allows us to reach our conclusion, given that $\theta > 0$.

□

All of these results are used systematically in this book, without necessarily referencing them.

5.2 Regularity in time

5.2.1 Weak time derivative

In the study of parabolic partial differential equations, one independent variable (usually time) plays a particular role with respect to the other variables (typically space variables). This is why we work in $L^p(]0, T[, X)$ spaces where X is the functional space in the space variables.

In this section, we therefore generalise the concept of weak derivatives for functions defined on an interval of $\mathbb{R}$ and with values in a Banach space. In a general way, it is possible to define and study distribution spaces with values in X, but this theory is not required here and we refer the reader to [113] for more details.

For reasons which become clear later, it is useful to construct a theory in which the weak derivative of the function being considered can exist in a space that is larger than the initial space.

Definition II.5.7. *Let I be an interval of $\mathbb{R}$, and $X \subset Y$ be two Banach spaces, $1 \leq p, q \leq +\infty$. We say that a function $u \in L^p(I, X)$ has a weak derivative in $L^q(I, Y)$ if there exists a function $g \in L^q(I, Y)$ such that*

$$\int_I \varphi'(t)u(t)\,dt = -\int_I \varphi(t)g(t)\,dt,\ \forall \varphi \in \mathcal{D}(I). \tag{II.22}$$

If such a function g exists, it is unique and we denote

$$\frac{du}{dt} = g(t).$$

We should note that in (II.22), the left-hand term is an element of X and the right-hand term is an element of Y. However, since $X \subset Y$ this equality makes sense.

Remark II.5.1. A priori this definition depends on the space $L^q(I, Y)$ in which we seek the weak derivative. We can show that if $Y \subset Z$ in a dense way, if Z' is separable and if g and h are the weak derivatives of u in $L^q(I, Y)$ and $L^r(I, Z)$, respectively, then $g = h$ almost everywhere. This, therefore, justifies the notation du/dt.

5.2.2 Weak continuity

Definition II.5.8. *Let Y be a Banach space; we say that a function $u : [0, T] \to Y$ is weakly continuous if for all $\psi \in Y'$, the function defined by $t \in [0, T] \mapsto \langle \psi, u(t) \rangle_{Y',Y} \in \mathbb{R}$ is continuous. We denote by $\mathcal{C}^0([0, T], Y_{\text{weak}})$, the set of functions defined on $[0, T]$ with values in Y which are weakly continuous.*

We will now show the following important result (see, e.g., [85]).

Lemma II.5.9. *Let X be a separable and reflexive Banach space, and let Y be a Banach space, such that $X \subset Y$ with continuous embedding. Then*

$$L^\infty(]0, T[, X) \cap \mathcal{C}^0([0, T], Y_{\text{weak}}) = \mathcal{C}^0([0, T], X_{\text{weak}}).$$

Proof.

The space X is embedded into Y in a continuous way, therefore the restrictions to X of elements of Y' are in X'.

- Let us show that $\mathcal{C}^0([0,T],X_{\text{weak}}) \subset L^\infty(]0,T[,X) \cap \mathcal{C}^0([0,T],Y_{\text{weak}})$.

  Let $u \in \mathcal{C}^0([0,T],X_{\text{weak}})$ and let $\psi \in Y'$. Since $\psi_{|X} \in X'$, the function $t \mapsto \langle \psi, u(t)\rangle$ is continuous by definition, which shows that $u \in \mathcal{C}^0([0,T],Y_{\text{weak}})$.
  Let us show that $u \in L^\infty(]0,T[,X)$. First, we note that u is measurable. Indeed, any sphere B of X which is closed for the strong topology is also closed for the weak topology of X, because it is convex (see [27]). Therefore, $u^{-1}(B)$ is a closed set and hence a Borel set of $]0,T[$, because u is continuous on $]0,T[$ with values in X for the weak topology. However, X being separable, any open set of X is a countable union of closed spheres. Indeed, if $(x_n)_n$ is a dense sequence in X, it is obvious that any open set U is the union of all closed spheres centred on a point of the sequence $(x_n)_n$, with a rational radius and contained in U. Hence, for any open set U, $u^{-1}(U)$ is a Borel set of $]0,T[$. This proves the measurability.
  Let us now introduce the family of elements of X'' indexed by $t \in [0,T]$, defined by

  $$\Phi_t : \psi \in X' \mapsto \langle \psi, u(t)\rangle.$$

  By hypothesis, for all $\psi \in X'$, the function $t \mapsto \Phi_t(\psi)$ is continuous on $[0,T]$ and therefore bounded. From the Banach–Steinhaus theorem (Theorem II.2.4), we know that the family of operators $(\Phi_t)_{t\in]0,T[}$ is bounded in the sense of the norm of X''. Alternatively, we can say that there exists $C > 0$ such that

  $$|\langle \psi, u(t)\rangle_{X',X}| = |\Phi_t(\psi)| \leq C\|\psi\|_{X'},\ \forall t \in]0,T[, \forall \psi \in X'.$$

  If we apply Proposition II.2.1, this gives

  $$\|u(t)\|_X = \sup_{\psi\in X',\psi\neq 0} \frac{|\langle \psi, u(t)\rangle_{X',X}|}{\|\psi\|_{X'}} \leq C,\ \forall t \in]0,T[.$$

  This demonstrates that $u \in L^\infty(]0,T[,X)$.
- Let us show that $L^\infty(]0,T[,X) \cap \mathcal{C}^0([0,T],Y_{\text{weak}}) \subset \mathcal{C}^0([0,T],X_{\text{weak}})$.

  Let $u \in L^\infty(]0,T[,X) \cap \mathcal{C}^0([0,T],Y_{\text{weak}})$. Let us first verify that for all $t \in [0,T]$, $u(t) \in X$. A priori, we know only that $u(t) \in Y$ for all t, and that $u(t) \in X$ for almost all t.
  First, let us extend u to all of $\mathbb{R}$ (e.g., by successive reflections performed by setting $u(t) = u(-t)$ for $t \in [-T,0]$, etc.). It is then obvious that $u \in L^\infty(\mathbb{R},X) \cap \mathcal{C}^0(\mathbb{R},Y_{\text{weak}})$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a mollifying kernel (see Definition II.2.23). We set $u_n = u \star \eta_{1/n}$ which is defined for all t and takes its values in X.
  Let $t_0 \in \mathbb{R}$ be fixed. For all $n \geq 1$, we have

  $$\|u_n(t_0)\|_X = \|(u \star \eta_{1/n})(t_0)\|_X \leq \|u\|_{L^\infty(\mathbb{R},X)}.$$

The sequence $(u_n(t_0))_n$ is bounded in X, which is reflexive, therefore we can extract a subsequence $(u_{n_k}(t_0))_k$ which weakly converges in X towards a certain $\tilde{u}(t_0)$ (Theorem II.2.7).
However, for all $\psi \in Y'$, we have

$$\langle \psi, (u \star \eta_{1/n})(t_0) - u(t_0)\rangle_{Y',Y} = \big((\langle \psi, u\rangle_{Y',Y} \star \eta_{1/n})(t_0) - \langle \psi, u\rangle_{Y',Y}(t_0)\big) \xrightarrow[n\to\infty]{} 0,$$

because, by hypothesis, $t \mapsto \langle \psi, u(t)\rangle_{Y',Y}$ is a continuous function on $\mathbb{R}$ (the extension by reflection preserves this property). We have hence shown that $(u_n(t_0))_n$ weakly converges in Y towards $u(t_0)$. Through the uniqueness of the weak limit in Y, we obtain

$$u(t_0) = \tilde{u}(t_0) \in X,$$

which indeed proves that the function u takes its values in X for all t and that there exists $C > 0$ such that

$$\|u(t)\|_X \leq C, \, \forall t \in \mathbb{R}. \tag{II.23}$$

We can now define the function $t \mapsto \langle \psi, u(t)\rangle_{X',X}$ for all $\psi \in X'$. Let us show that it is continuous. Let $(t_n)_n$ be a sequence of real numbers which converges towards $t \in \mathbb{R}$. From (II.23) the sequence $(u(t_n))_n$ is bounded in X, and we can therefore extract a subsequence which weakly converges towards a certain x in X. Furthermore $(u(t_n))_n$ weakly converges towards $u(t)$ in Y, and through the uniqueness of the weak limit in Y, we obtain $x = u(t)$. This proves that the sequence $(u(t_n))_n$ is relatively weakly compact in X and has only one accumulation point. We therefore know that all of the sequence weakly converges towards its unique accumulation point $u(t)$.

□

Remark II.5.2. If X and Y are two separable Banach spaces such that Y is embedded in a dense way into X, then we have

$$L^\infty(]0,T[, X') \cap \mathcal{C}^0([0,T], Y'_{\text{weak}-\star}) \subset \mathcal{C}^0([0,T], X'_{\text{weak}-\star}).$$

This result is proven in an equivalent way to the preceding one. The converse embedding is of course true from that which has gone before, if we add the reflexivity hypothesis.

5.2.3 Strong continuity

Let X and Y be two Banach spaces such that X is embedded in a continuous and dense way into Y, and let $T > 0$ and p, q satisfy $1 \leq p, q \leq +\infty$. We denote:

$$E_{p,q} = \left\{ u \in L^p(]0,T[,\, X),\; \frac{du}{dt} \in L^q(]0,T[,\, Y) \right\}.$$

Lemma II.5.10. *The space $E_{p,q}$ endowed with the norm*

$$\|u\|_{E_{p,q}} = \|u\|_{L^p(]0,T[,\, X)} + \left\| \frac{du}{dt} \right\|_{L^q(]0,T[,\, Y)},$$

is a Banach space. Moreover, if p and q are finite then $\mathcal{C}^\infty([0,T],X)$ is dense in $E_{p,q}$.

Proof.
Let θ_1, θ_2 be two nonnegative functions of $\mathcal{C}^\infty([0,T],\mathbb{R})$, having sum 1 with

$$\operatorname{supp}(\theta_1) \subset \left[0, \frac{2}{3}T\right], \text{ and} \operatorname{supp}(\theta_2) \subset \left[\frac{1}{3}T, T\right].$$

Then let $u \in E$. To approximate u by regular functions, it is sufficient to separately approximate $\theta_1 u$ and $\theta_2 u$, because $u = \theta_1 u + \theta_2 u$.

The function $v = \theta_1 u$ is an element of

$$\left\{ f \in L^p(]0,+\infty[,X),\; \frac{df}{dt} \in L^q(]0,+\infty[,Y) \right\}.$$

Let us set $v_h(t) = v(t+h)$; then $v_{h,\varepsilon} = v_h \star \eta_\varepsilon$ with $\varepsilon < h$ where $\eta : \mathbb{R} \to \mathbb{R}$ is a mollifying kernel. The claim follows from Corollary II.5.4. A similar argument holds for the function $\theta_2 u$.

□

Remark II.5.3. Let us assume that $p = +\infty$, $q < +\infty$, and that X is the dual of a Banach space E. Then we can easily see that the family of regular functions constructed in the preceding proof satisfies

$$v_{h,\epsilon} \xrightarrow[(h,\varepsilon)\to 0]{} v, \qquad \text{weakly-}\star \text{ in } L^\infty(]0,+\infty[,E'),$$

$$\frac{d}{dt}v_{h,\epsilon} \xrightarrow[(h,\varepsilon)\to 0]{} \frac{d}{dt}v, \quad \text{in } L^q(]0,T[,Y).$$

In other words, the density property of the regular functions still occurs by taking the weak-star topology on $L^\infty(]0,T[,E')$. We can deal, in the same way, with the case where p is finite, $q = +\infty$, and Y is the dual of F as well as the case where $p = q = +\infty$ and $X = E'$, $Y = F'$.

Proposition II.5.11. *Any element u of $E_{p,q}$ (defined almost everywhere) possesses a continuous representation on $[0,T]$ with values in Y, and the embedding of $E_{p,q}$ into $\mathcal{C}^0([0,T],Y)$ is continuous.*

Moreover, for all $t_1, t_2 \in [0,T]$, we have

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \frac{du}{dt}\, dt,$$

where it is understood that we have identified u and its continuous representation.

Proof.

This result is proven in an entirely similar way to those established in Section 4.

□

In the Hilbertian case we can improve the preceding result in the following way.

Theorem II.5.12 (Lions–Magenes [85]). *Let V and H be two Hilbert spaces such that V is embedded in a continuous and dense way into H. We then identify H with its dual such that we have $V \subset H \subset V'$, the duality bracket between V and V' being given by the scalar product of H. Let $1 \leq p, q \leq +\infty$ and let u, v be two functions such that*

$$u \in E_{p,q'} = \left\{ f \in L^p(]0,T[,V),\ \frac{df}{dt} \in L^{q'}(]0,T[,V') \right\},$$

$$v \in E_{q,p'} = \left\{ f \in L^q(]0,T[,V),\ \frac{df}{dt} \in L^{p'}(]0,T[,V') \right\}.$$

Then the function $t \mapsto (u(t),v(t))_H$ has a continuous representation on $[0,T]$ and we have for all $t_1, t_2 \in [0,T]$,

$$\begin{aligned}&(u(t_2),v(t_2))_H - (u(t_1),v(t_1))_H \\ &= \int_{t_1}^{t_2} \left\langle \frac{du}{dt}(t), v(t) \right\rangle_{V',V} + \left\langle \frac{dv}{dt}(t), u(t) \right\rangle_{V',V} dt.\end{aligned}$$

Proof.

Here, we give the proof of this result when $1 < p, q < +\infty$. The argument can be adapted to other cases by applying Remark II.5.3 and Proposition II.2.12.

Let us consider the following bilinear forms defined on $E_{p,q'} \times E_{q,p'}$ with values in $L^1(]0,T[)$ by

$$\begin{aligned}\Psi_1(f,g) &= \big(t \mapsto (f(t),g(t))_H\big),\\ \Psi_2(f,g) &= \left(t \mapsto \left\langle \frac{df}{dt}(t), g(t) \right\rangle_{V',V} + \left\langle \frac{dg}{dt}(t), f(t) \right\rangle_{V',V}\right).\end{aligned}$$

These two maps are well-defined because the exponents p, q and p', q' are conjugate, respectively. Moreover, we have for all $(f, g) \in E_{p,q'} \times E_{q,p'}$:

$$\begin{aligned}\|\Psi_1(f,g)\|_{L^1(]0,T[)} &\leq \|f\|_{L^p(]0,T[,V)}\|g\|_{L^{p'}(]0,T[,V')} \\ &\leq T^{1/p'}\|f\|_{L^p(]0,T[,V)}\|g\|_{\mathcal{C}^0(]0,T[,V')} \\ &\leq C\|f\|_{E_{p,q'}}\|g\|_{E_{q,p'}},\end{aligned}$$

and

$$\begin{aligned}\|\Psi_2(f,g)\|_{L^1(]0,T[)} \leq & \left\|\frac{df}{dt}\right\|_{L^{q'}(]0,T[,V')}\|g\|_{L^q(]0,T[,V)} \\ &+\left\|\frac{dg}{dt}\right\|_{L^{p'}(]0,T[,V')}\|f\|_{L^p(]0,T[,V)} \\ \leq & C\|f\|_{E_{p,q'}}\|g\|_{E_{q,p'}}.\end{aligned}$$

This proves that Ψ_1 and Ψ_2 are continuous bilinear forms. Using Lemma II.5.10, let us consider $(u_n)_n$ and $(v_n)_n$, two sequences of $\mathcal{C}^\infty([0,T],V)$ which converge towards u and v, respectively, in $E_{p,q'}$ and $E_{q,p'}$. The functions u_n and v_n being regular, we can differentiate the scalar product $(u_n(t), v_n(t))_H$ in the classic sense which implies, in particular, that for all functions $\varphi \in \mathcal{D}(]0,T[)$, we have

$$\begin{aligned}&-\int_0^T \varphi'(t)(u_n(t), v_n(t))_H\,dt \\ &=\int_0^T \left(\left\langle \frac{du_n}{dt}(t), v_n(t)\right\rangle_{V',V} + \left\langle \frac{dv_n}{dt}(t), u_n(t)\right\rangle_{V',V}\right)\varphi(t)\,dt.\end{aligned}$$

In other words, we have

$$-\int_0^T \varphi'(t)\Psi_1(u_n,v_n)\,dt = \int_0^T \varphi(t)\Psi_2(u_n,v_n)\,dt.$$

By continuity of Ψ_1 and Ψ_2, we can pass to the limit in this expression and obtain

$$\begin{aligned}&-\int_0^T \varphi'(t)(u(t), v(t))_H\,dt \\ &=\int_0^T \left(\left\langle \frac{du}{dt}(t), v(t)\right\rangle_{V',V} + \left\langle \frac{dv}{dt}(t), u(t)\right\rangle_{V',V}\right)\varphi(t)\,dt.\end{aligned}$$

The function $\Psi_1(u,v) : t \mapsto (u(t),v(t))_H$ belongs to $L^1(]0,T[)$, therefore this last expression, valid for all regular functions with compact support φ, shows us that $\Psi_1(u, v)$ belongs to $W^{1,1}(]0, T[)$ and that its weak derivative is $\Psi_2(u, v)$.

Corollary II.4.2 then allows us to conclude the proof.

□

Theorem II.5.13. *Let V and H be two Hilbert spaces satisfying the hypotheses of the preceding theorem; then the space*

$$E_{2,2} = \left\{ u \in L^2(]0, T[, V), \ \frac{du}{dt} \in L^2(]0, T[, V') \right\}$$

is continuously embedded into $\mathcal{C}^0([0, T], H)$.

Proof.

We apply the preceding theorem, with $p = q = 2$ and $u = v$, and we immediately obtain that the function $t \mapsto \frac{1}{2}\|u(t)\|_H^2$ is continuous on $[0, T]$ and that for all $t, s \in [0, T]$, we have

$$\begin{aligned} \frac{1}{2}\|u(t)\|_H^2 &= \frac{1}{2}\|u(s)\|_H^2 + \int_s^t \left\langle \frac{du}{dt}, u \right\rangle_{V',V} d\tau \\ &\leq \frac{1}{2}\|u(s)\|_H^2 + C\|u\|_{E_{2,2}}^2. \end{aligned}$$

By integrating this with respect to s, we find

$$\begin{aligned} \frac{1}{2}\|u(t)\|_H^2 &\leq \frac{1}{2}\|u\|_{L^2(]0,T[,H)}^2 + C\|u\|_{E_{2,2}}^2 \\ &\leq C\|u\|_{L^2(]0,T[,V)}^2 + C\|u\|_{E_{2,2}}^2 \leq C\|u\|_{E_{2,2}}^2. \end{aligned} \tag{II.24}$$

This proves that the function u lies in $L^\infty(]0, T[, H)$. Furthermore, Proposition II.5.11 shows that u is continuous with values in V'. We can then apply Lemma II.5.9, to obtain that u is weakly continuous with values in H (we should not forget that all Hilbert spaces are reflexive).

The strong continuity of u with values in H is now a consequence of the weak continuity in H, of the continuity of the function $t \mapsto \|u(t)\|_H^2$, and Proposition II.2.11. Moreover, the estimate (II.24) leads to

$$\|u\|_{\mathcal{C}^0([0,T],H)} \leq C\|u\|_{E_{2,2}}.$$

□

This situation is a special case of a more general result (see [85]) which is the following.

Theorem II.5.14. *Let V and W be two Hilbert spaces; then*

$$E_{2,2} = \left\{ v \in L^2(]0, T[, V), \ \frac{dv}{dt} \in L^2(]0, T[, W) \right\}$$

is continuously embedded in $\mathcal{C}^0([0,T],\,[V,W]_{\frac{1}{2}})$ *where* $[V,W]_{\frac{1}{2}}$ *is the interpolated space of order* $\frac{1}{2}$ *of* V *and* W*.*

We refer the reader to [85] for a precise definition of interpolated space $[V,W]_{\frac{1}{2}}$. In Chapter IV, we present the proof of a special case of this statement (see Theorem IV.5.11).

5.3 Compactness theorems

Let us start by proving a now classic lemma (due to J.-L. Lions [84]), which is the basis of a large part of all the following compactness results.

Lemma II.5.15. *Let* $B_0 \subset B_1$*and* $\subset B_2$ *be three Banach spaces. We assume that the embedding of* B_1 *in* B_2 *is continuous and the embedding of* B_0 *in* B_1 *is compact. Then, for all* $\varepsilon > 0$*, there exists a constant* $C(\varepsilon)$*, such that for all* $u \in B_0$*, we have*

$$\|u\|_{B_1} \leq \varepsilon \|u\|_{B_0} + C(\varepsilon)\|u\|_{B_2}.$$

Proof.
Let us assume that the claim is false; then there exists $\varepsilon_0 > 0$ and a sequence $(u_n)_n \subset B_0$, such that

$$\|u_n\|_{B_1} \geq \varepsilon_0 \|u_n\|_{B_0} + n\|u_n\|_{B_2},\ \forall n \geq 1.$$

By homogeneity we can take $\|u_n\|_{B_1} = 1$ in the above inequality. Hence, the sequence $(u_n)_n$ is bounded in B_0 and satisfies $\|u_n\|_{B_2} \leq 1/n$. The embedding of B_0 into B_1 is compact, thus we can extract a subsequence $(u_{n_k})_k$ which converges in B_1 towards an element denoted u_∞. Of course $\|u_\infty\|_{B_1} = 1$, and furthermore $\|u_\infty\|_{B_2} = 0$. This is the contradiction.

□

We can now prove one of the fundamental results of compactness in the study of nonlinear evolution problems.

Theorem II.5.16 (Aubin–Lions–Simon). *Let* $B_0 \subset B_1 \subset B_2$ *be three Banach spaces. We assume that the embedding of* B_1 *in* B_2 *is continuous and that the embedding of* B_0 *in* B_1 *is compact. Let* p, r *such that* $1 \leq p, r \leq +\infty$*. For* $T > 0$*, we define*

$$E_{p,r} = \left\{ v \in L^p(]0,T[,\,B_0),\ \frac{dv}{dt} \in L^r(]0,T[,\,B_2) \right\}.$$

i) If $p < +\infty$*, the embedding of* $E_{p,r}$ *in* $L^p(]0,T[,\,B_1)$ *is compact.*
ii) If $p = +\infty$ *and if* $r > 1$*, the embedding of* $E_{p,r}$ *in* $\mathcal{C}^0([0,T],\,B_1)$ *is compact.*

The proof which follows comes from [109] and does not assume that the spaces considered are reflexive, in contrast to the proof in [14]. It thus applies to the spaces L^1 and L^∞, for example.

Proof.

We only demonstrate point i) of the theorem the second point being treated in an entirely similar fashion. Furthermore, it is clear that we do not lose any generality by assuming that $r = 1$.

To demonstrate the theorem, we establish that if a sequence $(u_n)_n$ satisfies:

$$\begin{aligned} (u_n)_n \text{ is bounded in } \; & L^p(]0,T[,\, B_0), \\ \left(\frac{du_n}{dt}\right)_n \text{ is bounded in } \; & L^1(]0,T[,\, B_2), \end{aligned}$$

then we can extract a Cauchy subsequence in $L^p(]0,T[,\, B_1)$.

• **Step 1:**

From Lemma II.5.15, it suffices to find a Cauchy sequence in $L^p(]0,T[,\, B_2)$. Indeed, if a sequence $(v_n)_n$ satisfies the Cauchy criterion in $L^p(]0,T[,\, B_2)$ and is bounded in $L^p(]0,T[,\, B_0)$, then for all $\varepsilon > 0$ we have

$$\begin{aligned} \|v_n - v_m\|_{L^p(]0,T[,\, B_1)} &\leq \varepsilon \|v_n - v_m\|_{L^p(]0,T[,\, B_0)} + C(\varepsilon)\|v_n - v_m\|_{L^p(]0,T[,\, B_2)} \\ &\leq 2K\varepsilon + C(\varepsilon)\|v_n - v_m\|_{L^p(]0,T[,\, B_2)}, \end{aligned}$$

where K is some bound of $(v_n)_n$ in $L^p(]0,T[,\, B_0)$. Hence, if $(v_n)_n$ is a Cauchy sequence in $L^p(]0,T[,\, B_2)$, it follows

$$\limsup_{n,m\to\infty} \|v_n - v_m\|_{L^p(]0,T[,\, B_1)} \leq 2K\varepsilon,$$

which proves the claim, ε being arbitrary.

• **Step 2:**

Let $\theta \in \mathcal{C}^\infty([0,T],\mathbb{R}),\ \theta(T) = 0$, such that we have

$$u_n = \theta u_n + (1-\theta)u_n \equiv v_n + w_n.$$

We show that we are able to extract from $(v_n)_n$ a Cauchy sequence in $L^p(]0,T[, B_2)$. We would proceed in a similar way for the sequence $(w_n)_n$.

We extend v_n by continuity to $\mathbb{R}_+$ by setting $v_n(t) = 0,\ \forall t \geq T$, and for all $h > 0$ we break down v_n into

$$v_n(t) = \left(\frac{1}{h}\int_t^{t+h} v_n(s)ds\right) + \left(\frac{1}{h}\int_t^{t+h} (v_n(t) - v_n(s))ds\right) \equiv a_{n,h}(t) + b_{n,h}(t).$$

Let h be positive; then we show that the sequence $(a_{n,h}(t))_n$ is uniformly bounded and equicontinuous with values in a compact set of B_2 (we know that $a_{n,h}$ is continuous from Proposition II.5.11). For this, we have

$$\sup_{t\in\mathbb{R}^+} \|a_{n,h}(t)\|_{B_0} \leq \frac{1}{h} h^{1/p'} \|v_n\|_{L^p(]0,T[,\,B_0)},$$

which proves that $t \mapsto a_{n,h}(t)$ takes its values, independently from n, in a bounded set of B_0 (i.e., in a compact of B_2, because the embedding of B_0 in B_2 is compact). Moreover, we have

$$\frac{da_{n,h}}{dt} = \frac{1}{h}\left(v_n(t+h) - v_n(t)\right) = \frac{1}{h}\int_t^{t+h} \frac{dv_n}{dt}(\tau)d\tau,$$

from which, for all $t > 0$,

$$\begin{aligned}\left\|\frac{d}{dt}a_{n,h}(t)\right\|_{B_2} &\leq \frac{1}{h}\int_t^{t+h} \left\|\frac{dv_n}{dt}(\tau)\right\|_{B_2} d\tau,\\ &\leq \frac{1}{h}\left\|\frac{dv_n}{dt}(\tau)\right\|_{L^1(]0,T[,B_2)}.\end{aligned}$$

Hence, h being fixed, the sequence $(a_{n,h})_n$ is equicontinuous with values in a compact set of B_2. Ascoli's theorem (Theorem II.3.1) then shows that we can extract from the sequence $(a_{n,h})_n$ a subsequence which is convergent in $C^0([0,T],B_2)$ and thus in $L^p(]0,T[,B_2)$. Furthermore, we have

$$\begin{aligned}\|b_{n,h}(t)\|_{B_2} &\leq \frac{1}{h}\int_t^{t+h} \|v_n(t) - v_n(s)\|_{B_2}\, ds\\ &\leq \frac{1}{h}\int_t^{t+h}\left(\int_t^s \left\|\frac{dv_n}{dt}(\tau)\right\|_{B_2} d\tau\right) ds.\end{aligned}$$

Hence, by using Jensen's inequality (Proposition II.2.20) and Fubini's theorem we obtain

$$\begin{aligned}\int_0^T \|b_{n,h}(t)\|_{B_2}^p\, dt &\leq \int_0^T \frac{1}{h}\int_t^{t+h}\left(\int_t^s \left\|\frac{dv_n}{dt}(\tau)\right\|_{B_2} d\tau\right)^p ds\, dt\\ &\leq \left\|\frac{dv_n}{dt}\right\|_{L^1(]0,T[,B_2)}^{p-1} \int_0^T \frac{1}{h}\int_t^{t+h}\int_t^s \left\|\frac{dv_n}{dt}(\tau)\right\|_{B_2} d\tau\, ds\, dt\\ &\leq \left\|\frac{dv_n}{dt}\right\|_{L^1(]0,T[,B_2)}^{p-1} \int_0^T \frac{1}{h}\int_t^{t+h} \left\|\frac{dv_n}{dt}(\tau)\right\|_{B_2}(t+h-\tau)\, d\tau\, dt\\ &\leq h\left\|\frac{dv_n}{dt}\right\|_{L^1(]0,T[,B_2)}^{p},\end{aligned}$$

and hence

$$\|b_{n,h}\|_{L^p(]0,T[,B_2)} \leq C_T h^{1/p}. \tag{II.25}$$

- **Step 3:**

We now use a diagonal process to construct a convergent subsequence of $(v_n)_n$. For this, for $k \geq 1$, we set $h_k = 1/k$.

For $k = 1$ we have seen that we can extract a convergent subsequence $(a_{\varphi_1(n),h_1})_n$ from $(a_{n,h_1})_n$. For $k = 2$ we can again extract a subsequence of the sequence $(a_{\varphi_1(n),h_2})_n$, denoted $(a_{\varphi_1\circ\varphi_2(n),h_2})_n$, which converges. We hence proceed with successive extractions of subsequences such that for all $k \geq 1$, the sequence $(a_{\varphi_1\circ\cdots\circ\varphi_k(n),h_k})_n$ converges.

We now define $\psi(k) = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k(k)$. Let us verify that the sequence $(v_{\psi(k)})_k$ is indeed a Cauchy sequence. Let $\varepsilon > 0$; according to (II.25) there exists $k_0 \geq 1$ such that for all n and for all $k \geq k_0$, we have

$$\|b_{n,h_k}\|_{L^p(]0,T[,B_2)} \leq \varepsilon.$$

Let us now write

$$v_{\psi(k)} = a_{\psi(k),h_{k_0}} + b_{\psi(k),h_{k_0}}.$$

From the diagonal extraction process employed, the sequence $(a_{\psi(k),h_{k_0}})_{k\geq k_0}$ is a sequence extracted from the sequence $(a_{\varphi_1\circ\cdots\circ\varphi_{k_0}(n),h_{k_0}})_n$ which, by definition, is a convergent sequence and thus it satisfies the Cauchy criterion. Hence, there exists $k_1 \geq k_0$ such that for all $k, k' \geq k_1$, we have

$$\|a_{\psi(k),h_{k_0}} - a_{\psi(k'),h_{k_0}}\|_{L^p(]0,T[,B_2)} \leq \varepsilon.$$

Thus, finally, we have for all $k, k' \geq k_1$,

$$\begin{aligned}\|v_{\psi(k)} - v_{\psi(k')}\|_{L^p(]0,T[,B_2)} &\leq \|a_{\psi(k),h_{k_0}} - a_{\psi(k'),h_{k_0}}\|_{L^p(]0,T[,B_2)} \\ &\quad + \|b_{\psi(k),h_{k_0}}\|_{L^p(]0,T[,B_2)} + \|b_{\psi(k'),h_{k_0}}\|_{L^p(]0,T[,B_2)} \\ &\leq 3\varepsilon.\end{aligned}$$

This, indeed, proves that the sequence $(v_{\psi(k)})_k$ is a Cauchy sequence in $L^p(]0,T[,B_2)$.

$\square$

In certain cases, the preceding theorem does not apply and we need to use sharper results.

Let E be a Banach space. For $f \in L^1(]0,T[,E)$ we denote the translated function of f defined by (II.20) as $\tau_h f$. For $1 \leq q < +\infty$ and $0 < \sigma < 1$, we define the Nikolskii spaces $N_q^\sigma(]0,T[,\ E)$ by:

$$N_q^\sigma(]0,T[,\ E) = \left\{ f \in L^q(]0,T[,E),\ \sup_{0<h<T} \frac{\|\tau_h f - f\|_{L^q(]0,T-h[,E)}}{h^\sigma} < +\infty \right\}, \tag{II.26}$$

and for $f \in N_q^\sigma(]0,T[,E)$ we introduce the norm

$$\|f\|_{N_q^\sigma(]0,T[,E)} = \left(\|f\|^q_{L^q(]0,T[,E)} + \sup_{0<h<T} \left(\frac{1}{h^\sigma} \|\tau_h f - f\|_{L^q(]0,T-h[,E)} \right)^q \right)^{1/q}.$$

Intuitively, this leads to replacing a condition on the derivative with respect to time of f by a condition of the Hölder type, which is weaker but proves to be sufficient. We give, for example, the following theorem for which the reader will find a proof in [109].

Theorem II.5.17 (Simon). *Let B_0, B_1, B_2 be three Banach spaces with $B_0 \subset B_1 \subset B_2$. We assume that the embedding of B_1 into B_2 is continuous and that the embedding of B_0 into B_1 is compact.*

Then, for all $1 \leq q \leq +\infty$ and $0 < \sigma < 1$, the embedding

$$L^q(]0,T[,\ B_0) \cap N_q^\sigma(]0,T[,\ B_2) \hookrightarrow L^q(]0,T[,\ B_1),$$

is compact.

5.4 Banach-valued Fourier transform

To obtain compactness, we have seen that it is necessary to establish estimates of the derivatives with respect to time or of the translated functions with respect to time from the sequences of functions involved. As we show in the following, several methods are available to obtain these estimates. One of these consists in using the Fourier transform with respect to the time variable. We demonstrate, in this sense, the Proposition II.5.23 which gives a characterisation of the sequences of bounded functions in the Nikolskii spaces using the Fourier transform. We illustrate this technique in Section 1 of Chapter VII for the investigation of the Navier–Stokes equations with nonstandard boundary conditions.

Before this, we need to recall the definition of the essential properties of the Fourier transform of functions of the time variable with values in a Banach space.

Definition II.5.18. *Let X be a complex Banach space and let $f \in L^1(\mathbb{R}, X)$. We call the Fourier transform of f the function $\mathcal{F}(f) \in L^\infty(\mathbb{R}, X)$ defined by*

$$\mathcal{F}(f)(\tau) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\tau}\, dt.$$

In the case where X is a finite-dimensional space, we have the following classic and fundamental theorem, for which the reader will find a proof, for example, in [100].

Theorem II.5.19 (Hausdorff–Young). *We assume that $X = \mathbb{C}^n$.*

- *For all f in $L^1(\mathbb{R}, \mathbb{C}^n) \cap L^2(\mathbb{R}, \mathbb{C}^n)$ we have*

$$\mathcal{F}(f) \in L^2(\mathbb{R}, \mathbb{C}^n) \text{ and } \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2},$$

and hence $\mathcal{F}$ extends in a unique way as an isometry of $L^2(\mathbb{R},\mathbb{C}^n)$ on itself.

- *For any $p \in [1,2]$, there exists a $C > 0$ such that for all f in $L^1(\mathbb{R},\mathbb{C}^n) \cap L^p(\mathbb{R},\mathbb{C}^n)$ we have*

$$\mathcal{F}(f) \in L^{p'}(\mathbb{R},\mathbb{C}^n) \text{ and } \|\mathcal{F}(f)\|_{L^{p'}} \leq C\|f\|_{L^p}.$$

In our context, we shall need to use the Fourier transform in the case where X is a Banach space. In this case, we pay attention to the fact that the preceding theorem is not in general true! Nevertheless, in the particular case where X is a space containing integrable functions, we can find a suitable framework in which the Hausdorff–Young inequality holds. We refer to [8, 64, 96] for more complete and more precise results on this subject.

Theorem II.5.20 (Hausdorff–Young for Lebesgue spaces). *Let (E,μ) be a compact, locally-separated topological space equipped with a regular measure μ on its Borel sets. Let $q \in [1,+\infty]$ and we set $X = L^q(E,\mu)$.*

For all $p \in [1,2]$ such that $p \leq q \leq p'$ there exists a $C > 0$ such that for all f in $L^1(\mathbb{R}, L^q(E,\mu)) \cap L^p(\mathbb{R}, L^q(E,\mu))$ we have

$$\mathcal{F}(f) \in L^{p'}(\mathbb{R}, L^q(E,\mu)) \text{ and } \|\mathcal{F}(f)\|_{L^{p'}(\mathbb{R},L^q(E,\mu))} \leq C\|f\|_{L^p(\mathbb{R},L^q(E,\mu))}.$$

We note that the condition $p \leq q \leq p'$ is optimal (see [96] for a counter example).

Proof.

By density, it is sufficient to prove the result for smooth functions f. We then successively use the Minkowski inequality with $r = p'/q \geq 1$, followed by the Hausdorff inequality given by Theorem II.5.19 for the scalar function $t \mapsto f(t,x)$ with x fixed and finally use the Minkowski inequality again, with $r = q/p \geq 1$. This gives

$$\begin{aligned}
\|\mathcal{F}(f)\|^{p'}_{L^{p'}(\mathbb{R},L^q(E,\mu))} &= \int_{\mathbb{R}} \left(\int_E |\mathcal{F}(f)(\tau,x)|^q \, d\mu \right)^{p'/q} d\tau \\
&\leq \left(\int_E \left(\int_{\mathbb{R}} |\mathcal{F}(f)(\tau,x)|^{p'} \, d\tau \right)^{q/p'} d\mu \right)^{p'/q} \\
&\leq C \left(\int_E \left(\int_{\mathbb{R}} |f(t,x)|^p \, dt \right)^{q/p} d\mu \right)^{p'/q} \\
&\leq C \left(\int_{\mathbb{R}} \left(\int_E |f(t,x)|^q \, d\mu \right)^{p/q} dt \right)^{p'/p} \\
&= C\|f\|^{p'}_{L^p(\mathbb{R},L^q(E,\mu))},
\end{aligned}$$

which gives the claim.

□

Corollary II.5.21. *If $X = H$ is a Hilbert space then the Fourier transform $\mathcal{F}$ extends as an isometry on $L^2(\mathbb{R}, H)$. Moreover $\mathcal{F}$ is invertible on $L^2(\mathbb{R}, H)$ and its inverse is given by*

$$\mathcal{F}^{-1}(g)(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\tau) e^{i\tau t}\, d\tau,\ \forall g \in L^1(\mathbb{R}, H) \cap L^2(\mathbb{R}, H).$$

Proof.

This result is deduced from the preceding theorem because any Hilbert space is isomorphic with an $L^2(E, \mu)$ space for well-chosen (E, μ).

We note that this property characterises Hilbert spaces: if X is a Banach space such that $\mathcal{F}$ maps $L^2(\mathbb{R}, X)$ into itself then X is isomorphic to a Hilbert space.

□

Finally, we use the following result whose proof is a straightforward integration by parts.

Proposition II.5.22. *Let X be a Banach space and $f \in L^1(\mathbb{R}, X)$ such that $\frac{df}{dt} \in L^1(\mathbb{R}, X)$ (we say that $f \in W^{1,1}(\mathbb{R}, X)$); then we have*

$$\mathcal{F}\left(\frac{df}{dt}\right)(\tau) = i\tau \mathcal{F}(f)(\tau).$$

The first difficulty that we encounter is the fact that the functions with which we deal with are only defined on a bounded interval of time $]0, T[$ and not on the whole real axis. As a consequence, we proceed in the following manner.

Let $\tilde{f}$ be the extension of a function f of $W^{1,1}(]0, T[, X)$ by zero outside $]0, T[$. This function is not in $W^{1,1}(\mathbb{R}, X)$ but nevertheless we can derive it in the sense of distributions and obtain

$$\frac{\partial \tilde{f}}{\partial t} = \widetilde{\frac{\partial f}{\partial t}} + f(0)\delta_0 - f(T)\delta_T.$$

Let us recall that $f(0)$ and $f(T)$ are perfectly defined because the functions of $W^{1,1}(]0, T[, X)$ are continuous on $[0, T]$ with values in X (see Corollary II.4.2).

Furthermore, we can generalise the Fourier transform to the tempered distributions (in particular Dirac mass; see, e.g., [101]) and we show that we have

$$i\tau \mathcal{F}\left(\tilde{f}\right) = \mathcal{F}\left(\widetilde{\frac{\partial f}{\partial t}}\right) + \frac{1}{\sqrt{2\pi}} f(0) - \frac{e^{-i\tau T}}{\sqrt{2\pi}} f(T). \tag{II.27}$$

In view of the functions which interest us for analysis of nonlinear partial differential equations, the essential result is the following.

Proposition II.5.23. *Let H be a Hilbert space and let $f \in L^2(]0,T[,H)$ be a function such that for a certain $0 < \sigma \leq 1$ we have*

$$\int_{\mathbb{R}} |\tau|^{2\sigma} \|\mathcal{F}(\tilde{f})(\tau)\|_H^2 d\tau \leq C^2. \tag{II.28}$$

Then f belongs to the Nikolskii space $N_2^\sigma(]0,T[,H)$ defined by (II.26) and we have

$$\|f\|_{N_2^\sigma(]0,T[,H)} \leq M_\sigma(1+C),$$

where M_σ depends only on σ and T.

Proof.

Let f be a function satisfying (II.28), and let $h \in]0,T[$. We set $g_h(t) = \tilde{f}(t+h) - \tilde{f}(t)$ such that

$$\mathcal{F}(g_h)(\tau) = (e^{i\tau h} - 1)\mathcal{F}(\tilde{f})(\tau).$$

We then write for $\tau \neq 0$

$$\mathcal{F}(g_h)(\tau) = \frac{e^{i\tau h} - 1}{\tau^\sigma h^\sigma} h^\sigma \tau^\sigma \mathcal{F}(\tilde{f})(\tau).$$

However, the function $x \mapsto (e^{ix} - 1)/x^\sigma$ is bounded on $\mathbb{R}$ as soon as $\sigma \leq 1$. If K_σ is the bound of this function, we have

$$\|\mathcal{F}(g_h)(\tau)\|_H \leq K_\sigma h^\sigma |\tau|^\sigma \|\mathcal{F}(\tilde{f})(\tau)\|_H,$$

such that

$$\int_{\mathbb{R}} \|\mathcal{F}(g_h)(\tau)\|_H^2 d\tau \leq K_\sigma^2 h^{2\sigma} \int_{\mathbb{R}} |\tau|^{2\sigma} \|\mathcal{F}(\tilde{f})(\tau)\|_H^2 d\tau \leq K_\sigma^2 C^2 h^{2\sigma}.$$

From Corollary II.5.21 the Fourier transform is an isometry of $L^2(\mathbb{R}, H)$, therefore we obtain by the definition of g_h

$$\int_{\mathbb{R}} \|\tilde{f}(t+h) - \tilde{f}(t)\|_H^2 dt \leq K_\sigma^2 C^2 h^{2\sigma},$$

which implies that

$$\int_0^{T-h} \|f(t+h) - f(t)\|_H^2 dt \leq K_\sigma^2 C^2 h^{2\sigma},$$

and proves that f belongs in the Nikolskii space $N_2^\sigma(]0,T[,H)$ as well as the stated estimate.

□

Hence, to obtain compactness for a family of approximate solutions to an evolution problem, we can attempt to obtain uniform bounds on the Fourier transform with respect to time for these solutions of type (II.28). This ensures a bound in a Nikolskii space and hence the compactness property through Theorem II.5.17.

6 Some results in spectral analysis of unbounded operators

To generalise the well-known spectral theory in finite dimension to linear operators in infinite dimension, it is convenient to work with operators which map a Hilbert space H into itself (of course one could also work with Banach spaces; see [27, 101]). Indeed, to give meaning to the definition of an eigenvector

$$Au = \lambda u$$

it is clear that u and Au must coexist in the same space. Unfortunately, the common operators which appear in problems from physics are, in general, differential operators and these do not map the common Sobolev spaces $H^s(\Omega)$ onto themselves, because of the loss of derivatives (see Chapter III for the definition of Sobolev spaces).

In the sequel of this book, we particularly apply this theory to the Stokes operator (see Section 5.2 of Chapter IV).

6.1 Definitions

We need to look at these so-called *unbounded* operators A, as operators in H which are only defined on a subset of H, known as the *domain* of the operator and denoted by $D(A)$. For example, we can define the Laplace operator as an unbounded operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$.

Hence, to be given an unbounded operator is to be given:

- A Hilbert space, H.
- A linear subspace, $D(A) \subset H$.
- A linear mapping $A : D(A) \to H$.

In our context, we assume that all the unbounded operators have a dense domain in H and are closed, which means that the graph of A, defined by $G(A) = \{(u, Au),\, u \in D(A)\}$ is a closed subset of $H \times H$.

Remark II.6.1. The term “unbounded operator”, comes from the fact that if A is closed with a dense domain $D(A)$, which is not equal to H, then A cannot be bounded; that is, there does not exist some $C > 0$ such that

$$\|Au\|_H \le C\|u\|_H,\ \forall u \in D(A).$$

Indeed, let us suppose that this inequality is true for some $C > 0$. Any $u \in H$ is the limit of a sequence $(u_n)_n$ of elements of $D(A)$. This sequence being a Cauchy sequence, from the inequality above, $(Au_n)_n$ is also a Cauchy sequence in H and therefore is convergent. Hence $(u_n, Au_n)_n$ is a sequence of elements in the graph of A which converges, and, therefore, this graph being assumed to be closed, we obtain that $u \in D(A)$. We have therefore shown that $H = D(A)$, which is not the case.

We now define the fundamental concept of self-adjoint operator, which again generalises the usual concept in finite-dimensional spaces.

Definition II.6.1. *Let $A : D(A) \subset H \to H$, be an unbounded operator with a dense domain. We then introduce*

$$D(A^*) = \{u \in H,\ v \in D(A) \mapsto (Av, u)_H \text{ is continuous for the norm of } H\}.$$

For $u \in D(A^*)$, the mapping $v \mapsto (Av, u)_H$ can therefore be extended by a continuous linear functional on H which may be represented by an element denoted as $A^*u \in H$:

$$(Av, u)_H = (v, A^*u)_H, \forall u \in D(A^*), \forall v \in D(A).$$

The operator A^*, whose domain is $D(A^*)$, is called the adjoint operator of A.

This definition is consistent because the density of $D(A)$ into H ensures the uniqueness of the extension which is used in the definition. We can now define a fundamental class of operators.

Definition II.6.2. *An unbounded operator A is said to be self-adjoint if it satisfies*

$$D(A^*) = D(A) \text{ and } Au = A^*u, \forall u \in D(A).$$

Proposition II.6.3. *Let A be an unbounded self-adjoint operator. We assume that A is a bijection from $D(A)$ onto H and that A^{-1} is continuous from H into H. Then, the (bounded) operator A^{-1} is self-adjoint.*

Proof.

The domain of A^{-1} is H. We therefore first need to show that $D((A^{-1})^*)$ is also equal to H. Let $u \in H$; then, since A^{-1} is continuous, it is clear that $v \mapsto (A^{-1}v, u)_H$ is continuous on all of H and therefore by definition $u \in D((A^{-1})^*)$.

Let $u, v \in H$, then $A^{-1}u$ and $A^{-1}v$ are in $D(A)$, and since A is self-adjoint we have

$$\left(A(A^{-1}u), A^{-1}v\right)_H = \left(A^{-1}u, A(A^{-1}v)\right)_H;$$

in other words

$$(u, A^{-1}v)_H = (A^{-1}u, v)_H,$$

which shows that A^{-1} is indeed self-adjoint.

$\square$

In general, unbounded operators are not continuous from $D(A)$ equipped with the norm of H into H. This means that the norm of H is not the correct norm for us to set on $D(A)$. The following proposition, whose proof is straightforward, is the consequence of the fact that the graph of A is closed.

Proposition II.6.4. *Let A be a closed unbounded operator in a Hilbert space H of domain $D(A)$. We provide $D(A)$ with the following scalar product,*

$$(u, v)_{D(A)} = (u, v)_H + (Au, Av)_H, \ \forall u, v \in D(A), \tag{II.29}$$

and the associated norm. Then, $D(A)$ is a Hilbert space, the embedding from $D(A)$ into H is continuous and A is continuous from $D(A)$ into H.

Remark II.6.2. It is clear that the norm introduced above is equivalent to the norm known as the "graph norm" defined by

$$\|u\|_{\text{graph}} = \|u\|_H + \|Au\|_H, \ \forall u \in D(A).$$

6.2 Elementary results of spectral theory

The fundamental application of the concepts above, to the subject of interest to us, resides in the following result.

Theorem II.6.5 (Compact self-adjoint operators). *Let H be a separable (infinite-dimensional) Hilbert space and let T be a (bounded, i.e., defined and continuous on all H) compact self-adjoint operator from H to H. Then H has an orthonormal basis formed from eigenvectors of T. Moreover, the set of its eigenvalues (which are real numbers) can be ordered in a sequence tending towards* 0.

We do not give the proof here and we refer for instance to [27].

When A is an unbounded operator which is a bijection from $D(A)$ onto H, then the open mapping theorem tells us that A is an isomorphism of $D(A)$ (equipped with the graph norm) onto H. In this case A^{-1} is a continuous operator from H to $D(A)$. The embedding from $D(A)$ into H being continuous, we can also consider A^{-1} as a continuous operator from H into H.

If, moreover, the embedding from $D(A)$ into H is compact, which is often the case in the applications, then A^{-1}, seen as an operator from H to H, is compact (see Lemma II.3.5). We can therefore apply the preceding theorem and obtain the following result.

Theorem II.6.6 (Operators with compact inverse). *Let H be a separable, infinite-dimensional Hilbert space. Let $A : D(A) \subset H \to H$ be an unbounded* *operator. We assume that A is self-adjoint, bijective from $D(A)$ onto H and that the canonical embedding from $D(A)$ into H is compact.*

Then there exists an orthonormal basis $(w_k)_{k\geq 1}$ of H formed by eigenvectors of A, that is, such that for all $k \geq 1$,

$$w_k \in D(A), \text{ and } Aw_k = \lambda_k w_k,$$

where the eigenvalues $(\lambda_k)_{k\geq 1}$ of A are real numbers that we can order in such a way that $(|\lambda_k|)_k$ is increasing and tends towards $+\infty$ when k tends towards infinity.

Finally, the eigenvectors $(w_k)_{k\geq 1}$ form a complete orthogonal family of $D(A)$.

Proof.

As we have remarked above, under the hypotheses of the theorem, the operator A^{-1} can be viewed as a bounded, compact self-adjoint operator. From Theorem II.6.5, there exists an orthonormal basis $(w_k)_{k\geq 1}$ of H formed from eigenvectors of A^{-1} for the eigenvalues μ_k with, moreover, $\mu_k \to 0$.

Let us now note that, since A^{-1} is injective (Beware! It is not surjective on H), 0 cannot be an eigenvalue of A^{-1}. Hence, for all $k \geq 0$, $\mu_k \neq 0$. Moreover, since $A^{-1}w_k = \mu_k w_k$, we can clearly see that w_k belongs to the image of A^{-1}, that is to in $D(A)$.

If we now set $\lambda_k = 1/\mu_k$, we immediately obtain

$$w_k \in D(A) \text{ and } Aw_k = \lambda_k w_k.$$

The fact that $|\lambda_k| \to +\infty$ is a clear consequence of the fact that $\mu_k \to 0$. This demonstrates the first part of the theorem.

For $k, l \geq 0$, we have

$$(w_k, w_l)_{D(A)} = (w_k, w_l)_H + (Aw_k, Aw_l)_H = (1 + \lambda_k \lambda_l)(w_k, w_l)_H,$$

which shows, $(w_k)_{k\geq 1}$ being a Hilbertian basis of H, that when $k \neq l$, $(w_k, w_l)_{D(A)} = 0$ and that $\|w_k\|_{D(A)} = \sqrt{1+\lambda_k^2}$. The family $(w_k)_k$ is therefore an orthogonal family of $D(A)$.

To establish that this is also a complete family, it is necessary to show that the only vector of $D(A)$ orthogonal to all the w_k is the null vector. Therefore, let $u \in D(A)$ such that $(u, w_k)_{D(A)} = 0$ for all k. By using the self-adjoint characteristic of A, this gives

$$\begin{aligned}
0 = (u, w_k)_{D(A)} &= (u, w_k)_H + (Au, Aw_k)_H \\
&= (u, w_k)_H + \lambda_k (Au, w_k)_H \\
&= (u, w_k)_H + \lambda_k (u, Aw_k)_H \\
&= (1 + \lambda_k^2)(u, w_k)_H,
\end{aligned}$$

which demonstrates that $(u, w_k)_H = 0$ for all k. Since $(w_k)_k$ is an orthonormal basis of H, we have indeed shown that $u = 0$.

$\square$

We know that all $u \in H$ can be expressed uniquely in the form

$$u = \sum_{k \geq 1} u_k w_k,$$

with the convergence being taken in the sense of H and moreover, $u_k = (u, w_k)_H$. Using this expression we can recognise which of the elements of H are in $D(A)$.

Proposition II.6.7. *Let us take an operator A which satisfies the hypotheses of the preceding theorem. We then have*

$$D(A) = \left\{ u \in H, \text{ such that } \sum_{k \geq 1} \lambda_k^2 (u, w_k)_H^2 < +\infty \right\}.$$

Proof.

- Let $u \in D(A)$; $(w_k)_k$ is a complete orthogonal family of $D(A)$, thus we can see that
$$\left(\frac{w_k}{\|w_k\|_{D(A)}} \right)_k$$
is an orthonormal basis of $D(A)$. Hence, we know that
$$\sum_{k \geq 1} \frac{(u, w_k)_{D(A)}^2}{\|w_k\|_{D(A)}^2} < +\infty.$$
However, for all k, we have
$$\begin{aligned} (u, w_k)_{D(A)} &= (u, w_k)_H + (Au, Aw_k)_H = (u, w_k)_H + (u, A^2 w_k)_H \\ &= (1 + \lambda_k^2)(u, w_k)_H, \end{aligned}$$
and in particular
$$\|w_k\|_{D(A)}^2 = (1 + \lambda_k^2)\|w_k\|_H^2 = (1 + \lambda_k^2).$$
Hence, we have obtained
$$\sum_{k \geq 1} (1 + \lambda_k^2)(u, w_k)_H^2 < +\infty,$$
which proves the desired assertion.
- Now, let $u \in H$, we assume that

$$\sum_{k\geq 1} \lambda_k^2 (u, w_k)_H^2 < +\infty.$$

The hypothesis implies that

$$\sum_{k\geq 1} \|(u, w_k)_H w_k\|_{D(A)}^2 < +\infty,$$

and since the vectors $\big((u, w_k)_H w_k\big)_k$ are pairwise orthogonal in $D(A)$, this shows that the series

$$\sum_{k\geq 1} (u, w_k)_H w_k$$

converges towards a certain $\tilde{u} \in D(A)$, in the sense of the norm of $D(A)$. However, since the embedding from $D(A)$ to H is continuous, the convergence also takes place in H and hence we obtain

$$(\tilde{u}, w_k)_H = (u, w_k)_H,$$

which proves that $u - \tilde{u}$ is orthogonal to all w_k. Inasmuch as $(w_k)_k$ is complete in H, this shows that $u = \tilde{u}$ and therefore that $u \in D(A)$.

$\square$

We have shown in passing that for $u \in D(A)$, we have

$$\|u\|_H^2 = \sum_{k\geq 1} (u, w_k)_H^2 \text{ and } \|u\|_{D(A)}^2 = \sum_{k\geq 1} (1 + \lambda_k^2)(u, w_k)_H^2,$$

and moreover, since the absolute values of the eigenvalues $(|\lambda_k|)_k$ are bounded below by a positive real number, the norm in $D(A)$ is equivalent to the norm defined by

$$u \in D(A) \mapsto \left(\sum_{k\geq 1} \lambda_k^2 (u, w_k)_H^2 \right)^{1/2}.$$

We now wish to define the powers of the operator A. We could, for example, define the operator A^2 in the following natural way

$$D(A^2) = \{u \in D(A), \text{ such that } Au \in D(A)\}, \text{ and } A^2 u = A(Au), \forall u \in D(A^2).$$

However, we choose another definition which allows us to define the fractional powers of an operator. To this end, we have to assume that the operator is nonnegative (i.e., such that $(Au, u)_H \geq 0$ for all $u \in D(A)$). This is equivalent to assuming that the eigenvalues of A are nonnegative. From now on, we make this assumption and for all nonnegative real numbers s, we introduce

$$D(A^s) = \left\{ u \in H, \quad \text{such that} \ \sum_{k\geq 1} \lambda_k^{2s} (u, w_k)_H^2 < +\infty \right\},$$

and for all $u \in D(A^s)$, we set

$$A^s u = \sum_{k\geq 1} \lambda_k^s (u, w_k)_H w_k \in H.$$

Finally, we equip $D(A^s)$ with the natural scalar product defined by

$$(u, v)_{D(A^s)} = \sum_{k\geq 1} (1 + \lambda_k^{2s})(u, w_k)_H (v, w_k)_H.$$

We can easily verify the following properties.

Proposition II.6.8. *1. The operator A^1 is simply the operator A. Moreover, $D(A^0) = H$ and A^0 is the identity operator. The norms on $D(A^1)$ and $D(A^0)$ are equivalent to the usual norms on these spaces.*

2. For all $s > 0$, $D(A^s)$ is a Hilbert space and A^s is a nonnegative self-adjoint operator which is an isomorphism from $D(A^s)$ onto H. Moreover, $(w_k)_k$ is a complete family in $D(A^s)$.

3. For all $0 \leq s < s'$, we have $D(A^{s'}) \subset D(A^s)$, the inclusion being strict and the embedding being compact.

Proof.

The first two points are trivial, as is the strict inclusion $D(A^{s'}) \subset D(A^s)$. We only prove the compactness of the embedding of $D(A^{s'})$ in $D(A^s)$.

Let $(u^k)_k$ be a bounded sequence in $D(A^{s'})$, then there exists a $C > 0$ such that

$$\sum_n \lambda_n^{2s'} |u_n^k|^2 \leq C, \, \forall k \in \mathbb{N},$$

where we have denoted the coordinates of u^k in the basis $(w_n)_n$ as $u_n^k = (u^k, w_n)_H$.

We show that for all $\varepsilon > 0$, we can cover the sequence $(u^k)_k$ by a finite number of spheres with radius ε in $D(A^s)$. For any $\varepsilon > 0$, the sequence $(\lambda_n)_n$ tends towards $+\infty$, thus there exists a $n_0 \geq 0$ such that

$$\lambda_n \geq \left(\frac{2C}{\varepsilon^2}\right)^{1/(2(s'-s))}, \, \forall n \geq n_0.$$

Hence, for all k, we have

$$\sum_{n\geq n_0} \lambda_n^{2s} |u_n^k|^2 = \sum_{n\geq n_0} \lambda_n^{2(s-s')} \lambda_n^{2s'} |u_n^k|^2 \leq \frac{\varepsilon^2}{2C} \sum_{n\geq n_0} \lambda_n^{2s'} |u_n^k|^2 \leq \frac{\varepsilon^2}{2}. \quad \text{(II.30)}$$

Since n_0 is fixed, we see that for all $n < n_0$, we have

$$|u_n^k|^2 \leq \frac{C}{\lambda_n^{2s'}}, \quad \forall k.$$

Thus, the sequences of real numbers $(u_n^k)_k$ for $0 \leq n \leq n_0 - 1$ are bounded. The closed spheres of $\mathbb{R}^{n_0}$ are compact sets, thus there exists a finite family of elements of $\mathbb{R}^{n_0}$, denoted $(v^i)_{i\in I}$, with $v^i = (v_n^i)_{0\leq n\leq n_0-1}$ such that

$$\forall k \in \mathbb{N}, \exists i \in I, \text{ such that } |u_n^k - v_n^i|^2 \leq \frac{\varepsilon^2}{2\sum_{n=0}^{n_0-1}\lambda_n^{2s}}, \forall n \leq n_0 - 1. \tag{II.31}$$

For all $i \in I$, we consider the element $\tilde{v}^i$ of $D(A^s)$ defined by

$$\tilde{v}^i = \sum_{n=0}^{n_0-1} v_n^i w_n.$$

Let us now show that the sequence considered, $(u^k)_k$, is covered by the spheres of $D(A^s)$ with centres at $\tilde{v}^i$ and having radius ε. Indeed, if $k \in \mathbb{N}$ is fixed, we consider the index $i \in I$ given by (II.31), such that we have

$$\|u^k - \tilde{v^i}\|^2_{D(A^s)} = \sum_{n\leq n_0-1} \lambda_n^{2s}|u_n^k - v_n^i|^2 + \sum_{n\geq n_0} \lambda_n^{2s}|u_n^k|^{2s} \leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2,$$

the first term being bounded above by (II.31) and the second by (II.30).

□

We now wish to define similar concepts for $s < 0$. Unfortunately, in this case the preceding definitions do not apply (because the spaces $D(A^s)$ would all be equal to H and none would be complete). The spaces $D(A^s)$ with $s < 0$ must be larger than H. Hence, for all $u \in H$, we define

$$\|u\|^2_{D(A^s)} = \sum_{k\geq 1} \lambda_k^{2s}(u, w_k)_H^2.$$

This is a norm on H and if we define $D(A^s)$ as the completion of H for this norm, then the operator A^s is naturally defined. Indeed, $D(A^s)$ is a Hilbert space for the scalar product obtained by completion, and the preceding properties of positive real powers of A adapt without a problem. We then accept the following result.

Proposition II.6.9. *For all $s \geq 0$, if we identify H with its dual, we have*

$$D(A^s)' \approx D(A^{-s}),$$

and the duality can be written using the scalar product of H in the following way,

$$\langle u, v\rangle_{D(A^{-s}),D(A^s)} = \sum_{k\geq 1}(u, w_k)_H(v, w_k)_H, \ \forall u \in H, \forall v \in D(A^s).$$

6.3 Applications to the semigroup theory

Let $(A, D(A))$ be an unbounded operator in H which is self-adjoint, bijective from $D(A)$ onto H, nonnegative, and such that the canonical embedding of $D(A)$ into H is compact.

Our goal is to show, with the elements given above, how to solve the infinite-dimensional linear differential equation

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \\ u(0) = u_0. \end{cases} \tag{II.32}$$

Let $(w_k)_{k\geq 1}$ be the spectral basis associated with A and $(\lambda_k)_{k\geq 1}$ the (positive) eigenvalues of A. We assume that the sequence $(\lambda_k)_{k\geq 1}$ is sorted in a nondecreasing way.

Definition and Proposition II.6.10. *Let $u_0 \in H$ so that we write $u_0 = \sum_{k\geq 1} u_{0,k}w_k$. For any $t \geq 0$ we can define*

$$e^{-tA}u_0 = \sum_{k\geq 1} u_{0,k}e^{-t\lambda_k}w_k \in H.$$

Moreover, for any $s \geq 0$ and $t > 0$, we have $e^{-tA}u_0 \in D(A^s)$.

Remark II.6.3. We obviously have the property

$$e^{-(t+s)A} = e^{-tA}e^{-sA} = e^{-sA}e^{-tA}, \ \forall s, t \geq 0.$$

That's the reason why the family of continuous operator in H defined by $(e^{-tA})_{t\geq 0}$ is called *the semigroup associated with $-A$.*

Proof.

Since $\lambda_k \geq 0$ and $t \geq 0$, it is clear that this sum is well-defined in H. Moreover, for a fixed $t > 0$ and any $s > 0$, we have

$$\lambda_k^s e^{-t\lambda_k} = (t\lambda_k)^s e^{-t\lambda_k} t^{-s} \leq C_s t^{-s},$$

where $C_s = \sup_{[0,+\infty[} y^s e^{-y} < +\infty$.

It follows that

$$\sum_{k\geq 1} \left(\lambda_k^s u_{0,k} e^{-t\lambda_k}\right)^2 \leq C_s^2 t^{-2s} \sum_{k\geq 1} |u_{0,k}|^2 < +\infty,$$

and thus that $e^{-tA}u_0 \in D(A^s)$ for any $t > 0$ and any $s \geq 0$.

□

Theorem II.6.11. *For any $u_0 \in H$, there exists a unique $u \in \mathcal{C}^0([0,+\infty[, H) \cap \mathcal{C}^1(]0,+\infty[, D(A))$ which solves (II.32). It is defined by the formula*

$$u(t) = e^{-tA}u_0, \; \forall t \geq 0. \tag{II.33}$$

Proof.

- Let us first show the uniqueness property. The problem is linear, therefore it is enough to show that any solution u for the initial data $u_0 = 0$ is necessarily equal to 0.
  Let $0 < \varepsilon < T$ be given. We apply Theorem II.5.12 (and Corollary II.3.8) to obtain

$$\begin{aligned}\|u(T)\|_H^2 - \|u(\varepsilon)\|_H^2 &= 2\int_\varepsilon^T \left(\frac{du}{dt}(t), u(t)\right)_H dt \\ &= -2\int_\varepsilon^T (Au(t), u(t))_H \, dt \leq 0,\end{aligned}$$

  inasmuch as A is a nonnegative operator. It follows that

$$\|u(T)\|_H \leq \|u(\varepsilon)\|_H,$$

  but since u is continuous with values in H and satifies $u(0) = 0$, we can let ε go to 0 in the inequality above and obtain that $\|u(T)\|_H = 0$, which gives $u(T) = 0$. This being true for any $T > 0$, the claim is proved.
- We easily check that the function $t \in [0,+\infty[\mapsto u(t)$ defined by (II.33) satisfies the claimed regularity property (notice that u is not necessarily differentiable at $t = 0$ with values in H) and $u(0) = u_0$.
  For any $t > 0$, we can differentiate the series to obtain

$$\begin{aligned}\frac{du}{dt}(t) &= \sum_{k\geq 1}(-\lambda_k)u_{0,k}e^{-t\lambda_k}w_k \\ &= -\sum_{k\geq 1} u_{0,k}e^{-t\lambda_k}Aw_k = -A\left(\sum_{k\geq 1} u_{0,k}e^{-t\lambda_k}w_k\right) = -Au(t),\end{aligned}$$

  the last equality being true because the series converges in $D(A)$ and A is continuous from $D(A)$ in H.

□

When we add a source term f to the problem (II.32), the semigroup associated with $-A$ still allows us to solve the problem. More precisely, one can prove, for instance, the following result (see [48] or [95]) which is an infinite-dimensional version of a very standard result for ordinary differential equations.

Theorem II.6.12. *Let $u_0 \in H$ and $f \in \mathcal{C}^1([0,+\infty[, H)$. There exists a unique solution $u \in \mathcal{C}^0([0,+\infty[, H) \cap \mathcal{C}^1(]0,+\infty[, H) \cap \mathcal{C}^0(]0,+\infty[, D(A))$ to*

$$\begin{cases} \dfrac{du}{dt} + Au = f, \\ u(0) = u_0. \end{cases} \tag{II.34}$$

This solution is given by the Duhamel formula

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\, ds, \; \forall t \geq 0. \tag{II.35}$$

Proof.

Using the change of variable $s \to t - s$ in the integral in (II.35) we get

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-sA} f(t-s)\, ds.$$

Inasmuch as f is assumed to be of class $\mathcal{C}^1$ with values in H, we can justify the derivation in all the terms and then conclude by integration by parts.

□

Remark II.6.4. The above result, in particular Formula (II.35), still holds for less regular source terms but this needs to weaken the notion of solution we consider.

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