

Contents

Preface	v
Contents	ix
I The equations of fluid mechanics	1
1 Continuous description of a fluid	1
1.1 The continuous medium assumption. Density	1
1.2 Lagrangian and Eulerian coordinates	3
2 The transport theorem	5
3 Evolution equations	7
3.1 Balance equations	8
3.2 Cauchy's stress theorem	11
3.3 Evolution equations revisited	17
4 Fundamental laws: Newtonian fluids and thermodynamics laws	19
4.1 Fluids at rest	20
4.2 Newton's hypothesis	20
4.3 Consequences of the second law of thermodynamics	25
4.4 Equation for the specific internal energy	28
4.5 Formulation in entropy and temperature	29
5 Summary of the equations	30
6 Incompressible models	32
6.1 The incompressibility assumption	32
6.2 Overview of the incompressible models	37
7 Some exact steady solutions	42
7.1 Poiseuille flow in a pipe	43
7.2 Planar shear flow	44
7.3 Couette flow between two cylinders	46
II Analysis tools	49
1 Main notation	50
2 Fundamental results from functional analysis	51

2.1	Banach spaces	51
2.2	Weak and weak- \star convergences	52
2.3	Lebesgue spaces	56
2.4	Partitions of unity	68
2.5	A short introduction to distribution theory	71
2.6	Lipschitz continuous functions	75
3	Basic compactness results	77
3.1	Compact sets in function spaces	77
3.2	Compact maps	79
3.3	The Schauder fixed-point theorem	83
4	Functions of one real variable	84
4.1	Differentiation and antiderivatives	84
4.2	Differential inequalities and Gronwall's lemma	88
5	Spaces of Banach-valued functions	92
5.1	Definitions and main properties	92
5.2	Regularity in time	94
5.3	Compactness theorems	102
5.4	Banach-valued Fourier transform	106
6	Some results in spectral analysis of unbounded operators	110
6.1	Definitions	110
6.2	Elementary results of spectral theory	112
6.3	Applications to the semigroup theory	118
III	Sobolev spaces	121
1	Domains	122
1.1	General definitions	122
1.2	Lipschitz domains	123
2	Sobolev spaces on Lipschitz domains	135
2.1	Definitions	136
2.2	Mollifying operators and Friedrichs commutator estimates	138
2.3	Change of variables	149
2.4	Extension operator	150
2.5	Trace and trace lifting operators	153
2.6	Duality theory for Sobolev spaces	159
2.7	Translation estimates	164
2.8	Sobolev embeddings	167
2.9	Poincaré and Hardy inequalities	179
2.10	Domains of first-order differential operators	184
3	Calculus near the boundary of domains	189
3.1	Local charts description of the boundary	189
3.2	Distance to the boundary. Projection on the boundary	191
3.3	Regularised distance	194
3.4	Parametrisation of a neighborhood of $\partial\Omega$	200
3.5	Tangential Sobolev spaces	206
3.6	Differential operators in tangential/normal coordinates	217

4	The Laplace problem	222
4.1	Dirichlet boundary conditions	222
4.2	Neumann boundary conditions	226
IV	Steady Stokes equations	229
1	Nečas inequality	230
1.1	Proof of the inequality	231
1.2	Related Poincaré inequalities	238
2	Characterisation of gradient fields. De Rham's theorem	241
3	The divergence operator and related spaces	245
3.1	Right-inverse for the divergence	245
3.2	The space $H_{\text{div}}(\Omega)$	248
3.3	Divergence-free vector fields. Leray decomposition	249
4	The curl operator and related spaces	252
4.1	Poincaré's theorems	252
4.2	The space $H_{\text{curl}}(\Omega)$	257
4.3	Kernel and image of the curl operator	267
4.4	The div/curl problem	269
5	The Stokes problem	273
5.1	Well-posedness of the Stokes problem	273
5.2	Stokes operator	277
5.3	The unsteady Stokes problem	286
5.4	Penalty approximation of the Stokes problem	287
6	Regularity of the Stokes problem	290
6.1	First degree of regularity	290
6.2	Higher-order regularity	301
6.3	L^q theory of the Stokes problem	302
6.4	Regularity for the div/curl problem	304
7	The Stokes problem with stress boundary conditions	306
7.1	The Stokes–Neumann problem	307
7.2	Regularity properties	311
7.3	Stress boundary conditions	315
8	The interface Stokes problem	323
8.1	Existence and uniqueness	324
8.2	Regularity of the solution	326
9	The Stokes problem with vorticity boundary conditions	329
9.1	Preliminaries	330
9.2	A vector Laplace problem	332
9.3	The Stokes problem	340
V	Navier–Stokes equations for homogeneous fluids	345
1	Leray's theorem	346
1.1	Properties of the inertia term	346
1.2	Weak formulations of the Navier–Stokes equations	347
1.3	Existence and uniqueness of weak solutions	352

1.4	Kinetic energy evolution	363
1.5	Existence and regularity of the pressure	368
2	Strong solutions	370
2.1	New estimates	371
2.2	The two-dimensional case	373
2.3	The three-dimensional case	376
2.4	Parabolic regularity properties	384
2.5	Regularisation over time	389
3	The steady Navier–Stokes equations	391
3.1	The case of homogeneous boundary conditions	392
3.2	The case of nonhomogeneous boundary conditions	395
3.3	Uniqueness for small data	401
3.4	Asymptotic stability of steady solutions	402
VI	Nonhomogeneous fluids	409
1	Weak solutions of the transport equation	411
1.1	Setting of the problem	412
1.2	Trace theorem. Renormalisation property	413
1.3	The initial- and boundary-value problem	423
1.4	Stability theorem	427
2	The nonhomogeneous incompressible Navier–Stokes equations	434
2.1	Main result	434
2.2	Approximate problem	435
2.3	Estimates for the approximate solution	443
2.4	End of the proof of the existence theorem	447
2.5	The case without vacuum	452
VII	Boundary conditions modelling	453
1	Outflow boundary conditions	454
1.1	Setting up the model	454
1.2	Existence and uniqueness	458
2	Dirichlet boundary conditions through a penalty method	470
2.1	A simple example of a boundary layer	472
2.2	Statement of the main result	476
2.3	Formal asymptotic expansion	479
2.4	Well-posedness of profile equations	492
2.5	Convergence of the asymptotic expansion	497
A	Classic differential operators	507
1	The scalar and vector cases	507
1.1	Definitions	507
1.2	Useful formulas	509
2	Extension to second-order tensors	509

B	Thermodynamics supplement	511
1	Heat capacity	512
2	The first law of thermodynamics. Internal energy	512
3	The second law of thermodynamics	513
3.1	Entropy	513
3.2	Internal energy calculation	514
4	Specific variables	515
	References	517
	Index	523

Mathematical Tools for the Study of the Incompressible
Navier-Stokes Equations and Related Models

Boyer, F.; Fabrie, P.

2013, XIV, 526 p.,

ISBN: 978-1-4614-5975-0