

## 2.1 Introduction

In dealing with a function of a single variable,  $y = f(x)$ , in the ordinary calculus, we often find it of use to determine the values of  $x$  for which the function  $y$  is a *local maximum* or a *local minimum*. By a local maximum at position  $x_1$ , we mean that  $f$  at position  $x$  in the *neighborhood* of  $x_1$  is less than  $f(x_1)$  (see Fig. 2.1). Similarly for a local minimum of  $f$  to exist at position  $x_2$  (see Fig. 2.1) we require that  $f(x)$  be larger than  $f(x_2)$  for all values of  $x$  in the *neighborhood* of  $x_2$ . The values of  $x$  in the *neighborhood* of  $x_1$  or  $x_2$  may be called the *admissible values* of  $x$  relative to which  $x_1$  or  $x_2$  is a maximum or minimum position.

To establish the condition for a local extremum (maximum or minimum), let us expand the function  $f$  as a Taylor series about a position  $x = a$ . Thus assuming that  $f(x)$  has continuous derivatives at position  $x = a$  we have:

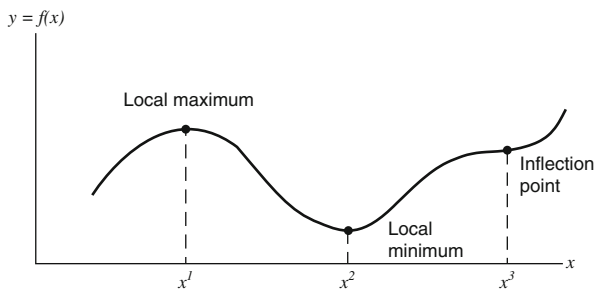
$$f(x) = f(a) + \left(\frac{df}{dx}\right)_{x=a} (x-a) + \frac{1}{2!} \left(\frac{d^2f}{dx^2}\right)_{x=a} (x-a)^2 + \frac{1}{3!} \left(\frac{d^3f}{dx^3}\right)_{x=a} (x-a)^3 + \dots$$

We next rearrange the series and rewrite it in the following more compact form:

$$f(x) - f(a) = [f'(a)](x-a) + \frac{1}{2!} [f''(a)](x-a)^2 + \frac{1}{3!} [f'''(a)](x-a)^3 + \dots \quad (2.1)$$

For  $f(a)$  to be a minimum it is necessary that  $[f(x) - f(a)]$  be a positive number for all values of  $x$  in the neighborhood of " $a$ ". Since  $(x-a)$  can be positive or negative for the admissible values of  $x$ , then clearly the term  $f'(a)$  must then be zero to prevent the dominant term in the series from yielding positive and negative values

Fig. 2.1



for the admissible values of  $x$ . That is, a necessary condition for a local minimum at “ $a$ ” is that  $f'(a) = 0$ . By similar reasoning we can conclude that the same condition prevails for a local maximum at “ $a$ ”. Considering the next term in the series, we see that there will be a constancy in sign for admissible values of  $x$  and so the sign of  $f''(a)$  will determine whether we have a local minimum or a local maximum at position “ $a$ ”. Thus with  $f'(a) = 0$ , the sign of  $f''(a)$  (assuming  $f''(a) \neq 0$ ) supplies the information for establishing a local minimum or a local maximum at position “ $a$ ”.

Suppose next that both  $f'(a)$  and  $f''(a)$  are zero but that  $f'''(a)$  does not equal zero. Then the third term of the series of Eq. (2.1) becomes the dominant term, and for admissible values of  $x$  there must be a change in sign of  $[f(x) - f(a)]$  as we move across point “ $a$ ”. Such a point is called an *inflection point* and is shown in Fig. 2.1 at position  $x_3$ .

Thus we see that point “ $a$ ”, for which  $f'(a) = 0$ , may correspond to a local minimum point, to a local maximum point, or to an inflection point. Such points as a group are often of much physical interest<sup>1</sup> and they are called *extremal positions* of the function.

We have presented a view of elements of the theory of local extrema in order to set the stage for the introduction of the calculus of variations<sup>2</sup> which will be of considerable use in the ensuing studies of elastic structures. In place of the function of the preceding discussion we shall be concerned now with *functionals* which are, plainly speaking, functions of functions. Specifically, a functional is an expression that takes on a particular value which is dependent on the function used in the functional. A form of functional that is employed in many areas of applied mathematics is the integral of  $F(x, y, y')$  between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  in two-dimensional space. Denoting this functional as  $I$  we have:

$$I = \int_{x_1}^{x_2} F(x, y, y') dx \quad (2.2)$$

<sup>1</sup>See Courant: “Differential and Integral Calculus,” Interscience Press.

<sup>2</sup>For a rigorous study of this subject refer to “Calculus of Variations” by Gelfand and Fomin. Prentice-Hall Inc., or to “An Introduction to the Calculus of Variations,” by Fox, Oxford University Press.

Clearly, the value of  $I$  for a given set of end points  $x_1$  and  $x_2$  will depend on the function  $y(x)$ . Thus, just as  $f(x)$  depends on the value of  $x$ , so does the value of  $I$  depend on the form of the function  $y(x)$ . And, just as we were able to set up necessary conditions for a local extreme of  $f$  at some point “ $a$ ” by considering *admissible values* of  $x$  (i.e.,  $x$  in the neighborhood of “ $a$ ”) so can we find necessary conditions for extremizing  $I$  with respect to an *admissible set of functions*  $y(x)$ . Such a procedure, forming one of the cornerstones of the calculus of variations, is considerably more complicated than the corresponding development in the calculus of functions and we shall undertake this in a separate section.

In this text we shall only consider necessary conditions for establishing an extreme. We usually know a priori whether this extreme is a maximum or a minimum by physical arguments. Accordingly the complex arguments<sup>3</sup> needed in the calculus of variations for giving sufficiency conditions for maximum or minimum states of  $I$  will be omitted.

We may generalize the functional  $I$  in the following ways:

- (a) The functional may have many independent variables other than just  $x$ ;
- (b) The functional may have many functions (dependent variables) of these independent variables other than just  $y(x)$ ;
- (c) The functional may have higher-order derivatives other than just first-order.

We shall examine such generalizations in subsequent sections.

In the next section we set forth some very simple functionals.

## 2.2 Examples of Simple Functionals

Historically the calculus of variations became an independent discipline of mathematics at the beginning of the eighteenth century. Much of the formulation of this mathematics was developed by the Swiss mathematician Leonhard Euler (1707–83). It is instructive here to consider three of the classic problems that led to the growth of the calculus of variations.

### (a) The Brachistochrone

In 1696 Johann Bernoulli posed the following problem. Suppose one were to design a frictionless chute between two points (1) and (2) in a vertical plane such that a body sliding under the action of its own weight goes from (1) to (2) in the *shortest* interval of time. The time for the descent from (1) to (2) we denote as  $I$  and it is given as follows

$$I = \int_{(1)}^{(2)} \frac{ds}{V} = \int_{(1)}^{(2)} \frac{\sqrt{dx^2 + dy^2}}{V} = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y')^2}}{V} dx$$

<sup>3</sup>See the references cited earlier.

where  $V$  is the speed of the body and  $s$  is the distance along the chute. Now employ the conservation of energy for the body. If  $V_1$  is the initial speed of the body we have at any position  $y$ :

$$\frac{mV_1^2}{2} + mgy_1 = \frac{mV^2}{2} + mgy$$

therefore

$$V = [V_1^2 - 2g(y - y_1)]^{1/2}$$

We can then give  $I$  as follows:

$$I = \int_1^2 \frac{\sqrt{1 + (y')^2}}{[V_1^2 - 2g(y - y_1)]^{1/2}} dx \quad (2.3)$$

We shall later show that the chute (i.e.,  $y(x)$ ) should take the shape of a *cycloid*.<sup>4</sup>

(b) **Geodesic Problem**

The problem here is to determine the curve on a given surface  $g(x, y, z) = 0$  having the *shortest length* between two points (1) and (2) on this surface. Such curves are called *geodesics*. (For a spherical surface the geodesics are segments of the so-called great circles.) The solution to this problem lies in determining the extreme values of the integral.

$$I = \int_{(1)}^{(2)} ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2 + (z')^2} dx \quad (2.4)$$

We have here an example where we have two functions  $y$  and  $z$  in the functional, although only one independent variable, i.e.,  $x$ . However,  $y$  and  $z$  are not independent of each other but must have values satisfying the equation:

$$g(x, y, z) = 0 \quad (2.5)$$

The extremizing process here is analogous to the *constrained* maxima or minima problems of the calculus of functions and indeed Eq. (2.5) is called a constraining equation in connection with the extremization problem.

If we are able to solve for  $z$  in terms of  $x$  and  $y$ , or for  $y$  in terms of  $z$  and  $x$  in Eq. (2.5), we can reduce Eq. (2.4) so as to have only one function of  $x$  rather than two. The extremization process with the one function of  $x$  is then no longer constrained.

---

<sup>4</sup>This problem has been solved by both Johann and Jacob Bernoulli, Sir Isaac Newton, and the French mathematician L'Hôpital.

**(c) Isoperimetric Problem**

The original isoperimetric problem is given as follows: of all the closed non-intersecting plane curves having a given fixed length  $L$ , which curve encloses the *greatest* area  $A$ ?

The area  $A$  is given from the calculus by the following line integral:

$$A = \frac{1}{2} \oint (x \, dy - y \, dx)$$

Now suppose that we can express the variables  $x$  and  $y$  parametrically in terms of  $\tau$ . Then we can give the above integral as follows

$$A = I = \frac{1}{2} \int_{\tau_1}^{\tau_2} \left( x \frac{dy}{d\tau} - y \frac{dx}{d\tau} \right) d\tau \quad (2.6)$$

where  $\tau_1$  and  $\tau_2$  correspond to beginning and end of the closed loop. The constraint on  $x$  and  $y$  is now given as follows:

$$L = \oint ds = \oint (dx^2 + dy^2)^{1/2}$$

Introducing the parametric representation we have:

$$L = \int_{\tau_1}^{\tau_2} \left[ \left( \frac{dx}{d\tau} \right)^2 + \left( \frac{dy}{d\tau} \right)^2 \right]^{1/2} d\tau \quad (2.7)$$

We have thus one independent variable,  $\tau$ , and two functions of  $\tau$ ,  $x$  and  $y$ , constrained this time by the *integral relationship* (2.7).

In the historical problems set forth here we have shown how functionals of the form given by Eq. (2.2) may enter directly into problems of interest. Actually the extremization of such functionals or their more generalized forms is equivalent to solving certain corresponding differential equations, and so this approach may give alternate viewpoints of various areas of mathematical physics. Thus in optics the extremization of the time required for a beam of light to go from one point to another in a vacuum relative to an admissible family of light paths is equivalent to satisfying Maxwell's equations for the radiation paths of light. This is the famous Fermat principle. In the study of particle mechanics the extremization of the difference between the kinetic energy and the potential energy (i.e., the Lagrangian) integrated between two points over an admissible family of paths yields the correct path as determined by Newton's law. This is Hamilton's principle.<sup>5</sup> In the theory of elasticity, which will be of primary concern to us in this text, we shall amongst other

---

<sup>5</sup>We shall consider Hamilton's principle in detail in Chap. 7.

things extremize the so-called total potential energy of a body with respect to an admissible family of displacement fields to satisfy the equations of equilibrium for the body. We can note similar dualities in other areas of mathematical physics and engineering science, notably electromagnetic theory and thermodynamics. Thus we conclude that the extremization of functionals of the form of Eq. (2.2) or their generalizations affords us a different view of many fields of study. We shall have ample opportunity in this text to see this new viewpoint as it pertains to solid mechanics. One important benefit derived by recasting the approach as a result of variational considerations is that some very powerful *approximate procedures* will be made available to us for the solution of problems of engineering interest. Such considerations will form a significant part of this text.

It is to be further noted that the series of problems presented required respectively: a *fastest* time of descent, a *shortest* distance between two points on a surface, and a *greatest* area to be enclosed by a given length. These problems are examples of what are called *optimization problems*.<sup>6</sup>

We now examine the extremization process for functionals of the type described in this section.

---

## 2.3 The First Variation

Consider a functional of the form

$$I = \int_{x_1}^{x_2} F(x, y, y') \, dx \quad (2.8)$$

where  $F$  is a known function, twice differentiable for the variables  $x$ ,  $y$ , and  $y'$ . As discussed earlier, the value of  $I$  between points  $(x_1, y_1)$  and  $(x_2, y_2)$  will depend on the path chosen between these points, i.e., it will depend on the function  $y(x)$  used. We shall assume the existence of a path, which we shall henceforth denote as  $y(x)$ , having the property of extremizing  $I$  with respect to other neighboring paths which we now denote collectively as  $\tilde{y}(x)$ .<sup>7</sup> We assume further that  $y(x)$  is twice differentiable. We shall for simplicity refer henceforth to  $y(x)$  as the *extremizing path* or the *extremizing function* and to  $\tilde{y}(x)$  as the *varied paths*.

We will now introduce a single-parameter family of varied paths as follows

$$\tilde{y}(x) = y(x) + \epsilon \eta(x) \quad (2.9)$$

---

<sup>6</sup>In seeking an *optimal solution* in a problem we strive to attain, subject to certain given constraints, that solution, amongst other possible solutions, that satisfies or comes closest to satisfying a certain criterion or certain criteria. Such a solution is then said to be optimal relative to this criterion or criteria, and the process of arriving at this solution is called optimization.

<sup>7</sup>Thus  $y(x)$  will correspond to “ $a$ ” of the early extremization discussion of  $f(x)$  while  $\tilde{y}(x)$  corresponds to the values of  $x$  in the neighborhood of “ $a$ ” of that discussion.

where  $\varepsilon$  is a small parameter and where  $\eta(x)$  is a differentiable function having the requirement that:

$$\eta(x_1) = \eta(x_2) = 0$$

We see that an infinity of varied paths can be generated for a given function  $\eta(x)$  by adjusting the parameter  $\varepsilon$ . All these paths pass through points  $(x_1, y_1)$  and  $(x_2, y_2)$ . Furthermore for any  $\eta(x)$  the varied path becomes coincident with the extremizing path when we set  $\varepsilon = 0$ .

With the agreement to denote  $y(x)$  as the extremizing function, then  $I$  in Eq. (2.8) becomes the extreme value of the integral

$$\int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}') dx.$$

We can then say:

$$\tilde{I} = \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}') dx = \int_{x_1}^{x_2} F(x, y + \varepsilon\eta, y' + \varepsilon\eta') dx \quad (2.10)$$

By having employed  $y + \varepsilon\eta$  as the admissible functions we are able to use the extremization criteria of simple function theory as presented earlier since  $\tilde{I}$  is now, for the desired extremal  $y(x)$ , a function of the parameter  $\varepsilon$  and thus it can be expanded as a power series in terms of this parameter. Thus

$$\tilde{I} = (\tilde{I})_{\varepsilon=0} + \left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} \varepsilon + \left( \frac{d^2\tilde{I}}{d\varepsilon^2} \right)_{\varepsilon=0} \frac{\varepsilon^2}{2!} + \dots$$

Hence:

$$\tilde{I} - I = \left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} \varepsilon + \left( \frac{d^2\tilde{I}}{d\varepsilon^2} \right)_{\varepsilon=0} \frac{\varepsilon^2}{2!} + \dots$$

For  $\tilde{I}$  to be extreme when  $\varepsilon = 0$  we know from our earlier discussion that

$$\left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} = 0$$

is a *necessary condition*. This, in turn, means that

$$\left\{ \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial \tilde{y}} \frac{d\tilde{y}}{d\varepsilon} + \frac{\partial F}{\partial \tilde{y}'} \frac{d\tilde{y}'}{d\varepsilon} \right) dx \right\}_{\varepsilon=0} = 0$$

Noting that  $d\tilde{y}/d\varepsilon = \eta$  and that  $d\tilde{y}'/d\varepsilon = \eta'$ , and realizing that deleting the tilde for  $\tilde{y}$  and  $\tilde{y}'$  in the derivatives of  $F$  is the same as setting  $\varepsilon = 0$  as required above, we may rewrite the above equation as follows:

$$\int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right) dx = 0 \quad (2.11)$$

We now integrate the second term by parts as follows:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta' dx = \left. \frac{\partial F}{\partial y'} \eta \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left[ \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \eta dx$$

Noting that  $\eta = 0$  at the end points, we see that the first expression on the right side of the above equation vanishes. We then get on substituting the above result into Eq. (2.11):

$$\int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \eta dx = 0 \quad (2.12)$$

With  $\eta(x)$  arbitrary between end points, a basic lemma of the calculus of variations<sup>8</sup> indicates that the bracketed expression in the above integrand is zero. Thus:

$$\boxed{\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0} \quad (2.13)$$

This is the famous *Euler–Lagrange equation*. It is the condition required for  $y(x)$  in the role we have assigned it of being the extremizing function. Substitution of  $F(x, y, y')$  will result in a second-order ordinary differential equation for the unknown function  $y(x)$ . In short, the variational procedure has resulted in an ordinary differential equation for getting the function  $y(x)$  which we have “tagged” and handled as the extremizing function.

We shall now illustrate the use of the Euler–Lagrange equation by considering the brachistochrone problem presented earlier.

**EXAMPLE 2.1.** Brachistochrone problem we have from the earlier study of the *brachistochrone* problem of Sect. 2.2 the requirement to extremize:

$$I = \int_1^2 \frac{\sqrt{1 + (y')^2}}{[V_1^2 - 2g(y - y_1)]^{1/2}} dx \quad (a)$$

<sup>8</sup>For a particular function  $\tilde{E}(x)$  continuous in the interval  $(x_1, x_2)$ , if  $\int_{x_1}^{x_2} \phi(x) \eta(x) dx = 0$  for every continuously differentiable function  $\eta(x)$  for which  $\eta(x_1) = \eta(x_2) = 0$ , then  $\phi \equiv 0$  for  $x_1 \leq x \leq x_2$ .



If we take the special case where the body is released from rest at the origin the above functional becomes:

$$I = \frac{1}{\sqrt{2g}} \int_1^2 \sqrt{\frac{1 + (y')^2}{y}} dx \quad (b)$$

The function  $F$  can be identified as  $\{[1 + (y')^2]/y\}^{1/2}$ . We go directly to the Euler–Lagrange equation to substitute for  $F$ . After some algebraic manipulation we obtain:

$$y'' = -\frac{1 + (y')^2}{2y}$$

Now make the substitution  $u = y'$ . Then we can say

$$u \frac{du}{dy} = -\frac{1 + u^2}{2y}$$

Separating variables and integrating we have:

$$y(1 + u^2) = C_1$$

therefore

$$y[1 + (y')^2] = C_1$$

We may arrange for another separation of variables and perform another quadrature as follows:

$$x = \int \frac{\sqrt{y}}{\sqrt{C_1 - y}} dy + x_0$$

Next make the substitution

$$y = C_1 \left[ \sin^2 \frac{t}{2} \right] \quad (c)$$

We then have:

$$x = C_1 \int \sin^2 \frac{t}{2} dt + x_0 = C_1 \left[ \frac{t - \sin t}{2} \right] + x_0 \quad (d)$$

Since at time  $t = 0$ , we have  $x = y = 0$  then  $x_0 = 0$  in the above equation. We then have as results

$$x = \frac{C_1}{2}(t - \sin t) \quad (d)$$

$$y = \frac{C_1}{2}(1 - \cos t) \quad (e)$$

wherein we have used the double-angle formula to arrive at Eq. [e](#) from Eq. [\(c\)](#). These equations represent a cycloid which is a curve generated by the motion of a point fixed to the circumference of a rolling wheel. The radius of the wheel here is  $C_1/2$ . ////

## 2.4 The Delta Operator

We now introduce an operator  $\delta$ , termed the *delta operator*, in order to give a certain formalism to the procedure of obtaining the first variation. We define  $\delta[y(x)]$  as follows:

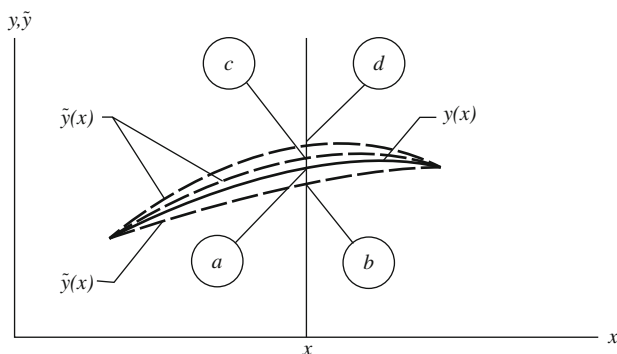
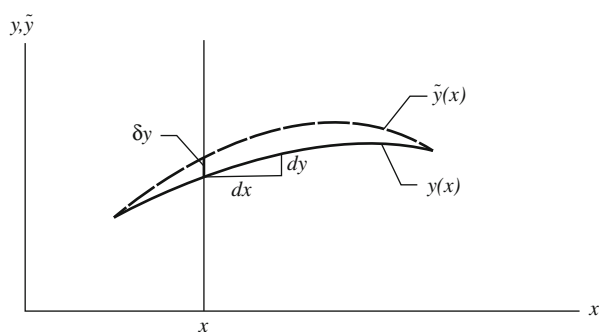
$$\delta[y(x)] = \tilde{y}(x) - y(x) \quad (2.14)$$

Notice that the delta operator represents a small arbitrary change in the dependent variable  $y$  for a *fixed* value of the independent variable  $x$ . Thus in Fig. [2.2](#) we have shown extremizing path  $y(x)$  and some varied paths  $\tilde{y}(x)$ . At the indicated position  $x$  any of the increments  $a-b$ ,  $a-c$  or  $a-d$  may be considered as  $\delta y$ —i.e., as a variation of  $y$ . Most important, note that we *do not associate a  $\delta x$  with each  $\delta y$* . This is in contrast to the differentiation process wherein a  $dy$  is associated with a given  $dx$ . We can thus say that  $\delta y$  is simply the vertical distance between points on different curves at the same value of  $x$  whereas  $dy$  is the vertical distance between points on the *same* curve at positions  $dx$  apart. This has been illustrated in Fig. [2.3](#).

We may generalize the delta operator to represent a small (usually infinitesimal) change of a function wherein the independent variable is kept fixed. Thus we may take the variation of the function  $dy/dx$ . We shall agree here, however, to use as varied function the derivatives  $d\tilde{y}/dx$  where the  $\tilde{y}$  are varied paths for  $y$ . We can then say:

$$\delta \left[ \frac{dy}{dx} \right] = \left( \frac{d\tilde{y}}{dx} \right) - \left( \frac{dy}{dx} \right) = \frac{d}{dx}(\tilde{y} - y) = \frac{d(\delta y)}{dx} \quad (2.15)$$

As a consequence of this arrangement we conclude that the  $\delta$  operator is *commutative* with the differential operator. In a similar way, by agreeing that  $\int \tilde{y}(x)dx$  is a varied function for  $\int y(x) dx$ , we can conclude that the variation operator is commutative with the integral operator.

**Fig. 2.2****Fig. 2.3**

Henceforth we shall make ample use of the  $\delta$  operator and its associated notation. It will encourage the development of mechanical skill in carrying out the variational process and will aid in developing “physical” feel in the handling of problems.

It will be well to go back now to Eq. (2.10) and re-examine the first variation using the delta-operator notation. First note that for the one-parameter family of varied functions,  $y + \epsilon\eta$ , it is clear that

$$\delta y = \tilde{y} - y = \epsilon\eta \quad (a)$$

and that:

$$\delta y' = \tilde{y}' - y' = \epsilon\eta' \quad (b) \quad (2.16)$$

Accordingly we can give  $F$  along a varied path as follows using the  $\delta$  notation:

$$F(x, y + \delta y, y' + \delta y')$$

Now at any position  $x$  we can expand  $F$  as a Taylor series about  $y$  and  $y'$  in the following manner

$$F(x, y + \delta y, y' + \delta y') = F(x, y, y') + \left[ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] + O(\delta^2)$$

therefore

$$F(x, y + \delta y, y' + \delta y') - F(x, y, y') + \left[ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] + O(\delta^2) \quad (2.17)$$

where  $O(\delta^2)$  refers to terms containing  $(\delta y)^2$ ,  $(\delta y')^2$ ,  $(\delta y)^3$ , etc., which are of negligibly higher order. We shall denote the left side of the equation as the *total variation* of  $F$  and denote it as  $\delta^{(T)}F$ . The bracketed expression on the right side of the equation with  $\delta$ 's to the *first power* we call the *first variation*,  $\delta^{(1)}F$ . Thus:

$$\delta^{(T)}F = F(x, y + \delta y, y' + \delta y') - F(x, y, y') \quad (a)$$

$$\delta^{(1)}F = \left[ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] \quad (b) \quad (2.18)$$

On integrating Eq. (2.17) from  $x_1$  to  $x_2$  we get:

$$\begin{aligned} & \int_{x_1}^{x_2} F(x, y + \delta y, y' + \delta y') dx - \int_{x_1}^{x_2} F(x, y, y') dx \\ &= \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] dx + O(\delta^2) \end{aligned}$$

This may be written as:

$$\tilde{I} - I = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right] dx + O(\delta^2)$$

We shall call  $\tilde{I} - I$  the total variation of  $I$  and denote it as  $\delta^{(T)}I$ , while as expected, the first expression on the right side of the equation is the first variation of  $I$ ,  $\delta^{(1)}I$ . Hence:

$$\begin{aligned} \delta^{(T)}I &= \tilde{I} - I \\ \delta^{(1)}I &= \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx \end{aligned}$$

Integrating the second expression in the above integral by parts we get for  $\delta^{(1)}I$ :

$$\delta^{(1)}I = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \delta y \, dx + \left[ \frac{\partial F}{\partial y'} \delta y \right] \Big|_{x_1}^{x_2}$$

Since all  $\tilde{y}(x)$  must take the specific values corresponding to those of  $y(x)$  at  $x_1$  and  $x_2$ , clearly the variation in  $y(x)$  must be zero at these points—i.e.,  $\delta y = 0$  at  $x_1$  and  $x_2$ . Thus we have for  $\delta^{(1)}I$ :

$$\delta^{(1)}I = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \delta y \, dx \quad (2.19)$$

We can then say for  $\delta^{(T)}I$ :

$$\begin{aligned} \delta^{(1)}I &= \delta^{(1)}I + O(\delta^2) + \dots \\ &= \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] \delta y \, dx + O(\delta^2) \end{aligned}$$

In order for  $I$  to be a maximum or a minimum it must retain the same sign for all possible variations  $\delta y$  over the interval. Thus  $\delta y$  at any position  $x$  could be  $\pm K$  where  $K$  is a small number. For this to be possible the bracketed expression in the integrand on the right side of the equation has to be zero which in turn leads to the familiar Euler–Lagrange equations. We may thus conclude that

$$\delta^{(1)}I = 0 \quad (2.20)$$

Accordingly, the requirement for extremization of  $I$  is that its *first variation be zero*. Suppose now that  $F = F(\varepsilon_{ij})$ . Then for independent variables  $x, y, z$  we have for the first variation of  $I$ , on extrapolating from Eq. (2.18(b))

$$\begin{aligned} \delta^{(1)}I &= \iiint_V \delta^{(1)}F \, dx \, dy \, dz \\ &= \iiint_V \frac{\partial F}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} \, dx \, dy \, dz \end{aligned}$$

where we observe the summation convention of the repeated indices. If  $F$  is a function of  $\varepsilon$  as a result of using a one-parameter family approach then one might suppose that:

$$\delta^{(1)}I = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varepsilon} \delta \varepsilon \, dx$$

But the dependent variable represents the extremal function in the development, and for the one-parameter development  $\varepsilon = 0$  corresponds to this condition. Hence we must compute  $\partial F / \partial \varepsilon$  at  $\varepsilon = 0$  in the above formulation and  $\delta \varepsilon$  can be taken as  $\varepsilon$  itself. Hence we get:

$$\delta^{(1)}I = \int_{x_2}^{x_1} \left( \frac{\partial \tilde{F}}{\partial \varepsilon} \right)_{\varepsilon=0} \varepsilon \, dx \quad (2.21)$$

If we use a two-parameter family (as we soon shall) then it should be clear for  $\varepsilon_1$  and  $\varepsilon_2$  as parameters that:

$$\delta^{(1)}I = \int_{x_1}^{x_2} \left[ \left( \frac{\partial \tilde{F}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_1 + \left( \frac{\partial \tilde{F}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_2 \right] dx \quad (2.22)$$

Now going back to Eq. (2.12) and its development we can say:

$$\left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \eta \, dx \quad (2.23)$$

Next rewriting Eq. (2.19) we have:

$$\delta^{(1)}I = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \delta y \, dx$$

Noting that  $\delta y = \varepsilon \eta$  for a single-parameter family approach we see by comparing the right sides of the above equations that:

$$\delta^{(1)}I \equiv \left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} \varepsilon \quad (2.24)$$

Accordingly setting  $(d\tilde{I}/d\varepsilon)_{\varepsilon=0} = 0$ , as we have done for a single-parameter family approach, is tantamount to setting the first variation of  $I$  equal to zero.<sup>9</sup> We shall use both approaches in this text for finding the extremal functions.

As a next step, we examine certain simple cases of the Euler–Lagrange equation to ascertain first integrals.

---

<sup>9</sup>Note that for a two parameter family we have from Eq. (2.22) the result:

$$\delta^{(1)}I = \left( \frac{\partial \tilde{I}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_1 + \left( \frac{\partial \tilde{I}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_2 \quad (2.25)$$

## 2.5 First Integrals of the Euler–Lagrange Equation

We now present four cases where we can make immediate statements concerning first integrals of the Euler–Lagrange equation as presented thus far.

**Case (a).  $F$  is not a function of  $y$ —i.e.,  $F = F(x, y')$**

In this case the Euler–Lagrange equation degenerates to the form:

$$\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0$$

Accordingly, we can say for a first integral that:

$$\frac{\partial F}{\partial y'} = \text{Const.} \quad (2.26)$$

**Case (b).  $F$  is only a function of  $y'$ —i.e.,  $F = F(y')$**

Equation 2.26 still applies because of the lack of presence of the variable  $y$ . However, now we know that the left side must be a function only of  $y'$ . Since this function must equal a constant at all times we conclude that  $y' = \text{const.}$  is a possible solution. This means that for this case an extremal path is simply that of a straight line.

**Case (c).  $F$  is independent of the independent variable  $x$ —i.e.,  $F = F(y, y')$**

For this case we begin by presenting an identity which you are urged to verify. Thus noting that  $d/dx$  may be expressed here as

$$\left( \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} \right)$$

we can say:

$$\frac{d}{dx} \left( y' \frac{\partial F}{\partial y'} - F \right) = -y' \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] - \frac{\partial F}{\partial x}$$

If  $F$  is not explicitly a function of  $x$ , we can drop the last term. A satisfaction of the Euler–Lagrange equation now means that the right side of the equation is zero so that we may conclude that

$$y' \frac{\partial F}{\partial y'} - F = C_1 \quad (2.27)$$

for the extremal function. We thus have next to solve a first-order differential equation to determine the extremal function.

**Case (d).** *F* is the total derivative of some function  $g(x,y)$ —i.e.,  $F = dg/dx$

It is easy to show that when  $F = dg/dx$ , it must satisfy identically the Euler–Lagrange equation. We first note that:

$$F = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' \quad (2.28)$$

Now substitute the above result into the Euler–Lagrange equation. We get:

$$\frac{\partial}{\partial y} \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' \right) - \frac{d}{dx} \left( \frac{\partial g}{\partial y} \right) = 0$$

Carry out the various differentiation processes:

$$\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 g}{\partial y^2} y' - \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y^2} y' = 0$$

The left side is clearly identically zero and so we have shown that a *sufficient* condition for  $F$  to satisfy the Euler–Lagrange equation identically is that  $F = dg(x,y)/dx$ . It can also be shown that this is a *necessary* condition for the identical satisfaction of the Euler–Lagrange equation. It is then obvious that we can always add a term of the form  $dg/dx$  to the function  $F$  in  $I$  without changing the Euler–Lagrange equations for the functional  $I$ . That is, for any Euler–Lagrange equation there are an *infinity of functionals differing from each other in these integrals by terms of the form  $dg/dx$* .

We shall have ample occasion to use these simple solutions. Now we consider the geodesic problem, presented earlier, for the case of the sphere.

**EXAMPLE 2.2.** Consider a sphere of radius  $R$  having its center at the origin of reference  $xyz$ . We wish to determine the shortest path between two points on this sphere. Using spherical coordinates  $R, \phi, \theta$  (see Fig. 2.4) a point  $P$  on the sphere has the following coordinates:

$$\begin{aligned} x &= R \sin \theta \cos \phi \\ y &= R \sin \theta \sin \phi \\ z &= R \cos \theta \end{aligned} \quad (a)$$

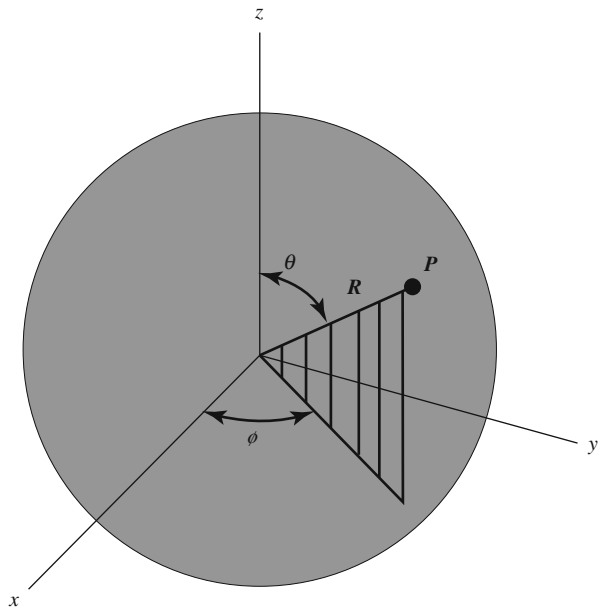
The increment of distance  $ds$  on the sphere may be given as follows:

$$ds^2 = dx^2 + dy^2 + dz^2 = R^2[d\theta^2 + \sin^2 \theta d\phi^2]$$

Hence the distance between point  $P$  and  $Q$  can be given as follows:

$$I = \int_P^Q ds = \int_P^Q R[d\theta^2 + \sin^2 \theta d\phi^2]^{1/2} \quad (b)$$



**Fig. 2.4**

With  $\phi$  and  $\theta$  as independent variables, the transformation equations (a) depict a sphere. If  $\dot{E}$  is related to  $\theta$ , i.e., it is a function of  $\theta$ , then clearly  $x$ ,  $y$ , and  $z$  in Eq. (a) are functions of a single parameter  $\theta$  and this must represent some curve on the sphere. Accordingly, since we are seeking a curve on the sphere we shall assume that  $\phi$  is a function of  $\theta$  in the above equation so that we can say:

$$I = R \int_P^Q \left[ 1 + \sin^2 \theta \left( \frac{d\phi}{d\theta} \right)^2 \right]^{1/2} d\theta \quad (c)$$

We have here a functional with  $\theta$  as the independent variable and  $\phi$  as the function, with the derivative of  $\phi$  appearing explicitly. With  $\phi$  not appearing explicitly in the integrand, we recognize this to be case (a) discussed earlier. We may then say, using Eq. (2.26):

$$\frac{\partial}{\partial \phi'} [1 + (\sin^2 \theta)(\phi')^2]^{1/2} = C_1 \quad (d)$$

This becomes:

$$\frac{(\sin^2 \theta)(\phi')}{[1 + (\sin^2 \theta)(\phi')^2]^{1/2}} = C_1$$

Solving for  $\phi'$  we get:

$$\phi' = \frac{C_1}{\sin \theta [\sin^2 \theta - C_1^2]^{1/2}}$$

Integrating we have:

$$\phi = C_1 \int \frac{d\theta}{\sin \theta [\sin^2 \theta - C_1^2]^{1/2}} + C_2 \quad (e)$$

We make next the following substitution for  $\theta$ :

$$\theta = \tan^{-1} \frac{1}{\eta} \quad (f)$$

This gives us:

$$\begin{aligned} \phi &= C_1 \int \frac{\frac{1}{1+\eta^2} d\eta}{\left(\frac{1}{1+\eta^2}\right)^{1/2} \left[\frac{1}{1+\eta^2} - C_1^2\right]^{1/2}} + C_2 \\ &= \int \frac{d\eta}{\left[\left(\frac{1}{C_1^2} - 1\right) - \eta^2\right]^{1/2}} + C_2 \end{aligned}$$

Denoting  $(1/C_1^2 - 1)^{1/2}$  as  $1/C_3$  we get on integrating:

$$\phi = \sin^{-1}(C_3 \eta) + C_2 \quad (g)$$

Replacing  $\eta$  from Eq. (f) we get:

$$\phi = \sin^{-1} \left[ C_3 \frac{1}{\tan \theta} \right] + C_2$$

Hence we can say:

$$\sin(\phi - C_2) = \frac{C_3}{\tan \theta}$$

This equation may next be written as follows:

$$\sin \phi \cos C_2 - \cos \phi \sin C_2 = C_3 \frac{\cos \theta}{\sin \theta}$$

Hence:

$$\sin \phi \sin \theta \cos C_2 - \sin \theta \cos \phi \sin C_2 = \cos \theta$$

Observing the transformation equations (a) we may express the above equation in terms of Cartesian coordinates in the following manner:

$$y \cos C_2 - x \sin C_2 = zC_3 \quad (h)$$

wherein we have cancelled the term  $R$ . This is the equation of a plane surface going through the origin. The intersection of this plane surface and the sphere then gives the proper locus of points on the sphere that forms the desired extremal path. Clearly this curve is the expected great circle. ////

## 2.6 First Variation with Several Dependent Variables

We now consider the case where we may have any number of functions with still a single *independent* variable. We shall denote the functions as  $q_1, q_2, \dots, q_n$  and the independent variable we shall denote as  $t$ . (This is a notation that is often used in particle mechanics.) The functional for this case then is given as follows:

$$I = \int_{t_1}^{t_2} F(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) dt \quad (2.29)$$

where  $\dot{q}_1 = dq_1/dt$ , etc. We wish to determine a set of functions  $q_1(t), q_2(t), \dots, q_n(t)$  which are twice differentiable and which extremize the functional  $I$  with respect to a broad class of admissible functions. We shall denote the varied functions as  $\tilde{q}_i(t)$  and, as before, we shall henceforth consider the notation  $q_i(t)$  to identify the extremizing functions we are seeking.

We shall use the following single-parameter family of admissible functions

$$\begin{aligned} \tilde{q}_1(t) &= q_1(t) + \varepsilon \eta_1(t) \\ \tilde{q}_2(t) &= q_2(t) + \varepsilon \eta_2(t) \\ &\vdots \\ \tilde{q}_n(t) &= q_n(t) + \varepsilon \eta_n(t) \end{aligned} \quad (2.30)$$

where  $\eta_1(t), \eta_2(t), \dots, \eta_n(t)$  are arbitrary functions having proper continuity and differentiability properties for the ensuing steps. Also, these functions are equal to zero at the end points  $t_1$  and  $t_2$ . Finally, when we set  $\varepsilon = 0$  we get back to the designated extremizing functions  $q_1(t), \dots, q_n(t)$ .

We now form  $\tilde{I}(\varepsilon)$  as follows:

$$\tilde{I}(\varepsilon) = \int_{t_1}^{t_2} F(\tilde{q}_1, \tilde{q}_2 \cdots \tilde{q}_n, \dot{\tilde{q}}_1, \dot{\tilde{q}}_2 \cdots \dot{\tilde{q}}_n, t) dt$$

In order for the  $q_i(t)$  to be the extremal paths we now require that:

$$\left[ \frac{d\tilde{I}(\varepsilon)}{d\varepsilon} \right]_{\varepsilon=0} = 0$$

Hence:

$$\left\{ \int_{t_1}^{t_2} \left( \frac{\partial F}{\partial \tilde{q}_1} \eta_1 + \cdots \frac{\partial F}{\partial \tilde{q}_n} \eta_n + \frac{\partial F}{\partial \dot{\tilde{q}}_1} \dot{\eta}_1 + \cdots \frac{\partial F}{\partial \dot{\tilde{q}}_n} \dot{\eta}_n \right) dt \right\}_{\varepsilon=0} = 0$$

Now setting  $\varepsilon = 0$  in the above expression is the same as removing the tildes from the  $q$ 's and  $\dot{q}$ 's. Thus we have:

$$\int_{t_1}^{t_2} \left( \frac{\partial F}{\partial q_1} \eta_1 + \cdots \frac{\partial F}{\partial q_n} \eta_n + \frac{\partial F}{\partial \dot{q}_1} \dot{\eta}_1 + \cdots \frac{\partial F}{\partial \dot{q}_n} \dot{\eta}_n \right) dt = 0$$

Now the functions  $\eta_i(t)$  are arbitrary and so we can take all  $\eta_i(t)$  except  $\eta_1(t)$  equal to zero. We then have on integrating  $(\partial F / \partial \dot{q}_1) \dot{\eta}_1$  by parts:

$$\int_{t_1}^{t_2} \left[ \frac{\partial F}{\partial q_1} - \frac{d}{dt} \frac{\partial F}{\partial \dot{q}_1} \right] \eta_1 dt = 0$$

Using the fundamental lemma of the calculus of variations, we conclude that:

$$\frac{\partial F}{\partial q_1} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{q}_1} \right) = 0$$

We may similarly assume that  $\eta_2$  is the only non-zero function and so forth to lead us to the conclusion that

$$\boxed{\frac{\partial F}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{q}_i} \right) = 0 \quad i = 1, 2, \cdots, n} \quad (2.31)$$

are necessary conditions for establishing the  $q_i(t)$  as the extremal functions. These are again the Euler–Lagrange equations which lead us on substitution for  $F$  to a system of second-order ordinary differential equations for establishing the  $q_i(t)$ . These equations may be coupled (i.e., simultaneous equations) or uncoupled, depending on the variables  $q_i$  chosen to be used in forming the functional  $I$  in Eq. (2.29).

We now illustrate the use of the Euler–Lagrange equations for a problem in particle mechanics.

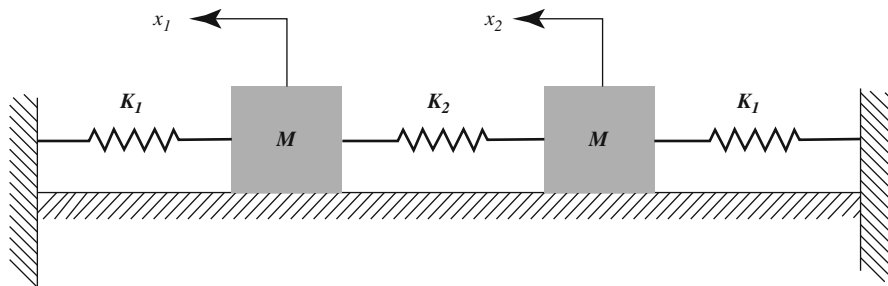


Fig. 2.5

**EXAMPLE 2.3.** We will present for use here the very important Hamilton principle in order to illustrate the multi-function problem of this section. Later we shall take the time to consider this principle in detail.

For a system of particles acted on by conservative forces, Hamilton's principle states that the proper paths taken from a configuration at time  $t_1$  to a configuration at time  $t_2$  are those that extremize the following functional

$$I = \int_{t_1}^{t_2} (T - V) dt \quad (a)$$

where  $T$  is the kinetic energy of the system and therefore a function of velocity  $\dot{x}_i$  of each particle while  $V$  is the potential energy of the system and therefore a function of the coordinates  $x_i$  of the particles. We thus have here a functional of many dependent variables  $x_i$  with the presence of a single independent variable,  $t$ .

Consider two identical masses connected by three springs as has been shown in Fig. 2.5. The masses can only move along a straight line as a result of frictionless constraints. Two independent coordinates are needed to locate the system; they are shown as  $x_1$  and  $x_2$ . The springs are unstretched when  $x_1 = x_2 = 0$ .

From elementary mechanics we can say for the system:

$$\begin{aligned} T &= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 \\ V &= \frac{1}{2}K_1x_1^2 + \frac{1}{2}K_2(x_2 - x_1)^2 + \frac{1}{2}K_1x_2^2 \end{aligned} \quad (b)$$

Hence we have for  $I$ :

$$I = \int_{t_1}^{t_2} \left\{ \frac{1}{2}M(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2} \left[ K_1x_1^2 + K_2(x_2 - x_1)^2 + K_1x_2^2 \right] \right\} dt \quad (c)$$

To extremize  $I$  we employ Eq. (2.31) as follows:

$$\begin{aligned}\frac{\partial F}{\partial x_1} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}_1} \right) &= 0 \\ \frac{\partial F}{\partial x_2} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}_2} \right) &= 0\end{aligned}\tag{d}$$

Substituting we get:

$$\begin{aligned}-K_1 x_1 + K_2(x_2 - x_1) - \frac{d}{dt}(M\dot{x}_1) &= 0 \\ -K_2 x_2 - K_2(x_2 - x_1) - \frac{d}{dt}(M\dot{x}_2) &= 0\end{aligned}\tag{e}$$

Rearranging we then have:

$$\begin{aligned}M\ddot{x}_1 + K_1 x_1 - K_2(x_2 - x_1) &= 0 \\ M\ddot{x}_2 + K_2 x_2 - K_2(x_2 - x_1) &= 0\end{aligned}\tag{f}$$

These are recognized immediately to be equations obtainable directly from Newton's laws. Thus the Euler–Lagrange equations lead to the basic equations of motion for this case. We may integrate these equations and, using initial conditions of  $\dot{x}_1(0), \dot{x}_2(0), x_1(0)$  and  $x_2(0)$ , we may then fully establish the subsequent motion of the system.

In this problem we could have more easily employed Newton's law directly. There are many problems, however, where it is easier to proceed by the variational approach to arrive at the equations of motion. Also, as will be seen in the text, other advantages accrue to the use of the variational method. ////

## 2.7 The Isoperimetric Problem

We now investigate the isoperimetric problem in its simplest form whereby we wish to extremize the functional

$$I = \int_{x_1}^{x_2} F(x, y, y') \, dx\tag{2.32}$$

subject to the restriction that  $y(x)$  have a form such that:

$$J = \int_{x_1}^{x_2} G(x, y, y') \, dx = \text{Const.}\tag{2.33}$$

where  $G$  is a given function.

We proceed essentially as in Sec. 2.3. We take  $y(x)$  from here on to represent the extremizing function for the functional of Eq. (2.32). We next introduce a

system of varied paths  $\tilde{y}(x)$  for computation of  $\tilde{I}$ . Our task is then to find conditions required to be imposed on  $y(x)$  so as to extremize  $\tilde{I}$  with respect to the admissible varied paths  $\tilde{y}(x)$  while satisfying the isoperimetric constraint of Eq. (2.33). To facilitate the computations we shall require that the admissible varied paths also satisfy Eq. (2.33). Thus we have:

$$\tilde{I} = \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}') dx \quad (a)$$

$$\tilde{J} = \int_{x_1}^{x_2} G(x, \tilde{y}, \tilde{y}') dx = \text{Const.} \quad (b)$$

Because of this last condition on  $y(x)$  we shall no longer use the familiar single-parameter family of admissible functions, since varying  $\varepsilon$  alone for a single parameter family of functions may mean that the constraint for the corresponding paths  $y(x)$  is not satisfied. To allow for enough flexibility to carry out extremization while maintaining *intact* the constraining equations, we employ a two-parameter family of admissible functions of the form,

$$\tilde{y}(x) = y(x) + \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x) \quad (2.35)$$

where  $\eta_1(x)$  and  $\eta_2(x)$  are arbitrary functions which vanish at the end points  $x_1, x_2$ , and where  $\varepsilon_1$  and  $\varepsilon_2$  are two small parameters. Using this system of admissible functions it is clear that  $\tilde{I}$  and  $\tilde{J}$  are functions of  $\varepsilon_1$  and  $\varepsilon_2$ . Thus:

$$\tilde{I}(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}') dx \quad (a)$$

$$\tilde{J}(\varepsilon_1, \varepsilon_2) = \int_{x_1}^{x_2} G(x, \tilde{y}, \tilde{y}') dx \quad (b)$$

To extremize  $\tilde{I}$  when  $\tilde{y} \rightarrow y$  we require (see Eq. 2.25) that:

$$\delta^{(1)}\tilde{I} = \left( \frac{\partial \tilde{I}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_1 + \left( \frac{\partial \tilde{I}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_2 = 0 \quad (2.37)$$

If  $\varepsilon_1$  and  $\varepsilon_2$  were independent of each other, we could then set each of the partial derivatives in the above equation equal to zero separately to satisfy the above equation. However,  $\varepsilon_1$  and  $\varepsilon_2$  are related by the requirement that  $\tilde{J}(\varepsilon_1, \varepsilon_2) = \text{const.}$  The first variation of  $\tilde{J}$  must be zero because of the constancy of its value and we have the equation:

$$\left( \frac{\partial \tilde{J}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_1 + \left( \frac{\partial \tilde{J}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \varepsilon_2 = 0 \quad (2.38)$$

At this time we make use of the method of the *Lagrange multiplier*. That is, we multiply Eq. (2.38) by an undetermined constant  $\lambda$  (the Lagrange multiplier) and add Eq. (2.37) and (2.38) to get:

$$\left[ \left( \frac{\partial \tilde{I}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} + \lambda \left( \frac{\partial \tilde{J}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \right] \varepsilon_1 + \left[ \left( \frac{\partial \tilde{I}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} + \lambda \left( \frac{\partial \tilde{J}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} \right] \varepsilon_2 = 0 \quad (2.39)$$

We now choose  $\lambda$  so that one of the two bracketed quantities is zero. We choose here the second bracketed quantity so that:

$$\left( \frac{\partial \tilde{I}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} + \lambda \left( \frac{\partial \tilde{J}}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = 0 \quad (2.40)$$

Now we may consider that  $\varepsilon_2$  is the dependent variable and that  $\varepsilon_1$  is the independent variable. We must then conclude from Eq. (2.39) that the coefficient of  $\varepsilon_1$  is zero. Thus:

$$\left( \frac{\partial \tilde{I}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} + \lambda \left( \frac{\partial \tilde{J}}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = 0 \quad (2.41)$$

Thus, Eq. (2.40) and (2.41) with the use of the multiplier  $\lambda$  give us the necessary conditions for an extreme of  $\tilde{I}$  while maintaining the constraint condition intact.

We can now shorten the notation by introducing  $\tilde{I}^*$  as follows:

$$\tilde{I}^* = \tilde{I} + \lambda \tilde{J} \quad (2.42)$$

so that Eqs. (2.40) and (2.41) become

$$\left( \frac{\partial \tilde{I}^*}{\partial \varepsilon_1} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = \left( \frac{\partial \tilde{I}^*}{\partial \varepsilon_2} \right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = 0 \quad (2.43)$$

In integral form we have for  $\tilde{I}^*$

$$\begin{aligned} \tilde{I}^* &= \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}') dx + \lambda \int_{x_1}^{x_2} G(x, \tilde{y}, \tilde{y}') dx \\ &= \int_{x_1}^{x_2} [F(x, \tilde{y}, \tilde{y}') + \lambda G(x, \tilde{y}, \tilde{y}')] dx = \int_{x_1}^{x_2} F^*(x, \tilde{y}, \tilde{y}') dx \end{aligned} \quad (2.44)$$

where:

$$F^* = F + \lambda G \quad (2.45)$$



We now apply the conditions given by Eq. (2.43) using Eq. (2.44) to replace  $\tilde{I}^*$ .

$$\left(\frac{\partial \tilde{I}^*}{\partial \varepsilon_i}\right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = \left\{ \int_{x_1}^{x_2} \left( \frac{\partial F^*}{\partial \tilde{y}} \eta_i + \frac{\partial F^*}{\partial \tilde{y}'} \eta'_i \right) dx \right\}_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = 0 \quad i = 1, 2 \quad (2.46)$$

We thus get a statement of the extremization process without the appearance of the constraint and this now becomes the starting point of the extremization process. Removing the tildes from  $\tilde{y}$  and  $\tilde{y}'$  is equivalent to setting  $\varepsilon_i = 0$  and so we may say:

$$\left(\frac{\partial \tilde{I}^*}{\partial \varepsilon_i}\right)_{\substack{\varepsilon_1=0 \\ \varepsilon_2=0}} = \int_{x_1}^{x_2} \left( \frac{\partial F^*}{\partial y} \eta_i + \frac{\partial F^*}{\partial y'} \eta'_i \right) dx = 0 \quad i = 1, 2 \quad (2.47)$$

Integrating the second expression in the integrand by parts and noting that  $\eta_i(x_1) = \eta_i(x_2) = 0$  we get:

$$\int_{x_1}^{x_2} \left( \frac{\partial F^*}{\partial y} - \frac{d}{dx} \frac{\partial F^*}{\partial y'} \right) \eta_i dx = 0 \quad i = 1, 2$$

Now using the fundamental lemma of the calculus of variations we conclude that:

$$\boxed{\frac{\partial F^*}{\partial y} - \frac{d}{dx} \left( \frac{\partial F^*}{\partial y'} \right) = 0} \quad (2.48)$$

Thus the Euler–Lagrange equation is again a necessary condition for the desired extremum, this time applied to  $F^*$  and thus including the Lagrange multiplier. This leads to a second-order differential equation for  $y(x)$ , the extremizing function. Integrating this equation then leaves us two constants of integration plus the Lagrange multiplier. These are determined from the specified values of  $y$  at the end points plus the constraint condition given by Eq. (2.33).

We have thus far considered only a single constraining integral. If we have “ $n$ ” such integrals

$$\int_{x_1}^{x_2} G_k(x, y, y') dx = C_k \quad k = 1, 2, \dots, n$$

then by using an  $n + 1$  parameter family of varied paths  $\tilde{y} = y + \varepsilon_1 \eta_1 + \varepsilon_2 \eta_2 + \dots + \varepsilon_{n+1} \eta_{n+1}$  we can arrive as before at the following requirement for extremization:

$$\frac{\partial F^*}{\partial y} - \frac{d}{dt} \left( \frac{\partial F^*}{\partial y'} \right) = 0$$

where:

$$F^* = F + \sum_{k=1}^n \lambda_k G_k$$

The  $\lambda$ 's are again the Lagrange multipliers. Finally, for  $p$  dependent variables, i.e.,

$$I = \int_{t_1}^{t_2} F(q_1, q_2, \dots, q_p, \dot{q}_1, \dots, \dot{q}_p, t) dt$$

with  $n$  constraints

$$\int_{t_1}^{t_2} G_k(q_1, \dots, q_p, \dot{q}_1, \dots, \dot{q}_p, t) dt = C_k \quad k = 1, 2, \dots, n$$

the extremizing process yields

$$\frac{\partial F^*}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial F^*}{\partial \dot{q}_i} \right) = 0 \quad i = 1, \dots, p \quad (2.49)$$

where

$$F^* = F + \sum_{k=1}^n \lambda_k G_k \quad (2.50)$$

We now illustrate the use of these equations by considering in detail the isoperimetric problem presented earlier.

**EXAMPLE 2.4.** Recall from Sec. 2.2 that the isoperimetric problem asks us to find the particular curve  $y(x)$  which for a given length  $L$  encloses the largest area  $A$ . Expressed parametrically we have a functional with two functions  $y$  and  $x$ ; the independent variable is  $\tau$ . Thus:

$$I = A = \frac{1}{2} \int_{\tau_1}^{\tau_2} \left( x \frac{dy}{d\tau} - y \frac{dx}{d\tau} \right) d\tau \quad (a)$$

The constraint relation is:

$$L = \int_{\tau_1}^{\tau_2} \sqrt{\left( \frac{dx}{d\tau} \right)^2 + \left( \frac{dy}{d\tau} \right)^2} d\tau \quad (b)$$

We now form  $F^*$  for this case. Thus:

$$F^* = \frac{1}{2}(x\dot{y} - y\dot{x}) + \lambda(\dot{x}^2 + \dot{y}^2)^{1/2}$$

where we use the dot superscript to represent  $(d/d\tau)$ . We now set forth the Euler–Lagrange equations:

$$\frac{\dot{y}}{2} - \frac{d}{d\tau} \left[ -\frac{y}{2} + \frac{\lambda \dot{x}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} \right] = 0 \quad (c)$$

$$-\frac{\dot{x}}{2} - \frac{d}{d\tau} \left[ \frac{x}{2} + \frac{\lambda \dot{y}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} \right] = 0 \quad (d)$$

We next integrate Eqs. (c) and (d) with respect to  $\tau$  to get:

$$y - \frac{\lambda \dot{x}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = C_1 \quad (e)$$

$$x + \frac{\lambda \dot{y}}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = C_2 \quad (f)$$

After eliminating  $\lambda$  between the equations we will reach the following result:

$$(x - C_2) dx + (y - C_1) dy = 0 \quad (g)$$

The last equation is easily integrated to yield:

$$\frac{(x - C_2)^2}{2} + \frac{(y - C_1)^2}{2} = C_3^2 \quad (h)$$

where  $C_3$  is a constant of integration. Thus we get as the required curve a circle—a result which should surprise no one. The radius of the circle is  $\sqrt{2}C_3$  which you may readily show (by eliminating  $C_1$  and  $C_2$  from Eq. (h) using Eqs. (e) and (f) and solving for  $\lambda$ ) is the value of the Lagrange multiplier<sup>10</sup>  $\lambda$ . The constants  $C_1$  and  $C_2$  merely position the circle. ///

## 2.8 Functional Constraints

We now consider the functional

$$I = \int_{t_1}^{t_2} F(q_1, q_2, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) dt \quad (2.51)$$

with the following  $m$  constraints on the  $n$  functions  $q_i$ :<sup>11</sup>

<sup>10</sup>The Lagrange multiplier is usually of physical significance.

<sup>11</sup>In dynamics of particles, if the constraining equations do not have derivatives the constraints are called *holonomic*.

$$\begin{aligned}
G_1(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) &= 0 \\
&\vdots \\
G_m(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) &= 0
\end{aligned} \tag{2.52}$$

We assume  $m < n$ . To extremize the functional  $I$  we may proceed by employing  $n$  one-parameter families of varied functions of the form

$$\tilde{q}_i(t) = q_i(t) + \varepsilon \eta_i(t) \quad i = 1, \dots, n \tag{2.53}$$

where  $\eta_i(t_1) = \eta_i(t_2) = 0$ . Furthermore we assume that the varied functions  $\tilde{q}_i$  satisfy the constraining equation (2.52). That is,

$$G_j(\tilde{q}_1, \dots, \tilde{q}_n, \dot{\tilde{q}}_1, \dots, \dot{\tilde{q}}_n, t) = 0 \quad j = 1, \dots, m \tag{2.54}$$

We now set the first variation of  $I$  equal to zero.

$$\delta^{(1)}I = 0 = \left( \frac{\partial \tilde{I}}{\partial \varepsilon} \right)_{\varepsilon=0} \varepsilon$$

Setting  $(\partial \tilde{I} / \partial \varepsilon)_{\varepsilon=0}$  equal to zero we have:

$$\left[ \int_{t_1}^{t_2} \left[ \frac{\partial F}{\partial \tilde{q}_1} \eta_1 + \dots + \frac{\partial F}{\partial \tilde{q}_n} \eta_n + \frac{\partial F}{\partial \dot{\tilde{q}}_1} \dot{\eta}_1 + \dots + \frac{\partial F}{\partial \dot{\tilde{q}}_n} \dot{\eta}_n \right] dt \right]_{\varepsilon=0} = 0$$

Dropping the tildes and subscript  $\varepsilon = 0$  we have:

$$\int_{t_1}^{t_2} \left( \frac{\partial F}{\partial q_1} \eta_1 + \dots + \frac{\partial F}{\partial q_n} \eta_n + \frac{\partial F}{\partial \dot{q}_1} \dot{\eta}_1 + \dots + \frac{\partial F}{\partial \dot{q}_n} \dot{\eta}_n \right) dt = 0 \tag{2.55}$$

Since the varied  $q$ 's satisfy the constraining equations, by assumption, we can conclude since  $G_i = 0$  that:

$$\delta^{(1)}(G_i) = 0 = \left( \frac{\partial G}{\partial \varepsilon} \right)_{\varepsilon=0} \varepsilon$$

Setting  $(\partial G_i / \partial \varepsilon)_{\varepsilon=0} = 0$  here we have:

$$\left\{ \frac{\partial G_i}{\partial \tilde{q}_1} \eta_1 + \dots + \frac{\partial G_i}{\partial \dot{\tilde{q}}_1} \dot{\eta}_1 + \dots + \frac{\partial G_i}{\partial \tilde{q}_n} \eta_n + \frac{\partial G_i}{\partial \dot{\tilde{q}}_n} \dot{\eta}_n \right\}_{\varepsilon=0} = 0$$

$i = 1, 2, \dots, m$

Dropping the tildes and  $\varepsilon = 0$  we then have:

$$\frac{\partial G_i}{\partial q_1} \eta_1 + \cdots + \frac{\partial G_i}{\partial q_n} \eta_n + \frac{\partial G_i}{\partial \dot{q}_1} \dot{\eta}_1 + \cdots + \frac{\partial G_i}{\partial \dot{q}_n} \dot{\eta}_n = 0 \quad i = 1, \dots, m$$

We now multiply each of the above  $m$  equations by an *arbitrary time function*,  $\lambda_i(t)$ , which we may call a *Lagrange multiplier function*. Adding the resulting equations we have:

$$\sum_{i=1}^m \lambda_i(t) \left[ \frac{\partial G_i}{\partial q_1} \eta_1 + \cdots + \frac{\partial G_i}{\partial q_n} \eta_n + \frac{\partial G_i}{\partial \dot{q}_1} \dot{\eta}_1 + \cdots + \frac{\partial G_i}{\partial \dot{q}_n} \dot{\eta}_n \right] = 0$$

Now integrate the above sum from  $t_1$  to  $t_2$  and then add the results to Eq. (2.55). We then have:

$$\int_{t_1}^{t_2} \left\{ \frac{\partial F}{\partial q_1} \eta_1 + \cdots + \frac{\partial F}{\partial q_n} \eta_n + \frac{\partial F}{\partial \dot{q}_1} \dot{\eta}_1 + \cdots + \frac{\partial F}{\partial \dot{q}_n} \dot{\eta}_n \right. \\ \left. + \sum_{i=1}^m \lambda_i \left( \frac{\partial G_i}{\partial q_1} \eta_1 + \cdots + \frac{\partial G_i}{\partial q_n} \eta_n + \frac{\partial G_i}{\partial \dot{q}_1} \dot{\eta}_1 + \cdots + \frac{\partial G_i}{\partial \dot{q}_n} \dot{\eta}_n \right) \right\} dt = 0$$

Integrating by parts the terms with  $\dot{\eta}_i$  and regrouping the results we then have:

$$\int_{t_1}^{t_2} \left[ \left\{ \frac{\partial F}{\partial q_1} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{q}_1} \right) + \sum_{i=1}^m \left[ \lambda_i \frac{\partial G_i}{\partial q_1} - \frac{d}{dt} \left( \lambda_i \frac{\partial G_i}{\partial \dot{q}_1} \right) \right] \right\} \eta_1 \right. \\ + \cdots \cdots \cdots \\ \left. + \left\{ \frac{\partial F}{\partial q_n} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{q}_n} \right) + \sum_{i=1}^m \left[ \lambda_i \frac{\partial G_i}{\partial q_n} - \frac{d}{dt} \left( \lambda_i \frac{\partial G_i}{\partial \dot{q}_n} \right) \right] \right\} \eta_n \right] dt = 0 \quad (2.56)$$

Now introduce  $F^*$  defined as:

$$F^* = F + \sum_{i=1}^m \lambda_i(t) G_i \quad (2.57)$$

We can then rewrite Eq. (2.56) as follows:

$$\int_{t_1}^{t_2} \left\{ \left( \frac{\partial F^*}{\partial q_1} - \frac{d}{dt} \frac{\partial F^*}{\partial \dot{q}_1} \right) \eta_1 + \cdots + \left( \frac{\partial F^*}{\partial q_n} - \frac{d}{dt} \frac{\partial F^*}{\partial \dot{q}_n} \right) \eta_n \right\} dt = 0$$

Now the  $\eta$ 's are not independent (they are related through  $m$  equations (2.54)) and so we cannot set the coefficients of the bracketed expressions equal to zero separately. However, we can say that  $(n - m)$  of the  $\eta$ 's are independent. For the remaining  $m$   $\eta$ 's we now assume that the time functions  $\lambda_i(t)$  are so chosen that the coefficients of the  $\eta$ 's are zero. Then, since we are left with only independent  $\eta$ 's, we can take the remaining coefficients equal to zero. In this way we conclude that:

$$\frac{\partial F^*}{\partial q_i} - \frac{d}{dt} \frac{\partial F^*}{\partial \dot{q}_i} = 0 \quad i = 1, 2, \dots, n$$

We thus arrive at the Euler–Lagrange equations once again. However, we have now  $m$  unknown time functions  $\lambda_i$  to be determined with the aid of the original  $m$  constraining equations.

We shall have ample opportunity of using the formulations of this section in the following chapter when we consider the Reissner functional.

---

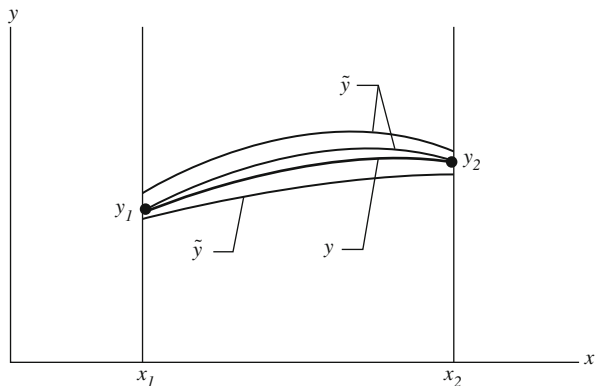
## 2.9 A Note on Boundary Conditions

In previous efforts at extremizing  $I$  we specified the end points  $(x_1, y_1)$  and  $(x_2, y_2)$  through which the extremizing function had to proceed. Thus, in Fig. 2.2 we asked for the function  $y(x)$  going through  $(x_1, y_1)$  and  $(x_2, y_2)$  to make it extremize  $I$  relative to neighboring varied paths *also* going through the aforesaid end points. The boundary conditions specifying  $y$  at  $x_1$  and at  $x_2$  are called the *kinematic* or *rigid boundary* conditions of the problem.

We now pose a different query. Suppose only  $x_1$  and  $x_2$  are given as has been shown in Fig. 2.6 and we ask what is the function  $y(x)$  that extremizes the functional  $\int_{x_1}^{x_2} F(x, y, y') dx$  between these limits. Thus, we *do not specify*  $y$  at  $x_1$  and  $x_2$  for the extremizing function.

As before, we denote the extremizing function in the discussion as  $y(x)$  and we have shown it so labeled at some position in Fig. 2.6. A system of nearby admissible “paths”  $\tilde{y}(x)$  has also been shown. Some of these paths go through the endpoints of the extremizing path while others do not. We shall extremize  $I$  with respect to such a family of admissible functions to obtain certain necessary requirements for  $y(x)$  to maintain the role of the extremizing function. Thus with the  $\delta$  operator we arrive at the following necessary condition using the same steps of earlier discussions and noting that  $\delta y$  need no longer always be zero at the end points:

$$\delta^{(1)}I = 0 = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \delta y dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x=x_1} - \left. \frac{\partial F}{\partial y'} \delta y \right|_{x=x_2} \quad (2.58)$$

**Fig. 2.6**

There are admissible functions that go through the end points  $(x_1, y_1)$  and  $(x_2, y_2)$  of the designated extremizing function  $y(x)$ . For such admissible functions  $\delta y_1 = \delta y_2 = 0$  and we conclude that a necessary condition for an extremum is:

$$\int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \right] \delta y \, dx = 0$$

Following the familiar procedures of previous computations we readily then arrive at the Euler–Lagrange equations as a necessary condition for  $y(x)$  to be an extreme:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = 0$$

There are now, however, additional necessary requirements for extremizing  $I$  if  $\delta y$  is not zero at the end points of the extremizing function. Accordingly from Eq. (2.58) we conclude that for such circumstances we require:

$$\left. \frac{\partial F}{\partial y'} \right|_{x=x_1} = 0 \quad (a)$$

$$\left. \frac{\partial F}{\partial y'} \right|_{x=x_2} = 0 \quad (b)$$

The conditions (2.59) are termed the *natural boundary conditions*. They are the boundary conditions that must be prescribed if both the values  $y(x_1)$  and  $y(x_2)$  are not specified.<sup>12</sup> However, it is also possible to assign one natural and one kinematic boundary condition to satisfy the requirements for  $y(x)$  to fulfill its assigned role as

<sup>12</sup>In problems of solid mechanics dealing with the total potential energy we will see that the kinematic boundary conditions involve displacement conditions of the boundary while natural boundary conditions involve force conditions at the boundary.

extremizing function. In more general functionals  $I$  set forth earlier and those to be considered in following sections, we find natural boundary conditions in much the same way as set forth in this section. In essence we proceed with the variation process *without* requiring the  $\eta$  functions (or by the same token the variations  $\delta y$ ) to be zero at the boundaries. Rather we set equal to zero all expressions established on the boundary by the integration by parts procedures. The resulting conditions are the natural boundary conditions.

Note that for all various possible boundary conditions, we work in any particular case with the same differential equation. However, the extremal function will eventually depend on the particular permissible combination of boundary conditions we choose to employ. Usually certain kinematic boundary conditions are imposed by the constraints present in a particular problem. We must use these boundary conditions or else our extremal functions will not correspond to the problem at hand. The remaining boundary conditions then are natural ones that satisfy the requirements for the extremizing process. We shall illustrate these remarks in Example 2.5 after a discussion of higher-order derivatives.

---

## 2.10 Functionals Involving Higher-Order Derivatives

We have thus far considered only first-order derivatives in the functionals. At this time we extend the work by finding extremal functions  $y(x)$  for functionals having higher-order derivatives. Accordingly we shall consider the following functional:

$$I = \int_{x_1}^{x_2} F(x, y, y', y'', y''') dx \quad (2.60)$$

The cases for lower or higher-order derivatives other than  $y'''$  are easily attainable from the procedure that we shall follow. Using the familiar one-parameter family of admissible functions for extremizing  $I$  we can then say:

$$\left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} = \left[ \frac{d}{d\varepsilon} \int_{x_1}^{x_2} F(x, \tilde{y}, \tilde{y}', \tilde{y}'', \tilde{y}''') dx \right]_{\varepsilon=0} = 0$$

This becomes:

$$\left\{ \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial \tilde{y}} \eta + \frac{\partial F}{\partial \tilde{y}'} \eta' + \frac{\partial F}{\partial \tilde{y}''} \eta'' + \frac{\partial F}{\partial \tilde{y}'''} \eta''' \right] dx \right\}_{\varepsilon=0} = 0$$

We may drop the subscript  $\varepsilon = 0$  along with the tildes as follows:

$$\int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' + \frac{\partial F}{\partial y''} \eta'' + \frac{\partial F}{\partial y'''} \eta''' \right] dx = 0$$



We now carry out the following series of integration by parts:

$$\begin{aligned}
 \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta' dx &= \left. \frac{\partial F}{\partial y'} \eta \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left[ \frac{\partial F}{\partial y'} \right] \eta dx \\
 \int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \eta'' dx &= \left. \frac{\partial F}{\partial y''} \eta' \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left[ \frac{\partial F}{\partial y''} \right] \eta' dx \\
 &= \left. \frac{\partial F}{\partial y''} \eta' \right|_{x_1}^{x_2} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \eta \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) \eta dx \\
 \int_{x_1}^{x_2} \frac{\partial F}{\partial y'''} \eta''' dx &= \left. \frac{\partial F}{\partial y'''} \eta'' \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left[ \frac{\partial F}{\partial y'''} \right] \eta'' dx \\
 &= \left. \frac{\partial F}{\partial y'''} \eta'' \right|_{x_1}^{x_2} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'''} \right) \eta' \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y'''} \right) \eta' dx \\
 &= \left. \frac{\partial F}{\partial y'''} \eta'' \right|_{x_1}^{x_2} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'''} \right) \eta' \Big|_{x_1}^{x_2} + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y'''} \right) \eta \Big|_{x_1}^{x_2} \\
 &\quad - \int_{x_1}^{x_2} \frac{d^3}{dx^3} \left( \frac{\partial F}{\partial y'''} \right) \eta dx
 \end{aligned}$$

Now combining the results we get:

$$\begin{aligned}
 & - \int_{x_1}^{x_2} \left[ \frac{d^3}{dx^3} \left( \frac{\partial F}{\partial y'''} \right) - \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) + \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} \right] \eta dx \\
 & + \left. \frac{\partial F}{\partial y'''} \eta'' \right|_{x_1}^{x_2} - \left\{ \frac{d}{dx} \frac{\partial F}{\partial y'''} - \frac{\partial F}{\partial y''} \right\} \eta' \Big|_{x_1}^{x_2} \\
 & + \left\{ \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y'''} \right) - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) + \frac{\partial F}{\partial y'} \right\} \eta \Big|_{x_1}^{x_2} = 0
 \end{aligned} \tag{2.61}$$

Since the function  $\eta$  having the properties such that  $\eta(x_1) = \eta'(x_1) = \eta''(x_1) = \eta(x_2) = \eta'(x_2) = \eta''(x_2) = 0$  are admissible no matter what the boundary conditions may be, it is clear that a necessary condition for an extremum is:

$$\int_{x_1}^{x_2} \left[ \frac{d^3}{dx^3} \left( \frac{\partial F}{\partial y'''} \right) - \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) + \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} \right] \eta dx = 0$$

And we can then arrive at the following Euler-Lagrange equation:

$$\frac{d^3}{dx^3} \left( \frac{\partial F}{\partial y'''} \right) - \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) + \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0 \tag{2.62}$$

If, specifically, the conditions  $y(x_1)$ ,  $y(x_2)$ ,  $y'(x_1)$ ,  $y'(x_2)$ ,  $y''(x_1)$  and  $y''(x_2)$  are *specified* (these are the kinematic boundary conditions of this problem), the admissible functions must have these prescribed values and these prescribed derivatives at the end points. Then clearly we have for this case:

$$\eta(x_1) = \eta(x_2) = \eta'(x_1) = \eta'(x_2) = \eta''(x_1) = \eta''(x_2) = 0$$

Thus we see that giving the kinematic end conditions and satisfying the Euler–Lagrange equations permits the resulting  $y(x)$  to be an extremal. If the kinematic conditions are not specified then we may satisfy the *natural boundary conditions* for this problem. Summarizing, we may use either of the following overall sets of requirements (see Eq. (2.61)) at the end points:

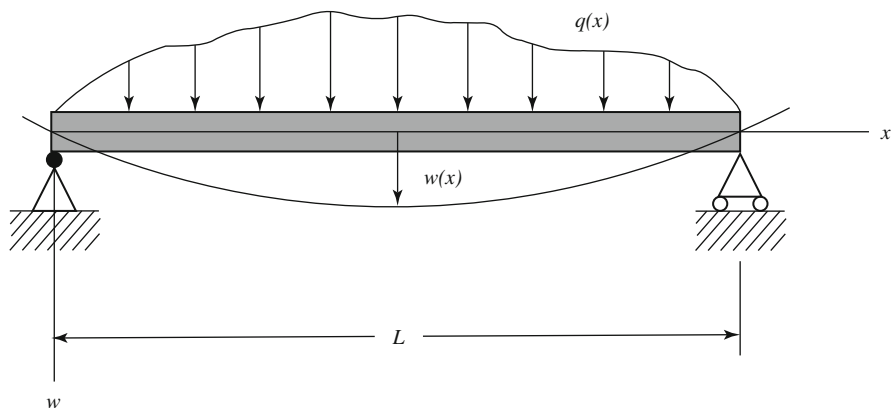
<i>Kinematic</i>	or	<i>Natural</i>	
$y''$ specified	or	$\frac{\partial F}{\partial y'''} = 0$	(a)
$y'$ specified	or	$\frac{d}{dx} \left( \frac{\partial F}{\partial y'''} \right) - \frac{\partial F}{\partial y''} = 0$	(b)
$y$ specified	or	$\frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y'''} \right) - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) + \frac{\partial F}{\partial y'} = 0$	(c) (2.63)

Clearly combinations of kinematic and natural boundary conditions are also permissible.

It should be apparent that by disregarding the terms involving  $y'''$  we may use the results for a functional whose highest-order derivative is second-order. And, by disregarding both terms with  $y'''$  and with  $y''$ , we get back to the familiar expressions used earlier where  $y'$  was the highest-order term in the functional. Furthermore the pattern of growth from first order derivatives on up is clearly established by the results so that one can readily extrapolate the formulations for orders higher than three. Finally we may directly extend the results of this problem to include function  $I$  with more than one function  $y$ . We simply get equations of the form (2.62) for each function. Thus for  $t$  as the independent variable and  $q_1, q_2, \dots, q_n$  as the functions in  $I$  we have for derivatives up to order three for all functions:

$$\begin{aligned} & \frac{d^3}{dt^3} \left( \frac{\partial F}{\partial q_1'''} \right) - \frac{d^2}{dt^2} \left( \frac{\partial F}{\partial q_1''} \right) + \frac{d}{dt} \left( \frac{\partial F}{\partial q_1'} \right) - \frac{\partial F}{\partial q_1} = 0 \\ & \vdots \\ & \frac{d^3}{dt^3} \left( \frac{\partial F}{\partial q_n'''} \right) - \frac{d^2}{dt^2} \left( \frac{\partial F}{\partial q_n''} \right) + \frac{d}{dt} \left( \frac{\partial F}{\partial q_n'} \right) - \frac{\partial F}{\partial q_n} = 0 \end{aligned}$$

Similarly for each  $q$  we have a set of conditions corresponding to Eq. (2.63).

**Fig. 2.7**

**EXAMPLE 2.5.** We now consider the deflection  $w$  of the centerline of a simply-supported beam having a rectangular cross section and loaded in the plane of symmetry of the cross section (see Fig. 2.7). In Chap. 4 we will show through the principal of minimum total potential energy that the following functional is to be extremized in order to insure equilibrium:

$$I = \int_0^L \left[ \frac{EI}{2} \left( \frac{d^2 w}{dx^2} \right)^2 - qw \right] dx \quad (a)$$

where  $I$  is the second moment of the cross section about the horizontal centroidal axis, and  $q$  is the transverse loading function.

The Euler–Lagrange equation for this case may be formed from Eq. (2.62) as follows:

$$-\frac{d^2}{dx^2} \left( \frac{\partial F}{\partial w''} \right) + \frac{d}{dx} \left( \frac{\partial F}{\partial w'} \right) - \frac{\partial F}{\partial w} = 0 \quad (b)$$

where

$$F = \frac{EI}{2} (w'')^2 - qw \quad (c)$$

We then get for the Euler–Lagrange equation:

$$-\frac{d^2}{dx^2} (EI w'') + \frac{d}{dx} (0) + q = 0$$

therefore

$$EIw^{IV} = q \quad (d)$$

The possible boundary conditions needed for extremization of functional (a) can be readily deduced from Eq. (2.63). Thus at the end points:

$$w \text{ specified or } \frac{\partial F}{\partial w''} = 0 \quad (e)$$

$$w \text{ specified or } -\frac{d}{dx} \left( \frac{\partial F}{\partial w''} \right) + \frac{\partial F}{\partial w'} = 0 \quad (f)$$

It is clear from Fig. 2.7 that we *must* employ the kinematic boundary condition  $w = 0$  at the ends. In order then to carry out the extremization process properly we must require additionally that:

$$\frac{\partial F}{\partial w''} = 0 \quad \text{at ends} \quad (g)$$

Substituting for  $F$  we get:

$$w''(0) = w''(L) = 0 \quad (h)$$

You may recall that this indicates that the bending moments are zero at the ends—a condition needed for the frictionless pins there. We thus see here that the natural boundary conditions have to do with forces in structural problems. ///

In the following section we consider the case where we have higher-order derivatives involving more than one independent variable.

---

## 2.11 A Further Extension

As we shall demonstrate later, the total potential energy in a plate is generally expressible as a double integral of the displacement function  $w$  and partial derivatives of  $w$  in the following form:

$$I = \iint_S F \left( x, y, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y \partial x}, \frac{\partial^2 w}{\partial y^2} \right) dx dy \quad (2.65)$$

where  $S$  is the area over which the integration is carried out. Using the notation  $\partial w / \partial x = w_x$ ,  $\partial^2 w / \partial x^2 = w_{xx}$ , etc., the above functional may be given as follows:

$$I = \iint_S F(x, y, w, w_x, w_y, w_{xy}, w_{yx}, w_{yy}, w_{xx}) dx dy \quad (2.66)$$

We have here a functional with *more than one independent variable*. The procedure for finding the extremizing function, which we consider now to be represented as  $w(x, y)$  is very similar to what we have done in the past. We use as admissible functions,  $\tilde{w}(x, y)$ , a one parameter family of functions defined as follows:

$$\tilde{w}(x, y) = w(x, y) + \varepsilon \eta(x, y) \quad (2.67)$$

Accordingly we have for the extremization process:

$$\left( \frac{d\tilde{I}}{d\varepsilon} \right)_{\varepsilon=0} = 0 = \left\{ \iint_S \left\{ \frac{\partial F}{\partial \tilde{w}} \eta + \frac{\partial F}{\partial \tilde{w}_x} \eta_x + \frac{\partial F}{\partial \tilde{w}_y} \eta_y + \frac{\partial F}{\partial \tilde{w}_{xx}} \eta_{xx} + \frac{\partial F}{\partial \tilde{w}_{xy}} \eta_{xy} \right. \right. \\ \left. \left. + \frac{\partial F}{\partial \tilde{w}_{yx}} \eta_{yx} + \frac{\partial F}{\partial \tilde{w}_{yy}} \eta_{yy} \right\} dx dy \right\}_{\varepsilon=0}$$

We may remove the tildes and the subscript notation  $\varepsilon = 0$  simultaneously to get the following statement:

$$\iint_S \left[ \frac{\partial F}{\partial w} \eta + \frac{\partial F}{\partial w_x} \eta_x + \frac{\partial F}{\partial w_y} \eta_y + \frac{\partial F}{\partial w_{xx}} \eta_{xx} \right. \\ \left. + \frac{\partial F}{\partial w_{xy}} \eta_{xy} + \frac{\partial F}{\partial w_{yx}} \eta_{yx} + \frac{\partial F}{\partial w_{yy}} \eta_{yy} \right] dx dy = 0 \quad (2.68)$$

The familiar integration by parts process can be carried out here by employing Green's theorem (see Appendix I) in two dimensions which is given as follows:

$$\iint_S G \frac{\partial \eta}{\partial x} dx dy = - \iint_S \frac{\partial G}{\partial x} \eta dx dy + \oint \eta G \cos(n, x) dl \quad (2.69)$$

Recall that  $n$  is the normal to the bounding curve  $C$  of the domain  $S$ . We apply the above theorem once to the second and third terms in the integrand of Eq. (2.68) and twice to the last four terms to get:

$$\iint_S \left[ \frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial w_y} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_{xx}} \right) \right. \\ \left. + \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial F}{\partial w_{xy}} \right) + \frac{\partial^2}{\partial y \partial x} \left( \frac{\partial F}{\partial w_{yx}} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial F}{\partial w_{yy}} \right) \right] \eta dx dy \\ + \text{System of Line Integrals} = 0$$

We can conclude, as in earlier cases, that a necessary requirement is that:

$$\begin{aligned} \frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial w_y} \right) + \frac{\partial^2}{\partial x^2} \left( \frac{\partial F}{\partial w_{xx}} \right) \\ + \frac{\partial^2}{\partial y \partial x} \left( \frac{\partial F}{\partial w_{yx}} \right) + \frac{\partial^2}{\partial x \partial y} \left( \frac{\partial F}{\partial w_{xy}} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial F}{\partial w_{yy}} \right) = 0 \end{aligned} \quad (2.70)$$

This is the Euler–Lagrange equation for this case. We may deduce the natural boundary conditions from the line integrals. However, we shall exemplify this step later when we study the particular problem of classical plate theory, since the results require considerable discussion.

If we have several functions in  $F$ , we find that additional equations of the form given by Eq. (2.70) then constitute additional Euler–Lagrange equations for the problem.

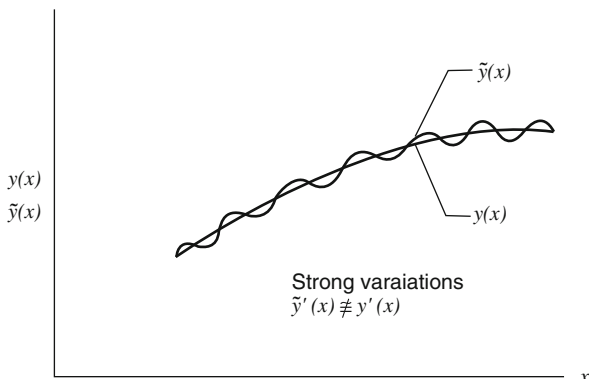
## 2.12 Closure

We have set forth a brief introduction in this chapter into elements of the calculus of variations. In short, we have considered certain classes of functionals with the view toward establishing necessary conditions for finding functions that extremize the functionals. The results were ordinary or partial differential equations for the extremizing functions (the Euler–Lagrange equations) as well as the establishment of the dualities of kinematic (or rigid) and natural boundary conditions. We will see later that the natural boundary conditions are not easily established without the use of the variational approach. And since these conditions are often important for properly posing particular boundary value problems, we can then conclude that the natural boundary conditions are valuable products of the variational approach.

We also note that by the nature of the assumption leading to Eq. (2.9) we have been concerned with varied paths,  $\tilde{y}(x)$ , which not only are close to the extremal  $y(x)$  but which have derivatives  $\tilde{y}'(x)$  close to  $y'(x)$ . Such variations are called *weak variations*. There are, however, variations which do not require the closeness of the derivatives of the varied paths to that of the extremal function (see Fig. 2.8 showing such a varied path). When this is the case we say that we have *strong variations*. A more complex theory is then needed beyond the level of this text. The reader is referred to more advanced books such as those given in the footnote on page 71.

In the remainder of the text we shall employ the formulations presented in this chapter for Euler–Lagrange equations and boundary conditions when there is only a single independent variable present. The equations, you should recall, are then ordinary differential equations and the boundary conditions are prescriptions at end points. For more than one independent variable, we shall work out the Euler–Lagrange equations (now partial differential equations) as well as the boundary conditions (now line or surface integrals) from first principles using the formulation  $\delta^{(1)}(I) = 0$  or  $(\partial \tilde{I} / \partial \varepsilon)_\varepsilon = 0$  as the basis of evaluations.

Fig. 2.8



It is to be pointed out that this chapter does not terminate the development of the calculus of variations. In the later chapters we shall investigate the process of going from a boundary value problem to a functional for which the differential equation corresponds to the Euler–Lagrange equations. This is inverse to the process set forth in this chapter and leads to the useful quadratic functional. In Chap. 7 we shall set forth two basic theorems, namely the maximum theorem and the “mini-max” theorem which form the basis for key approximation procedures used for estimating eigenvalues needed in the study of vibrations and stability of elastic bodies. Additionally, in Chap. 9 we shall examine the second variation. Clearly there will be a continued development of the calculus of variations as we proceed.

Our immediate task in Chap. 3 is to set forth functionals whose Euler–Lagrange equation and boundary conditions form important boundary value problems in solid mechanics. We shall illustrate the use of such functionals for problems of simple trusses; we thus launch the study of structural mechanics as a by-product. Additionally we shall introduce certain valuable approximation techniques associated with the variational process that form the cornerstone of much such work throughout the text.

It may be apparent from these remarks that Chap. 3 is one of the key chapters in this text.

## Problems

2.1 Given the functional:

$$I = \int_{x_1}^{x_2} (3x^2 + 2x(y')^2 + 10xy)dx$$

What is the Euler–Lagrange equation?

- 2.2 What is the first variation of  $F$  in the preceding problem? What is the first variation of  $I$  in this problem?
- 2.3 What is the Euler–Lagrange equation for the functional

$$I = \int_{x_1}^{x_2} (3x^2 + 2xy' + 10)dx$$

- 2.4 Justify the identity given at the outset of Case (c) in Sec. 2.5.
- 2.5 Consider the functional:

$$I = \int_{x_1}^{x_2} [3(y')^2 + 4x] dx$$

What is the extremal function? Take  $y = 0$  at the end points.

- 2.6 Fermat's principle states that the path of light is such as to minimize the time passage from one point to another. What is the proper path of light in a plane medium from point (1) to point (2) (Fig. 2.9) wherein the speed of light is given as: (a)  $v = Cy$  (b)  $v = C/y$ ?
- 2.7 Demonstrate that the shortest distance between two points in a plane is a straight line.
- 2.8 Consider a body of revolution moving with speed  $V$  through rarefied gas outside the earth's atmosphere (Fig. 2.10). We wish to design the shape of the body, i.e., get  $y(x)$ , so as to minimize the drag. Assume that there is no friction on the body at contact with the molecules of the gas. Then one can readily show that the pressure on the body is given as:

$$p = 2\rho V^2 \sin^2 \theta$$

(This means that the molecules are reflected *specularly*.) Show that the drag  $F_D$  for a length  $L$  of the body is given as:

$$F_D = \int_0^L 4\pi\rho V^2 (\sin^3 \theta) y \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx$$

Assume now that  $dy/dx$  is small so that

$$\sin \theta = \frac{dy/dx}{[1 + (dy/dx)^2]^{1/2}} \cong \left( \frac{dy}{dx} \right)$$

Show that

$$F_D = 4\pi\rho V^2 \int_0^L \left( \frac{dy}{dx} \right)^3 y dx$$



Fig. 2.9

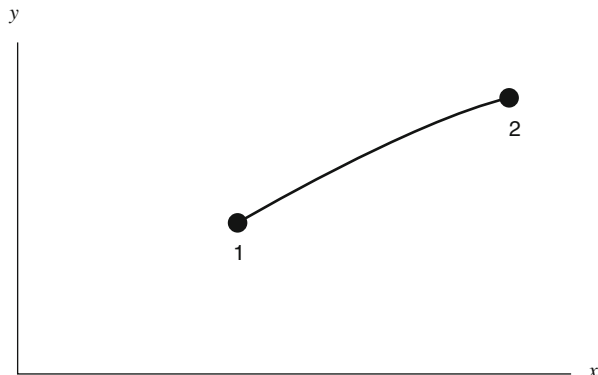
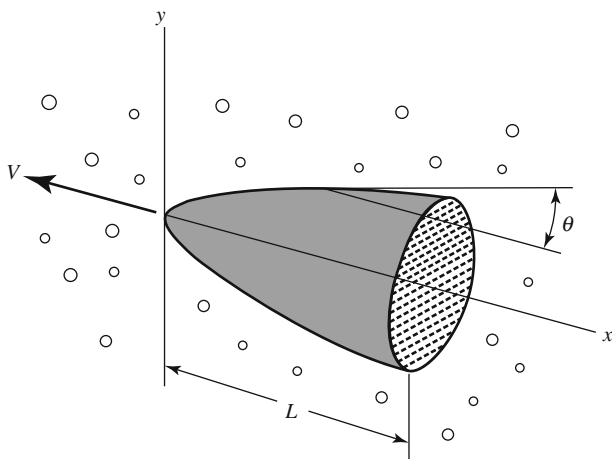


Fig. 2.10



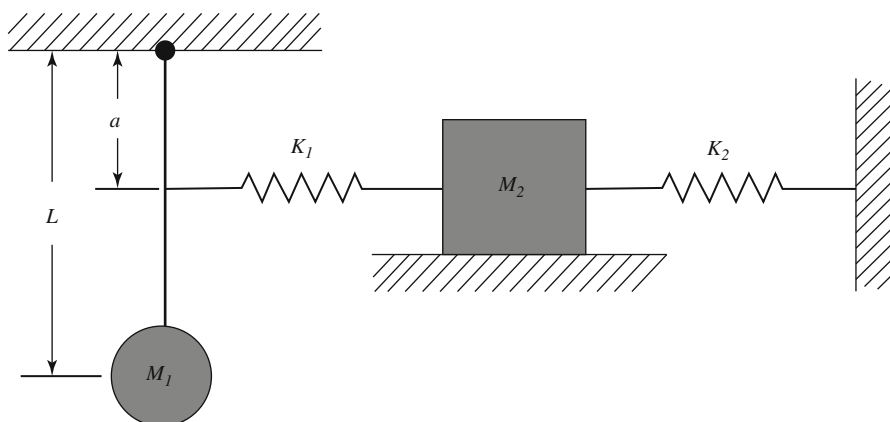
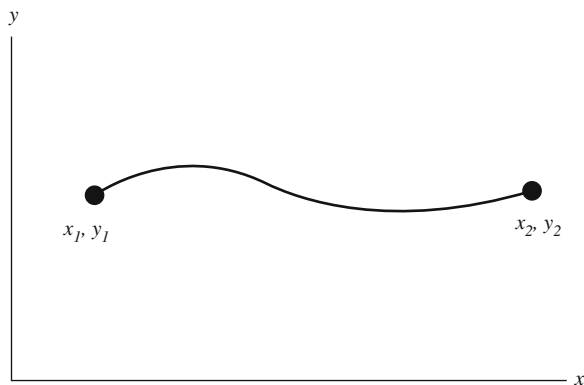
What is the Euler–Lagrange equation for  $y(x)$  in order to minimize the drag?

- 2.9 Consider the problem wherein a curve is desired between points  $(x_1, y_1)$  and  $(x_2, y_2)$  (see Fig. 2.11) which upon rotation about the  $x$  axis produces a surface of revolution having a minimum area. Show that the solution is of the form:

$$y = C_1 \cosh t, \quad t = \frac{x + C_2}{C_1}$$

which is the parametric representation of a *catenary*.

- 2.10 Use the  $\delta$  operator on the functional given in the text to find the Euler–Lagrange equation for the brachistochrone problem.

**Fig. 2.11****Fig. 2.12**

- 2.11 Using Hamilton's principle find the equations of motion for the system shown in Fig. 2.12 for small oscillations.
- 2.12 Using Hamilton's principle show that

$$(mR^2\dot{\phi} \sin^2\theta) = \text{Const.}$$

for the *spherical pendulum* shown in Fig. 2.13. Gravity acts in the  $-z$  direction. Neglect friction and the weight of the connecting link  $R$ . Use spherical coordinates  $\theta$  and  $\phi$  as shown in the diagram and take the lowest position of  $m$  as the datum for potential energy.

- 2.13 Use the  $\delta$  operator on the functional given in the text to find the Euler-Lagrange equations for the spring-mass system in Example 2.3

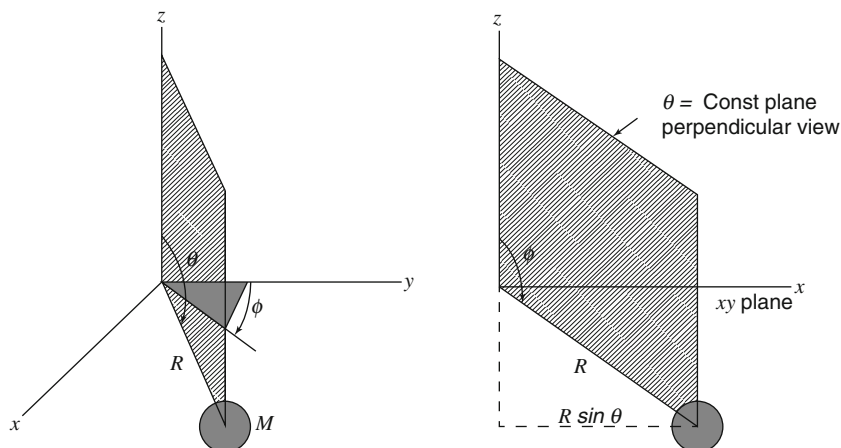
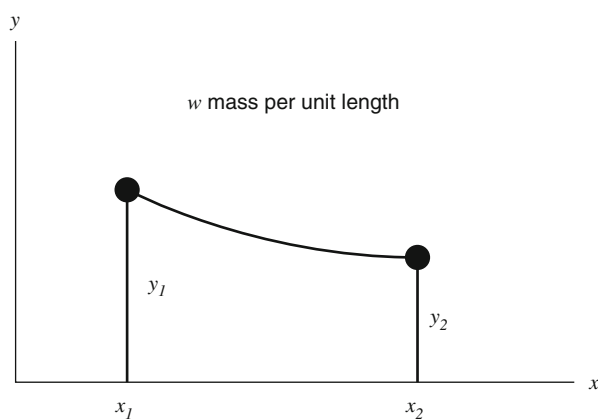


Fig. 2.13

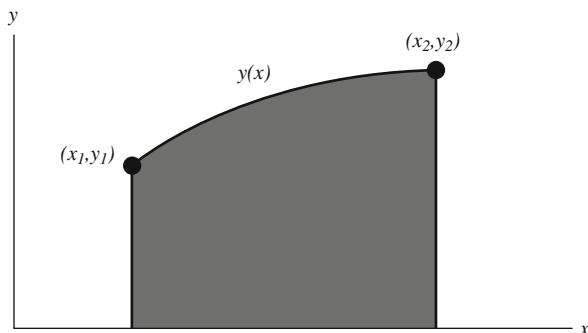
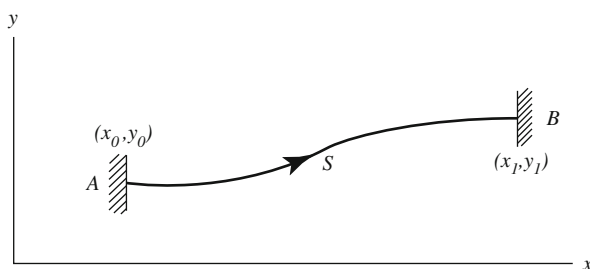
Fig. 2.14



- 2.14 Extend the results of Problem 2.7 to any two points in three-dimensional space.
- 2.15 Consider the case of a rope hanging between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  as shown in Fig. 2.14. Find the Euler–Lagrange equation for the rope by minimizing the potential energy of the rope for a given length  $L$ . The rope is perfectly flexible. Show that

$$y = \frac{-\lambda}{gw} + \frac{C_1}{wg} \cosh \frac{(x + C_2)}{C_1}$$

How are  $C_1$ ,  $C_2$ , and  $\lambda$  obtained?

**Fig. 2.15****Fig. 2.16**

- 2.16 Show that the curve (Fig. 2.15) from position  $(x_1, y_1)$  to  $(x_2, y_2)$  which has a given length  $L$  and which maximizes the first moment of inertia of the cross-hatched area about the  $x$  axis has a differential equation given as:

$$\frac{y^2}{2} + \frac{\lambda}{\sqrt{1 + (y')^2}} = C_1$$

where  $\lambda$  is a Lagrange multiplier. Solve for  $y(x)$  in the form of a quadrature but do not carry out the integration.

- 2.17 Consider two points  $A$  and  $B$  and a straight line joining these points. Of all the curves of length  $L$  connecting these points what is the curve that maximizes the area between the curve and the line  $AB$ ? What is the physical interpretation of the Lagrange multiplier  $\lambda$ ?
- 2.18 Consider a uniform rod *fixed* at points  $(x_0, y_0)$  and  $(x_1, y_1)$  as has been shown in Fig. 2.16. The distance  $s$  measured along the rod will be considered a coordinate and the length  $\int_A^B ds$  of the rod is  $L$ . Let  $\theta(s)$  be the angle between the tangent to the curve and the  $x$  axis. Clearly  $\theta(0) = \theta(L) = 0$  due to the constraints. We will later show that the strain energy is proportional to the integral.

$$\int_0^L \left( \frac{d\theta}{ds} \right)^2 ds$$

Express two isoperimetric constraints that link the length  $L$  and the positions  $(x_0, y_0)$  and  $(x_1, y_1)$ . Extremize the above functional using the aforementioned constraints and show for a first integral of Euler's equation that we get:

$$(\theta')^2 = C + \lambda_1 \cos \theta + \lambda_2 \sin \theta$$

We may thus determine the centerline curve of the rod constrained as shown in Fig. 2.16.

2.19 Consider the problem of extremizing the following functional

$$I = \int_{t_1}^{t_2} F(y, \dot{y}, \ddot{y}, t) dt \quad (a)$$

where  $y$  and  $\dot{y}$  are specified at the end points. Reformulate the functional as one having only *first-order* derivatives and a *constraining equation*. Using Lagrange multiplier functions express the Euler–Lagrange equations. Eliminate the Lagrange multiplier function and write the resulting Euler–Lagrange equation.

2.20 In the preceding problem, take

$$F = [y + 2(\ddot{y})^2 - ty]$$

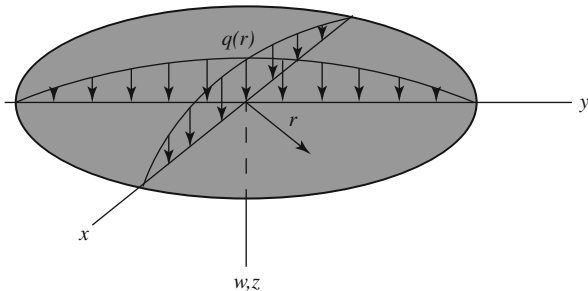
What is the extremal function if  $y = \dot{y} = 0$  at  $t = 0$  and  $y = \dot{y} = 1$  at 1? What is the Lagrange multiplier function?

2.21 Consider the brachistochrone problem in a *resistive medium* where the drag force is expressed as a given function of speed,  $R(V)$ , per unit mass of particle. Note there are now two functions  $y$  and  $V$  to be considered. What is the constraining equation between these functions? Show that using  $\lambda(x)$ , a Lagrange multiplier function, that one of the Euler–Lagrange equations is

$$\frac{V\lambda'(x)}{\sqrt{1 + (y')^2}} = \frac{dH}{dV}$$

and an integral of the other is

$$\frac{Hy'}{\sqrt{1 + (y')^2}} = C + \lambda(x)g$$

**Fig. 2.17**

where  $C$  is a constant of integration,  $g$  is the acceleration of gravity, and  $H$  is given as:

$$H = \frac{1}{V} + \lambda(x)R(V)$$

2.22 Derive the Euler–Lagrange equation and the natural boundary conditions for the following functional:

$$I = \int_{x_1}^{x_2} F(x, d^4y/dx^4) dx$$

2.23 Do Problem 2.19 using the given functional  $F$  (i.e., with second-order derivative).

2.24 Derive the Euler–Lagrange equation and natural and geometric boundary conditions of beam theory using the  $\delta$  operator on the functional given in the text (Example 2.5).

2.25 In our study of plates we will see (Chap. 6) that to find the equation of equilibrium for a circular plate (see Fig. 2.17) loaded symmetrically by a force per unit area  $q(r)$ , and fixed at the edges, we must extremize the following functional:

$$I = D\pi \int_0^a \left[ r \left( \frac{d^2w}{dr^2} \right)^2 + \frac{1}{r} \left( \frac{dw}{dr} \right)^2 + 2v \frac{dw}{dr} \frac{d^2w}{dr^2} - \frac{2q}{D} rw \right] dr$$

where  $D$  and  $v$  are elastic constants, and  $w$  is the deflection (in the  $z$  direction) of the center-plane of the plate. Show that the following is the proper governing differential equation for  $w$ :

$$r \frac{d^4w}{dr^4} + 2 \frac{d^3w}{dr^3} - \frac{1}{r} \frac{d^2w}{dr^2} + \frac{1}{r^2} \frac{dw}{dr} = \frac{qr}{D}$$

Solid Mechanics

A Variational Approach, Augmented Edition

Dym, C.L.; Shames, I.H.

2013, XVII, 685 p., Hardcover

ISBN: 978-1-4614-6033-6