Chapter 2
The Postulates of Quantum Mechanics

It is impossible to present quantum mechanics in a few pages. Since the goal of this book is to describe quantum algorithms, we limit ourselves to the *principles* of quantum mechanics and describe them as "game rules." Suppose you have played checkers for many years and know several strategies, but you really do not know chess. Suppose now that someone describes the chess rules. Soon you will be playing a new game. Certainly, you will not master many chess strategies, but you will be able to play. This chapter has a similar goal. The postulates of a theory are its game rules. If you break the rules, you will be out of the game.

At best, we can focus on four postulates. The first describes the arena where the game goes on. The second describes the dynamics of the process. The third describes how we adjoin various systems. The fourth describes the process of physical measurement. All these postulates are described in terms of linear algebra. It is essential to have a solid understanding of the basic results in this area. Moreover, the postulate of composite systems uses the concept of tensor product, which is a method of combining two vector spaces to build a larger vector space. It is also important to be familiar with this concept.

2.1 State Space

The *state* of a physical system describes its physical characteristics at a given time. Usually we describe some of the possible features that the system can have, because otherwise, the physical problems would be too complex. For example, the spin state of a billiard ball can be characterized by a vector in $\mathbb{R}^3$. In this example, we disregard the linear velocity of the billiard ball, its color or any other characteristics that are not directly related to its rotation. The spin state is completely characterized by the axis direction, the rotation direction and rotation intensity. The spin state can be described by three real numbers that are the components of a vector, whose direction characterizes the rotation axis, whose sign describes to which side of the billiard

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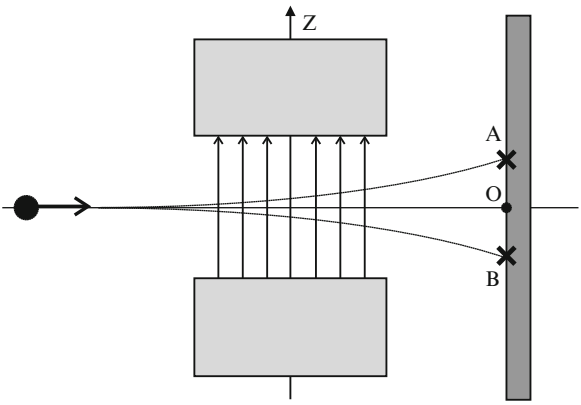

Fig. 2.1 Scheme of an experimental device to measure the spin state of an electron. The electron passes through a magnetic field having vertical direction. It hits A or B depending on the rotation direction. The distance of the points A and B from point O depends on the rotation speed. The results of this experiment are quite different from what we expect classically

ball is spinning and whose length characterizes the speed of rotation. In classical physics, the direction of the rotation axis can vary continuously, as well as the rotation intensity.

Does an *electron*, which is considered an elementary particle, *i.e.* not composed of other smaller particles, rotates like a billiard ball? The best way to answer this is by experimenting in real settings to check whether the electron in fact rotates and whether it obeys the laws of classical physics. Since the electron has charge, its rotation would produce magnetic fields that could be measured. Experiments of this kind were performed at the beginning of quantum mechanics, with beams of silver atoms, later on with beams of hydrogen atoms, and today they are performed with individual particles (instead of beams), such as electrons or photons. The results are different from what is expected by the laws of the classical physics.

We can send the electron through a magnetic field in the vertical direction (direction z), according to the scheme of Fig. 2.1. The possible results are shown. Either the electron hits the screen at the point A or point B. One never finds the electron at point O, which means no rotation. This experiment shows that the *spin* of the electron only admits two values: *spin up* and *spin down* both with the same intensity of "rotation." This result is quite different from classical, since the direction of the rotation axis is quantized, admitting only two values. The rotation intensity is also quantized.

Quantum mechanics describes the electron spin as a unit vector in the Hilbert space $\mathbb{C}^2$. The *spin up* is described by the vector

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and *spin down* by the vector

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This seems a paradox, because vectors $|0\rangle$ and $|1\rangle$ are orthogonal. Why use orthogonal vectors to describe *spin up* and *spin down*? In $\mathbb{R}^3$, if we add *spin up* and *spin down*, we obtain a rotationless particle, because the sum of two opposite vectors of equal length gives the zero vector, which describes the absence of rotation. In the classical world, you cannot rotate a billiard ball to both sides at the same time. We have two mutually excluded situations. It is the *law of excluded middle*. The notions of *spin up* and *spin down* refer to $\mathbb{R}^3$, whereas quantum mechanics describes the behavior of the electron before the observation, that is, before entering the magnetic field, which aims to determine its state of rotation.

If the electron has not entered the magnetic field and if it is somehow isolated from the macroscopic environment, its spin state is described by a linear combination of vectors $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle, \tag{2.1}$$

where the coefficients a_0 and a_1 are complex numbers that satisfy the constraint

$$|a_0|^2 + |a_1|^2 = 1. \tag{2.2}$$

Since vectors $|0\rangle$ and $|1\rangle$ are orthogonal, the sum does not result in the zero vector. Excluded situations in classical physics can coexist in quantum mechanics. This coexistence is destroyed when we try to observe it using the device shown in Fig. 2.1. In the classical case, the spin state of an object is independent of the choice of the measuring apparatus and, in principle, has not changed after the measurement. In the quantum case, the spin state of the particle is a mathematical idealization which depends on the choice of the measuring apparatus to have a physical interpretation and, in principle, suffers irreversible changes after the measurement. The quantities $|a_0|^2$ and $|a_1|^2$ are interpreted as the probability of detection of spin up or down, respectively.

2.1.1 State–Space Postulate

An *isolated physical system* has an associated Hilbert space, called the *state space*. The state of the system is fully described by a unit vector, called the *state vector* in that Hilbert space.

Notes

1. The postulate does not tell us the Hilbert space we should use for a given physical system. In general, it is not easy to determine the dimension of the Hilbert space of the system. In the case of electron spin, we use the Hilbert space of dimension 2, because there are only two possible results when we perform an experiment to determine the vertical electron spin. More complex physical systems admit more possibilities, which can be an infinite number.

2. A system is isolated or *closed* if it does not influence and is not influenced by the outside. In principle, the system need not be small, but it is easier to isolate small systems with few atoms. In practice, we can only deal with approximate isolated systems, so the state–space postulate is an idealization.

The state–space postulate is impressive, on the one hand, but deceiving, on the other hand. The postulate admits that classically incompatible states coexist in superposition, such as rotating to both sides simultaneously, but this occurs only in isolated systems, *i.e.* we cannot see this phenomenon, as we are on the outside of the insulation (let us assume that we are not *Schrödinger's cat*). A second restriction demanded by the postulate is that quantum states must have unit norm. The postulate constraints show that the quantum superposition is not absolute, *i.e.* is not the way we understand the classical superposition. If quantum systems admit a kind of superposition that could be followed classically, the quantum computer would have available an exponential amount of parallel processors with enough computing power to solve the problems in *class NP-complete*.[1] It is believed that the quantum computer is exponentially faster than the classical computer only in a restricted class of problems.

2.2 Unitary Evolution

The goal of physics is not simply to describe the state of a physical system at a given time, rather the main objective is to determine the state of this system in future. A theory makes predictions that can be verified or falsified by physical experiments. This is equivalent to determining the dynamical laws the system obeys. Usually, these laws are described by differential equations, which govern the time evolution of the system.

2.2.1 Evolution Postulate

The *time evolution* of an isolated quantum system is described by a *unitary transformation*. If the state of the quantum system at time t_1 is described by vector $|\psi_1\rangle$, the system state $|\psi_2\rangle$ at time t_2 is obtained from $|\psi_1\rangle$ by a unitary transformation U, which depends only on t_1 and t_2, as follows:

$$|\psi_2\rangle = U|\psi_1\rangle. \tag{2.3}$$

[1] The class NP-complete consists of the most difficult problems in the class NP (*Non-deterministic Polynomial*). The class NP is defined as the class of computational problems that have solutions whose correctness can be "quickly" verified.

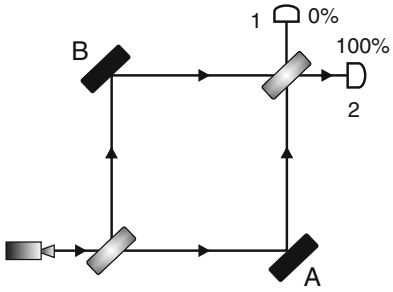

Fig. 2.2 Schematic drawing of an experimental device, which consists of a light source, two half-silvered mirrors A and B, fully reflective mirrors, detectors 1 and 2. The interference produced by the last half-silvered mirror makes all light to go to the detector 2

Notes

1. The action of a unitary operator on a vector preserves its norm. Thus, if $|\psi\rangle$ is a unit vector, $U|\psi\rangle$ is also a unit vector.
2. A *quantum algorithm* is a prescription of a sequence of unitary operators applied to an initial state takes the form

$$|\psi_n\rangle = U_n \cdots U_1 |\psi_1\rangle.$$

   The qubits in state $|\psi_n\rangle$ are measured, returning the result of the algorithm. Before measurement, we can obtain the initial state from the final state because unitary operators are invertible.
3. The evolution postulate is to be written in the form of a differential equation, called *Schrödinger equation*. This equation provides a method to obtain operator U once given the physical context. Since the goal of physics is to describe the dynamics of physical systems, the Schrödinger equation plays a fundamental role. The goal of computer science is to analyze and implement algorithms, so the computer scientist wants to know if it is possible to implement some form of a unitary operator previously chosen. Equation (2.3) is useful for the area of quantum algorithms.

Let us analyze a second experimental device. It will help to clarify the role of unitary operators in quantum systems. This device uses *half-silvered mirrors* with 45° incident light, which transmit 50% of incident light and reflect 50%. If a single photon hits the mirror at 45°, with probability 1/2, it keeps the direction unchanged and with probability 1/2, it is reflected. These half-silvered mirrors have a layer of glass that can change the phase of the wave by 1/2 wavelength. The complete device consists of a source that can emit one photon at a time, two half-silvered mirrors, two fully reflective mirrors and two photon detectors, as shown in Fig. 2.2. By tuning the device, the result of the experiment shows that 100% of the light reaches detector 2.

There is no problem explaining the result using the interference of electromagnetic waves in the context of the *classical physics*, because there is a phase change in the light beam that goes through one of the paths producing a destructive

interference with the beam going to the detector 1 and constructive interference with the beam going to the detector 2. However, if the light intensity emitted by the source is decreased such that one photon is emitted at a time, this explanation fails. If we insist on using classical physics in this situation, we predict that 50% of the photons would be detected by detector 1 and 50% by detector 2, because the photon either goes through the mirror A or goes through B, and it is not possible to interfere since it is a single photon.

In *quantum mechanics*, if the set of mirrors is isolated from the environment, the two possible paths are represented by two orthonormal vectors $|0\rangle$ and $|1\rangle$, which generate the state space that describes the possible paths to reach the photon detector. Therefore, a photon can be in superposition of "*path A*," described by $|0\rangle$, together with "*path B*," described by $|1\rangle$. This is the application of the first postulate. The next step is to describe the dynamics of the process. How is this done and what are the unitary operators in the process? In this experiment, the dynamics is produced by the half-silvered mirrors, since they generate the paths. The action of the half-silvered mirrors on the photon must be described by a unitary operator U. This operator must be chosen so that the two possible paths are created in a balanced way, *i.e.*

$$U|0\rangle = \frac{|0\rangle + \mathrm{e}^{\mathrm{i}\phi}|1\rangle}{\sqrt{2}}. \tag{2.4}$$

This is the most general case where paths A and B have the same probability to be followed, because the coefficients have the same modulus. To complete the definition of operator U, we need to know its action on state $|1\rangle$. There are many possibilities, but the most natural choice that reflects the experimental device is $\phi = \pi/2$ and

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}. \tag{2.5}$$

The state of the photon after passing through the second half-silvered mirror is

$$\begin{aligned} U(U|0\rangle) &= \frac{(|0\rangle + i|1\rangle) + i(i|0\rangle + |1\rangle)}{2} \\ &= i|1\rangle. \end{aligned} \tag{2.6}$$

The intermediate step of the calculation was displayed on purpose. We can see that the paths described by $|0\rangle$ algebraically cancel, which can be interpreted as a destructive interference, while the $|1\rangle$-paths interfere constructively. The final result shows that the photon that took path B remains, going directly to the detector 2. Therefore, quantum mechanics predicts that 100% of the photons will be detected by detector 2.

2.3 Composite Systems

The *postulate of composite systems* states that the state space of a *composite system* is the *tensor product* of the state space of the components. If $|\psi_1\rangle, \ldots, |\psi_n\rangle$ describe the states of n isolated quantum systems, the state of the composite system is $|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$.

An example of a composite system is the memory of a *n-qubit* quantum computer. Usually, the memory is divided into sets of qubits, called *registers*. The state space of the computer memory is the tensor product of the state space of the registers, which is obtained by repeated tensor product of the Hilbert space $\mathbb{C}^2$ of each *qubit*.

The state space of the memory of a 2-qubit quantum computer is $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. Therefore, any unit vector in $\mathbb{C}^4$ represents the quantum state of two qubits. For example, the vector

$$|0,0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.7}$$

which can be written as $|0\rangle \otimes |0\rangle$, represents the state of two electrons both with spin up. Analogous interpretation applies to $|0,1\rangle$, $|1,0\rangle$, and $|1,1\rangle$. Consider now the unit vector in $\mathbb{C}^4$ given by

$$|\psi\rangle = \frac{|0,0\rangle + |1,1\rangle}{\sqrt{2}}. \tag{2.8}$$

What is the spin state of each electron in this case? To answer this question, we have to factor $|\psi\rangle$ as follows:

$$\frac{|0,0\rangle + |1,1\rangle}{\sqrt{2}} = \big(a|0\rangle + b|1\rangle\big) \otimes \big(c|0\rangle + d|1\rangle\big). \tag{2.9}$$

We can expand the right-hand side and match the coefficients setting up a system of equations to find a, b, c, and d. The state of the first qubit will be $a|0\rangle + b|1\rangle$ and second will be $c|0\rangle + d|1\rangle$. But there is a big problem: the system of equations has no solution, *i.e.* there are no coefficients a, b, c, and d satisfying (2.9). Every state of a composite system that cannot be factored is called *entangled*. The quantum state is well defined when we look at the composite system as a whole, but we cannot attribute the states to the parts.

A single qubit can be in a superposed state, but it cannot be entangled, because its state is not composed of subsystems. The qubit should not be taken as a synonym of a particle, because it is confusing. The state of a single particle can be entangled when we are analyzing more than a physical quantity related to it. For example, we may describe both the position and the rotation state. The position state may be entangled with the rotation state.

Exercise 2.1. Consider the states

$$|\psi_1\rangle = \frac{1}{2}\big(|0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle\big),$$

$$|\psi_2\rangle = \frac{1}{2}\big(|0,0\rangle + |0,1\rangle + |1,0\rangle - |1,1\rangle\big).$$

Show that $|\psi_1\rangle$ is not entangled and $|\psi_2\rangle$ is entangled.

Exercise 2.2. Show that if $|\psi\rangle$ is an entangled state of two qubits, then the application of a unitary operator of the form $U_1 \otimes U_2$ necessarily generates an entangled state.

2.4 Measurement Process

In general, measuring a quantum system that is in the state $|\psi\rangle$ seeks to obtain classical information about this state. In practice, measurements are performed in laboratories using devices such as lasers, magnets, scales, and chronometers. In theory, we describe the process mathematically in a way that is consistent with what occurs in practice. Measuring a physical system that is in an unknown state, in general, disturbs this state irreversibly. In those cases, there is no way to know or recover the state before the measurement. If the state was not disturbed, no new information about it is obtained. Mathematically, the disturbance is described by a *orthogonal projector*. If the projector is over an one-dimensional space, it is said that the quantum state *collapsed* and is now described by the unit vector belonging to the one-dimensional space. In the general case, the projection is over a vector space of dimension greater than 1, and it is said that the collapse is partial or, in extreme cases, there is no change at all in the quantum state of the system.

The measurement requires the interaction between the quantum system with a macroscopic device, which violates the *state–space postulate*, because the quantum system is not isolated at this moment. We do not expect the evolution of the quantum state during the measurement process to be described by a unitary operator.

2.4.1 Measurement Postulate

A *projective measurement* is described by a Hermitian operator O, called *observable* in the state space of the system being measured. The observable O has a *diagonal representation*

$$O = \sum_{\lambda} \lambda P_{\lambda}, \tag{2.10}$$

where P_λ is the projector on the eigenspace of O associated with the eigenvalue λ. The possible results of measurement of the observable O are the eigenvalues λ.

If the system state at the time of measurement is $|\psi\rangle$, the probability of obtaining the result λ will be $\|P_\lambda|\psi\rangle\|^2$ or, equivalently,

$$p_\lambda = \langle\psi|P_\lambda|\psi\rangle. \tag{2.11}$$

If the result of the measurement is λ, the state of the quantum system immediately after the measurement will be

$$\frac{1}{\sqrt{p_\lambda}} P_\lambda|\psi\rangle. \tag{2.12}$$

Notes

1. There is a correspondence between the physical layout of the devices in a physics lab and the observable O. When an experimental physicist measures a quantum system, she or he gets real numbers as result. Those numbers correspond to the eigenvalues λ of the Hermitean operator O.
2. The states $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ have the same *probability distribution* p_λ when one measures the same observable O. The states after measurement differ by the same factor $e^{i\phi}$. The term $e^{i\phi}$ multiplying a quantum state is called *global phase factor* whereas a term $e^{i\phi}$ multiplying a vector of a sum of vectors, such as $|0\rangle + e^{i\phi}|1\rangle$, is called *relative phase factor*. The real number ϕ is called *phase*.

Since the possible outcomes of a measurement of observable O obey a probability distribution, we can define the *expected value* of a measurement as

$$\langle O\rangle = \sum_\lambda p_\lambda \lambda, \tag{2.13}$$

and the *standard deviation* as

$$\Delta O = \sqrt{\langle O^2\rangle - \langle O\rangle^2}. \tag{2.14}$$

It is important to remember that the mean and standard deviation of an observable depends on the state that the physical system was in just before the measurement.

Exercise 2.3. Show that $\langle O\rangle = \langle\psi|O|\psi\rangle$.

Exercise 2.4. Show that if the physical system is in a state $|\psi\rangle$ that is an eigenvector of O, then $\Delta O = 0$, that is, there is no uncertainty about the result of the measurement of the observable O. What is the result of the measurement?

Exercise 2.5. Show that $\sum_\lambda p_\lambda = 1$ for any observable O and any state $|\psi\rangle$.

Exercise 2.6. Suppose that the physical system is in generic state $|\psi\rangle$. Show that $\sum_\lambda p_\lambda^2 = 1$ to an observable O, if and only if $\Delta O = 0$.

2.4.2 Measurement in Computational Basis

The *computational basis* of space $\mathbb{C}^2$ is the set $\{|0\rangle, |1\rangle\}$. For one qubit, the observable of the *measurement in the computational basis* is Pauli matrix Z, whose spectral decomposition is

$$Z = (+1)P_{+1} + (-1)P_{-1}, \tag{2.15}$$

where $P_{+1} = |0\rangle\langle 0|$ and $P_{-1} = |1\rangle\langle 1|$. The possible results of the measurement are ± 1. If the state of the qubit is given by (2.1), the probabilities associated with possible outcomes are

$$p_{+1} = |a_0|^2, \tag{2.16}$$

$$p_{-1} = |a_1|^2, \tag{2.17}$$

whereas the states immediately after the measurement are $|0\rangle$ and $|1\rangle$, respectively. In fact, each of these states has a global phase that can be discarded. Note that

$$p_{+1} + p_{-1} = 1,$$

because state $|\psi\rangle$ have unit norm.

Before generalizing to n qubits, it is interesting to reexamine the process of measurement of a qubit with another observable given by

$$O = \sum_{k=0}^{1} k|k\rangle\langle k|. \tag{2.18}$$

Since the eigenvalues of O are 0 and 1, the above analysis holds if we replace $+1$ by 0 and -1 by 1. With this new observable, there is a one-to-one correspondence in the nomenclature of the measurement result and the final state. If the result is 0, the state after the measurement is $|0\rangle$. If the result is 1, the state after the measurement is $|1\rangle$.

The *computational basis* of the Hilbert space of n qubits in decimal notation is the set $\{|0\rangle, \ldots, |2^n - 1\rangle\}$. The *measurement in the computational basis* is associated with observable

$$O = \sum_{k=0}^{2^n-1} k\, P_k, \tag{2.19}$$

where $P_k = |k\rangle\langle k|$. A generic state of n qubits is given by

$$|\psi\rangle = \sum_{k=0}^{2^n-1} a_k|k\rangle, \tag{2.20}$$

where amplitudes a_k satisfying the constraint

$$\sum_k |a_k|^2 = 1. \tag{2.21}$$

The measurement result is an integer value k in the range $0 \leq k \leq 2^n - 1$ with a probability distribution given by

$$\begin{aligned} p_k &= \langle \psi | P_k | \psi \rangle \\ &= | \langle k | \psi \rangle |^2 \\ &= |a_k|^2. \end{aligned} \tag{2.22}$$

Equation (2.21) ensures that the sum of the probabilities is 1. The n-qubit state immediately after the measurement is

$$\frac{P_k|\psi\rangle}{\sqrt{p_k}} \simeq |k\rangle. \tag{2.23}$$

For example, suppose that the state of two qubits is given by

$$|\psi\rangle = \frac{1}{\sqrt{3}} \left(|0,0\rangle - i\,|0,1\rangle + |1,1\rangle \right). \tag{2.24}$$

The probability that the result is 00, 01 or 11 in binary notation is $1/3$. Result 10 is never obtained, because the associated probability is 0. If the measurement result is 00, the system state immediately after will be $|0,0\rangle$. Similarly for 01 and 11. For the measurement in the computational basis, it makes sense that the result is *state* $|0,0\rangle$, because there is a correspondence between eigenvalue 00 and state $|0,0\rangle$.

The result of the measurement specifies to which vector of the computational basis state $|\psi\rangle$ is projected. The result does not provide the value of coefficient a_k, that is, none of the 2^n amplitudes a_k describing state $|\psi\rangle$ are revealed. Suppose we want to find number k as a result of an algorithm. This result should be encoded as one of the vectors of the computational basis, which spans the vector space to which state $|\psi\rangle$ belongs. It is undesirable, in principle, that the result itself is associated with one of the amplitudes. If the desired result is a non-integer real number, then the k most significant digits should be coded as a vector of the computational basis. After a measurement, we have a chance to get closer to k. A technique used in quantum algorithms is to amplify the value of a_k making it as close to 1 as possible. A measurement at this point will return the value k with high probability. Therefore, the number that specifies a *ket*, for example number k of $|k\rangle$ is a possible outcome of the algorithm, while the amplitudes of the quantum state are associated with the probability of obtaining a result.

The description of the measurement process of observable (2.19) is equivalent to simultaneous measurements or in a cascade of observables Z, *i.e.* one observable Z for each qubit. The possible results of measuring Z are ± 1. Simultaneous measurements, or in a cascade of n qubits, result in a sequence of values ± 1. The relationship between a result of this kind and the one described before is obtained by replacing $+1$ by 0 and -1 by 1. We will have a binary number that can be converted into a decimal number which is one of the values k of (2.19).

For example, for 3 qubits the result may be $(-1, +1, +1)$, which is equivalent to $(1, 0, 0)$. Converting to base ten, we get number 4. The state after the measurement will be obtained using the projector

$$\begin{aligned} P_{-1,+1,+1} &= |1\rangle\langle 1| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ &= |1,0,0\rangle\langle 1,0,0| \end{aligned} \tag{2.25}$$

over the state system of the three qubits followed by *renormalization*. The renormalization in this case replaces the coefficient by 1. The state after measurement will be $|1,0,0\rangle$. So far using the computational basis, for both observables (2.19) and Z's, we can simply say that the result is $|1,0,0\rangle$, because we automatically know that the eigenvalues of Z in question are $(-1,+1,+1)$ and the number k is 4.

A simultaneous measurement of n observables Z is not equivalent to measure observable $Z \otimes \cdots \otimes Z$. The latter observable returns a single value, which can be $+1$ or -1, whereas with n observables Z, simultaneously or not, we obtain n values ± 1. Measurements on a cascade are performed with observable $Z \otimes I \otimes \cdots \otimes I$, $I \otimes Z \otimes \cdots \otimes I$, and so on. They can also be performed simultaneously. Usually, we use a more compact notation, Z_1, Z_2, successively, where Z_1 means that observable Z was used for the first qubit and the identity operator for the remaining qubits. Since these observables commute, the order is irrelevant and the limits imposed by the *uncertainty principle* do not apply. Measurement of observables of this kind is called *partial measurement* in the computational basis.

Exercise 2.7. Suppose that the state of a qubit is $|1\rangle$.

1. What is the mean value and standard deviation of the measurement of observable X?
2. What is the mean value and standard deviation of the measurement of observable Z? Compare with Exercise 2.4.

2.4.3 Partial Measurement in Computational Basis

The term *measurement in the computational basis* of n qubits implies a measurement of all n qubits. However, it is possible to perform a *partial measurement*, *i.e.* to measure some qubits. The result in this case is not necessarily a state of the computational basis. For example, we can measure the first qubit of system described by the state $|\psi\rangle$ of (2.24). It is convenient to rewrite that state as follows:

$$|\psi\rangle = \sqrt{\frac{2}{3}}|0\rangle \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}} + \frac{1}{\sqrt{3}}|1\rangle \otimes |1\rangle. \tag{2.26}$$

We can see that the measurement result is either 0 or 1. The probability of obtaining 1 is $1/3$, because the only way to get 1 for a measurement of the first qubit is to obtain 1 as well, for the second qubit. Therefore, the probability of obtaining 0 is $2/3$, and the state immediately after the measurement in this case is

$$|0\rangle \otimes \frac{|0\rangle - i|1\rangle}{\sqrt{2}}.$$

Only the qubits involved in the measurement are projected on the computational basis. The states that have 0 in the first qubit survive and the final state must be renormalized. The remaining qubits may be in superposition. In this example, when the result is 0, the state of the second qubit is a superposition, when the result is 1, the state of the second qubit is $|1\rangle$.

If we have a system composed of subsystems A and B, a partial measurement is a measurement of an observable of the type $O_A \otimes I_B$, where O_A is an observable of system A and I_B is the identity operator of system B. Physically, this means that the measuring apparatus interacted only with the subsystem A. Equivalently, a partial measurement interacting only with subsystem B is a measurement of an observable of the type $I_A \otimes O_B$.

If we have a register of m qubits together with a register of n qubits, we can represent the computational basis in a compact form $\left\{|i, j\rangle : 0 \leq i \leq 2^m - 1, 0 \leq j \leq 2^n - 1\right\}$, where i and j are both represented in base ten. A generic state will be represented by

$$|\psi\rangle = \sum_{i=0}^{2^m-1} \sum_{j=0}^{2^n-1} a_{ij} |i, j\rangle. \tag{2.27}$$

Suppose we measure all qubits of the first register in the computational basis, *i.e.* we measure observable $O_A \otimes I_B$, where

$$O_A = \sum_{k=0}^{2^m-1} k P_k. \tag{2.28}$$

The probability of obtaining value $0 \leq k \leq 2^m - 1$ is

$$\begin{aligned} p_k &= \langle\psi| \left(P_k \otimes I\right) |\psi\rangle \\ &= \sum_{j=0}^{2^n-1} \left|a_{kj}\right|^2. \end{aligned} \tag{2.29}$$

The set $\{p_0, \ldots, p_{2^m-1}\}$ is a probability distribution and therefore satisfies

$$\sum_{k=0}^{2^m-1} p_k = 1. \tag{2.30}$$

If the measurement result is k, the state immediately after the measurement will be

$$\frac{1}{\sqrt{p_k}} (P_k \otimes I) |\psi\rangle = \frac{1}{\sqrt{p_k}} |k\rangle \left(\sum_{j=0}^{2^n-1} a_{kj} |j\rangle \right). \tag{2.31}$$

Note that the state after the measurement is a superposition of the second register. A measurement of observable (2.28) is equivalent to measure observables $Z_1, \ldots, Z_m$.

Exercise 2.8. Suppose that the state of two qubits is given by

$$|\psi\rangle = \frac{3}{5\sqrt{2}}|0,0\rangle - \frac{3\,i}{5\sqrt{2}}|0,1\rangle + \frac{2\sqrt{2}}{5}|1,0\rangle - \frac{2\sqrt{2}\,i}{5}|1,1\rangle. \tag{2.32}$$

1. Describe completely the measurement process of observable Z_1, that is, obtain the probability of each outcome and the corresponding states after the measurement. Suppose that, after measuring Z_1, we measure Z_2. Describe all resulting cases.
2. Now invert the order of the observable and describe the whole process.
3. If the intermediate quantum states are disregarded, is there a difference when we invert the order of the observable? Note that the measurement of Z_1 and Z_2 may be performed simultaneously. One can move the qubits without changing the quantum state, which may be entangled or not, and put each of them into a measuring device, both adjusted to measure observable Z, as in Fig. 2.1.
4. For two qubits, the state after the measurement of the first qubit in the computational basis can be either $|0\rangle|\alpha\rangle$ or $|1\rangle|\beta\rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ are states of the second qubit. In general, we have $|\alpha\rangle \neq |\beta\rangle$. Why is this not the case in previous items?

Further Reading

The amount of good books about quantum mechanics is very large. For the first contact, we suggest [31, 66, 69]. For a more complete approach, we suggest [23]. For a more conceptual approach, we suggest [25, 65]. For those who are only interested in the application of quantum mechanics to quantum computing, we suggest [41, 57, 64, 67, 68].

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