Chapter 2
Univariate Hardy-Type Fractional Inequalities

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Abstract Here we present integral inequalities for convex and increasing functions applied to products of functions. As applications we derive a wide range of fractional inequalities of Hardy type. They involve the left and right Riemann-Liouville fractional integrals and their generalizations, in particular the Hadamard fractional integrals. Also inequalities for left and right Riemann-Liouville, Caputo, Canavati and their generalizations fractional derivatives. These application inequalities are of L_p type, $p \geq 1$, and exponential type, as well as their mixture.

2.1 Introduction

We start with some facts about fractional derivatives needed in the sequel; for more details, see, for instance, [1, 9].

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the Riemann–Liouville fractional integrals; see [12]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

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$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}f(t)(x-t)^{\alpha-1}dt,\quad (x>a), \tag{2.1}$$

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}f(t)(t-x)^{\alpha-1}dt,\quad (x<b), \tag{2.2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha>0$; see also [13]. The first result yields that the fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$, $1\leq p\leq\infty$, that is,

$$\left\|I_{a+}^{\alpha}f\right\|_{p}\leq K\|f\|_{p},\quad \left\|I_{b-}^{\alpha}f\right\|_{p}\leq K\|f\|_{p}, \tag{2.3}$$

where

$$K=\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{2.4}$$

Inequality (2.3), which is the result involving the left-sided fractional integral, was proved by H.G. Hardy in one of his first papers; see [10]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Next we follow [11].

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, and let $k:\Omega_1\times\Omega_2\to\mathbb{R}$ be a nonnegative measurable function, $k(x,\cdot)$ measurable on Ω_2 and

$$K(x)=\int_{\Omega_2}k(x,y)\,d\mu_2(y),\quad x\in\Omega_1. \tag{2.5}$$

We suppose that $K(x)>0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g:\Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int_{\Omega_2}k(x,y)f(y)\,d\mu_2(y), \tag{2.6}$$

where $f:\Omega_2\to\mathbb{R}$ is a measurable function.

Theorem 2.1 ([11]). *Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1\times\Omega_2$, and K be defined on Ω_1 by (2.5). Assume that the function $x\mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$. Define ν on Ω_2 by*

$$\nu(y):=\int_{\Omega_1}u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)<\infty. \tag{2.7}$$

If $\Phi:[0,\infty)\to\mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1}u(x)\,\Phi\left(\left|\frac{g(x)}{K(x)}\right|\right)d\mu_1(x)\leq\int_{\Omega_2}\nu(y)\,\Phi(|f(y)|)\,d\mu_2(y) \tag{2.8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ *such that:*

(i) $f, \Phi(|f|)$ *are both* $k(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\nu(y)\, \Phi(|f|)$ *is* μ_2*-integrable, and for all corresponding functions g given by (2.6).*

Important assumptions (i) and (ii) are missing from Theorem 2.1 of [11].

In this article we generalize Theorem 2.1 for products of several functions and we give wide applications to fractional calculus.

2.2 Main Results

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x,y)\, d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{2.9}$$

$i = 1, \ldots, m$. We assume that $K_i(x) > 0$ a.e. on Ω_1 and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y)\, f_i(y)\, d\mu_2(y), \tag{2.10}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, \ldots, m$.

Here u stands for a weight function on Ω_1.

The first introductory result is proved for $m = 2$.

Theorem 2.2. *Assume that the function* $x \mapsto \left(\frac{u(x) k_1(x,y) k_2(x,y)}{K_1(x) K_2(x)} \right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_2 *on* Ω_2 *by*

$$\lambda_2(y) := \int_{\Omega_1} \frac{u(x)\, k_1(x,y)\, k_2(x,y)}{K_1(x)\, K_2(x)}\, d\mu_1(x) < \infty. \tag{2.11}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, *are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x)\, \Phi_1\left(\left| \frac{g_1(x)}{K_1(x)} \right| \right) \Phi_2\left(\left| \frac{g_2(x)}{K_2(x)} \right| \right) d\mu_1(x) \leq$$
$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|)\, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1(|f_1(y)|)\, \lambda_2(y)\, d\mu_2(y) \right), \tag{2.12}$$

true for all measurable functions, $i = 1, 2$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_2 \Phi_1(|f_1|)$, $\Phi_2(|f_2|)$, *are both* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

Proof. Notice here that Φ_1, Φ_2 are continuous functions. Here we use Jensen's inequality and Fubini's theorem and that Φ_i are increasing. We have

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right)d\mu_1(x) =$$

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{1}{K_1(x)}\int_{\Omega_2} k_1(x,y)\,f_1(y)\,d\mu_2(y)\right|\right)\cdot \tag{2.13}$$

$$\Phi_2\left(\left|\frac{1}{K_2(x)}\int_{\Omega_2} k_2(x,y)\,f_2(y)\,d\mu_2(y)\right|\right)d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\frac{1}{K_1(x)}\int_{\Omega_2} k_1(x,y)\,|f_1(y)|\,d\mu_2(y)\right)\cdot$$

$$\Phi_2\left(\frac{1}{K_2(x)}\int_{\Omega_2} k_2(x,y)\,|f_2(y)|\,d\mu_2(y)\right)d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x)\,\frac{1}{K_1(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)\cdot$$

$$\frac{1}{K_2(x)}\left(\int_{\Omega_2} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)d\mu_1(x) =$$

(calling $\gamma_1(x) := \int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)$)

$$\int_{\Omega_1}\int_{\Omega_2} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_2(y)\,d\mu_1(x) =$$

$$\int_{\Omega_2}\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,\Phi_2(|f_2(y)|)\,d\mu_1(x)\,d\mu_2(y) = \tag{2.14}$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\left(\int_{\Omega_1} \frac{u(x)\,\gamma_1(x)}{K_1(x)\,K_2(x)} k_2(x,y)\,d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\cdot$$

$$\left(\int_{\Omega_1} \frac{u(x)\,k_2(x,y)}{K_1(x)\,K_2(x)}\left(\int_{\Omega_2} k_1(x,y)\,\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right)d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_2(|f_2(y)|)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2} \frac{u(x)\,k_1(x,y)\,k_2(x,y)}{K_1(x)\,K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y) = \tag{2.15}$$

$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_1}\left(\int_{\Omega_2}\frac{u(x)k_1(x,y)k_2(x,y)}{K_1(x)K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right]=$$

$$\left(\int_{\Omega_2}\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_2}\left(\int_{\Omega_1}\frac{u(x)k_1(x,y)k_2(x,y)}{K_1(x)K_2(x)}\Phi_1(|f_1(y)|)\,d\mu_1(x)\right)d\mu_2(y)\right]=$$

$$\left(\int_{\Omega_2}\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot$$

$$\left[\int_{\Omega_2}\Phi_1(|f_1(y)|)\left(\int_{\Omega_1}\frac{u(x)k_1(x,y)k_2(x,y)}{K_1(x)K_2(x)}d\mu_1(x)\right)d\mu_2(y)\right]= \tag{2.16}$$

$$\left(\int_{\Omega_2}\Phi_2(|f_2(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\Phi_1(|f_1(y)|)\,\lambda_2(y)\,d\mu_2(y)\right],$$

proving the claim. □

When $m=3$, the corresponding result follows.

Theorem 2.3. *Assume that the function* $x\mapsto\left(\frac{u(x)k_1(x,y)k_2(x,y)k_3(x,y)}{K_1(x)K_2(x)K_3(x)}\right)$ *is integrable on* Ω_1*, for each* $y\in\Omega_2$*. Define* λ_3 *on* Ω_2 *by*

$$\lambda_3(y):=\int_{\Omega_1}\frac{u(x)k_1(x,y)k_2(x,y)k_3(x,y)}{K_1(x)K_2(x)K_3(x)}d\mu_1(x)<\infty. \tag{2.17}$$

Here $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$*,* $i=1,2,3$*, are convex and increasing functions.*
Then

$$\int_{\Omega_1}u(x)\prod_{i=1}^{3}\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)d\mu_1(x)\le \tag{2.18}$$

$$\left(\prod_{i=2}^{3}\int_{\Omega_2}\Phi_i(|f_i(y)|)\,d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_1(|f_1(y)|)\,\lambda_3(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i=1,2,3$*,* $f_i:\Omega_2\to\mathbb{R}$ *such that:*

(i) f_i*,* $\Phi_i(|f_i|)$*, are both* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$*.*
(ii) $\lambda_3\Phi_1(|f_1|)\,,\Phi_2(|f_2|)\,,\Phi_3(|f_3|)$*, are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

Proof. Here we use Jensen's inequality, Fubini's theorem, and that Φ_i are increasing. We have

$$\int_{\Omega_1}u(x)\prod_{i=1}^{3}\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)d\mu_1(x)=$$

$$\int_{\Omega_1}u(x)\prod_{i=1}^{3}\Phi_i\left(\left|\frac{1}{K_i(x)}\int_{\Omega_2}k_i(x,y)f_i(y)\,d\mu_2(y)\right|\right)d\mu_1(x)\le \tag{2.19}$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \Phi_i \left(\frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) \, |f_i(y)| \, d\mu_2(y) \right) d\mu_1(x) \leq$$

$$\int_{\Omega_1} u(x) \prod_{i=1}^{3} \left(\frac{1}{K_i(x)} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\frac{u(x)}{\prod_{i=1}^{3} K_i(x)} \right) \left(\prod_{i=1}^{3} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) d\mu_1(x) =$$

(calling $\theta(x) := \frac{u(x)}{\prod_{i=1}^{3} K_i(x)}$)

$$\int_{\Omega_1} \theta(x) \left(\prod_{i=1}^{3} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) d\mu_1(x) = \tag{2.20}$$

$$\int_{\Omega_1} \theta(x) \left[\int_{\Omega_2} \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \, \Phi_3(|f_3(y)|) \, d\mu_2(y) \right] d\mu_1(x) =$$

$$\int_{\Omega_1} \left(\int_{\Omega_2} \theta(x) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \, \Phi_3(|f_3(y)|) \, d\mu_2(y) \right) d\mu_1(x) =$$

$$\int_{\Omega_2} \left(\int_{\Omega_1} \theta(x) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right.$$

$$\left. k_3(x,y) \, \Phi_3(|f_3(y)|) \, d\mu_1(x) \right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3(|f_3(y)|) \left(\int_{\Omega_1} \theta(x) \, k_3(x,y) \left(\prod_{i=1}^{2} \int_{\Omega_2} k_i(x,y) \, \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \right. \tag{2.21}$$

$$\left. d\mu_1(x) \right) d\mu_2(y) =$$

$$\int_{\Omega_2} \Phi_3(|f_3(y)|) \left[\int_{\Omega_1} \theta(x) \, k_3(x,y) \left(\int_{\Omega_2} \left\{ \int_{\Omega_2} k_1(x,y) \, \Phi_1(|f_1(y)|) \, d\mu_2(y) \right\} \cdot \right. \right.$$

$$
\begin{aligned}
& k_2(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\,d\mu_2(y)\Bigg)\,d\mu_1(x)\Bigg]\,d\mu_2(y) = \\
& \int_{\Omega_2} \boldsymbol{\Phi}_3(|f_3(y)|)\left[\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\cdot\right.\right. \quad (2.22)\\
& \left.\left.\left\{\int_{\Omega_2} k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right\}d\mu_2(y)\right)d\mu_1(x)\right]d\mu_2(y) = \\
& \left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\cdot\right.\right.\\
& \left.\left.\left\{\int_{\Omega_2} k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right\}d\mu_2(y)\right)d\mu_1(x)\right] = \\
& \left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\left(\int_{\Omega_1}\theta(x)\,k_2(x,y)\,k_3(x,y)\,\boldsymbol{\Phi}_2(|f_2(y)|)\cdot\right.\right.\\
& \left.\left.\left\{\int_{\Omega_2} k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right\}d\mu_1(x)\right)d\mu_2(y)\right] = \quad (2.23)\\
& \left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\boldsymbol{\Phi}_2(|f_2(y)|)\left(\int_{\Omega_1}\theta(x)\,k_2(x,y)\,k_3(x,y)\cdot\right.\right.\\
& \left.\left.\left(\int_{\Omega_2} k_1(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right)d\mu_2(y)\right] = \\
& \left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left[\int_{\Omega_2}\boldsymbol{\Phi}_2(|f_2(y)|)\left\{\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\prod_{i=1}^{3}k_i(x,y)\cdot\right.\right.\right.\\
& \left.\left.\left.\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right\}d\mu_2(y)\right] = \\
& \left(\int_{\Omega_2}\boldsymbol{\Phi}_3(|f_3(y)|)\,d\mu_2(y)\right)\left(\int_{\Omega_2}\boldsymbol{\Phi}_2(|f_2(y)|)\,d\mu_2(y)\right)\cdot\\
& \left(\int_{\Omega_1}\left(\int_{\Omega_2}\theta(x)\prod_{i=1}^{3}k_i(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_2(y)\right)d\mu_1(x)\right) = \quad (2.24)\\
& \left(\prod_{i=2}^{3}\int_{\Omega_2}\boldsymbol{\Phi}_i(|f_i(y)|)\,d\mu_2(y)\right)\cdot\\
& \left(\int_{\Omega_2}\left(\int_{\Omega_1}\theta(x)\prod_{i=1}^{3}k_i(x,y)\,\boldsymbol{\Phi}_1(|f_1(y)|)\,d\mu_1(x)\right)d\mu_2(y)\right) = \\
& \left(\prod_{i=2}^{3}\int_{\Omega_2}\boldsymbol{\Phi}_i(|f_i(y)|)\,d\mu_2(y)\right)\cdot
\end{aligned}
$$

$$\left(\int_{\Omega_2}\Phi_1\left(|f_1(y)|\right)\left(\int_{\Omega_1}\theta(x)\prod_{i=1}^{3}k_i(x,y)\,d\mu_1(x)\right)d\mu_2(y)\right)=$$

$$\left(\prod_{i=2}^{3}\int_{\Omega_2}\Phi_i\left(|f_i(y)|\right)d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_1\left(|f_1(y)|\right)\lambda_3(y)\,d\mu_2(y)\right), \tag{2.25}$$

proving the claim. □

For general $m\in\mathbb{N}$, the following result is valid.

Theorem 2.4. *Assume that the function* $x\mapsto\left(\frac{u(x)\prod_{i=1}^{m}k_i(x,y)}{\prod_{i=1}^{m}K_i(x)}\right)$ *is integrable on* Ω_1, *for each* $y\in\Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y):=\int_{\Omega_1}\left(\frac{u(x)\prod_{i=1}^{m}k_i(x,y)}{\prod_{i=1}^{m}K_i(x)}\right)d\mu_1(x)<\infty. \tag{2.26}$$

Here $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,\ldots,m$, *are convex and increasing functions.*
Then

$$\int_{\Omega_1}u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right)d\mu_1(x)\leq \tag{2.27}$$

$$\left(\prod_{i=2}^{m}\int_{\Omega_2}\Phi_i\left(|f_i(y)|\right)d\mu_2(y)\right)\left(\int_{\Omega_2}\Phi_1\left(|f_1(y)|\right)\lambda_m(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i=1,\ldots,m$, $f_i:\Omega_2\to\mathbb{R}$ *such that:*

(i) f_i, $\Phi_i(|f_i|)$, *are both* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$.
(ii) $\lambda_m\Phi_1(|f_1|),\Phi_2(|f_2|),\Phi_3(|f_3|),\ldots,\Phi_m(|f_m|)$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10).*

When $k(x,y)=k_1(x,y)=k_2(x,y)=\ldots=k_m(x,y)$, then $K(x):=K_1(x)=K_2(x)=\ldots=K_m(x)$. Then from Theorem 2.4 we get:

Corollary 2.5. *Assume that the function* $x\mapsto\left(\frac{u(x)k^m(x,y)}{K^m(x)}\right)$ *is integrable on* Ω_1, *for each* $y\in\Omega_2$. *Define* U_m *on* Ω_2 *by*

$$U_m(y):=\int_{\Omega_1}\left(\frac{u(x)k^m(x,y)}{K^m(x)}\right)d\mu_1(x)<\infty. \tag{2.28}$$

Here $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,\ldots,m$, *are convex and increasing functions.*

Then

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{g_i(x)}{K(x)} \right| \right) d\mu_1(x) \leq \tag{2.29}$$

$$\left(\prod_{i=2}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1(|f_1(y)|) \, U_m(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $i = 1, \ldots, m$, $f_i : \Omega_2 \to \mathbb{R}$ such that:

(i) f_i, $\Phi_i(|f_i|)$, are both $k(x,y)\,d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$.
(ii) $U_m \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \Phi_m(|f_m|)$, are all μ_2 -integrable,

and for all corresponding functions g_i given by (2.10).

When $m = 2$ from Corollary 2.5, we obtain

Corollary 2.6. *Assume that the function $x \mapsto \left(\frac{u(x) k^2(x,y)}{K^2(x)} \right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define U_2 on Ω_2 by*

$$U_2(y) := \int_{\Omega_1} \left(\frac{u(x) k^2(x,y)}{K^2(x)} \right) d\mu_1(x) < \infty. \tag{2.30}$$

Here $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, are convex and increasing functions.
Then

$$\int_{\Omega_1} u(x) \, \Phi_1 \left(\left| \frac{g_1(x)}{K(x)} \right| \right) \Phi_2 \left(\left| \frac{g_2(x)}{K(x)} \right| \right) d\mu_1(x) \leq \tag{2.31}$$

$$\left(\int_{\Omega_2} \Phi_2(|f_2(y)|) \, d\mu_2(y) \right) \left(\int_{\Omega_2} \Phi_1(|f_1(y)|) \, U_2(y) \, d\mu_2(y) \right),$$

true for all measurable functions, $f_1, f_2 : \Omega_2 \to \mathbb{R}$ such that:

(i) f_1, f_2, $\Phi_1(|f_1|)$, $\Phi_2(|f_2|)$ are all $k(x,y)\,d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$.
(ii) $U_2 \Phi_1(|f_1|), \Phi_2(|f_2|)$, are both μ_2 -integrable,

and for all corresponding functions g_1, g_2 given by (2.10).

For $m \in \mathbb{N}$, the following more general result is also valid.

Theorem 2.7. *Let $j \in \{1, \ldots, m\}$ be fixed. Assume that the function $x \mapsto \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m on Ω_2 by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x) \prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)} \right) d\mu_1(x) < \infty. \tag{2.32}$$

Here $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, are convex and increasing functions.

Then

$$I := \int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq \tag{2.33}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m} \int_{\Omega_2} \Phi_i(|f_i(y)|)\, d\mu_2(y)\right)\left(\int_{\Omega_2} \Phi_j\left(\left|f_j(y)\right|\right) \lambda_m(y)\, d\mu_2(y)\right) := I_j,$$

true for all measurable functions, $i = 1,\ldots,m$, $f_i : \Omega_2 \to \mathbb{R}$ such that:

(i) *f_i, $\Phi_i(|f_i|)$, are both $k_i(x,y)\, d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$.*
(ii) *$\lambda_m \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \Phi_2(|f_2|), \Phi_3(|f_3|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$, are all μ_2 -integrable,*

and for all corresponding functions g_i given by (2.10). Above $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means missing item.

We make

Remark 2.8. In the notations and assumptions of Theorem 2.7, replace assumption (ii) by the assumption,

(iii) $\Phi_1(|f_1|), \ldots, \Phi_m(|f_m|); \lambda_m \Phi_1(|f_1|), \ldots, \lambda_m \Phi_m(|f_m|)$, are all μ_2 -integrable functions.

Then, clearly it holds,

$$I \leq \frac{\sum_{j=1}^{m} I_j}{m}. \tag{2.34}$$

An application of Theorem 2.7 follows.

Theorem 2.9. *Let $j \in \{1,\ldots,m\}$ be fixed. Assume that the function $x \mapsto \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right)$ is integrable on Ω_1, for each $y \in \Omega_2$. Define λ_m on Ω_2 by*

$$\lambda_m(y) := \int_{\Omega_1} \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right) d\mu_1(x) < \infty. \tag{2.35}$$

Then

$$\int_{\Omega_1} u(x)\, e^{\sum_{i=1}^{m}\left|\frac{g_i(x)}{K_i(x)}\right|} d\mu_1(x) \leq \tag{2.36}$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{m} \int_{\Omega_2} e^{|f_i(y)|} d\mu_2(y)\right)\left(\int_{\Omega_2} e^{\left|f_j(y)\right|} \lambda_m(y)\, d\mu_2(y)\right),$$

true for all measurable functions, $i = 1,\ldots,m$, $f_i : \Omega_2 \to \mathbb{R}$ such that:

(i) f_i, $e^{|f_i|}$, *are both* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_m e^{|f_j|}; e^{|f_1|}, e^{|f_2|}, e^{|f_3|}, \ldots, \widehat{e^{|f_j|}}, \ldots, e^{|f_m|}$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10). Above* $\widehat{e^{|f_j|}}$ *means absent item.*

Another application of Theorem 2.7 follows.

Theorem 2.10. *Let* $j \in \{1,\ldots,m\}$ *be fixed,* $\alpha \geq 1$. *Assume that the function* $x \mapsto \left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right)$ *is integrable on* Ω_1, *for each* $y \in \Omega_2$. *Define* λ_m *on* Ω_2 *by*

$$\lambda_m(y) := \int_{\Omega_1}\left(\frac{u(x)\prod_{i=1}^{m} k_i(x,y)}{\prod_{i=1}^{m} K_i(x)}\right) d\mu_1(x) < \infty. \tag{2.37}$$

Then

$$\int_{\Omega_1} u(x)\left(\prod_{i=1}^{m}\left|\frac{g_i(x)}{K_i(x)}\right|^{\alpha}\right) d\mu_1(x) \leq \tag{2.38}$$

$$\left(\prod_{\substack{i=1\\ i\neq j}}^{m}\int_{\Omega_2}|f_i(y)|^{\alpha}\,d\mu_2(y)\right)\left(\int_{\Omega_2}|f_j(y)|^{\alpha}\lambda_m(y)\,d\mu_2(y)\right),$$

true for all measurable functions, $i = 1,\ldots,m$, $f_i : \Omega_2 \to \mathbb{R}$ *such that:*

(i) $|f_i|^{\alpha}$ *is* $k_i(x,y)\,d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$.
(ii) $\lambda_m |f_j|^{\alpha}; |f_1|^{\alpha}, |f_2|^{\alpha}, |f_3|^{\alpha}, \ldots, \widehat{|f_j|^{\alpha}}, \ldots, |f_m|^{\alpha}$, *are all* μ_2 *-integrable,*

and for all corresponding functions g_i *given by (2.10). Above* $\widehat{|f_j|^{\alpha}}$ *means absent item.*

We make

Remark 2.11. Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+}^{\alpha_i}(|f_i|)\right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1,\ldots,m$, e.g., when $f_i \in L_\infty(a,b)$.
Consider

$$g_i(x) = \left(I_{a+}^{\alpha_i} f_i\right)(x), \quad x \in (a,b),\, i = 1,\ldots,m, \tag{2.39}$$

we remind

$$\left(I_{a+}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)}\int_a^x (x-t)^{\alpha_i - 1} f_i(t)\,dt.$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.
We see that

$$\left(I_{a+}^{\alpha_i} f\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(t)\,dt, \tag{2.40}$$

where χ stands for the characteristic function.

So, we pick here

$$k_i(x,t) := \frac{\chi_{(a,x]}(t)(x-t)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1, \ldots, m. \tag{2.41}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & a < y \leq x, \\ 0, & x < y < b. \end{cases} \tag{2.42}$$

Clearly it holds

$$K_i(x) = \int_{(a,b)} \frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)} dy = \frac{(x-a)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{2.43}$$

$a < x < b$, $i = 1, \ldots, m$.

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i + 1)}{(x-a)^{\alpha_i}} \right) =$$

$$\prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)(x-y)^{\alpha_i - 1} \alpha_i}{(x-a)^{\alpha_i}} \right) = \frac{\chi_{(a,x]}(y)(x-y)^{\left(\sum\limits_{i=1}^m \alpha_i - m\right)} \left(\prod\limits_{i=1}^m \alpha_i \right)}{(x-a)^{\left(\sum\limits_{i=1}^m \alpha_i\right)}}. \tag{2.44}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \;\; \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.45}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{(a,x]}(y)(x-y)^{\alpha - m} \gamma}{(x-a)^{\alpha}}. \tag{2.46}$$

Therefore, for (2.32), we get for appropriate weight u that

$$\lambda_m(y) = \gamma \int_y^b u(x) \frac{(x-y)^{\alpha - m}}{(x-a)^{\alpha}} dx < \infty, \tag{2.47}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{(x-a)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \le$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i (|f_i(x)|)\, dx \right) \left(\int_a^b \Phi_j \left(\left| f_j(x) \right| \right) \lambda_m(x)\, dx \right), \tag{2.48}$$

with $j \in \{1, \ldots, m\}$, true for measurable f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite ($i = 1, \ldots, m$) and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $\lambda_m \Phi_j \left(\left| f_j \right| \right); \Phi_1(|f_1|), \Phi_2(|f_2|), \ldots, \widehat{\Phi_j \left(\left| f_j \right| \right)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions,

where $\widehat{\Phi_j \left(\left| f_j \right| \right)}$ means absent item.
Let now

$$u(x) = (x-a)^\alpha, \ \ x \in (a,b). \tag{2.49}$$

Then

$$\lambda_m(y) = \gamma \int_y^b (x-y)^{\alpha - m}\, dx = \frac{\gamma (b-y)^{\alpha - m + 1}}{\alpha - m + 1}, \tag{2.50}$$

$y \in (a,b)$, where $\alpha > m - 1$.
Hence (2.48) becomes

$$\begin{aligned} &\int_a^b (x-a)^\alpha \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{(x-a)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \le \\ &\left(\frac{\gamma}{\alpha - m + 1} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right) \left(\int_a^b (b-x)^{\alpha - m + 1} \Phi_j \left(\left| f_j(x) \right| \right) dx \right) \le \\ &\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|)\, dx \right), \end{aligned} \tag{2.51}$$

where $\alpha > m - 1$, f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions (i), (ii) following (2.48).
If $\Phi_i = id$, then (2.51) turns to

$$\int_a^b \prod_{i=1}^m \left| \left(I_{a+}^{\alpha_i} f_i\right)(x) \right| dx \le$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^{m}\Gamma\left(\alpha_i+1\right)\right)\left(\alpha-m+1\right)}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\left|f_i\left(x\right)\right|dx\right)\cdot$$

$$\left(\int_a^b\left(b-x\right)^{\alpha-m+1}\left|f_j\left(x\right)\right|dx\right)\leq$$

$$\left(\frac{\gamma\left(b-a\right)^{\alpha-m+1}}{\left(\prod_{i=1}^{m}\Gamma\left(\alpha_i+1\right)\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_a^b\left|f_i\left(x\right)\right|dx\right), \tag{2.52}$$

where $\alpha>m-1$, f_i with $I_{a+}^{\alpha_i}\left(\left|f_i\right|\right)$ finite and f_i Lebesgue integrable, $i=1,\ldots,m$.

Next let $p_i>1$, and $\Phi_i\left(x\right)=x^{p_i}$, $x\in\mathbb{R}_+$. These Φ_i are convex, increasing, and continuous on $\mathbb{R}_+$.

Then, by (2.48), we get

$$I_1:=\int_a^b\left(x-a\right)^{\alpha}\prod_{i=1}^{m}\left|\frac{\left(I_{a+}^{\alpha_i}f_i\right)\left(x\right)}{\left(x-a\right)^{\alpha_i}}\right|^{p_i}dx\leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b\left|f_i\left(x\right)\right|^{p_i}dx\right)\cdot$$

$$\left(\int_a^b\left(b-x\right)^{\alpha-m+1}\left|f_j\left(x\right)\right|^{p_j}dx\right)\leq$$

$$\left(\frac{\gamma\left(b-a\right)^{\alpha-m+1}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_i+1\right)\right)^{p_i}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_a^b\left|f_i\left(x\right)\right|^{p_i}dx\right). \tag{2.53}$$

Notice that $\sum_{i=1}^{m}\alpha_ip_i>\alpha$; thus, $\beta:=\alpha-\sum_{i=1}^{m}\alpha_ip_i<0$. Since $0<x-a<b-a$ ($x\in(a,b)$), then $(x-a)^{\beta}>(b-a)^{\beta}$.

Therefore

$$I_1:=\int_a^b\left(x-a\right)^{\beta}\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx\geq$$

$$\left(b-a\right)^{\beta}\int_a^b\prod_{i=1}^{m}\left|\left(I_{a+}^{\alpha_i}f_i\right)\left(x\right)\right|^{p_i}dx. \tag{2.54}$$

Consequently, by (2.53) and (2.54), it holds

$$\int_a^b \prod_{i=1}^m \left|\left(I_{a+}^{\alpha_i} f_i\right)(x)\right|^{p_i} dx \leq \tag{2.55}$$

$$\left(\frac{\gamma(b-a)^{\left(\left(\sum\limits_{i=1}^m \alpha_i p_i\right)-m+1\right)}}{\left(\prod\limits_{i=1}^m (\Gamma(\alpha_i+1))^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i} dx\right),$$

where $p_i > 1$, $i = 1,..,m$, $\alpha > m-1$, true for measurable f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, with the properties ($i = 1,\ldots,m$):

(i) $|f_i|^{p_i}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

If $p = p_1 = p_2 = \ldots = p_m > 1$, then by (2.55), we get

$$\left\|\prod_{i=1}^m \left(I_{a+}^{\alpha_i} f_i\right)\right\|_{p,(a,b)} \leq \tag{2.56}$$

$$\left(\frac{\gamma^{\frac{1}{p}}(b-a)^{\left(\alpha-\frac{m}{p}+\frac{1}{p}\right)}}{\left(\prod\limits_{i=1}^m (\Gamma(\alpha_i+1))\right)(\alpha-m+1)^{\frac{1}{p}}}\right)\left(\prod_{i=1}^m \|f_i\|_{p,(a,b)}\right),$$

$\alpha > m-1$, true for measurable f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, and such that ($i = 1,\ldots,m$):

(i) $|f_i|^p$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^p$ is Lebesgue integrable on (a,b).

Using (ii) and if $\alpha_i > \frac{1}{p}$, by Hölder's inequality we derive that $I_{a+}^{\alpha_i}(|f_i|)$ is finite on (a,b). If we set $p = 1$ to (2.56) we get (2.52).

If $\Phi_i(x) = e^x$, $x \in \mathbb{R}_+$, then from (2.51) we get

$$\int_a^b (x-a)^\alpha e^{\sum\limits_{i=1}^m \left(\left|\frac{\left(I_{a+}^{\alpha_i} f_i\right)(x)}{(x-a)^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)} dx \leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\left(\prod_{i=1}^m \left(\int_a^b e^{|f_i(x)|} dx\right)\right), \tag{2.57}$$

where $\alpha > m-1$, f_i with $I_{a+}^{\alpha_i}(|f_i|)$ finite, $i = 1,\ldots,m$, under the assumptions:

(i) $e^{|f_i|}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $e^{|f_i|}$ is Lebesgue integrable on (a,b).

We continue with

Remark 2.12. Let f_i be Lebesgue measurable functions : $(a,b) \to \mathbb{R}$, such that $I_{b-}^{\alpha_i}(|f_i|)(x) < \infty$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1,\ldots,m$, e.g., when $f_i \in L_\infty(a,b)$.

Consider

$$g_i(x) = \left(I_{b-}^{\alpha_i} f_i\right)(x), \quad x \in (a,b), i = 1, \ldots, m, \tag{2.58}$$

we remind

$$\left(I_{b-}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b f_i(t)(t-x)^{\alpha_i - 1} dt, \tag{2.59}$$

$(x < b)$.

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{b-}^{\alpha_i} f_i\right)(x) = \int_a^b \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_i(t) dt. \tag{2.60}$$

So, we pick here

$$k_i(x,t) := \chi_{[x,b)}(t) \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, \quad i = 1, \ldots, m. \tag{2.61}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)}, & x \le y < b, \\ 0, & a < y < x. \end{cases} \tag{2.62}$$

Clearly it holds

$$K_i(x) = \int_{(a,b)} \chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} dy = \frac{(b-x)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{2.63}$$

$a < x < b$, $i = 1, \ldots, m$.

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i + 1)}{(b-x)^{\alpha_i}} \right) =$$

$$\prod_{i=1}^m \left(\chi_{[x,b)}(y) \frac{(y-x)^{\alpha_i - 1} \alpha_i}{(b-x)^{\alpha_i}} \right) = \chi_{[x,b)}(y) \frac{(y-x)^{\left(\sum_{i=1}^m \alpha_i - m\right)} \left(\prod_{i=1}^m \alpha_i\right)}{(b-x)^{\left(\sum_{i=1}^m \alpha_i\right)}}. \tag{2.64}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \ \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.65}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{[x,b)}(y)(y-x)^{\alpha - m} \gamma}{(b-x)^{\alpha}}. \tag{2.66}$$

Therefore, for (2.32), we get for appropriate weight u that

$$\lambda_m(y) = \gamma \int_a^y u(x) \frac{(y-x)^{\alpha-m}}{(b-x)^{\alpha}} dx < \infty, \tag{2.67}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\left| \frac{\left(I_{b-}^{\alpha_i} f_i\right)(x)}{(b-x)^{\alpha_i}} \right| \Gamma(\alpha_i + 1) \right) dx \leq$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|) \, dx \right) \left(\int_a^b \Phi_j\left(\left|f_j(x)\right|\right) \lambda_m(x) \, dx \right), \tag{2.68}$$

with $j \in \{1, \ldots, m\}$,

true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite ($i = 1, \ldots, m$) and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\lambda_m \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions,

where $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means absent item.

Let now

$$u(x) = (b-x)^{\alpha}, \quad x \in (a,b). \tag{2.69}$$

Then

$$\lambda_m(y) = \gamma \int_a^y (y-x)^{\alpha-m} dx = \frac{\gamma(y-a)^{\alpha-m+1}}{\alpha - m + 1}, \tag{2.70}$$

$y \in (a,b)$, where $\alpha > m - 1$.

Hence (2.68) becomes

$$\int_a^b (b-x)^{\alpha} \prod_{i=1}^m \Phi_i \left(\frac{\left|\left(I_{b-}^{\alpha_i} f_i\right)(x)\right|}{(b-x)^{\alpha_i}} \Gamma(\alpha_i + 1) \right) dx \leq$$

$$\left(\frac{\gamma}{\alpha - m + 1} \right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|) \, dx \right) \left(\int_a^b (x-a)^{\alpha-m+1} \Phi_j\left(\left|f_j(x)\right|\right) dx \right) \leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|) \, dx \right), \tag{2.71}$$

where $\alpha > m - 1$, f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions (i), (ii) following (2.68).

If $\Phi_i = id$, then (2.71) turns to

$$\int_a^b \prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right| dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^m \Gamma(\alpha_i+1) \right)(\alpha-m+1)} \right) \left(\prod_{\substack{i=1 \\ i\neq j}}^m \int_a^b |f_i(x)|\, dx \right) \cdot$$

$$\left(\int_a^b (x-a)^{\alpha-m+1} \left| f_j(x) \right| dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\left(\prod_{i=1}^m \Gamma(\alpha_i+1) \right)(\alpha-m+1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|\, dx \right), \tag{2.72}$$

where $\alpha > m-1$, f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite and f_i Lebesgue integrable, $i = 1, \ldots, m$.

Next let $p_i > 1$, and $\Phi_i(x) = x^{p_i}$, $x \in \mathbb{R}_+$.

Then, by (2.68), we get

$$I_2 := \int_a^b (b-x)^{\alpha} \frac{\left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right)}{(b-x)^{\sum_{i=1}^m \alpha_i p_i}} dx \leq$$

$$\left(\frac{\gamma}{\left(\prod_{i=1}^m (\Gamma(\alpha_i+1))^{p_i} \right)(\alpha-m+1)} \right) \left(\prod_{\substack{i=1 \\ i\neq j}}^m \int_a^b |f_i(x)|^{p_i}\, dx \right) \cdot$$

$$\left(\int_a^b (x-a)^{\alpha-m+1} \left| f_j(x) \right|^{p_j} dx \right) \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha-m+1}}{\left(\prod_{i=1}^m (\Gamma(\alpha_i+1))^{p_i} \right)(\alpha-m+1)} \right) \left(\prod_{i=1}^m \int_a^b |f_i(x)|^{p_i}\, dx \right). \tag{2.73}$$

Notice here that $\beta := \alpha - \sum_{i=1}^m \alpha_i p_i < 0$. Since $0 < b-x < b-a$ ($x \in (a,b)$), then $(b-x)^{\beta} > (b-a)^{\beta}$.

Therefore

$$I_2 := \int_a^b (b-x)^{\beta} \left(\prod_{i=1}^m \left| \left(I_{b-}^{\alpha_i} f_i \right)(x) \right|^{p_i} \right) dx \geq$$

$$(b-a)^{\beta}\int_a^b\left(\prod_{i=1}^m\left|\left(I_{b-}^{\alpha_i}f_i\right)(x)\right|^{p_i}\right)dx. \tag{2.74}$$

Consequently, by (2.73) and (2.74), it holds

$$\int_a^b\prod_{i=1}^m\left|\left(I_{b-}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx\leq \tag{2.75}$$

$$\left(\frac{\gamma(b-a)^{\left(\left(\sum_{i=1}^m\alpha_ip_i\right)-m+1\right)}}{\left(\prod_{i=1}^m(\Gamma(\alpha_i+1))^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^m\int_a^b|f_i(x)|^{p_i}dx\right),$$

where $p_i>1$, $i=1,..,m$, $\alpha>m-1$,

true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, with the properties ($i=1,\dots,m$):

(i) $|f_i|^{p_i}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

If $p:=p_1=p_2=\ldots=p_m>1$, then by (2.75), we get

$$\left\|\prod_{i=1}^m\left(I_{b-}^{\alpha_i}f_i\right)\right\|_{p,(a,b)}\leq \tag{2.76}$$

$$\left(\frac{\gamma^{\frac{1}{p}}(b-a)^{\left(\alpha-\frac{m}{p}+\frac{1}{p}\right)}}{\left(\prod_{i=1}^m(\Gamma(\alpha_i+1))\right)(\alpha-m+1)^{\frac{1}{p}}}\right)\left(\prod_{i=1}^m\|f_i\|_{p,(a,b)}\right),$$

$\alpha>m-1$, true for measurable f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, and such that ($i=1,\dots,m$):

(i) $|f_i|^p$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$.
(ii) $|f_i|^p$ is Lebesgue integrable on (a,b).

Using (ii) and if $\alpha_i>\frac{1}{p}$, by Hölder's inequality, we derive that $I_{b-}^{\alpha_i}(|f_i|)$ is finite on (a,b).

If we set $p=1$ to (2.76) we obtain (2.72).

If $\Phi_i(x)=e^x$, $x\in\mathbb{R}_+$, then from (2.71), we obtain

$$\int_a^b(b-x)^{\alpha}e^{\sum_{i=1}^m\left(\left|\frac{\left(I_{b-}^{\alpha_i}f_i\right)(x)}{(b-x)^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)}dx\leq$$

$$\left(\frac{\gamma(b-a)^{\alpha-m+1}}{\alpha-m+1}\right)\left(\prod_{i=1}^m\left(\int_a^b e^{|f_i(x)|}dx\right)\right), \tag{2.77}$$

where $\alpha>m-1$, f_i with $I_{b-}^{\alpha_i}(|f_i|)$ finite, $i=1,\dots,m$, under the assumptions:

(i) $e^{|f_i|}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$.
(ii) $e^{|f_i|}$ is Lebesgue integrable on (a,b).

We mention

Definition 2.13 ([1], p. 448). The left generalized Riemann–Liouville fractional derivative of f of order $\beta > 0$ is given by

$$D_a^{\beta} f(x) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dx}\right)^n \int_a^x (x-y)^{n-\beta-1} f(y)\, dy, \tag{2.78}$$

where $n = [\beta] + 1$, $x \in [a,b]$.

For $a, b \in \mathbb{R}$, we say that $f \in L_1(a,b)$ has an L_∞ fractional derivative $D_a^{\beta} f$ ($\beta > 0$) in $[a,b]$, if and only if:

(1) $D_a^{\beta-k} f \in C([a,b])$, $k = 2, \ldots, n = [\beta] + 1$
(2) $D_a^{\beta-1} f \in AC([a,b])$
(3) $D_a^{\beta} f \in L_\infty(a,b)$

Above we define $D_a^0 f := f$ and $D_a^{-\delta} f := I_{a+}^{\delta} f$, if $0 < \delta \leq 1$.

From [1, p. 449] and [9] we mention and use

Lemma 2.14. *Let $\beta > \alpha \geq 0$ and let $f \in L_1(a,b)$ have an L_∞ fractional derivative $D_a^{\beta} f$ in $[a,b]$ and let $D_a^{\beta-k} f(a) = 0$, $k = 1, \ldots, [\beta] + 1$, then*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D_a^{\beta} f(y)\, dy, \tag{2.79}$$

for all $a \leq x \leq b$.

Here $D_a^{\alpha} f \in AC([a,b])$ for $\beta - \alpha \geq 1$, and $D_a^{\alpha} f \in C([a,b])$ for $\beta - \alpha \in (0,1)$.

Notice here that

$$D_a^{\alpha} f(x) = \left(I_{a+}^{\beta-\alpha}\left(D_a^{\beta} f\right)\right)(x), \quad a \leq x \leq b. \tag{2.80}$$

We give

Theorem 2.15. *Let $f_i \in L_1(a,b)$, $\alpha_i, \beta_i : \beta_i > \alpha_i \geq 0$, $i = 1, \ldots, m$. Here (f_i, α_i, β_i) fulfill terminology and assumptions of Definition 2.13 and Lemma 2.14. Let $\overline{\alpha} := \sum_{i=1}^{m} (\beta_i - \alpha_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\beta_i - \alpha_i)$, assume $\overline{\alpha} > m - 1$, and $p \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(D_a^{\alpha_i} f_i\right) \right\|_{p,(a,b)} \leq \tag{2.81}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p}\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\beta_i - \alpha_i + 1)\right)\right) (\overline{\alpha} - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| D_a^{\beta_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (2.52) and (2.56). □

We continue with

Theorem 2.16. *All here as in Theorem 2.15. Then*

$$\int_a^b (x-a)^{\overline{\alpha}} e^{\sum_{i=1}^m \left(\left| \frac{\left(D_a^{\alpha_i} f_i\right)(x)}{(x-a)^{\left(\beta_i - \alpha_i\right)}} \right| \Gamma(\beta_i - \alpha_i + 1) \right)} dx \leq$$

$$\left(\frac{\overline{\gamma}(b-a)^{\overline{\alpha}-m+1}}{\overline{\alpha}-m+1} \right) \left(\prod_{i=1}^m \left(\int_a^b e^{\left| \left(D_a^{\beta_i} f_i\right)(x) \right|} dx \right) \right). \tag{2.82}$$

Proof. By (2.57), assumptions there (i) and (ii) are easily fulfilled. □

We need

Definition 2.17 ([6], p. 50, [1], p. 449). Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt$$

$$= \left(I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{2.83}$$

and it exists almost everywhere for $x \in [a,b]$, in fact $D_{*a}^{\nu} f \in L_1(a,b)$, ([1], p. 394).

We have $D_{*a}^{n} f = f^{(n)}$, $n \in \mathbb{Z}_+$.

We also need

Theorem 2.18 ([4]). *Let $\nu \geq \rho + 1$, $\rho > 0$, $\nu, \rho \notin \mathbb{N}$. Call $n := \lceil \nu \rceil$, $m^* := \lceil \rho \rceil$. Assume $f \in AC^n([a,b])$, such that $f^{(k)}(a) = 0$, $k = m^*, m^*+1, \ldots, n-1$, and $D_{*a}^{\nu} f \in L_\infty(a,b)$. Then $D_{*a}^{\rho} f \in AC([a,b])$ (where $D_{*a}^{\rho} f = \left(I_{a+}^{m^*-\rho} f^{(m^*)} \right)(x)$), and*

$$D_{*a}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_a^x (x-t)^{\nu-\rho-1} D_{*a}^{\nu} f(t)\, dt$$

$$= \left(I_{a+}^{\nu-\rho} \left(D_{*a}^{\nu} f \right) \right)(x), \tag{2.84}$$

$\forall\, x \in [a,b]$.

We give

Theorem 2.19. *Let (f_i, ν_i, ρ_i), $i = 1, \ldots, m$, $m \geq 2$, as in the assumptions of Theorem 2.18. Set $\overline{\alpha} := \sum_{i=1}^m (\nu_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^m (\nu_i - \rho_i)$, and let $p \geq 1$. Here $a, b \in \mathbb{R}$, $a < b$. Then*

$$\left\| \prod_{i=1}^m \left(D_{*a}^{\rho_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.85}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}}\left(b-a\right)^{\left(\overline{\alpha}-\frac{m}{p}+\frac{1}{p}\right)}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\nu_i-\rho_i+1\right)\right)\right)\left(\overline{\alpha}-m+1\right)^{\frac{1}{p}}}\right)\left(\prod_{i=1}^{m}\left\|D_{*a}^{\nu_i}f_i\right\|_{p,(a,b)}\right).$$

Proof. By (2.52) and (2.56), see here $\overline{\alpha}\geq m>m-1$. □

We also give

Theorem 2.20. *Here all as in Theorem 2.19, let $p_i\geq 1$, $i=1,\ldots,l$; $l<m$. Then*

$$\int_a^b (x-a)^{\left(\overline{\alpha}-\sum_{i=1}^{l}p_i(\nu_i-\rho_i)\right)}\left(\prod_{i=1}^{l}\left|D_{*a}^{\rho_i}f_i(x)\right|^{p_i}\right)\cdot$$

$$e^{\left(\sum_{i=l+1}^{m}\left|D_{*a}^{\rho_i}f_i(x)\right|\left(\frac{\Gamma(\nu_i-\rho_i+1)}{(x-a)^{(\nu_i-\rho_i)}}\right)\right)}dx\leq$$

$$\left(\frac{\overline{\gamma}(b-a)^{\overline{\alpha}-m+1}}{\left(\prod_{i=1}^{l}\left(\Gamma\left(\nu_i-\rho_i+1\right)\right)^{p_i}\right)\left(\overline{\alpha}-m+1\right)}\right)\left(\prod_{i=1}^{l}\int_a^b\left|D_{*a}^{\nu_i}f_i(x)\right|^{p_i}dx\right)\cdot \tag{2.86}$$

$$\left(\prod_{i=l+1}^{m}\int_a^b e^{\left|D_{*a}^{\nu_i}f_i(x)\right|}dx\right).$$

Proof. By (2.51). □

We need

Definition 2.21 ([2, 7, 8]). Let $\alpha\geq 0$, $n:=\lceil\alpha\rceil$, $f\in AC^n([a,b])$. We define the right Caputo fractional derivative of order $\alpha\geq 0$, by

$$\overline{D}_{b-}^{\alpha}f(x):=(-1)^n I_{b-}^{n-\alpha}f^{(n)}(x), \tag{2.87}$$

we set $\overline{D}_{-}^{0}f:=f$, i.e.,

$$\overline{D}_{b-}^{\alpha}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (J-x)^{n-\alpha-1}f^{(n)}(J)\,dJ. \tag{2.88}$$

Notice that $\overline{D}_{b-}^{n}f=(-1)^n f^{(n)}$, $n\in\mathbb{N}$.

In [3] we introduced a balanced fractional derivative combining both right and left fractional Caputo derivatives.

We need

Theorem 2.22 ([4]). *Let $f\in AC^n([a,b])$, $\alpha>0$, $n\in\mathbb{N}$, $n:=\lceil\alpha\rceil$, $\alpha\geq\rho+1$, $\rho>0$, $r=\lceil\rho\rceil$, $\alpha,\rho\notin\mathbb{N}$. Assume $f^{(k)}(b)=0$, $k=r,r+1,\ldots,n-1$, and $\overline{D}_{b-}^{\alpha}f\in L_\infty([a,b])$. Then*

$$\overline{D}_{b-}^{\rho} f(x) = \left(I_{b-}^{\alpha-\rho} \left(\overline{D}_{b-}^{\alpha} f \right) \right)(x) \in AC([a,b]), \tag{2.89}$$

that is,

$$\overline{D}_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\alpha-\rho)} \int_x^b (t-x)^{\alpha-\rho-1} \left(\overline{D}_{b-}^{\alpha} f \right)(t)\, dt, \tag{2.90}$$

$\forall\, x \in [a,b]$.

We give

Theorem 2.23. *Let (f_i, α_i, ρ_i), $i = 1, \ldots, m$, $m \geq 2$, as in the assumptions of Theorem 2.22. Set $\overline{\alpha} := \sum_{i=1}^{m} (\alpha_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^{m} (\alpha_i - \rho_i)$, and let $p \geq 1$. Here $a, b \in \mathbb{R}$, $a < b$. Then*

$$\left\| \prod_{i=1}^{m} \left(\overline{D}_{b-}^{\rho_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.91}$$

$$\left(\frac{\overline{\gamma}^{\frac{1}{p}} (b-a)^{\left(\overline{\alpha} - \frac{m}{p} + \frac{1}{p} \right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\alpha_i - \rho_i + 1) \right) \right) (\overline{\alpha} - m + 1)^{\frac{1}{p}}} \right) \left(\prod_{i=1}^{m} \left\| \overline{D}_{b-}^{\nu_i} f_i \right\|_{p,(a,b)} \right).$$

Proof. By (2.72) and (2.76), see here $\overline{\alpha} \geq m > m-1$. $\square$

We make

Remark 2.24. Let $r_1, r_2 \in \mathbb{N}$; $A_j > 0$, $j = 1, \ldots, r_1$; $B_j > 0$, $j = 1, \ldots, r_2$; $x \geq 0$, $p \geq 1$. Clearly $e^{A_j x^p}, e^{B_j x^p} \geq 1$, and $\sum_{j=1}^{r_1} e^{A_j x^p} \geq r_1$, $\sum_{j=1}^{r_2} e^{B_j x^p} \geq r_2$. Hence, $\varphi_1(x) := \ln\left(\sum_{j=1}^{r_1} e^{A_j x^p} \right)$, $\varphi_2(x) := \ln\left(\sum_{j=1}^{r_2} e^{B_j x^p} \right) \geq 0$. Clearly here $\varphi_1, \varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are increasing, convex, and continuous.

We give

Theorem 2.25. *Let (f_i, α_i, ρ_i), $i = 1, 2$, as in the assumptions of Theorem 2.22. Set $\overline{\alpha} := \sum_{i=1}^{2} (\alpha_i - \rho_i)$, $\overline{\gamma} := \prod_{i=1}^{2} (\alpha_i - \rho_i)$. Here $a, b \in \mathbb{R}$, $a < b$, and φ_1, φ_2 as in Remark 2.24. Then*

$$\int_a^b (b-x)^{\overline{\alpha}} \prod_{i=1}^{2} \varphi_i \left(\frac{\left| \overline{D}_{b-}^{\rho_i} f_i(x) \right|}{(b-x)^{(\alpha_i - \rho_i)}} \Gamma(\alpha_i - \rho_i + 1) \right) dx \leq \tag{2.92}$$

$$\left(\frac{\overline{\gamma} (b-a)^{\overline{\alpha}-1}}{\overline{\alpha} - 1} \right) \left(\prod_{i=1}^{2} \int_a^b \varphi_i \left(\left| \overline{D}_{b-}^{\alpha_i} f_i(x) \right| \right) dx \right),$$

under the assumptions ($i = 1, 2$):

(i) $\varphi_i\left(\left|\overline{D}_{b-}^{\alpha_i} f_i(t)\right|\right)$ *is* $\left(\chi_{[x,b)}(t)\,\frac{(t-x)^{\alpha_i-\rho_i-1}}{\Gamma(\alpha_i-\rho_i)}dt\right)$ *-integrable, a.e. in* $x\in(a,b)$.
(ii) $\varphi_i\left(\left|\overline{D}_{b-}^{\alpha_i} f_i\right|\right)$ *is Lebesgue integrable on* (a,b).

We make

Remark 2.26. (i) Let now $f\in C^n([a,b])$, $n=\lceil\nu\rceil$, $\nu>0$. Clearly $C^n([a,b])\subset AC^n([a,b])$. Assume $f^{(k)}(a)=0$, $k=0,1,\ldots,n-1$. Given that $D_{*a}^{\nu}f$ exists, then there exists the left generalized Riemann–Liouville fractional derivative $D_a^{\nu}f$ (see (2.78)) and $D_{*a}^{\nu}f=D_a^{\nu}f$ (see also [6], p. 53). In fact here $D_{*a}^{\nu}f\in C([a,b])$, see [6], p. 56.
So Theorems 2.19 and 2.20 can be true for left generalized Riemann–Liouville fractional derivatives.

(ii) Let also $\alpha>0$, $n:=\lceil\alpha\rceil$, and $f\in C^n([a,b])\subset AC^n([a,b])$. From [2] we derive that $\overline{D}_{b-}^{\alpha}f\in C([a,b])$. By [2], we obtain that the right Riemann–Liouville fractional derivative $D_{b-}^{\alpha}f$ exists on $[a,b]$. Furthermore if $f^{(k)}(b)=0$, $k=0,1,\ldots,n-1$, we get that $\overline{D}_{b-}^{\alpha}f(x)=D_{b-}^{\alpha}f(x)$, $\forall\ x\in[a,b]$; hence $D_{b-}^{\alpha}f\in C([a,b])$.

So Theorems 2.23 and 2.25 can be valid for right Riemann–Liouville fractional derivatives. To keep this article short we avoid details.

We give

Definition 2.27. Let $\nu>0$, $n:=[\nu]$, $\alpha:=\nu-n$ $(0\le\alpha<1)$. Let $a,b\in\mathbb{R}$, $a\le x\le b$, $f\in C([a,b])$. We consider $C_a^{\nu}([a,b]):=\{f\in C^n([a,b]):I_{a+}^{1-\alpha}f^{(n)}\in C^1([a,b])\}$. For $f\in C_a^{\nu}([a,b])$, we define the left generalized ν-fractional derivative of f over $[a,b]$ as

$$\Delta_a^{\nu}f:=\left(I_{a+}^{1-\alpha}f^{(n)}\right)'; \tag{2.93}$$

see [1], p. 24, and Canavati derivative in [5].
Notice here $\Delta_a^{\nu}f\in C([a,b])$.
So that

$$(\Delta_a^{\nu}f)(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{-\alpha}f^{(n)}(t)\,dt, \tag{2.94}$$

$\forall\ x\in[a,b]$.
Notice here that

$$\Delta_a^n f=f^{(n)},\quad n\in\mathbb{Z}_+. \tag{2.95}$$

We need

Theorem 2.28 ([4]). *Let* $f\in C_a^{\nu}([a,b])$, $n=[\nu]$, *such that* $f^{(i)}(a)=0$, $i=r,r+1,\ldots,n-1$, *where* $r:=[\rho]$, *with* $0<\rho<\nu$. *Then*

$$(\Delta_a^{\rho}f)(x)=\frac{1}{\Gamma(\nu-\rho)}\int_a^x(x-t)^{\nu-\rho-1}(\Delta_a^{\nu}f)(t)\,dt, \tag{2.96}$$

i.e.,

$$\left(\Delta_a^{\rho} f\right) = I_{a+}^{\nu-\rho}\left(\Delta_a^{\nu} f\right) \in C\left([a,b]\right). \tag{2.97}$$

Thus $f \in C_a^{\rho}\left([a,b]\right)$.

We present

Theorem 2.29. *Let* (f_i, ν_i, ρ_i), $i = 1, \ldots, m$, *as in Theorem 2.28 and fractional derivatives as in Definition 2.27. Let* $\alpha := \sum_{i=1}^{m} (\nu_i - \rho_i)$, $\gamma := \prod_{i=1}^{m} (\nu_i - \rho_i)$, $p_i \geq 1$, $i = 1, \ldots, m$, *assume* $\alpha > m - 1$. *Then*

$$\int_a^b \prod_{i=1}^{m} \left|\Delta_a^{\rho_i} f_i(x)\right|^{p_i} dx \leq \tag{2.98}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^{m} (\nu_i - \rho_i) p_i\right) - m + 1\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma\left(\nu_i - \rho_i + 1\right)\right)^{p_i}\right)(\alpha - m + 1)}\right)\left(\prod_{i=1}^{m} \int_a^b \left|\Delta_a^{\nu_i} f_i(x)\right|^{p_i} dx\right).$$

Proof. By (2.52) and (2.55). $\square$

We continue with

Theorem 2.30. *Let all here as in Theorem 2.29. Consider* λ_i, $i = 1, \ldots, m$, *distinct prime numbers. Then*

$$\int_a^b (x-a)^{\alpha} \prod_{i=1}^{m} \lambda_i^{\left(\left|\Delta_a^{\rho_i} f_i(x)\right| \frac{\Gamma(\nu_i - \rho_i + 1)}{(x-a)^{(\nu_i - \rho_i)}}\right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1}\right)\left(\prod_{i=1}^{m} \int_a^b \lambda_i^{\left|\Delta_a^{\nu_i} f_i(x)\right|} dx\right). \tag{2.99}$$

Proof. By (2.51). $\square$

We need

Definition 2.31 ([2]). Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $f \in C\left([a,b]\right)$. Consider

$$C_{b-}^{\nu}\left([a,b]\right) := \{f \in C^n\left([a,b]\right) : I_{b-}^{1-\alpha} f^{(n)} \in C^1\left([a,b]\right)\}. \tag{2.100}$$

Define the right generalized ν-fractional derivative of f over $[a,b]$, by

$$\Delta_{b-}^{\nu} f := (-1)^{n-1}\left(I_{b-}^{1-\alpha} f^{(n)}\right)'. \tag{2.101}$$

We set $\Delta_{b-}^{0} f = f$. Notice that

$$\left(\Delta_{b-}^{\nu} f\right)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (J-x)^{-\alpha} f^{(n)}(J)\, dJ, \tag{2.102}$$

and $\Delta_{b-}^{\nu} f \in C([a,b])$.

We also need

Theorem 2.32 ([4]). *Let $f \in C_{b-}^{\nu}([a,b])$, $0 < \rho < \nu$. Assume $f^{(i)}(b) = 0$, $i = r, r+1, \ldots, n-1$, where $r := [\rho]$, $n := [\nu]$. Then*

$$\Delta_{b-}^{\rho} f(x) = \frac{1}{\Gamma(\nu-\rho)} \int_x^b (J-x)^{\nu-\rho-1} \left(\Delta_{b-}^{\nu} f\right)(J)\, dJ, \tag{2.103}$$

$\forall\, x \in [a,b]$, i.e.,

$$\Delta_{b-}^{\rho} f = I_{b-}^{\nu-\rho}\left(\Delta_{b-}^{\nu} f\right) \in C([a,b]), \tag{2.104}$$

and $f \in C_{b-}^{\rho}([a,b])$.

We give

Theorem 2.33. *Let (f_i, ν_i, ρ_i), $i = 1, \ldots, m$, and fractional derivatives as in Theorem 2.32 and Definition 2.31. Let $\alpha := \sum_{i=1}^{m} (\nu_i - \rho_i)$, $\gamma := \prod_{i=1}^{m} (\nu_i - \rho_i)$, $p_i \geq 1$, $i = 1, \ldots, m$, and assume $\alpha > m-1$. Then*

$$\int_a^b \prod_{i=1}^{m} \left|\Delta_{b-}^{\rho_i} f_i(x)\right|^{p_i} dx \leq \tag{2.105}$$

$$\left(\frac{\gamma (b-a)^{\left(\left(\sum_{i=1}^{m} (\nu_i - \rho_i) p_i\right) - m + 1\right)}}{\left(\prod_{i=1}^{m} \left(\Gamma(\nu_i - \rho_i + 1)\right)^{p_i}\right)(\alpha - m + 1)} \right) \left(\prod_{i=1}^{m} \int_a^b \left|\Delta_{b-}^{\nu_i} f_i(x)\right|^{p_i} dx \right).$$

Proof. By (2.72) and (2.75). $\square$

We continue with

Theorem 2.34. *Let all here as in Theorem 2.33. Consider λ_i, $i = 1, \ldots, m$, distinct prime numbers. Then*

$$\int_a^b (b-x)^{\alpha} \prod_{i=1}^{m} \lambda_i^{\left(\left|\Delta_{b-}^{\rho_i} f_i(x)\right| \frac{\Gamma(\nu_i - \rho_i + 1)}{(b-x)^{(\nu_i - \rho_i)}} \right)} dx \leq$$

$$\left(\frac{\gamma (b-a)^{\alpha - m + 1}}{\alpha - m + 1} \right) \left(\prod_{i=1}^{m} \int_a^b \lambda_i^{\left|\Delta_{b-}^{\nu_i} f_i(x)\right|} dx \right). \tag{2.106}$$

Proof. By (2.71). $\square$

We make

Definition 2.35. [12, p. 99] The fractional integrals of a function f with respect to given function g are defined as follows:

Let $a,b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here g is an increasing function on $[a,b]$ and $g \in C^1([a,b])$. The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{(g(x)-g(t))^{1-\alpha}}, \quad x > a, \tag{2.107}$$

$$\left(I_{b-;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{(g(t)-g(x))^{1-\alpha}}, \quad x < b, \tag{2.108}$$

respectively.

We make

Remark 2.36. Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{a+;g}^{\alpha_i}(|f_i|)\right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1,\ldots,m$.

Consider

$$g_i(x) := \left(I_{a+;g}^{\alpha_i} f_i\right)(x), \quad x \in (a,b), i = 1,\ldots,m, \tag{2.109}$$

where

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_a^x \frac{g'(t) f_i(t) dt}{(g(x)-g(t))^{1-\alpha_i}}, \quad x > a. \tag{2.110}$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.

We see that

$$\left(I_{a+;g}^{\alpha_i} f_i\right)(x) = \int_a^b \frac{\chi_{(a,x]}(t) g'(t) f_i(t)}{\Gamma(\alpha_i)(g(x)-g(t))^{1-\alpha_i}} dt, \tag{2.111}$$

where χ is the characteristic function.

So, we pick here

$$k_i(x,t) := \frac{\chi_{(a,x]}(t) g'(t)}{\Gamma(\alpha_i)(g(x)-g(t))^{1-\alpha_i}}, \quad i = 1,\ldots,m. \tag{2.112}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}}, & a < y \leq x, \\ 0, & x < y < b. \end{cases} \tag{2.113}$$

Clearly it holds

$$K_i(x) = \int_a^b \frac{\chi_{(a,x]}(y) g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}} dy =$$

$$\int_a^x \frac{g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}}dy = \frac{1}{\Gamma(\alpha_i)}\int_a^x (g(x)-g(y))^{\alpha_i-1}dg(y) = \tag{2.114}$$
$$\frac{1}{\Gamma(\alpha_i)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha_i-1}dz = \frac{(g(x)-g(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

So for $a<x<b$, $i=1,\ldots,m$, we get

$$K_i(x) = \frac{(g(x)-g(a))^{\alpha_i}}{\Gamma(\alpha_i+1)}. \tag{2.115}$$

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{(a,x]}(y)\,g'(y)}{\Gamma(\alpha_i)(g(x)-g(y))^{1-\alpha_i}} \cdot \frac{\Gamma(\alpha_i+1)}{(g(x)-g(a))^{\alpha_i}}\right) =$$
$$\frac{\chi_{(a,x]}(y)\,(g(x)-g(y))^{\left(\sum_{i=1}^m \alpha_i - m\right)}(g'(y))^m\left(\prod_{i=1}^m \alpha_i\right)}{(g(x)-g(a))^{\left(\sum_{i=1}^m \alpha_i\right)}}. \tag{2.116}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0,\ \ \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.117}$$

we have that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{(a,x]}(y)\,(g(x)-g(y))^{\alpha-m}(g'(y))^m\gamma}{(g(x)-g(a))^{\alpha}}. \tag{2.118}$$

Therefore, for (2.32), we get for appropriate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y) = \gamma\big(g'(y)\big)^m \int_y^b u(x)\,\frac{(g(x)-g(y))^{\alpha-m}}{(g(x)-g(a))^{\alpha}}dx < \infty, \tag{2.119}$$

for all $a<y<b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i=1,\ldots,m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\left|\frac{\left(I_{a+;g}^{\alpha_i}f_i\right)(x)}{(g(x)-g(a))^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx \leq$$
$$\left(\prod_{\substack{i=1\\ i\neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right)\left(\int_a^b \Phi_j\left(\left|f_j(x)\right|\right)\lambda_m^g(x)\,dx\right), \tag{2.120}$$

with $j\in\{1,\ldots,m\}$,

true for measurable f_i with $I^{\alpha_i}_{a+;g}(|f_i|)$ finite, $i = 1, \ldots, m$, and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\lambda_m^g \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \Phi_2(|f_2|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions, where $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means absent item.

Let now

$$u(x) = (g(x) - g(a))^{\alpha} g'(x), \ \ x \in (a,b). \tag{2.121}$$

Then

$$\lambda_m^g(y) = \gamma\left(g'(y)\right)^m \int_y^b (g(x) - g(y))^{\alpha - m} g'(x)\,dx =$$

$$\gamma\left(g'(y)\right)^m \int_{g(y)}^{g(b)} (z - g(y))^{\alpha - m}\,dz = \tag{2.122}$$

$$\gamma\left(g'(y)\right)^m \frac{(g(b) - g(y))^{\alpha - m + 1}}{\alpha - m + 1},$$

with $\alpha > m - 1$. That is,

$$\lambda_m^g(y) = \gamma\left(g'(y)\right)^m \frac{(g(b) - g(y))^{\alpha - m + 1}}{\alpha - m + 1}, \tag{2.123}$$

$\alpha > m - 1$, $y \in (a,b)$.

Hence (2.120) becomes

$$\int_a^b g'(x)(g(x) - g(a))^{\alpha} \prod_{i=1}^m \Phi_i\left(\left|\frac{\left(I^{\alpha_i}_{a+;g} f_i\right)(x)}{(g(x) - g(a))^{\alpha_i}}\right| \Gamma(\alpha_i + 1)\right) dx \leq$$

$$\left(\frac{\gamma}{\alpha - m + 1}\right) \left(\prod_{\substack{i=1 \\ i \neq j}}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right) \cdot$$

$$\left(\int_a^b \left(g'(x)\right)^m (g(b) - g(x))^{\alpha - m + 1} \Phi_j\left(\left|f_j(x)\right|\right) dx\right) \leq \tag{2.124}$$

$$\left(\frac{\gamma (g(b) - g(a))^{\alpha - m + 1} \|g'\|_{\infty}^m}{\alpha - m + 1}\right) \left(\prod_{i=1}^m \int_a^b \Phi_i(|f_i(x)|)\,dx\right),$$

where $\alpha > m - 1$, f_i with $I^{\alpha_i}_{a+;g}(|f_i|)$ finite, $i = 1, \ldots, m$, under the assumptions:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\Phi_i(|f_i|)$ is Lebesgue integrable on (a,b).

If $\Phi_i(x) = x^{p_i}$, $p_i \geq 1$, $x \in \mathbb{R}_+$, then by (2.124), we have

$$\int_a^b g'(x)(g(x) - g(a))^{\left(\alpha - \sum_{i=1}^m p_i \alpha_i\right)} \prod_{i=1}^m \left|\left(I^{\alpha_i}_{a+;g} f_i\right)(x)\right|^{p_i} dx \leq \tag{2.125}$$

$$\left(\frac{\gamma\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-m+1}\left\|g'\right\|_{\infty}^{m}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_{i}+1\right)\right)^{p_{i}}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_{a}^{b}\left|f_{i}\left(x\right)\right|^{p_{i}}dx\right),$$

but we see that

$$\int_{a}^{b}g'\left(x\right)\left(g\left(x\right)-g\left(a\right)\right)^{\left(\alpha-\sum_{i=1}^{m}p_{i}\alpha_{i}\right)}\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_{i}}f_{i}\right)\left(x\right)\right|^{p_{i}}dx\geq$$

$$\left(g\left(b\right)-g\left(a\right)\right)^{\left(\alpha-\sum_{i=1}^{m}p_{i}\alpha_{i}\right)}\int_{a}^{b}g'\left(x\right)\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_{i}}f_{i}\right)\left(x\right)\right|^{p_{i}}dx. \tag{2.126}$$

By (2.125) and (2.126) we get

$$\int_{a}^{b}g'\left(x\right)\prod_{i=1}^{m}\left|\left(I_{a+;g}^{\alpha_{i}}f_{i}\right)\left(x\right)\right|^{p_{i}}dx\leq \tag{2.127}$$

$$\left(\frac{\gamma\left(g\left(b\right)-g\left(a\right)\right)^{\left(\sum_{i=1}^{m}p_{i}\alpha_{i}-m+1\right)}\left\|g'\right\|_{\infty}^{m}}{\left(\prod_{i=1}^{m}\left(\Gamma\left(\alpha_{i}+1\right)\right)^{p_{i}}\right)\left(\alpha-m+1\right)}\right)\left(\prod_{i=1}^{m}\int_{a}^{b}\left|f_{i}\left(x\right)\right|^{p_{i}}dx\right),$$

$\alpha>m-1$, f_i with $I_{a+;g}^{\alpha_i}\left(\left|f_i\right|\right)$ finite, $i=1,\ldots,m$, under the assumptions:

(i) $\left|f_i\right|^{p_i}$ is $k_i\left(x,y\right)dy$ -integrable, a.e. in $x\in\left(a,b\right)$.
(ii) $\left|f_i\right|^{p_i}$ is Lebesgue integrable on $\left(a,b\right)$.

We need

Definition 2.37 ([11]). Let $0<a<b<\infty$, $\alpha>0$. The left- and right-sided Hadamard fractional integrals of order α are given by

$$\left(J_{a+}^{\alpha}f\right)\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left(\ln\frac{x}{y}\right)^{\alpha-1}\frac{f\left(y\right)}{y}dy,\quad x>a, \tag{2.128}$$

and

$$\left(J_{b-}^{\alpha}f\right)\left(x\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{b}\left(\ln\frac{y}{x}\right)^{\alpha-1}\frac{f\left(y\right)}{y}dy,\quad x<b, \tag{2.129}$$

respectively.

Notice that the Hadamard fractional integrals of order α are special cases of left- and right-sided fractional integrals of a function f with respect to another function, here $g\left(x\right)=\ln x$ on $\left[a,b\right]$, $0<a<b<\infty$.

Above f is a Lebesgue measurable function from $\left(a,b\right)$ into $\mathbb{R}$, such that $\left(J_{a+}^{\alpha}\left(\left|f\right|\right)\right)\left(x\right)$ and/or $\left(J_{b-}^{\alpha}\left(\left|f\right|\right)\right)\left(x\right)\in\mathbb{R}$, $\forall\, x\in\left(a,b\right)$.

We give

Theorem 2.38. *Let* (f_i, α_i), $i = 1, \ldots, m$; $J_{a+}^{\alpha_i} f_i$ *as in Definition 2.37. Set* $\alpha := \sum_{i=1}^{m} \alpha_i$, $\gamma := \prod_{i=1}^{m} \alpha_i$; $p_i \geq 1$, $i = 1, \ldots, m$, *assume* $\alpha > m - 1$. *Then*

$$\int_a^b \prod_{i=1}^{m} \left| \left(J_{a+}^{\alpha_i} f_i \right)(x) \right|^{p_i} dx \leq \tag{2.130}$$

$$\left(\frac{b\gamma \left(\ln \left(\frac{b}{a} \right) \right)^{\left(\sum_{i=1}^{m} p_i \alpha_i - m + 1 \right)}}{a^m (\alpha - m + 1) \left(\prod_{i=1}^{m} (\Gamma(\alpha_i + 1))^{p_i} \right)} \right) \left(\prod_{i=1}^{m} \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $J_{a+}^{\alpha_i}(|f_i|)$ *is finite,* $i = 1, \ldots, m$, *under the assumptions:*

(i) $|f_i(y)|^{p_i}$ *is* $\left(\frac{\chi_{(a,x]}(y) dy}{\Gamma(\alpha_i) y \left(\ln\left(\frac{x}{y} \right) \right)^{1-\alpha_i}} \right)$ *-integrable, a.e. in* $x \in (a,b)$.

(ii) $|f_i|^{p_i}$ *is Lebesgue integrable on* (a,b).

We also present

Theorem 2.39. *Let all as in Theorem 2.38. Consider* $p := p_1 = p_2 = \ldots = p_m \geq 1$. *Then*

$$\left\| \prod_{i=1}^{m} \left(J_{a+}^{\alpha_i} f_i \right) \right\|_{p,(a,b)} \leq \tag{2.131}$$

$$\left(\frac{(b\gamma)^{\frac{1}{p}} \left(\ln \left(\frac{b}{a} \right) \right)^{\left(\alpha - \frac{m}{p} + \frac{1}{p} \right)}}{a^{\frac{m}{p}} (\alpha - m + 1)^{\frac{1}{p}} \left(\prod_{i=1}^{m} (\Gamma(\alpha_i + 1)) \right)} \right) \left(\prod_{i=1}^{m} \|f_i\|_{p,(a,b)} \right),$$

where $J_{a+}^{\alpha_i}(|f_i|)$ *is finite,* $i = 1, \ldots, m$, *under the assumptions:*

(i) $|f_i(y)|^p$ *is* $\left(\frac{\chi_{(a,x]}(y) dy}{\Gamma(\alpha_i) y \left(\ln\left(\frac{x}{y} \right) \right)^{1-\alpha_i}} \right)$ *-integrable, a.e. in* $x \in (a,b)$.

(ii) $|f_i|^p$ *is Lebesgue integrable on* (a,b).

We make

Remark 2.40. Let f_i be Lebesgue measurable functions from (a,b) into $\mathbb{R}$, such that $\left(I_{b-;g}^{\alpha_i}(|f_i|) \right)(x) \in \mathbb{R}$, $\forall\, x \in (a,b)$, $\alpha_i > 0$, $i = 1, \ldots, m$.

Consider

$$g_i(x) := \left(I_{b-;g}^{\alpha_i} f_i \right)(x), \; x \in (a,b), i = 1, \ldots, m, \tag{2.132}$$

where

$$\left(I_{b-;g}^{\alpha_i} f_i \right)(x) = \frac{1}{\Gamma(\alpha_i)} \int_x^b \frac{g'(t) f(t) dt}{(g(t) - g(x))^{1-\alpha_i}}, \;\; x < b. \tag{2.133}$$

Notice that $g_i(x) \in \mathbb{R}$ and it is Lebesgue measurable.

We pick $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(y) = dy$, the Lebesgue measure.
We see that

$$\left(I_{b-;g}^{\alpha_i} f_i\right)(x) = \int_a^b \frac{\chi_{[x,b)}(t)\, g'(t)\, f(t)\, dt}{\Gamma(\alpha_i)(g(t)-g(x))^{1-\alpha_i}}, \tag{2.134}$$

where χ is the characteristic function.
So, we pick here

$$k_i(x,y) := \frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}}, \quad i = 1,\ldots,m. \tag{2.135}$$

In fact

$$k_i(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}}, & x \le y < b, \\ 0, \quad a < y < x. \end{cases} \tag{2.136}$$

Clearly it holds

$$K_i(x) = \int_a^b \frac{\chi_{[x,b)}(y)\, g'(y)\, dy}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}} =$$

$$\frac{1}{\Gamma(\alpha_i)} \int_x^b g'(y)(g(y)-g(x))^{\alpha_i - 1} dy = \tag{2.137}$$

$$\frac{1}{\Gamma(\alpha_i)} \int_{g(x)}^{g(b)} (z-g(x))^{\alpha_i-1} dg(y) = \frac{(g(b)-g(x))^{\alpha_i}}{\Gamma(\alpha_i+1)}.$$

So for $a < x < b$, $i = 1,\ldots,m$, we get

$$K_i(x) = \frac{(g(b)-g(x))^{\alpha_i}}{\Gamma(\alpha_i+1)}. \tag{2.138}$$

Notice that

$$\prod_{i=1}^m \frac{k_i(x,y)}{K_i(x)} = \prod_{i=1}^m \left(\frac{\chi_{[x,b)}(y)\, g'(y)}{\Gamma(\alpha_i)(g(y)-g(x))^{1-\alpha_i}} \cdot \frac{\Gamma(\alpha_i+1)}{(g(b)-g(x))^{\alpha_i}} \right) =$$

$$\frac{\chi_{[x,b)}(y)\,(g'(y))^m (g(y)-g(x))^{\left(\sum_{i=1}^m \alpha_i - m\right)} \prod_{i=1}^m \alpha_i}{(g(b)-g(x))^{\sum_{i=1}^m \alpha_i}}. \tag{2.139}$$

Calling

$$\alpha := \sum_{i=1}^m \alpha_i > 0, \;\; \gamma := \prod_{i=1}^m \alpha_i > 0, \tag{2.140}$$

we have that

$$\prod_{i=1}^{m} \frac{k_i(x,y)}{K_i(x)} = \frac{\chi_{[x,b)}(y)(g'(y))^m (g(y)-g(x))^{\alpha-m}\gamma}{(g(b)-g(x))^{\alpha}}. \tag{2.141}$$

Therefore, for (2.32), we get for appropriate weight u that (denote λ_m by λ_m^g)

$$\lambda_m^g(y) = \gamma\left(g'(y)\right)^m \int_a^y u(x) \frac{(g(y)-g(x))^{\alpha-m}}{(g(b)-g(x))^{\alpha}} dx < \infty, \tag{2.142}$$

for all $a < y < b$.

Let $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, be convex and increasing functions. Then by (2.33) we obtain

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\left| \frac{\left(I_{b-;g}^{\alpha_i} f_i\right)(x)}{(g(b)-g(x))^{\alpha_i}} \right| \Gamma(\alpha_i+1) \right) dx \leq$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{m} \int_a^b \Phi_i(|f_i(x)|)\, dx \right) \left(\int_a^b \Phi_j\left(\left|f_j(x)\right|\right) \lambda_m^g(x)\, dx \right), \tag{2.143}$$

with $j \in \{1, \ldots, m\}$,

true for measurable f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i = 1, \ldots, m$, and with the properties:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\, dy$ -integrable, a.e. in $x \in (a,b)$.

(ii) $\lambda_m^g \Phi_j\left(\left|f_j\right|\right); \Phi_1(|f_1|), \ldots, \widehat{\Phi_j\left(\left|f_j\right|\right)}, \ldots, \Phi_m(|f_m|)$ are all Lebesgue integrable functions, where $\widehat{\Phi_j\left(\left|f_j\right|\right)}$ means absent item.

Let now

$$u(x) = (g(b)-g(x))^{\alpha} g'(x), \quad x \in (a,b). \tag{2.144}$$

Then

$$\lambda_m^g(y) = \gamma\left(g'(y)\right)^m \int_a^y g'(x)(g(y)-g(x))^{\alpha-m} dx =$$

$$\gamma\left(g'(y)\right)^m \int_a^y (g(y)-g(x))^{\alpha-m} dg(x) = \gamma\left(g'(y)\right)^m \int_{g(a)}^{g(y)} (g(y)-z)^{\alpha-m} dz =$$

$$\gamma\left(g'(y)\right)^m \frac{(g(y)-g(a))^{\alpha-m+1}}{\alpha-m+1}, \tag{2.145}$$

with $\alpha > m-1$. That is,

$$\lambda_m^g(y) = \gamma\left(g'(y)\right)^m \frac{(g(y)-g(a))^{\alpha-m+1}}{\alpha-m+1}, \tag{2.146}$$

$\alpha > m-1$, $y \in (a,b)$.

Hence (2.143) becomes

$$\int_a^b g'(x)(g(b)-g(x))^{\alpha}\prod_{i=1}^{m}\Phi_i\left(\left|\frac{\left(I_{b-;g}^{\alpha_i}f_i\right)(x)}{(g(b)-g(x))^{\alpha_i}}\right|\Gamma(\alpha_i+1)\right)dx \leq$$

$$\left(\frac{\gamma}{\alpha-m+1}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{m}\int_a^b \Phi_i(|f_i(x)|)\,dx\right)\cdot$$

$$\left(\int_a^b \Phi_j\left(\left|f_j(x)\right|\right)\left(g'(x)\right)^m(g(x)-g(a))^{\alpha-m+1}dx\right) \leq \tag{2.147}$$

$$\left(\frac{\gamma(g(b)-g(a))^{\alpha-m+1}\|g'\|_{\infty}^{m}}{\alpha-m+1}\right)\left(\prod_{i=1}^{m}\int_a^b \Phi_i(|f_i(x)|)\,dx\right),$$

where $\alpha > m-1$, f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i=1,\ldots,m$, under the assumptions:

(i) $\Phi_i(|f_i|)$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x\in(a,b)$.
(ii) $\Phi_i(|f_i|)$ is Lebesgue integrable on (a,b).

If $\Phi_i(x) = x^{p_i}$, $p_i \geq 1$, $x\in\mathbb{R}_+$, then by (2.147), we have

$$\int_a^b g'(x)(g(b)-g(x))^{\left(\alpha-\sum_{i=1}^{m}\alpha_i p_i\right)}\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx \leq \tag{2.148}$$

$$\left(\frac{\gamma(g(b)-g(a))^{\alpha-m+1}(\|g'\|_{\infty})^m}{(\alpha-m+1)\prod_{i=1}^{m}(\Gamma(\alpha_i+1))^{p_i}}\right)\left(\prod_{i=1}^{m}\int_a^b |f_i(x)|^{p_i}dx\right),$$

but we see that

$$\int_a^b g'(x)(g(b)-g(x))^{\left(\alpha-\sum_{i=1}^{m}\alpha_i p_i\right)}\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx \geq$$

$$(g(b)-g(a))^{\left(\alpha-\sum_{i=1}^{m}\alpha_i p_i\right)}\int_a^b g'(x)\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx. \tag{2.149}$$

Hence by (2.148) and (2.149) we derive

$$\int_a^b g'(x)\prod_{i=1}^{m}\left|\left(I_{b-;g}^{\alpha_i}f_i\right)(x)\right|^{p_i}dx \leq \tag{2.150}$$

$$\left(\frac{\gamma(g(b)-g(a))^{\left(\sum_{i=1}^{m}p_i\alpha_i-m+1\right)}\|g'\|_{\infty}^{m}}{\left(\prod_{i=1}^{m}(\Gamma(\alpha_i+1))^{p_i}\right)(\alpha-m+1)}\right)\left(\prod_{i=1}^{m}\int_a^b |f_i(x)|^{p_i}dx\right),$$

$\alpha > m-1$, f_i with $I_{b-;g}^{\alpha_i}(|f_i|)$ finite, $i=1,\ldots,m$, under the assumptions:

(i) $|f_i|^{p_i}$ is $k_i(x,y)\,dy$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

We give

Theorem 2.41. *Let (f_i, α_i), $i = 1, \ldots, m$; $J_{b-}^{\alpha_i} f_i$ as in Definition 2.37. Set $\alpha := \sum_{i=1}^{m} \alpha_i$, $\gamma := \prod_{i=1}^{m} \alpha_i$; $p_i \geq 1$, $i = 1, \ldots, m$, assume $\alpha > m-1$. Then*

$$\int_a^b \prod_{i=1}^{m} \left| \left(J_{b-}^{\alpha_i} f_i\right)(x) \right|^{p_i} dx \leq \tag{2.151}$$

$$\left(\frac{b\gamma \left(\ln\left(\frac{b}{a}\right)\right)^{\left(\sum_{i=1}^{m} p_i\alpha_i - m + 1\right)}}{a^m (\alpha - m + 1) \left(\prod_{i=1}^{m} (\Gamma(\alpha_i + 1))^{p_i} \right)} \right) \left(\prod_{i=1}^{m} \int_a^b |f_i(x)|^{p_i} dx \right),$$

where $J_{b-}^{\alpha_i}(|f_i|)$ is finite, $i = 1, \ldots, m$, under the assumptions:

(i) $|f_i(y)|^{p_i}$ is $\left(\frac{\chi_{[x,b)}(y)dy}{\Gamma(\alpha_i)y\left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_i}} \right)$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p_i}$ is Lebesgue integrable on (a,b).

We finish with

Theorem 2.42. *Let all as in Theorem 2.41. Take $p := p_1 = p_2 = \ldots = p_m \geq 1$. Then*

$$\left\| \prod_{i=1}^{m} \left(J_{b-}^{\alpha_i} f_i\right) \right\|_{p,(a,b)} \leq \tag{2.152}$$

$$\left(\frac{(b\gamma)^{\frac{1}{p}} \left(\ln\left(\frac{b}{a}\right)\right)^{\left(\alpha - \frac{m}{p} + \frac{1}{p}\right)}}{a^{\frac{m}{p}} (\alpha - m + 1)^{\frac{1}{p}} \left(\prod_{i=1}^{m} (\Gamma(\alpha_i + 1)) \right)} \right) \left(\prod_{i=1}^{m} \|f_i\|_{p,(a,b)} \right),$$

where $J_{b-}^{\alpha_i}(|f_i|)$ is finite, $i = 1, \ldots, m$, under the properties:

(i) $|f_i(y)|^{p}$ is $\left(\frac{\chi_{[x,b)}(y)dy}{\Gamma(\alpha_i)y\left(\ln\left(\frac{y}{x}\right)\right)^{1-\alpha_i}} \right)$ -integrable, a.e. in $x \in (a,b)$.
(ii) $|f_i|^{p}$ is Lebesgue integrable on (a,b).

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