Transcendental Methods in the Study of Algebraic Cycles with a Special Emphasis on Calabi–Yau Varieties

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Abstract We review the transcendental aspects of algebraic cycles, and explain how this relates to Calabi–Yau varieties. More precisely, after presenting a general overview, we begin with some rudimentary aspects of Hodge theory and algebraic cycles. We then introduce Deligne cohomology, as well as the generalized higher cycles due to Bloch that are connected to higher K-theory, and associated regulators. Finally, we specialize to the Calabi–Yau situation, and explain some recent developments in the field.

Key words: Calabi–Yau variety, Algebraic cycle, Abel–Jacobi map, Regulator, Deligne cohomology, Chow group

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1 Introduction

These notes concern that part of Calabi–Yau geometry that involves algebraic cycles—typically built up from special subvarieties, such as rational points and rational curves. From these algebraic cycles, one forms various doubly indexed groups, called higher Chow groups, that mimic simplicial homology theory in algebraic topology. These Chow groups come equipped with various maps whose target space is a certain transcendental cohomology theory called Deligne cohomology.

More precisely these maps are called regulators, from the higher cycle groups of S. Bloch, denoted by $\mathrm{CH}^k(X,m)$, of a projective algebraic manifold X, to Deligne cohomology, viz.:

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$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m) \to H^{2r-m}_{\mathscr{D}}(X,\mathbf{A}(r)), \tag{1}$$

where $\mathbf{A} \subseteq \mathbf{R}$ is a subring, $\mathbf{A}(r) := \mathbf{A}(2\pi\mathrm{i})^r$ is called the "Tate twist", and as we will indicate below, some striking evidence that these regulator maps become highly interesting in the case where X is Calabi–Yau. As originally discussed in [39], we are interested in the following case scenarios, with the intention of also providing an update on new developments. For the moment we will consider $\mathbf{A} = \mathbf{Z}$; however we will also consider $\mathbf{A} = \mathbf{Q},\ \mathbf{R}$ later on.

When $m = 0$, the objects of interest are the null homologous codimension 2 (= dimension 1) cycles $\mathrm{CH}^2_{\mathrm{hom}}(X) = \mathrm{CH}_{1,\mathrm{hom}}(X)$ on a projective threefold X, and where in this case, (1) becomes the Abel–Jacobi map:

$$\Phi_2 : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X) = \frac{H^3(X,\mathbf{C})}{F^2H^3(X,\mathbf{C}) + H^3(X,\mathbf{Z}(2))} \simeq \frac{\{H^{3,0}(X) \oplus H^{2,1}(X)\}^\vee}{H_3(X,\mathbf{Z}(1))}, \tag{2}$$

defined by a process of integration, $J^2(X)$ being the Griffiths jacobian of X. One of the reasons for introducing the Abel–Jacobi map is to study the Griffiths group $\mathrm{Griff}^2(X) \otimes \mathbf{Q}$. If we put $\mathrm{CH}^r_{\mathrm{alg}}(X)$ to be codimension r cycles algebraically equivalent to zero, then the Griffiths group is given by $\mathrm{Griff}^r(X) := \mathrm{CH}^r_{\mathrm{hom}}(X)/\mathrm{CH}^r_{\mathrm{alg}}(X)$.

When $m = 1$, the object of interest is the group

$$\mathrm{CH}^2(X,1) = \frac{\left\{\sum_{j,\mathrm{cd}_X Z_j = 1} (f_j, Z_j) \,\middle|\, \begin{matrix} f_j \in \mathbf{C}(Z_j)^\times \\ \sum_j \mathrm{div}(f_j) = 0 \end{matrix}\right\}}{\mathrm{Image}(\text{Tame symbol})},$$

on a projective algebraic surface X. If we mod out by the subgroup of $\mathrm{CH}^2(X,1)$ where the f_j's $\in \mathbf{C}^\times$, then we arrive at the quotient group of indecomposables $\mathrm{CH}^2_{\mathrm{ind}}(X,1)$ which plays an analogous role to the Griffiths group above. Moreover if we assume that the torsion part of $H^3(X,\mathbf{Z})$ is zero, then in this case (1) becomes a map:

$$\underline{\mathrm{cl}}_{2,1} : \mathrm{CH}^2_{\mathrm{ind}}(X,1) \to \frac{[H^{2,0}(X) \oplus H^{1,1}_{\mathrm{tr}}(X)]^\vee}{H_2(X,\mathbf{Z})}, \tag{3}$$

where $H^{1,1}_{\mathrm{tr}}(X)$ is the transcendental part of $H^{1,1}(X)$, being the orthogonal complement to the subgroup of algebraic cocycles.

In the case $m = 2$, the objects of interest are the group of symbols:

$$\mathrm{CH}^2(X,2) = \left\{\xi := \prod_j \{f_j, g_j\} \,\middle|\, \begin{matrix} f_j, g_j \in \mathbf{C}(X)^\times \\ \sum_{j,p\in X}\left((-1)^{\nu_p(f_j)\nu_p(g_j)}\left(\frac{f_j^{\nu_p(g_j)}}{g_j^{\nu_p(f_j)}}\right)(p), p\right) = 0 \end{matrix}\right\},$$

(ν_p = order of vanishing at p), on a smooth projective curve X. In this case (1) becomes the regulator:

$$\mathrm{cl}_{2,2} : \mathrm{CH}^2(X,2) \to H^1(X,\mathbf{C}/\mathbf{Z}(2)). \tag{4}$$

As first pointed out in [39], if X is a smooth projective variety of dimension d, where $1 \leq d \leq 3$, then the maps and objects

- $\mathrm{cl}_{2,2}$ in (4) and $\mathrm{CH}^2(X, 2) \otimes \mathbf{Q}$ for $d = 1$
- $\underline{\mathrm{cl}}_{2,1}$ in (3) and $\mathrm{CH}^2_{\mathrm{ind}}(X, 1) \otimes \mathbf{Q}$ for $d = 2$
- Φ_2 in (2) and $\mathrm{Griff}^2(X) \otimes \mathbf{Q}$ for $d = 3$

become especially interesting and generally nontrivial in the case where X is a Calabi–Yau variety; moreover, in a sense that will be specified later, these maps are essentially "trivial" when restricted to indecomposables, for X either of "lower or higher order" to its Calabi–Yau counterpart. Cycle constructions involving nodal rational curves and torsion points, play a prominent role here.

Several recent developments in the context of algebraic cycles are included in these notes since the appearance of [39], which should be of interest to specialists. Having said this, these notes are prepared with the expressed interest in enticing a wider group of researchers into the subject.

We have benefited from conversations with Matt Kerr, Bruno Kahn and Xi Chen. We are also grateful to Bruno for sharing with us his preprint [29]. We owe the referee a debt of gratitude for doing a splendid job in recommending improvements and catching errors in an earlier version of this paper. We are also pleased that the referee made us aware of the interesting work of Friedman–Laza [20], and for raising the very interesting question of how to construct normal functions over the Calabi–Yau variations of Hodge structure that they construct.

2 Notation

Throughout these notes, and unless otherwise specified, $X = X/\mathbf{C}$ is a projective algebraic manifold, of dimension d. A projective algebraic manifold is the same thing as a smooth complex projective variety. If $V \subseteq X$ is an irreducible subvariety of X, then $\mathbf{C}(V)$ is the rational function field of V, with multiplicative group $\mathbf{C}(V)^\times$. Depending on the context (which will be made abundantly clear in the text), $\mathcal{O}_X$ will either be the sheaf of germs of holomorphic functions on X in the analytic topology, or the sheaf of germs of regular functions in the Zariski topology.

3 Some Hodge Theory

Some useful reference material for this section is [25, 36].

Let E^k_X = $\mathbf{C}$-valued C^∞ k-forms on X. (One could also use the common notation of $A^k(X)$ for C^∞ forms, but let's not.) We have the decomposition:

$$E^k_X = \bigoplus_{p+q=k} E^{p,q}_X, \quad \overline{E^{p,q}_X} = E^{q,p}_X,$$

where $E_X^{p,q}$ are the C^∞ (p,q)-forms which in local holomorphic coordinates $z = (z_1, \dots, z_n) \in X$, are of the form:

$$\sum_{|I|=p,|J|=q} f_{IJ} dz_I \wedge d\overline{z}_J, \quad f_{IJ} \textit{ are } \mathbf{C} - \textit{valued } C^\infty \textit{functions},$$

$$I = 1 \le i_1 < \cdots < i_p \le d, \quad J = 1 \le j_1 < \cdots < j_q \le d,$$

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

One has the differential $d : E_X^k \to E_X^{k+1}$, and we define

$$H^k_{\mathrm{DR}}(X, \mathbf{C}) = \frac{\ker d : E_X^k \to E_X^{k+1}}{dE_X^{k-1}}.$$

The operator d decomposes into $d = \partial + \overline{\partial}$, where $\partial : E_X^{p,q} \to E_X^{p+1,q}$ and $\overline{\partial} : E_X^{p,q} \to E_X^{p,q+1}$. Further $d^2 = 0 \Rightarrow \partial^2 = \overline{\partial}^2 = 0 = \partial\overline{\partial} + \overline{\partial}\partial$, by (p,q) type.

The above decomposition descends to the cohomological level, viz.,

Theorem 3.1 (Hodge decomposition).

$$H^k_{\mathrm{sing}}(X, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} \simeq H^k_{\mathrm{DR}}(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) = d$*-closed* (p,q)*-forms (modulo coboundaries), and*

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

Furthermore:

$$H^{p,q}(X) \simeq \frac{E^{p,q}_{X,d-\mathrm{closed}}}{\partial\overline{\partial}E_X^{p-1,q-1}}.$$

Some more terminology: *Hodge filtration.* Put

$$F^k H^i(X, \mathbf{C}) = \bigoplus_{p \ge k} H^{p,i-p}(X).$$

Now recall $\dim X = d$.

Theorem 3.2 (Poincaré and Serre duality). *The following pairings induced by*

$$(w_1, w_2) \mapsto \int_X w_1 \wedge w_2,$$

are non-degenerate:

$$H^k_{\mathrm{DR}}(X, \mathbf{C}) \times H^{2d-k}_{\mathrm{DR}}(X, \mathbf{C}) \to \mathbf{C},$$

$$H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbf{C}.$$

Therefore $H^k(X) \simeq H^{2d-k}(X)^\vee,\ H^{p,q}(X) \simeq H^{d-p,d-q}(X)^\vee$

Corollary 3.3.

$$\frac{H^i(X,\mathbf{C})}{F^rH^i(X,\mathbf{C})} \simeq F^{d-r+1}H^{2d-i}(X,\mathbf{C})^\vee.$$

3.4 Formalism of Mixed Hodge Structures

Definition 3.5. Let $\mathbf{A} \subset \mathbf{R}$ be a subring. An $\mathbf{A}$-Hodge structure (HS) of weight $N \in \mathbf{Z}$ is given by the following datum:

• A finitely generated $\mathbf{A}$-module V, and either of the two equivalent statements below:

$\bullet_1$ A decomposition

$$V_{\mathbf{C}} = \bigoplus_{p+q=N} V^{p,q}, \quad \overline{V^{p,q}} = V^{q,p},$$

where $^-$ is complex conjugation induced from conjugation on the second factor $\mathbf{C}$ of $V_{\mathbf{C}} := V \otimes \mathbf{C}$.

$\bullet_2$ A finite descending filtration

$$V_{\mathbf{C}} \supset \cdots \supset F^r \supset F^{r-1} \supset \cdots \supset \{0\},$$

satisfying

$$V_{\mathbf{C}} = F^r \bigoplus \overline{F^{N-r+1}}, \ \forall\, r \in \mathbf{Z}.$$

Remark 3.6. The equivalence of $\bullet_1$ and $\bullet_2$ can be seen as follows. Given the decomposition in $\bullet_1$, put

$$F^rV_{\mathbf{C}} = \bigoplus_{p+q=N, p\geq r} V^{p,q}.$$

Conversely, given $\{F^r\}$ in $\bullet_2$, put $V^{p,q} = F^p \bigcap \overline{F^q}$.

Example 3.7. $X/\mathbf{C}$ smooth projective. Then $H^i(X,\mathbf{Z})$ is a $\mathbf{Z}$-Hodge structure of weight i.

Example 3.8. $\mathbf{A}(k) := (2\pi \mathrm{i})^k\mathbf{A}$ is an $\mathbf{A}$-Hodge structure of weight $-2k$ and of pure Hodge type $(-k,-k)$, called the Tate twist.

Example 3.9. $X/\mathbf{C}$ smooth projective. Then $H^i(X,\mathbf{Q}(k)) := H^i(X,\mathbf{Q}) \otimes \mathbf{Q}(k)$ is a $\mathbf{Q}$-Hodge structure of weight $i-2k$.

To extend these ideas to singular varieties, one requires the following terminology.

Definition 3.10. An $\mathbf{A}$-mixed Hodge structure ($\mathbf{A}$-MHS) is given by the following datum:

- A finitely generated $\mathbf{A}$-module $V_{\mathbf{A}}$,
- A finite descending "Hodge" filtration on $V_{\mathbf{C}} := V_{\mathbf{A}} \otimes \mathbf{C}$,

$$V_{\mathbf{C}} \supset \cdots \supset F^r \supset F^{r-1} \supset \cdots \supset \{0\},$$

- An increasing "weight" filtration on $V_{\mathbf{A}} \otimes \mathbf{Q} := V_{\mathbf{A}} \otimes_{\mathbf{Z}} \mathbf{Q}$,

$$\{0\} \subset \cdots \subset W_{\ell-1} \subset W_\ell \subset \cdots \subset V_{\mathbf{A}} \otimes \mathbf{Q},$$

such that $\{F^r\}$ induces a (pure) HS of weight ℓ on $Gr_\ell^W := W_\ell / W_{\ell-1}$.

Theorem 3.11 (Deligne [16]). *Let Y be a complex variety. Then $H^i(Y, \mathbf{Z})$ has a canonical and functorial $\mathbf{Z}$-MHS.*

Remark 3.12. (i) A morphism $h : V_{1,\mathbf{A}} \to V_{2,\mathbf{A}}$ of $\mathbf{A}$-MHS is an $\mathbf{A}$-linear map satisfying:

- $h(W_\ell V_{1,\mathbf{A}\otimes\mathbf{Q}}) \subseteq W_\ell V_{2,\mathbf{A}\otimes\mathbf{Q}}, \quad \forall\, \ell,$
- $h(F^r V_{1,\mathbf{C}}) \subseteq F^r V_{2,\mathbf{C}}, \quad \forall\, r.$

Deligne ([16] (Theorem 2.3.5)) shows that the category of $\mathbf{A}$-MHS is abelian; in particular if $h : V_{1,\mathbf{A}} \to V_{2,\mathbf{A}}$ is a morphism of $\mathbf{A}$-MHS, then $\ker(h)$, $\mathrm{coker}(h)$ are endowed with the induced filtrations. Let us further assume that $\mathbf{A} \otimes \mathbf{Q}$ is a field. Then Deligne (*op. cit.*) shows that h is strictly compatible[1] with the filtrations $W_\bullet$ and $F^\bullet$, and that the functors $V \mapsto Gr_\ell^W V$, $V \mapsto Gr_F^r V$ are exact.

(ii) Roughly speaking, the functoriality of the MHS in Deligne's theorem translates to the following yoga: the "standard" exact sequences in singular (co)homology, together with push-forwards and pullbacks by morphisms (wherever permissible) respect MHS. In particular for a subvariety $Y \subset X$, the localization cohomology sequence associated to the pair (X, Y) is a long exact sequence of MHS. Here is where the Tate twist comes into play: Suppose that $Y \subset X$ is an inclusion of projective algebraic manifolds with $\mathrm{codim}_X Y = r \geq 1$. One has a Gysin map $H^{i-2r}(Y, \mathbf{Q}) \to H^i(X, \mathbf{Q})$ which involves Hodge structures of different weights. To remedy this, one considers the induced map $H^{i-2r}(Y, \mathbf{Q}(-r)) \to H^i(X, \mathbf{Q}(0)) = H^i(X, \mathbf{Q})$ via (twisted) Poincaré duality (see (5) below), which is a morphism of pure Hodge structures (hence of MHS). A simple proof of this fact can be found in Sect. 7 of [36]. Note that the morphism $H_Y^i(X, \mathbf{Q}) \to H^i(X, \mathbf{Q})$ is a morphism of MHS, and that accordingly $H_Y^i(X, \mathbf{Q}) \simeq H^{i-2r}(Y, \mathbf{Q}(-r))$ is an isomorphism of MHS (with Y still smooth).

[1] Strict compatibility means that $h(F^r V_{1,\mathbf{C}}) = h(V_{1,\mathbf{C}}) \cap F^r V_{2,\mathbf{C}}$ and $h(W_\ell V_{1,\mathbf{A}\otimes\mathbf{Q}}) = h(V_{1,\mathbf{A}\otimes\mathbf{Q}}) \cap W_\ell V_{2,\mathbf{A}\otimes\mathbf{Q}}$ for all r and ℓ. A nice explanation of Deligne's proof of this fact can be found in [44], where a quick summary goes as follows: For any $\mathbf{A}$-MHS V, $V_{\mathbf{C}}$ has a $\mathbf{C}$-splitting into a bigraded direct sum of complex vector spaces $I^{p,q} := F^p \cap W_{p+q} \cap [\overline{F^q} \cap W_{p+q} + \sum_{i\geq 2} \overline{F^{q-i+1}} \cap W_{p+q-i}]$, where one shows that $F^r V_{\mathbf{C}} = \oplus_{p \geq r} \oplus_q I^{p,q}$ and $W_\ell V_{\mathbf{C}} = \oplus_{p+q \leq \ell} I^{p,q}$. Then by construction of $I^{p,q}$, one has $h(I^{p,q}(V_{1,\mathbf{C}}) \subseteq I^{p,q}(V_{2,\mathbf{C}})$. Hence h preserves both the Hodge and complexified weight filtrations. Now use the fact that $\mathbf{A} \otimes \mathbf{Q}$ is a field to deduce that h preserves the weight filtration over $\mathbf{A} \otimes \mathbf{Q}$.

Example 3.13. Let $\overline{U}$ be a compact Riemann surface, $\Sigma \subset \overline{U}$ a finite set of points, and put $U := \overline{U}\backslash\Sigma$. According to Deligne, $H^1(U, \mathbf{Z}(1))$ carries a $\mathbf{Z}$-MHS. The Hodge filtration on $H^1(U, \mathbf{C})$ is defined in terms of a filtered complex of holomorphic differentials on U with logarithmic poles along Σ ([16], but also see (10) below). One can "observe" the MHS as follows. Poincaré duality gives us $H^1_{\Sigma}(\overline{U}, \mathbf{Z}) \simeq H_1(\Sigma, \mathbf{Z}) = 0$, and the localization sequence in cohomology below is a sequence of MHS:

$$0 \to H^1(\overline{U}, \mathbf{Z}(1)) \to H^1(U, \mathbf{Z}(1)) \to H^0(\Sigma, \mathbf{Z}(0))^{\circ} \to 0,$$

where

$$H^0(\Sigma, \mathbf{Z}(0))^{\circ} := \ker\left(H^2_{\Sigma}(\overline{U}, \mathbf{Z}(1)) \to H^2(\overline{U}, \mathbf{Z}(1))\right) \simeq \mathbf{Z}(0)^{|\Sigma|-1}.$$

Put $W_0 = H^1(U, \mathbf{Z}(1))$, $W_{-1} = \mathrm{Im}(H^1(\overline{U}, \mathbf{Z}(1)) \to H^1(U, \mathbf{Z}(1)))$, $W_{-2} = 0$. Then $Gr^W_{-1} H^1(U, \mathbf{Z}(1)) \simeq H^1(\overline{U}, \mathbf{Z}(1))$ has pure weight -1 and $Gr^W_0 H^1(U, \mathbf{Z}(1)) \simeq \mathbf{Z}(0)^{|\Sigma|-1}$ has pure weight 0.

The following notation will be introduced:

Definition 3.14. Let V be an $\mathbf{A}$-MHS. We put

$$\Gamma_{\mathbf{A}} V := \hom_{\mathbf{A}-\mathrm{MHS}}(\mathbf{A}(0), V),$$

and

$$J_{\mathbf{A}}(V) = \mathrm{Ext}^1_{\mathbf{A}-\mathrm{MHS}}(\mathbf{A}(0), V).$$

In the case where $\mathbf{A} = \mathbf{Z}$ or $\mathbf{A} = \mathbf{Q}$, we simply put $\Gamma = \Gamma_{\mathbf{A}}$ and $J = J_{\mathbf{A}}$.

Example 3.15. Suppose that $V = V_{\mathbf{Z}}$ is a $\mathbf{Z}$ (pure) HS of weight $2r$. Then $V(r) := V \otimes \mathbf{Z}(r)$ is of weight 0, and (up to the twist) one can identify ΓV with $V_{\mathbf{Z}} \cap F^r V_{\mathbf{C}} = V_{\mathbf{Z}} \cap V^{r,r} := \epsilon^{-1}(V^{r,r})$, where $\epsilon : V \to V_{\mathbf{C}}$.

Example 3.16. Let V be a $\mathbf{Z}$-MHS. There is the identification due to J. Carlson (see [8, 28]),

$$J(V) \simeq \frac{W_0 V_{\mathbf{C}}}{F^0 W_0 V_{\mathbf{C}} + W_0 V},$$

where in the denominator term, $V := V_{\mathbf{Z}}$ is identified with its image $V_{\mathbf{Z}} \to V_{\mathbf{C}}$ (viz., quotienting out torsion). For example, if $\{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbf{Z}(0), V)$ corresponds to the short exact sequence of MHS:

$$0 \to V \to E \xrightarrow{\alpha} \mathbf{Z}(0) \to 0,$$

then one can find $x \in W_0 E$ and $y \in F^0 W_0 E_{\mathbf{C}}$ such that $\alpha(x) = \alpha(y) = 1$. Then $x - y \in V_{\mathbf{C}}$ descends to a class in $W_0 V_{\mathbf{C}} / \{F^0 W_0 V_{\mathbf{C}} + W_0 V\}$, which defines the map from $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbf{Z}(0), V)$ to $W_0 V_{\mathbf{C}} / \{F^0 W_0 V_{\mathbf{C}} + W_0 V\}$.

4 Algebraic Cycles

For the next two sections, the reader may find it helpful to consult [37]. Recall $X/\mathbf{C}$ smooth projective, $\dim X = d$. For $0 \leq r \leq d$, put $z^r(X)$ $(= z_{d-r}(X))$ = free abelian group generated by subvarieties of codim r (= dim $d - r$) in X.

Example 4.1. (i) $z^d(X) = z_0(X) = \{\sum_{j=1}^M n_j p_j \mid n_j \in \mathbf{Z},\ p_j \in X\}$.
(ii) $z^0(X) = z_d(X) = \mathbf{Z}\{X\} \simeq \mathbf{Z}$.
(iii) Let $X_1 := V(z_2^2 z_0 - z_1^3 - z_0 z_1^2) \subset \mathbf{P}^2$, and $X_2 := V(z_2^2 z_0 - z_1^3 - z_1 z_0^2) \subset \mathbf{P}^2$. Then $3X_1 - 5X_2 \in z^1(\mathbf{P}^2) = z_1(\mathbf{P}^2)$.
(iv) $\operatorname{codim}_X V = r - 1$, $f \in \mathbf{C}(V)^\times$. $\operatorname{div}(f) := (f) := (f)_0 - (f)_\infty \in z^r(X)$ (principal divisor). (Note: $\operatorname{div}(f)$ is easy to define, by first passing to a normalization $\tilde{V}$ of V, then using the fact that the local ring $\mathscr{O}_{\tilde{V},\wp}$ of regular functions at $\wp$ is a discrete valuation ring for a codimension one "point" $\wp$ on $\tilde{V}$, together with the proper push-forward associated to $\tilde{V} \to V$.)

Divisors in (iv) generate a subgroup,

$$z^r_{\text{rat}}(X) \subset z^r(X),$$

which defines the rational equivalence relation on $z^r(X)$.

Definition 4.2.

$$\mathrm{CH}^r(X) := z^r(X)/z^r_{\text{rat}}(X),$$

is called the r-th Chow group of X.

Remark 4.3. On can show that $\xi \in z^r_{\text{rat}}(X) \Leftrightarrow\ \exists\, w \in z^r(\mathbf{P}^1 \times X)$, each component of the support $|w|$ flat over $\mathbf{P}^1$, such that $\xi = w[0] - w[\infty]$. (Here $w[t] := \mathrm{pr}_{2,*}(\langle \mathrm{pr}_1^*(t) \bullet w\rangle_{\mathbf{P}^1 \times X})$.) If one replaces $\mathbf{P}^1$ by any choice of smooth connected curve Γ (not fixed!) and $0,\ \infty$ by any 2 points $P,\ Q \in \Gamma$, then one obtains the subgroup $z^r_{\text{alg}}(X) \subset z^r(X)$ of cycles that are algebraically equivalent to zero.[2] There is the fundamental class map (described later) $z^r(X) \to H^{2r}(X, \mathbf{Z})$ whose kernel is denoted by $z^r_{\text{hom}}(X)$. More precisely, the target space and map requires some twisting, viz.,

$$z^r(X) \to H^{2r}(X, \mathbf{Z}(r)).$$

To explain the role of twisting here, we illustrate this with three case scenarios.

- Let $f : Y \to X$ be a morphism of smooth projective varieties, where $\dim Y = \dim X - 1$. One has a commutative diagram of cycle class maps:

$$\begin{array}{ccc} z^{r-1}(Y) & \to & H^{2(r-1)}(Y, \mathbf{Z}(r-1)) \\ f_*\big\downarrow & & \big\downarrow f_* \\ z^r(X) & \to & H^{2r}(X, \mathbf{Z}(r)) \end{array}$$

[2] The fact that a smooth connected Γ will suffice (as opposed to a [connected] chain of curves) in the definition of algebraic equivalence follows from the transitive property of algebraic equivalence (see [36] (p. 180)).

Thus from the perspective of (mixed) Hodge theory, this diagram is "natural", as the right hand vertical arrow is a morphism of (M)HS.

- Let $U/\mathbf{C}$ be a smooth quasi-projective variety of dimension d, and $Y \subset U$ a closed algebraic subset. Using the twisted Poincaré duality theory formalism in this situation (see [28] (p. 82, p. 92)), Poincaré duality gives us an isomorphism of MHS:

$$H^i_Y(U, \mathbf{Z}(j)) \simeq H_{2d-i}(Y, \mathbf{Z}(d-j)) := H_{2d-i}(Y, \mathbf{Z})(j-d),$$

where $H_i(Y, \mathbf{Z}) := H_i^{BM}(Y, \mathbf{Z})$ is Borel–Moore homology.[3] For example if $U = Y = X$ is smooth projective, then $H^i(X, \mathbf{Z}(j))$ is a pure HS of weight $i-2j$, and $H_a(X, \mathbf{Z}(b)) := H_a(X, \mathbf{Z})(-b)$ is known to be a pure HS of weight $2b-a$, hence $H_{2d-i}(Y, \mathbf{Z}(d-j))$ has weight $2(d-j)-(2d-i) = i-2j$. Thus

$$H^i(X, \mathbf{Z}(j)) \simeq H_{2d-i}(X, \mathbf{Z}(d-j)), \tag{5}$$

is an isomorphism of HS.

Remark 4.4. Although tempting, from a "purist" point of view, it would be a mistake to interpret $H_a(X, \mathbf{Z}(b)) = H_a(X, \mathbf{Z})(b)$. This would imply that the Poincaré duality isomorphism in (5) would not preserve weights, and hence not an isomorphism of (M)HS in the sense given in Remark 3.12.

- Let $\mathscr{O}_X$ be the sheaf of analytic functions on X. Recall the exponential short exact sequence of sheaves

$$0 \to \mathbf{Z} \to \mathscr{O}_X \xrightarrow{\exp(2\pi \mathrm{i}\cdot(-))} \mathscr{O}_X^\times \to 0,$$

where $\mathscr{O}_X^\times \subset \mathscr{O}_X$ is the sheaf of units. It is well-known that $H^1(X, \mathscr{O}_X^\times) \simeq \mathrm{CH}^1(X)$, and hence there is an induced Chern class map $\mathrm{CH}^1(X) \to H^2(X, \mathbf{Z})$. But this is not so natural as there is no canonical choice of i. Instead, one considers

$$0 \to \mathbf{Z}(1) \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 0,$$

and accordingly the induced cycle class map $\mathrm{CH}^1(X) \to H^2(X, \mathbf{Z}(1))$.

One has inclusions:

$$z^r_{\mathrm{rat}}(X) \subseteq z^r_{\mathrm{alg}}(X) \subseteq z^r_{\mathrm{hom}}(X) \subset z^r(X).$$

Definition 4.5. Put

(i) $\mathrm{CH}^r_{\mathrm{alg}}(X) := z^r_{\mathrm{alg}}(X)/z^r_{\mathrm{rat}}(X)$,
(ii) $\mathrm{CH}^r_{\mathrm{hom}}(X) := z^r_{\mathrm{hom}}(X)/z^r_{\mathrm{rat}}(X)$,
(iii) $\mathrm{Griff}^r(X) := z^r_{\mathrm{hom}}(X)/z^r_{\mathrm{alg}}(X) = \mathrm{CH}^r_{\mathrm{hom}}(X)/\mathrm{CH}^r_{\mathrm{alg}}(X)$, called the Griffiths group.

The Griffiths group is known to be trivial in the cases $r = 0,\ 1,\ d$.

[3] We remind the reader that for singular homology $H_*^{sing}(U, \mathbf{Z})$ and ignoring twists, Poincaré duality gives the isomorphism $H^i_c(U, \mathbf{Z}) \simeq H^{sing}_{2d-i}(U, \mathbf{Z})$, where $H^i_c(U, \mathbf{Z})$ is cohomology with compact support; whereas $H^i(U, \mathbf{Z}) \simeq H^{BM}_{2d-i}(U, \mathbf{Z})$.

4.6 Generalized Cycles

The basic idea is this:

$$\mathrm{CH}^r(X) = \mathrm{Coker}\Big(\bigoplus_{\mathrm{cd}_X V=r-1} \mathbf{C}(V)^\times \xrightarrow{\mathrm{div}} z^r(X)\Big).$$

In the context of Milnor K-theory, this is just

$$\underbrace{\Big(\to \cdots \bigoplus_{\mathrm{cd}_X V=r-2} K_2^M(\mathbf{C}(V))\Big) \xrightarrow{\mathrm{Tame}} \bigoplus_{\mathrm{cd}_X V=r-1}}_{\text{building a complex on the left}} K_1^M(\mathbf{C}(V)) \xrightarrow{\mathrm{div}} \bigoplus_{\mathrm{cd}_X V=r} K_0^M(\mathbf{C}(V)).$$

For a field $\mathbf{F}$, one has the Milnor K-groups $K_\bullet^M(\mathbf{F})$, where $K_0^M(\mathbf{F}) = \mathbf{Z}$, $K_1^M(\mathbf{F}) = \mathbf{F}^\times$ and

$$K_2^M(\mathbf{F}) = \Big\{\text{Symbols } \{a,b\} \;\Big|\; a,b \in \mathbf{F}^\times\Big\}\Big/\!\!\Big/\left\{\begin{matrix}\text{Steinberg relations}\\ \{a_1a_2,b\} = \{a_1,b\}\{a_2,b\}\\ \{a,b\} = \{b,a\}^{-1}\\ \{a,1-a\} = 1,\ a \neq 1\end{matrix}\right\}.$$

One has a Gersten–Milnor resolution of a sheaf of Milnor K-groups on X, which leads to a complex whose last three terms and corresponding homologies (indicated at $\updownarrow$) for $0 \le m \le 2$ are:

$$\begin{matrix}\bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbf{C}(Z)) & \xrightarrow{T} & \bigoplus_{\mathrm{cd}_X Z=r-1} \mathbf{C}(Z)^\times & \xrightarrow{\mathrm{div}} & \bigoplus_{\mathrm{cd}_X Z=r} \mathbf{Z}\\ \updownarrow & & \updownarrow & & \updownarrow\\ \mathrm{CH}^r(X,2) & & \mathrm{CH}^r(X,1) & & \mathrm{CH}^r(X,0)\end{matrix} \tag{6}$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map. The Tame symbol map

$$T : \bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{cd}_X D=r-1} K_1^M(\mathbf{C}(D)),$$

is defined as follows. First $K_2^M(\mathbf{C}(Z))$ is generated by symbols $\{f,g\}$, $f,\ g \in \mathbf{C}(Z)^\times$.

For $f,\ g \in \mathbf{C}(Z)^\times$,

$$T(\{f,g\}) = \sum_D (-1)^{\nu_D(f)\nu_D(g)}\Big(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\Big)_D,$$

where $(\cdots)_D$ means restriction to the generic point of D, and ν_D represents order of a zero or pole along an irreducible divisor $D \subset Z$.

Example 4.7. Taking cohomology of the complex in (6), we have:

(i) $\mathrm{CH}^r(X,0) = z^r(X)/z^r_{\mathrm{rat}}(X) =: \mathrm{CH}^r(X)$.
(ii) $\mathrm{CH}^r(X,1)$ is represented by classes of the form $\xi = \sum_j (f_j, D_j)$, where $\mathrm{codim}_X D_j = r-1$, $f_j \in \mathbf{C}(D_j)^\times$, and $\sum \mathrm{div}(f_j) = 0$; modulo the image of the Tame symbol.
(iii) $\mathrm{CH}^r(X,2)$ is represented by classes in the kernel of the Tame symbol; modulo the image of a higher Tame symbol.

Example 4.8. (i) $X = \mathbf{P}^2$, with homogeneous coordinates $[z_0,z_1,z_2]$. $\mathbf{P}^1 = \ell_j := V(z_j)$, $j = 0,1,2$. Let $P = [0,0,1] = \ell_0 \cap \ell_1$, $Q = [1,0,0] = \ell_1 \cap \ell_2$, $R = [0,1,0] = \ell_0 \cap \ell_2$. Introduce $f_j \in \mathbf{C}(\ell_j)^\times$, where $(f_0) = P - R$, $(f_1) = Q - P$, $(f_2) = R - Q$. Explicitly, $f_0 = z_1/z_2$, $f_1 = z_2/z_0$ and $f_2 = z_0/z_1$. Then $\xi := \sum_{j=0}^2 (f_j, \ell_j) \in \mathrm{CH}^2(\mathbf{P}^2, 1)$ represents a higher Chow cycle.

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       ℓ1/\ℓ0
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This cycle turns out to be nonzero.[4] Consider the line $\mathbf{P}^1_0 := V(z_0+z_1+z_2) \subset \mathbf{P}^2$, and set $q_j = \mathbf{P}^1_0 \cap \ell_j$, $j = 0,1,2$. Then $q_0 = [0,1,-1]$, $q_1 = [1,0,-1]$, $q_2 = [1,-1,0]$, and accordingly $f_j(q_j) = -1$. These Chow groups are known to satisfy a projective bundle formula (see [6], p. 269) which implies that

$$\mathrm{CH}^2(\mathbf{P}^2,1) \simeq \{\mathbf{P}^1\} \otimes \mathrm{CH}^1(\mathrm{Spec}(\mathbf{C}),1),$$
$$\mathrm{CH}^2(\mathbf{P}^1_0,1) \simeq \{\mathbf{P}^1 \cap \mathbf{P}^1_0\}_{\mathbf{P}^2} \otimes \mathrm{CH}^1(\mathrm{Spec}(\mathbf{C}),1),$$

where $\mathbf{P}^2 \to \mathrm{Spec}(\mathbf{C})$, and $\mathbf{P}^1_0 \to \mathrm{Spec}(\mathbf{C})$ are the structure maps, and $\mathbf{P}^1 \subset \mathbf{P}^2$ is a choice of line. It is well-known that $\mathrm{CH}^1(\mathrm{Spec}(\mathbf{C}),1) = \mathbf{C}^\times$ ([6], see Example 5.3 below), and thus via restriction we have the isomorphisms:

$$\mathrm{CH}^2(\mathbf{P}^2,1) \simeq \mathrm{CH}^2(\mathbf{P}^1_0,1) \simeq \mathbf{C}^\times;$$

moreover under this isomorphism,

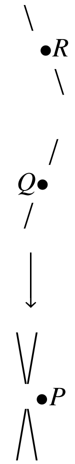
$$\xi \mapsto \prod_{j=0}^{2} f_j(q_j) = -1 \in \mathbf{C}^\times.$$

Hence $\xi \in \mathrm{CH}^2(\mathbf{P}^2,1)$ is a nonzero 2-torsion class.[5]

[4] A special thanks to Rob de Jeu for supplying us this idea.

[5] Matt Kerr informed us of an alternate and slick approach to this example via the definition given in Example 4.7(ii). Namely one need only add $\mathrm{Tame}\{z_1/z_0, z_2/z_0\} = (-f_0^{-1}, \ell_0) + (f_1^{-1}, \ell_1) + (f_2^{-1}, \ell_2)$ to ξ to get the 2-torsion class $(-1, \ell_0)$, which is the same as ξ in $\mathrm{CH}^2(\mathbf{P}^2,1)$.

(ii) Again $X = \mathbf{P}^2$. Let $C \subset X$ be the nodal rational curve given by $z_2^2 z_0 = z_1^3 + z_0 z_1^2$ (in affine coordinates $(x, y) = (z_1/z_0, z_2/z_0) \in \mathbf{C}^2$, C is given by $y^2 = x^3 + x^2$). Let $\tilde{C} \simeq \mathbf{P}^1$ be the normalization of C, with morphism $\pi : \tilde{C} \to C$. Put $P = (0, 0) \in C$ (node) and let $\{R, Q\} = \pi^{-1}(P)$. Choose $f \in \mathbf{C}(\tilde{C})^\times = \mathbf{C}(C)^\times$, such that $(f)_{\tilde{C}} = R - Q$. Then $(f)_C = 0$ and hence $(f, C) \in \mathrm{CH}^2(\mathbf{P}^2, 1)$ defines a higher Chow cycle.



Exercise 4.9. Show that $(f, C) = 0 \in \mathrm{CH}^2(\mathbf{P}^2, 1)$.

5 A Short Detour via Milnor K-Theory

This section provides some of the foundations for the previous section. In the first part of this section, we follow closely the treatment of Milnor K-theory provided in [2], which allows us to provide an abridged definition of the higher Chow groups $\mathrm{CH}^r(X, m)$, for $0 \le m \le 2$. The reader with pressing obligations who prefers to work with concrete examples may skip this section, without losing sight of the main ideas presented in this paper.

Let $\mathbf{F}$ be a field, with multiplicative group $\mathbf{F}^\times$, and put $T(\mathbf{F}^\times) = \bigoplus_{n\ge 0} T^n(\mathbf{F}^\times)$, the tensor product of the $\mathbf{Z}$-module $\mathbf{F}^\times$. Here $T^0(\mathbf{F}^\times) := \mathbf{Z}$, $\mathbf{F}^\times = T^1(\mathbf{F}^\times)$, $a \mapsto [a]$. If $a \ne 0, 1$, set $r_a = [a] \otimes [1 - a] \in T^2(\mathbf{F}^\times)$. The two-sided ideal R generated by the $\{r_a\}$'s is graded, and we put:

$$K_\bullet^M \mathbf{F} = \frac{T(\mathbf{F}^\times)}{R} = \bigoplus_{n\ge 0} K_n^M \mathbf{F}, \quad \text{(Milnor } K\text{-theory).}$$

For example, $K_0(\mathbf{F}) = \mathbf{Z}$, $K_1(\mathbf{F}) = \mathbf{F}^\times$, and $K_2^M(\mathbf{F})$ is the abelian group generated by symbols $\{a, b\}$, subject to the Steinberg relations:

$$\{a_1 a_2, b\} = \{a_1, b\}\{a_2, b\}$$

$$\{a, 1 - a\} = 1, \ \text{for } a \ne 0, 1$$

$$\{a, b\} = \{b, a\}^{-1}$$

Furthermore, one can also show that:

$$\{a, a\} = \{-1, a\} = \{a, a^{-1}\} = \{a^{-1}, a\}, \text{and } \{a, -a\} = 1. \tag{7}$$

Quite generally, one can argue that $K_n^M(\mathbf{F})$ is generated $\{a_1, \ldots, a_n\}$, $a_1, \ldots, a_n \in \mathbf{F}^\times$, subject to:

$$\text{(i)} \qquad (a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\},$$

is a multilinear function from $\mathbf{F}^\times \times \cdots \times \mathbf{F}^\times \to K_n^M(\mathbf{F})$,

$$\text{(ii)} \qquad \{a_1, \ldots, a_n\} = 0,$$

if $a_i + a_{i+1} = 1$ for some $i < n$.

Next, let us assume given a field $\mathbf{F}$ with discrete valuation $\nu : \mathbf{F}^\times \to \mathbf{Z}$, with corresponding discrete valuation ring $\mathrm{O}_\mathbf{F} := \{a \in \mathbf{F} \mid \nu(a) \geq 0\}$, and residue field $\mathbf{k}(\nu)$. One has maps $T : K_\bullet^M(\mathbf{F}) \to K_{\bullet-1}^M(\mathbf{k}(\nu))$. Choose $\pi \in \mathbf{F}^\times$ such that $\nu(\pi) = 1$, and note that $\mathbf{F}^\times = \mathrm{O}_\mathbf{F}^\times \cdot \pi^\mathbf{Z}$. For example, if we write $a = a_0\pi^i$, $b = b_0\pi^j \in K_1^M(\mathbf{F})$, then $T(a) = i \in \mathbf{Z} = K_0^M(\mathbf{k}(\nu))$ and

$$T\{a, b\} = (-1)^{ij} \frac{a^j}{b^i} \in \mathbf{k}(\nu)^\times = K_1^M(\mathbf{k}(\nu)) \quad \text{(Tame symbol)}.$$

5.1 The Gersten–Milnor Complex

The reader may find [41] particularly useful regarding the discussion in this subsection. Let $\mathscr{O}_X$ be the sheaf of regular functions on X, with sheaf of units $\mathscr{O}_X^\times$. As in [30], we put

$$\mathscr{K}_{r,X}^M := (\underbrace{\mathscr{O}_X^\times \otimes \cdots \otimes \mathscr{O}_X^\times}_{r \text{ times}})/\mathscr{J}, \quad \text{(Milnor sheaf)},$$

where $\mathscr{J}$ is the subsheaf of the tensor product generated by sections of the form:

$$\{\tau_1 \otimes \cdots \otimes \tau_r \,\big|\, \tau_i + \tau_j = 1, \quad \text{for some } i \text{ and } j,\ i \neq j\}.$$

For example, $\mathscr{K}_{1,X}^M = \mathscr{O}_X^\times$. Introduce the Gersten–Milnor complex (a flasque resolution of $\mathscr{K}_{r,X}^M$, see [17, 33]):

$$\mathscr{K}_{r,X}^M \to K_r^M(\mathbf{C}(X)) \to \bigoplus_{\mathrm{cd}_X Z=1} K_{k-1}^M(\mathbf{C}(Z)) \to \cdots$$

$$\to \bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{cd}_X Z=r-1} K_1^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{cd}_X Z=r} K_0^M(\mathbf{C}(Z)) \to 0.$$

We have

$$K_0^M(\mathbf{C}(Z)) = \mathbf{Z}, \qquad K_1^M(\mathbf{C}(Z)) = \mathbf{C}(Z)^\times,$$

$$K_2^M(\mathbf{C}(Z)) = \{\text{symbols } \{f,g\}/\text{Steinberg relations}\}.$$

The last three terms of this complex then are:

$$\bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbf{C}(Z)) \xrightarrow{T} \bigoplus_{\mathrm{cd}_X Z=r-1} \mathbf{C}(Z)^\times \xrightarrow{\mathrm{div}} \bigoplus_{\mathrm{cd}_X Z=r} \mathbf{Z} \to 0$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map

$$T : \bigoplus_{\mathrm{codim}_X Z=r-2} K_2^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D=r-1} K_1^M(\mathbf{C}(D)),$$

defined earlier.

Definition 5.2. For $0 \le m \le 2$,

$$\mathrm{CH}^r(X,m) = H_{\mathrm{Zar}}^{r-m}(X, \mathscr{K}_{r,X}^M).$$

Example 5.3.

$$\mathrm{CH}^1(X,1) \simeq H_{\mathrm{Zar}}^0(X, \mathscr{K}_{1,X}^M) \simeq H_{\mathrm{Zar}}^0(X, \mathscr{O}_X^\times) \simeq \mathbf{C}^\times.$$

Remark 5.4. The higher Chow groups $\mathrm{CH}^r(W,m)$ were introduced in [6], and are defined for any non-negative integers r and m, and quasi-projective variety W over a field k. The formula in Definition 5.2 is only for smooth varieties X.

6 Hypercohomology

An excellent reference for this is the chapter on spectral sequences in [25].

The reader familiar with hypercohomology can obviously skip this section. Let $(\mathscr{S}^{\bullet\ge 0}, d)$ be a (bounded) complex of sheaves on X. One has a Cech double complex

$$(C^\bullet(\mathscr{U}, \mathscr{S}^\bullet),\ d,\ \delta),$$

where $\mathscr{U}$ is an open cover of X. The k-th hypercohomology is given by the k-th total cohomology of the associated single complex

$$(M^\bullet := \oplus_{i+j=\bullet} C^i(\mathscr{U}, \mathscr{S}^j),\ D = d \pm \delta),$$

viz.,

$$\mathbf{H}^k(\mathscr{S}^\bullet) := \lim_{\vec{\mathscr{U}}} H^k(M^\bullet).$$

Associated to the double complex are two filtered subcomplexes of the associated single complex, with two associated Grothendieck spectral sequences abutting to $\mathbf{H}^k(\mathscr{S}^\bullet)$ (where $p + q = k$):

$$'E_2^{p,q} := H_\delta^p(X, \mathscr{H}_d^q(\mathscr{S}^\bullet))$$

$$''E_2^{p,q} := H_d^p(H_\delta^q(X, \mathscr{S}^\bullet))$$

The first spectral sequence shows that quasi-isomorphic complexes yield the same hypercohomology:

Alternate take. Two complexes of sheaves $\mathscr{K}_1^\bullet, \mathscr{K}_2^\bullet$ are said to be quasi-isomorphic if there is a morphism $h : \mathscr{K}_1^\bullet \to \mathscr{K}_2^\bullet$ inducing an isomorphism on cohomology $h_* : \mathscr{H}^\bullet(\mathscr{K}_1^\bullet) \xrightarrow{\sim} \mathscr{H}^\bullet(\mathscr{K}_2^\bullet)$. Take a complex of acyclic sheaves $(\mathscr{K}^\bullet, d)$ (viz., $H^{i>0}(X, \mathscr{K}^j) = 0$ for all j) quasi-isomorphic to $\mathscr{S}^\bullet$. Then by the second spectral sequence:

$$\mathbf{H}^k(\mathscr{S}^\bullet) := H^i(H^0(X, \mathscr{K}^\bullet)).$$

For example if $\mathcal{L}^{\bullet,\bullet}$ is an acyclic resolution of $\mathscr{S}^\bullet$, then the associated single complex $\mathscr{K}^\bullet = \oplus_{i+j=\bullet} \mathcal{L}^{i,j}$ is acyclic and quasi-isomorphic to $\mathscr{S}^\bullet$.

Example 6.1. Let $(\Omega_X^\bullet, d)$, $(\mathscr{E}_X^\bullet, d)$ be the complexes of sheaves of holomorphic and $\mathbf{C}$-valued C^∞ forms respectively. By the holomorphic and C^∞ Poincaré lemmas, one has quasi-isomorphisms:

$$(\mathbf{C} \to 0 \to \cdots) \overset{\approx}{\to} (\Omega_X^\bullet, d) \overset{\approx}{\to} (\mathscr{E}_X^\bullet, d),$$

where the latter two are Hodge filtered, using an argument similar to that in (12) below. The first spectral sequence of hypercohomology shows that

$$H^k(X, \mathbf{C}) \simeq \mathbf{H}^k(\mathbf{C} \to 0 \to \cdots) \simeq \mathbf{H}^k((F^p)\Omega_X^\bullet) \simeq \mathbf{H}^k((F^p)\mathscr{E}_X^\bullet).$$

The second spectral sequence of hypercohomology applied to the latter term, using the known acyclicity of $\mathscr{E}_X^\bullet$, yields

$$\mathbf{H}^k(F^p\mathscr{E}_X^\bullet) \simeq \frac{\ker d : F^pE_X^k \to F^pE_X^k}{dF^pE_X^{k-1}} \simeq F^pH_{\mathrm{DR}}^k(X),$$

where the latter isomorphism is due to the Hodge to de Rham spectral sequence.

7 Deligne Cohomology

A standard reference for this section is [19] (also see [27]). For a subring $\mathbf{A} \subseteq \mathbf{R}$, we introduce the Deligne complex

$$\mathbf{A}_{\mathscr{D}}(r) : \quad \mathbf{A}(r) \to \underbrace{\mathscr{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{r-1}}_{\text{call this } \Omega_X^{\bullet<r}}.$$

Definition 7.1. Deligne cohomology is given by the hypercohomology:

$$H^i_{\mathscr{D}}(X, \mathbf{A}(r)) = \mathbf{H}^i(\mathbf{A}_{\mathscr{D}}(r)).$$

Example 7.2. When $\mathbf{A} = \mathbf{Z}$, we have a quasi-isomorphism

$$\mathbf{Z}_{\mathscr{D}}(1) \approx \mathscr{O}_X^{\times}[-1],$$

hence

$$H^2_{\mathscr{D}}(X, \mathbf{Z}(1)) \simeq H^1(X, \mathscr{O}_X^{\times}) =: \mathrm{Pic}(X) \simeq \mathrm{CH}^1(X).$$

$$H^1_{\mathscr{D}}(X, \mathbf{Z}(1)) \simeq H^0(X, \mathscr{O}_X^{\times}) \simeq \mathbf{C}^{\times} \simeq \mathrm{CH}^1(X, 1).$$

Example 7.3. Alternate take on $H^1_{\mathscr{D}}(X, \mathbf{Z}(1))$. Look at the Cech double complex:

$$\begin{array}{ccc} \mathscr{C}^0(\mathscr{U}, \mathbf{Z}(1)) & \to & \mathscr{C}^0(\mathscr{U}, \mathscr{O}_X) \\ \delta\downarrow & & \downarrow\delta \\ \mathscr{C}^1(\mathscr{U}, \mathbf{Z}(1)) & \to & \mathscr{C}^1(\mathscr{U}, \mathscr{O}_X) \end{array}$$

So a class in $H^1_{\mathscr{D}}(X, \mathbf{Z}(1))$ is represented (after a suitable refinement) by $(\lambda := \{\lambda_{\alpha\beta}\}, f := \{f_\gamma\}) \in (\Gamma(U_\alpha \cap U_\beta, \mathbf{Z}(1)), \Gamma(U_\gamma, \mathscr{O}_X))$, with $f_\beta - f_\alpha =: \delta(f)_{\alpha\beta} = \lambda_{\alpha\beta}$. Note that $\exp(f) \in H^0(X, \mathscr{O}_X^{\times})$ determines the isomorphism $H^1_{\mathscr{D}}(X, \mathbf{Z}(1)) \simeq H^0(X, \mathscr{O}_X^{\times}) \simeq \mathbf{C}^{\times}$.

Definition 7.4. The product structure on Deligne cohomology

$$H^k_{\mathscr{D}}(X, \mathbf{Z}(i)) \otimes H^l_{\mathscr{D}}(X, \mathbf{Z}(j)) \to H^{k+l}_{\mathscr{D}}(X, \mathbf{Z}(i+j)),$$

is induced by the multiplication of complexes $\mu : \mathbf{Z}_{\mathscr{D}}(i) \otimes \mathbf{Z}_{\mathscr{D}}(j) \to \mathbf{Z}_{\mathscr{D}}(i+j)$ defined by

$$\mu(x, y) := \begin{cases} x \cdot y, & \text{if } \deg x = 0, \\ x \wedge dy, & \text{if } \deg x > 0 \text{ and } \deg y = j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 7.5. For example, this product structure implies that

$$H^1_{\mathscr{D}}(X, \mathbf{Z}(1)) \cup H^1_{\mathscr{D}}(X, \mathbf{Z}(1)) = \{0\} \subset H^2_{\mathscr{D}}(X, \mathbf{Z}(2)).$$

Recall from Hodge theory, one has the isomorphism:

$$\mathbf{H}^i(\Omega_X^{\bullet \geq r}) \simeq F^r H^i(X, \mathbf{C}).$$

This together with the short exact sequence of complexes:

$$0 \to \Omega_X^{\bullet \geq r} \to \Omega_X^{\bullet} \to \Omega_X^{\bullet < r} \to 0,$$

implies that

$$\mathbf{H}^i(\Omega_X^{\bullet < r}) \simeq \frac{H^i(X, \mathbf{C})}{F^r H^i(X, \mathbf{C})}.$$

Thus applying $\mathbf{H}^{\bullet}(-)$ to the short exact sequence:

$$0 \to \Omega_X^{\bullet<r}[-1] \to \mathbf{A}_{\mathscr{D}}(r) \to \mathbf{A}(r) \to 0,$$

yields the short exact sequence:

$$0 \to \underbrace{\frac{H^{i-1}(X,\mathbf{C})}{H^{i-1}(X,\mathbf{A}(r)) + F^rH^{i-1}(X,\mathbf{C})}}_{=J_{\mathbf{A}}\left(H^{i-1}(X,\mathbf{A}(r))\right)} \to H^i_{\mathscr{D}}(X,\mathbf{A}(r)) \to \Gamma_{\mathbf{A}}(H^i(X,\mathbf{A}(r))) \to 0. \quad (8)$$

If we consider $\mathbf{A} = \mathbf{Z}$, and $i = 2r$, then (8) becomes:

$$0 \to J^r(X) \to H^{2r}_{\mathscr{D}}(X,\mathbf{Z}(r)) \to \mathrm{Hg}^r(X) \to 0,$$

where $J^r(X) = J(H^{2r-1}(X,\mathbf{Z}(r)))$ is the Griffiths jacobian, and where the Hodge group $\mathrm{Hg}^r(X)$ in untwisted form can be identified with:

$$\{w \in H^{2r}(X,\mathbf{Z}) \mid w \in H^{r,r}(X,\mathbf{C})\},$$

via $\ker\left(H^{2r}(X,\mathbf{Z}) \to \mathbf{H}^{2r}(\Omega_X^{\bullet<r})\right)$. In particular, $\mathrm{Hg}^r(X)$ includes the torsion classes in $H^{2r}(X,\mathbf{Z}(r))$.

Next, if $\mathbf{A} = \mathbf{Z}$ and $i \le 2r-1$, then from Hodge theory, $H^i(X,\mathbf{Z}(r)) \cap F^rH^i(X,\mathbf{C})$ is torsion. In particular, there is a short exact sequence:

$$0 \to \frac{H^{i-1}(X,\mathbf{C})}{F^rH^{i-1}(X,\mathbf{C}) + H^{i-1}(X,\mathbf{Z}(r))} \to H^i_{\mathscr{D}}(X,\mathbf{Z}(r)) \to H^i_{\mathrm{tor}}(X,\mathbf{Z}(r)) \to 0,$$

where $H^i_{\mathrm{tor}}(X,\mathbf{Z}(r))$ is the torsion subgroup of $H^i(X,\mathbf{Z}(r))$. The compatibility of Poincaré and Serre duality yields the isomorphism:

$$\frac{H^{i-1}(X,\mathbf{C})}{F^rH^{i-1}(X,\mathbf{C}) + H^{i-1}(X,\mathbf{Z}(r))} \simeq \frac{F^{d-r+1}H^{2d-i+1}(X,\mathbf{C})^{\vee}}{H_{2d-i+1}(X,\mathbf{Z}(d-r))}.$$

Next, if $\mathbf{A} = \mathbf{R}$ and $i = 2r-1$, then $H^i_{\mathrm{tor}}(X,\mathbf{R}(r)) = 0$; moreover if we set

$$\pi_{r-1} : \mathbf{C} = \mathbf{R}(r) \oplus \mathbf{R}(r-1) \to \mathbf{R}(r-1)$$

to be the projection, then we have the isomorphisms:

$$\begin{aligned} H^{2r-1}_{\mathscr{D}}(X,\mathbf{R}(r)) &\simeq \frac{H^{2r-2}(X,\mathbf{C})}{F^rH^{2r-2}(X,\mathbf{C}) + H^{2r-2}(X,\mathbf{R}(r))} \\ &\underset{\simeq}{\xrightarrow{\pi_{r-1}}} \frac{H^{2r-2}(X,\mathbf{R}(r-1))}{\pi_{r-1}(F^rH^{2r-2}(X,\mathbf{C}))} \\ &=: H^{r-1,r-1}(X,\mathbf{R}(r-1)) \\ &\simeq \{H^{d-r+1,d-r+1}(X,\mathbf{R}(d-r+1))\}^{\vee}. \end{aligned}$$

7.6 Alternate Take on Deligne Cohomology

Let $h : (A^\bullet, d) \to (B^\bullet, d)$ be a morphism of complexes. We define

$$\mathrm{Cone}(A^\bullet \xrightarrow{h} B^\bullet)$$

by the formula

$$[\mathrm{Cone}(A^\bullet \xrightarrow{h} B^\bullet)]^q := A^{q+1} \oplus B^q, \quad \delta(a,b) = (-da, h(a) + db).$$

Example 7.7. $\mathrm{Cone}(\mathbf{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet)[-1]$ is given by:

$$\mathbf{A}(r) \to \mathscr{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-2} \xrightarrow{(0,d)} (\Omega_X^r \oplus \Omega_X^{r-1})$$

$$\xrightarrow{\delta} (\Omega_X^{r+1} \oplus \Omega_X^r) \xrightarrow{\delta} \cdots \xrightarrow{\delta} (\Omega_X^d \oplus \Omega_X^{d-1}) \to \Omega_X^d$$

Using the holomorphic Poincaré lemma, one can show that the natural map

$$\mathbf{A}_{\mathscr{D}}(r) \to \mathrm{Cone}(\mathbf{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet)[-1],$$

is a quasi-isomorphism.[6] Thus

$$H_{\mathscr{D}}^k(X, \mathbf{A}(r)) \simeq \mathbf{H}^r(\mathrm{Cone}(\mathbf{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet)[-1]).$$

Let $\mathscr{D}_X^\bullet$ be the sheaf of currents acting on C^∞ compactly supported $(2d - \bullet)$-forms. Further, let $\mathscr{D}_X^{p,q}$ be the sheaf of currents acting on C^∞ compactly supported$(d-p, d-q)$-forms. One has a decomposition

$$\mathscr{D}_X^\bullet = \bigoplus_{p+q=\bullet} \mathscr{D}_X^{p,q},$$

with a morphism of complexes $\mathscr{E}_X^\bullet \hookrightarrow \mathscr{D}_X^\bullet$ (induced by $\omega \mapsto (2\pi \mathrm{i})^{-d} \int_X \omega \wedge (-)$), and with $\mathscr{E}_X^{p,q} \hookrightarrow \mathscr{D}_X^{p,q}$, compatible with both ∂ and $\bar{\partial}$. Likewise, let $\mathscr{C}_X^\bullet = \mathscr{C}_{2d-\bullet,X}(\mathbf{A}(r))$ be the sheaf of (Borel–Moore) chains of real codimension $\bullet$. Identifying the constant sheaf $\mathbf{A}(r)$ with the complex $\mathbf{A}(r) \to 0 \to \cdots \to 0$, we have quasi-isomorphisms

$$\mathbf{A}(r) \xrightarrow{\approx} \mathscr{C}_X^\bullet(\mathbf{A}(r)), \quad \mathscr{E}_X^\bullet \xrightarrow{\approx} \mathscr{D}_X^\bullet$$

where the latter is (Hodge) filtered.

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[6] Indeed first consider $(a,b) \in \Omega_X^r \oplus \Omega_X^{r-1} \overset{\delta}{\mapsto} (-da, db - a) \in \Omega_X^{r+1} \oplus \Omega_X^r$. Then $\delta(a,b) = (0,0) \Leftrightarrow da = 0 \ \&\ a = db \Leftrightarrow a = db$. Therefore $\ker\delta/\mathrm{Im}(0,d) \simeq \Omega_X^{r-1}/d\Omega_X^{r-2} = \mathscr{H}^{r-1}(\mathbf{A}_{\mathscr{D}}(r))$. Next, for $j \geq 1$, $(a,b) \in \Omega_X^{r+j} \oplus \Omega_X^{r+j-1}$, $\delta(a,b) = 0 \Leftrightarrow (a,b) = \delta(-b, 0)$.

Observe that $\mathscr{D}_X^\bullet(X))[-1]$ is a subcomplex of $\mathrm{Cone}(\mathscr{C}_X^\bullet(X,\mathbf{A}(r))\oplus F^r\mathscr{D}_X^\bullet(X)\xrightarrow{\epsilon-l}\mathscr{D}_X^\bullet(X))[-1]$. Hence the cone complex description of:

$$H^i_{\mathscr{D}}(X,\mathbf{A}(r))\simeq H^i(\mathrm{Cone}(\mathscr{C}_X^\bullet(X,\mathbf{A}(r))\oplus F^r\mathscr{D}_X^\bullet(X)\xrightarrow{\epsilon-l}\mathscr{D}_X^\bullet(X))[-1]),$$

yields the exact sequence[7]:

$$\begin{aligned}\cdots\to H^{i-1}(X,\mathbf{A}(r))\oplus F^rH^{i-1}(X,\mathbf{C})\to H^{i-1}(X,\mathbf{C})\\ \to H^i_{\mathscr{D}}(X,\mathbf{A}(r))\to H^i(X,\mathbf{A}(r))\oplus F^rH^i(X,\mathbf{C})\to\cdots\end{aligned}\tag{9}$$

7.8 Deligne–Beilinson Cohomology

The formulation of Deligne cohomology in Definition 7.1 above, which incidentally can be defined in the same way for any complex manifold (and is also called analytic Deligne cohomology), works well for projective algebraic manifolds X, but not so well for smooth open $U\subset X$. First of all, the naive Hodge filtration on U, viz., $\Omega_U^{\bullet\ge r}$ is the *wrong* choice. For example, if W is a Stein manifold, then $H^q(W,\Omega^i_W)=0$ for all i and where $q\ge 1$. This tells us, via the Grothendieck spectral sequences associated to hypercohomology, that

$$H^j(W,\mathbf{C})\simeq\frac{H^0(W,\Omega^j_W)_{d-\mathrm{closed}}}{dH^0(W,\Omega^{j-1}_W)}.$$

(Note: If W is a smooth affine variety, then by Grothendieck, one can use algebraic differential forms.) We hardly expect $H^j(W,\mathbf{C})=F^jH^j(W,\mathbf{C})$ to be the case in general. Secondly, analytic Deligne cohomology fails to take into consideration the underlying algebraic structure of U. For instance $H^1_{\mathscr{D}}(U,\mathbf{Z}(1))=H^0(U,\mathscr{O}^\times_{U,an})$, i.e. the non-zero analytic functions on U. It would be preferable to recover the non-zero algebraic functions on U instead. Beilinson's remedy is to incorporate Deligne's logarithmic complex into the picture. By a standard reduction, we may assume that $j:U=X\backslash Y\hookrightarrow X$, where Y is a normal crossing divisor[8] with smooth components. We define $\Omega_X^\bullet\langle Y\rangle$ to be the de Rham complex of meromorphic forms on X, holomorphic on U, with at most logarithmic poles along Y. One has a filtered complex

$$F^r\Omega_X^\bullet\langle Y\rangle=\Omega_X^{\bullet\ge r}\langle Y\rangle,$$

with Hodge to de Rham spectral sequence degenerating at E_1. This gives

$$F^rH^i(U,\mathbf{C})=\mathbf{H}^i(F^r\Omega_X^\bullet\langle Y\rangle)\subset\mathbf{H}^i(\Omega_X^\bullet\langle Y\rangle)=H^i(U,\mathbf{C}),\tag{10}$$

[7] The reader familiar with Deligne homology will see this definition as the same thing up to twist. Indeed this definition already incorporates Poincaré duality.

[8] Y is a normal crossing divisor, which in local analytic coordinates $(z_1,\ldots,z_d)$ on X, Y is given by $z_1\cdots z_\ell=0$, and so $\Omega^1_X\langle Y\rangle$ has local frame $\{dz_1/z_1,\ldots,dz_\ell/z_\ell,dz_{\ell+1},\ldots,dz_d\}$.

and defines the correct Hodge filtration. The weight filtration is characterized in terms of differentials with residues along $Y^{[\bullet]}$, where $Y^{[\bullet]}$ is the simplicial complex made up of the intersections of the irreducible components of Y.

Definition 7.9. Deligne–Beilinson cohomology is given by

$$H^i_{\mathscr{D}}(U,\mathbf{A}(r)) := \mathbf{H}^i(\mathbf{A}_{\mathscr{D}}(r)),$$

where

$$\mathbf{A}_{\mathscr{D}}(r) := \text{Cone}(Rj_*\mathbf{A}(r) \bigoplus F^r\Omega^\bullet_X\langle Y\rangle \xrightarrow{\epsilon - l} Rj_*\Omega^\bullet_U)[-1].$$

Here ϵ and l are the natural maps obtained after a choice of (the direct image of) injective resolutions of $\mathbf{A}(r)$ and $\Omega^\bullet_U$. One shows that this is independent of the good compactifications of U. One has a short exact sequence:

$$0 \to \frac{H^{i-1}(U,\mathbf{C})}{F^rH^{i-1}(U,\mathbf{C}) + H^{i-1}(U,\mathbf{A}(r))} \to H^i_{\mathscr{D}}(U,\mathbf{A}(r)) \to F^r \bigcap H^i(U,\mathbf{A}(r)) \to 0, \tag{11}$$

where

$$F^r \bigcap H^i(U,\mathbf{A}(r)) := \ker\left(F^rH^i(U,\mathbf{C}) \oplus H^i(U,\mathbf{A}(r)) \xrightarrow{\epsilon - l} H^i(U,\mathbf{C})\right).$$

We would like a more earthly description of $H^i_{\mathscr{D}}(U,\mathbf{A}(r))$. First observe that there are filtered quasi-isomorphisms

$$(F^r,\Omega^\bullet_X\langle Y\rangle) \hookrightarrow (F^r,\mathscr{E}^\bullet_X\langle Y\rangle) \hookrightarrow (F^r,\mathscr{D}^\bullet_X\langle Y\rangle), \tag{12}$$

where

$$F^r\mathscr{D}^\bullet_X\langle Y\rangle = \{F^r\Omega^\bullet_X\langle Y\rangle\} \otimes_{\Omega^\bullet_X} \mathscr{D}^\bullet_X.$$

To see this, one uses the argument in [27]. By a spectral sequence argument, it is enough to show that the associated graded pieces in (12) are quasi-isomorphic, viz.,

$$\Omega^r_X\langle Y\rangle \approx \Omega^r_X\langle Y\rangle \otimes_{\mathscr{O}_X} \mathscr{E}^{0,\bullet}_X \approx \Omega^r_X\langle Y\rangle \otimes_{\mathscr{O}_X} \mathscr{D}^{0,\bullet}_X,$$

where the differential is now $1 \otimes \overline{\partial}$. One now applies the $\overline{\partial}$ lemma together with the flatness of $\Omega^r_X\langle Y\rangle$ over $\mathscr{O}_X$, and using $\mathscr{O}_X$ as $\overline{\partial}$-linear. According to [34], $\mathscr{D}^\bullet_X\langle Y\rangle$ admits the interpretation of the space of currents acting on those (compactly supported) forms on X which "vanish holomorphically" on Y. Let $\mathscr{C}^i(X,\mathbf{A}(r))$ be the chains of real codimension i in X, and $\mathscr{C}^i_Y(X,\mathbf{A}(r))$ the subspace of chains supported on Y. Put

$$\mathscr{C}^i(X,Y,\mathbf{A}(r)) := \frac{\mathscr{C}^i(X,\mathbf{A}(r))}{\mathscr{C}^i_Y(X,\mathbf{A}(r))}.$$

One has a map of complexes:

$$(\mathscr{C}^\bullet(X,Y,\mathbf{A}(r)),d) \to (\mathscr{D}^\bullet_X\langle Y\rangle(X),d),$$

which induces a quasi-isomorphism

$$\mathscr{C}^\bullet(X, Y, \mathbf{A}(r)) \otimes \mathbf{C} \to \mathscr{D}_X^\bullet\langle Y\rangle(X).$$

Definition 7.10. Deligne–Beilinson cohomology is given by

$$H^i_{\mathscr{D}}(U, \mathbf{A}(r)) := H^i(\mathrm{Cone}(\mathscr{C}^\bullet(X, Y, \mathbf{A}(r)) \bigoplus F^r\mathscr{D}_X^\bullet\langle Y\rangle(X) \xrightarrow{\epsilon - l} (\mathscr{D}_X^\bullet\langle Y\rangle(X))[-1]).$$

Example 7.11. Let us compute $H^1_{\mathscr{D}}(U, \mathbf{Z}(1))$. First of all $\{\xi\} \in H^1_{\mathscr{D}}(U, \mathbf{Z}(1))$ is represented by a D-closed triple:

$$\xi = (a, b, c) \in (\mathscr{C}^1(X, Y, \mathbf{Z}(1)) \bigoplus F^1\mathscr{D}_X^1\langle Y\rangle(X) \bigoplus \mathscr{D}_X^0\langle Y\rangle(X)),$$

where $da = 0$, $db = 0$ and $a - b = dc$. Note that $\overline{\partial}$-regularity implies that $b \in \Omega_X^1\langle Y\rangle(X)_{d-\mathrm{closed}}$. Let $\hat{\Omega}_U^1$ be the sheaf of d-closed holomorphic 1-forms on U, and let's make the identification $\mathbf{C}^\times = \mathbf{C}/\mathbf{Z}(1)$. From the short exact sequence:

$$0 \to \mathbf{C}^\times \to \mathscr{O}_U^\times \xrightarrow{d\log} \hat{\Omega}_U^1 \to 0,$$

and the relation $a - b = dc$, it follows that

$$b \in \ker\left(H^0(U, \hat{\Omega}_U^1) \to H^1(U, \mathbf{C}^\times)\right),$$

and hence $b = d\log f$ for some $f \in \mathscr{O}_U^\times(U)$. Since $b \in \Omega_X^1\langle Y\rangle(X)$, it follows that $f \in \mathscr{O}_{U,\mathrm{alg}}^\times(U)$. Thus in Deligne cohomology[9]

$$\{\xi\} = (2\pi \mathrm{i} T_\xi, \Omega_\xi, R_\xi),$$

where $T_\xi := \delta_{f^{-1}(\mathbf{R}^-)}$ is given by integration along $f^{-1}[-\infty, 0]$, and $\Omega_\xi = [d\log f]$, $R_\xi = [\log f]$ are the obvious defined currents.

Corollary 7.12.

$$\mathrm{cl}_{1,1} : \mathrm{CH}^1(U, 1) := \mathscr{O}_{U,\mathrm{alg}}^\times(U) \xrightarrow{\sim} H^1_{\mathscr{D}}(U, \mathbf{Z}(1)),$$

is an isomorphism.

Remark 7.13. We observe in passing the following. We deduce from (11) the short exact sequence:

$$0 \to \frac{H^0(U, \mathbf{C})}{F^1H^0(U, \mathbf{C}) + H^0(U, \mathbf{Z}(1))} \to H^1_{\mathscr{D}}(U, \mathbf{Z}(1)) \to \Gamma(H^1(U, \mathbf{Z}(1))) \to 0,$$

---

[9] For compactly supported $\omega \in E_{U,c}^{2d-1}$, and $f \in \mathscr{O}_U^\times(U)$,

$$\int_U \frac{df}{f} \wedge \omega = \int_U d(\log f \wedge \omega) - \int_{U \backslash f^{-1}[-\infty,0]} \log f \wedge d\omega = 2\pi \mathrm{i} \int_{f^{-1}[-\infty,0]} \omega + d[\log f](\omega),$$

where we use the principal branch of log.

which in turn from Corollary 7.12 yields the short exact sequence:

$$0 \to \mathbf{C}^\times \to \mathrm{CH}^1(U,1) \xrightarrow{d\log} \Gamma(H^1(U,\mathbf{Z}(1))) \to 0.$$

Remark 7.14. Let $U/\mathbf{C}$ be a smooth quasi-projective variety. If $H^\bullet_{\mathscr{D},an}(U,\mathbf{Z}(\bullet))$ denotes that analytic Deligne cohomology, then we know that $H^2_{\mathscr{D},an}(U,\mathbf{Z}(1)) \simeq H^1(U,\mathscr{O}^\times_{U,an})$, the holomorphic isomorphism classes of holomorphic line bundles over U. For Deligne–Beilinson cohomology, and using the fact that $H^1(U,\mathbf{Z}(1)) = W_0H^1(U,\mathbf{Z}(1))$, it follows that there is a short exact sequence:

$$0 \to J(H^1(U,\mathbf{Z}(1))) \to H^2_{\mathscr{D}}(U,\mathbf{Z}(1)) \xrightarrow{\alpha} F^1 \cap H^2(U,\mathbf{Z}(1)) \to 0,$$

but in general

$$\Gamma(H^2(U,\mathbf{Z}(1))) = F^0 \cap W_0H^2(U,\mathbf{Z}(1)) \subsetneq F^1 \cap H^2(U,\mathbf{Z}(1)),$$

where the shift $F^0 \mapsto F^1$ is really the same filtration, but the latter is in "untwisted" terminology. To remedy this, let us put $H^2_{\mathscr{H}}(U,\mathbf{Z}(1)) = \alpha^{-1}(\Gamma(H^2(U,\mathbf{Z}(1))))$. This turns out to be the same thing as the image $H^2_{\mathscr{D}}(\overline{U},\mathbf{Z}(1)) \to H^2_{\mathscr{D}}(U,\mathbf{Z}(1))$, where $\overline{U}$ is any smooth projective compactification of U. Then $H^2_{\mathscr{H}}(U,\mathbf{Z}(1))$ amounts to a special instance of Beilinson's absolute Hodge cohomology (see [3]). We then have the following:

Proposition 7.15. *Let $U/\mathbf{C}$ be a smooth quasi-projective variety. Then:*

$$H^2_{\mathscr{H}}(U,\mathbf{Z}(1)) \simeq H^1_{\mathrm{Zar}}(U,\mathscr{O}^\times_{U,\mathrm{alg}})$$

Proof. First recall that

$$H^1_{\mathrm{Zar}}(U,\mathscr{O}^\times_{U,\mathrm{alg}}) = H^1_{\mathrm{Zar}}(U,\mathscr{K}^M_{1,U}) = \mathrm{CH}^1(U).$$

There is a commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{CH}^1_{\mathrm{hom}}(U) & \to & \mathrm{CH}^1(U) & \to & \frac{\mathrm{CH}^1(U)}{\mathrm{CH}^1_{\mathrm{hom}}(U)} & \to 0 \\
& {\scriptstyle\Phi_1}\downarrow & & {\scriptstyle\mathrm{cl}_1}\downarrow & & \downarrow{\scriptstyle\wr} & \\
0 \to & J(H^1(U,\mathbf{Z}(1))) & \to & H^2_{\mathscr{H}}(U,\mathbf{Z}(1)) & \to & \Gamma(H^2(U,\mathbf{Z}(1))) & \to 0
\end{array}$$

It suffices to show that Φ_1 is an isomorphism. Let $\overline{U}$ be a smooth projective compactification of U. We may assume that $Y := \overline{U}\backslash U$ is a divisor. With regard to the short exact sequence:

$$0 \to H^1(\overline{U},\mathbf{Z}(1)) \to H^1(U,\mathbf{Z}(1)) \to H^2_Y(\overline{U},\mathbf{Z}(1))^\circ \to 0,$$

it is clear that $J(H^2_Y(\overline{U}, \mathbf{Z}(1))^\circ) = 0$, and hence the following diagram finishes the proof:

$$\begin{array}{ccccccc}
\mathrm{CH}^1_Y(\overline{U})^\circ & \to & \mathrm{CH}^1_{\mathrm{hom}}(\overline{U}) & \to & \mathrm{CH}^1_{\mathrm{hom}}(U) & \to 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow \Phi_1 & \\
\Gamma H^2_Y(\overline{U}, \mathbf{Z}(1))^\circ & \to & J(H^1(\overline{U}, \mathbf{Z}(1))) & \to & J(H^1(U, \mathbf{Z}(1))) & \to 0
\end{array}$$

□

8 Examples of $H^{r-m}_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X})$ and Corresponding Regulators

The reader is encouraged to consult for example [35, 43] (as well as works due to Bloch, Beilinson, Esnault and Goncharov), for various earlier incarnations of regulator type currents for higher Chow cycles. A complete description of the Beilinson/Bloch regulator in terms of polylogarithmic type currents for complex varieties can be found in [31, 32].

8.1 *Case $m = 0$ and CY Threefolds*

In this case we recall that

$$H^r_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) = \mathrm{CH}^r(X).$$

The fundamental class map:

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \to H^{2r}_{\mathrm{DR}}(X, \mathbf{C}) \simeq H^{2d-2r}_{\mathrm{DR}}(X, \mathbf{C})^\vee,$$

can be defined in a number of equivalent ways:

(i) (See [18].) The $d\log$ map $\mathcal{K}^M_{r,X} \to \Omega^r_X$, $\{f_1, \ldots, f_r\} \mapsto \bigwedge_j d\log f_j$, induces a morphism of complexes in the Zariski topology $\{\mathcal{K}^M_{r,X} \to 0\} \to \Omega^{\bullet \geq r}_X[r]$, and thus using GAGA,

$$\mathrm{CH}^r(X) = H^r_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) = \mathbf{H}^r(\{\mathcal{K}^M_{r,X} \to 0\}) \to \mathbf{H}^r(\Omega^{\bullet \geq r}_X[r])$$

$$= \mathbf{H}^{2r}(\Omega^{\bullet \geq r}_X) = F^r H^{2r}_{\mathrm{DR}}(X, \mathbf{C}).$$

(ii) Let $V \subset X$ be a subvariety of codimension r in X, and $\{w\} \in H^{2d-2r}_{\mathrm{DR}}(X, \mathbf{C})$, (de Rham cohomology). Define

$$\mathrm{cl}_r(V)(w) = \frac{1}{(2\pi \mathrm{i})^{d-r}} \delta_V := \frac{1}{(2\pi \mathrm{i})^{d-r}} \int_{V^*} w,$$

and extend to $\mathrm{CH}^r(X)$ by linearity, where $V^* = V\backslash V_{\rm sing}$. Note that $\dim_{\mathbf{R}} V = 2d-2r$. The easiest way to show that cl_r is well-defined (finite volume, closed current) is to first pass to a desingularization of V above, and apply a Stokes' theorem argument. The proof of a more direct approach can be found, for example, in [25].

(One way to connected (i) and (ii) is as follows. If we write Γ for $H^0(X,-)$, then there is a diagram that commutes up to sign:

$$\begin{array}{ccccc} \Gamma K_r^M(\mathbf{C}(X)) \to & \Gamma \bigoplus_{\mathrm{cd}_X Y=1} K_{r-1}^M(\mathbf{C}(Y)) & \to \cdots \to & \Gamma \bigoplus_{\mathrm{cd}_X V=r} K_0^M(\mathbf{C}(X)) \\ \int_X \frac{d\log_r}{(2\pi \mathrm{i})^d}\Big\downarrow & \int_Y \frac{d\log_{r-1}}{(2\pi \mathrm{i})^{d-1}}\Big\downarrow & \cdots & \int_V \frac{d\log_0}{(2\pi \mathrm{i})^{d-r}}\Big\downarrow \\ \Gamma F^r \mathscr{D}_X^r \xrightarrow{d} & \Gamma F^r \mathscr{D}_X^{r+1} & \xrightarrow{d} \cdots \xrightarrow{d} & \Gamma F^r \mathscr{D}_X^{2r} \end{array}$$

where

$$d\log_r(\{f_1,\ldots,f_r\}) = \bigwedge_{j=1}^r d\log f_j, \quad \int_V \frac{d\log_0}{(2\pi \mathrm{i})^{d-r}} = \frac{1}{(2\pi \mathrm{i})^{d-r}}\delta_V.$$

From the aforementioned filtered quasi-isomorphism $\Omega_X^\bullet \hookrightarrow \mathscr{D}_X^\bullet$, the prescriptions in (i) and (ii) can be seen as almost tautologies.)

(iii) Thirdly one has a fundamental class generator $\{V\} \in H_{2d-2r}(V,\mathbf{Z}(d-r)) \simeq H_V^{2r}(X,\mathbf{Z}(r)) \to H_{2d-2r}(X,\mathbf{Z}((d-r)) \simeq H^{2r}(X,\mathbf{Z}(r))$. In summary we have $\mathrm{cl}_r : \mathrm{CH}^r(X) \to \mathrm{Hg}^r(X)$. This map fails to be surjective in general for $r > 1$ (see [36]).

Conjecture 8.2 (Hodge$_{\mathbf{Q}}$).

$$\mathrm{cl}_r : \mathrm{CH}^r(X)\otimes\mathbf{Q} \to \mathrm{Hg}^r(X)\otimes\mathbf{Q}, \ \textit{is surjective}.$$

Next, the Abel–Jacobi map:

$$\Phi_r : \mathrm{CH}^r_{\rm hom}(X) \to J^r(X),$$

is defined as follows. Recall that

$$J^r(X) = \frac{H^{2r-1}(X,\mathbf{C})}{F^r H^{2r-1}(X,\mathbf{C}) + H^{2r-1}(X,\mathbf{Z}(r))} \simeq \frac{F^{d-r+1}H^{2d-2r+1}(X,\mathbf{C})^\vee}{H_{2d-2r+1}(X,\mathbf{Z}(d-r))},$$

is a compact complex torus, called the Griffiths jacobian.

Prescription for Φ_r: Let $\xi \in \mathrm{CH}^r_{\rm hom}(X)$. Then $\xi = \partial\zeta$ bounds a $2d-2r+1$ real dimensional chain ζ in X. Let $\{w\} \in F^{d-r+1}H^{2d-2r+1}(X,\mathbf{C})$. Define:

$$\Phi_r(\xi)(\{w\}) = \frac{1}{(2\pi \mathrm{i})^{d-r}}\int_\zeta w \quad \text{(modulo periods)}.$$

That Φ_r is well-defined follows from the fact that $F^\ell H^i(X, \mathbf{C})$ depends only on the complex structure of X, namely

$$F^\ell H^i(X, \mathbf{C}) \simeq \frac{F^\ell E^i_{X,d-\text{closed}}}{d(F^\ell E^{i-1}_X)},$$

where we recall that E^i_X are the C^∞ complex-valued i-forms on X.

Alternate take for Φ_r: Let $\xi \in \mathrm{CH}^r_{\hom}(X)$. First observe that

$$H^{2r-1}_{|\xi|}(X, \mathbf{Z}) \simeq H_{2d-2r+1}(|\xi|, \mathbf{Z}) = 0,$$

as $\dim_{\mathbf{R}} |\xi| = 2d - 2r$. Secondly there is a fundamental class map $\xi \mapsto \{\xi\} \in H_{2d-2r}(|\xi|, \mathbf{Z}(d-r)) \simeq H^{2r}_{|\xi|}(X, \mathbf{Z}(r))$ (Poincaré duality). Further, since ξ is nulhomologous, we have by duality

$$[\xi] \in H^{2r}_{|\xi|}(X, \mathbf{Z}(r))^\circ := \ker\big(H^{2r}_{|\xi|}(X, \mathbf{Z}(r)) \to H^{2r}(X, \mathbf{Z}(r))\big).$$

Hence ξ determines a morphism of MHS, $\mathbf{Z}(0) \to H^{2r}_{|\xi|}(X, \mathbf{Z}(r))^\circ$. From the short exact sequence of MHS,

$$0 \to H^{2r-1}(X, \mathbf{Z}(r)) \to H^{2r-1}(X \backslash |\xi|, \mathbf{Z}(r)) \to H^{2r}_{|\xi|}(X, \mathbf{Z}(r))^\circ \to 0,$$

we can pullback via this morphism to obtain another short exact sequence of MHS,

$$0 \to H^{2r-1}(X, \mathbf{Z}(r)) \to E \to \mathbf{Z}(0) \to 0.$$

Then $\Phi_r(\xi) := \{E\} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbf{Z}(0), H^{2r-1}(X, \mathbf{Z}(r)))$. This class $\{E\}$ is easy to calculate in $J^r(X)$, in terms of a membrane integral. Note that via duality,

$$E \subset H^{2r-1}(X \backslash |\xi|, \mathbf{Z}(r)) \simeq H_{2d-2r+1}(X, |\xi|, \mathbf{Z}(d-r)),$$

and that if ζ is a real $2d - 2r + 1$ chain such that $\partial\zeta = \xi$ on X, then $\{\zeta\} \in H_{2d-2r+1}(X, |\xi|, \mathbf{Z})$. One can show that the class $x \in W_0 E$ corresponding to the current

$$\frac{1}{(2\pi \mathrm{i})^{d-r}} \int_\zeta,$$

maps to $1 \in \mathbf{Z}(0)$. Now choose $y \in F^0 W_0 E_{\mathbf{C}}$ also mapping to $1 \in \mathbf{Z}(0)$. By Hodge type alone, the current corresponding to $x - y$ in the Poincaré dual description of $J^r(X)$ is the same as for $x = \frac{1}{(2\pi \mathrm{i})^{d-r}} \int_\zeta$, which is precisely the classical description of the Griffiths Abel–Jacobi map. This next result is a consequence of the work of Griffiths (see [26], as well as Sect. 14 of [36]).

Theorem 8.3. *If $F^{r-1}H^{2r-1}(X, \mathbf{C}) \cap H^{2r-1}(X, \mathbf{Q}(r)) = 0$, then there is an induced map*

$$\underline{\Phi}_r : \mathrm{Griff}^r(X) \to J^r(X).$$

In particular $\Phi_r(\mathrm{CH}^r_{\mathrm{alg}}(X)) = 0 \in J^r(X)$. *This is the case for a general CY threefold with* $r = 2$.

Example 8.4. We define the cycle class map $\mathrm{cl}_r : \mathrm{CH}^r(X) \to H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r))$. Recall the short exact sequence:

$$0 \to J^r(X) \to H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r)) \to Hg^r(X) \to 0.$$

Let $\xi \in \mathrm{CH}^r(X)$ with support $|\xi|$. One has a similar LES as in (9):

$$\cdots \to H^{2r-1}_{|\xi|}(X, \mathbf{Z}(r)) \oplus F^r H^{2r-1}_{|\xi|}(X, \mathbf{C}) \to H^{2r-1}_{|\xi|}(X, \mathbf{C})$$

$$\to H^{2r}_{\mathscr{D},|\xi|}(X, \mathbf{Z}(r)) \to H^{2r}_{|\xi|}(X, \mathbf{Z}(r)) \oplus F^r H^{2r}_{|\xi|}(X, \mathbf{C}) \xrightarrow{x-y} H^{2r}_{|\xi|}(X, \mathbf{C}) \to \cdots$$

Via Poincaré duality, one has cycle class maps

$$\xi \mapsto [(2\pi \mathrm{i})^{r-d}(\{\xi\}, \delta_\xi)] \in \ker\left(H^{2r}_{|\xi|}(X, \mathbf{Z}(r)) \oplus F^r H^{2r}_{|\xi|}(X, \mathbf{C}) \to H^{2r}_{|\xi|}(X, \mathbf{C})\right);$$

moreover recall that $H^{2r-1}_{|\xi|}(X, \mathbf{C}) = 0$ (weak purity). Thus we have a class $[\xi] \in H^{2r}_{\mathscr{D},|\xi|}(X, \mathbf{Z}(r))$. Now use the forgetful map

$$H^{2r}_{\mathscr{D},|\xi|}(X, \mathbf{Z}(r)) \to H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r)),$$

to define $\mathrm{cl}_r(\xi) \in H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r))$. From the injection

$$H^{2r}_{\mathscr{D},|\xi|}(X, \mathbf{Z}(r)) \hookrightarrow H^{2r}_{|\xi|}(X, \mathbf{Z}(r)) \oplus F^r H^{2r}_{|\xi|}(X, \mathbf{C}),$$

and the aforementioned forgetful map, in terms of the cone complex, $\mathrm{cl}_r(\xi)$ is represented by $((2\pi \mathrm{i})^{r-d}\{\xi\}, (2\pi \mathrm{i})^{r-d}\delta_\xi, 0)$. If $\xi \sim_{\mathrm{hom}} 0$, then $\xi = \partial\zeta$, $(2\pi \mathrm{i})^{r-d}\delta_\xi = dS$ for some $S \in F^r \mathscr{D}^{2r-1}_X(X)$. So

$$D((2\pi \mathrm{i})^{r-d}\zeta, S, 0) + ((2\pi \mathrm{i})^{r-d}\{\xi\}, (2\pi \mathrm{i})^{r-d}\delta_\xi, 0) = \left((0, 0, (2\pi \mathrm{i})^{r-d} \int_\zeta - S\right).$$

For $\omega \in F^{d-r+1}H^{2d-2r+1}(X, \mathbf{C})$,

$$(2\pi \mathrm{i})^{r-d} \int_\zeta \omega \; - \; S(\omega) = \frac{1}{(2\pi \mathrm{i})^{d-r}} \int_\zeta \omega,$$

by Hodge type. This is the Griffiths Abel–Jacobi map.

Both maps $(\mathrm{cl}_r, \ \Phi_r)$ can be combined to give

$$\mathrm{cl}_{r,0} : \mathrm{CH}^r(X) = \mathrm{CH}^r(X, 0) \to H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r)),$$

with commutative diagram:

$$\begin{array}{ccccccc}
0 \to & \mathrm{CH}^r_{\mathrm{hom}}(X) & \to \mathrm{CH}^r(X) \to & \frac{\mathrm{CH}^r(X)}{\mathrm{CH}^r_{\mathrm{hom}}(X)} & \to 0 \\
& \Phi_r \downarrow & \mathrm{cl}_{r,0} \downarrow & \mathrm{cl}_r \downarrow & \\
0 \to & J^r(X) & \to H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r)) \to & \mathrm{Hg}^r(X) & \to 0.
\end{array}$$

8.5 Deligne Cohomology and Normal Functions

Suppose that $\xi \in \mathrm{CH}^r(X)$ is given and that $Y \subset X$ is a smooth hypersurface. Then there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^r(X) & \to & \mathrm{CH}^r(Y) \\
\big\downarrow & & \big\downarrow \\
H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r)) & \to & H^{2r}_{\mathscr{D}}(Y, \mathbf{Z}(r));
\end{array}$$

Further, if we assume that the restriction $\xi_Y \in \mathrm{CH}_{\mathrm{hom}}(Y)$ is null-homologous, then $\mathrm{cl}_{r,0}(\xi) \in H^{2r}_{\mathscr{D}}(X, \mathbf{Z}(r)) \mapsto J^r(Y) \subset H^{2r}_{\mathscr{D}}(Y, \mathbf{Z}(r))$. Next, if $Y = X_0 \in \{X_t\}_{t\in S}$ is a family of smooth hypersurfaces of X, then such a ξ determines a holomorphically varying map $\nu_\xi(t) \in J^r(X_t)$, called a normal function. The class $\mathrm{cl}_r(\xi) = \delta(\nu_\xi) \in \mathrm{Hg}^r(X)$ is called the topological invariant of ν_ξ, i.e. ν_ξ determines $\mathrm{cl}_r(\xi)$. In [31], these ideas are extended in complete generality to the situation of the higher Chow groups, where the notion of "arithmetic normal functions" are introduced.

Example 8.6 (Griffiths' famous example ([26])). Let:

$$X = V(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5) \subset \mathbf{P}^5$$

be the Fermat quintic fourfold. Consider these three copies of $\mathbf{P}^2 \subset X$:

$$L_1 := V(z_0 + z_1, z_2 + z_3, z_4 + z_5),$$

$$L_2 := V(z_0 + \xi z_2, z_2 + \xi z_3, z_4 + z_5),$$

$$L_3 := V(z_0 + \xi z_1, z_2 + \xi z_3, z_4 + \xi z_5).$$

where ξ is a primitive 5-th root of unity. Then $L_1 \bullet (L_2 - L_3) = 1 \neq 0$, hence $\xi := [L_2 - L_3]$ is a non-zero class in $H^{2,2}(X, \mathbf{Z}(2))$. Further, if $\{X_t\}_{t\in U\subset \mathbf{P}^1}$ is a general pencil of smooth hyperplane sections of X, and if $t \in U$, then it is well known that $\xi_t \in \mathrm{CH}^2_{\mathrm{hom}}(X_t)$ by a theorem of Lefschetz. Since $\delta(\nu_\xi) = [L_2 - L_3] \neq 0$, it follows that $\nu_\xi(t)$ is non-zero for most $t \in U$. Therefore for general $t \in U$, $\mathrm{Griff}^2(X_t)$ contains an infinite cyclic group by Theorem 8.3. The upshot is that if:

$$Y = V\Big(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \big(\sum_{j=0}^{4} a_j z_j\big)^5\Big) \subset \mathbf{P}^4,$$

for general $a_0, \dots, a_4 \in \mathbf{C}$, then $\mathrm{Griff}^2(Y) \neq 0$ contains an infinite cyclic subgroup. H. Clemens was the first to show that the Griffiths group of a general quintic threefold in $\mathbf{P}^4$ is (countably) infinite dimensional, when tensored over $\mathbf{Q}$. Later it was shown by C. Voisin that the same holds for general CY threefolds. The idea is to make use of the rational curves on such threefolds.

Theorem 8.7 (See [7, 13, 22, 24, 26, 45]). *Let $X \subset \mathbf{P}^4$ be a (smooth) threefold of degree d. If $d \le 4$, then $\Phi_2 : \mathrm{CH}^2_{\mathrm{hom}}(X) \xrightarrow{\sim} J^2(X)$ is an isomorphism. Now assume that X is general. If $d \ge 6$ then $\mathrm{Im}(\Phi_2)$ is torsion. If $d = 5$, then $\mathrm{Im}(\Phi_2) \otimes \mathbf{Q}$ is countably infinite dimensional.*

Theorem 8.8 ([45]). *If X is a general Calabi–Yau threefold, then $\mathrm{Im}(\Phi_2)$ is countably infinite dimensional, when tensored over $\mathbf{Q}$. In particular, since $\Phi_2(\mathrm{CH}^2_{\mathrm{alg}}(X)) = 0$, it follows that $\mathrm{Griff}^2(X;\mathbf{Q})$ is (countably) infinite dimensional over $\mathbf{Q}$.*

8.9 Case $m = 1$ and $K3$ Surfaces

Recall the Tame symbol map

$$T : \bigoplus_{\mathrm{codim}_X Z = r-2} K_2^M(\mathbf{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D = r-1} K_1^M(\mathbf{C}(D)).$$

Then:

$$\mathrm{CH}^r(X,1) = H^{r-1}_{\mathrm{Zar}}(X, \mathscr{K}^M_{r,X}) \simeq \left\{ \frac{\sum_j (f_j, D_j) \mid \sum_j \mathrm{div}(f_j) = 0}{T(\Gamma(\bigoplus_{\mathrm{codim}_X Z = r-2} K_2^M(\mathbf{C}(Z))))} \right\}.$$

We recall:

Definition 8.10. The subgroup of $\mathrm{CH}^r(X,1)$ represented by $\mathbf{C}^\times \otimes \mathrm{CH}^{r-1}(X)$ is called the subgroup of decomposables $\mathrm{CH}^r_{\mathrm{dec}}(X,1) \subset \mathrm{CH}^r(X,1)$. The space of indecomposables is given by

$$\mathrm{CH}^r_{\mathrm{ind}}(X,1) := \frac{\mathrm{CH}^r(X,1)}{\mathrm{CH}^r_{\mathrm{dec}}(X,1)}.$$

The map

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r_{\mathrm{hom}}(X,1) \to H^{2r-1}_{\mathscr{D}}(X, \mathbf{Z}(r)),$$

is given by a map

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r_{\mathrm{hom}}(X,1) \to \frac{F^{d-r+1} H^{2d-2r+2}(X,\mathbf{C})^\vee}{H_{2d-2r+2}(X, \mathbf{Z}(d-r))},$$

defined as follows. Assume given a higher Chow cycle $\xi = \sum_{i=1}^N (f_i, Z_i)$ representing a class in $\mathrm{CH}^r_{\mathrm{hom}}(X,1)$. Then via a proper modification, we can view $f_i : Z_i \to \mathbf{P}^1$ as a morphism, and consider the $2d-2r+1$-chain $\gamma_i = f_i^{-1}([-\infty, 0])$. Then

$\sum_{i=1}^{N}\operatorname{div}(f_i)=0$ implies that $\gamma:=\sum_{i=1}^{N}\gamma_i$ defines a $2d-2r+1$-cycle. Since ξ is null-homologous, it is easy to show that γ bounds some real dimensional $2d-2r+2$-chain ζ in X, viz., $\partial\zeta=\gamma$. For $\omega\in F^{d-r+1}H^{2d-2r+2}(X,\mathbf{C})$, the current defining $\mathrm{cl}_{r,1}(\xi)$ is given by:

$$\mathrm{cl}_{r,1}(\xi)(\omega) = \frac{1}{(2\pi\mathrm{i})^{d-r+1}}\Big[\sum_{i=1}^{N}\int_{Z_i\setminus\gamma_i}\omega\log f_i - 2\pi\mathrm{i}\int_{\zeta}\omega\Big],$$

where we choose the principal branch of the log function. (This is different branch from the one chosen in [35], for this regulator.) One can easily check that the current defined above is d-closed. Namely, if we write $\omega=d\eta$ for some $\eta\in F^{d-r+1}E_X^{2d-2r}$, then by a Stokes' theorem argument, both integrals above contribute to "periods" which cancel. The details of this argument can be found in [21], but quite generally can be found in [32].

Using the description of real Deligne cohomology given above, and the regulator formula, we arrive at the formula for the real regulator $r_{r,1}:\mathrm{CH}^r(X,1)\to H_{\mathscr{D}}^{2r-1}(X,\mathbf{R}(r))=H^{r-1,r-1}(X,\mathbf{R}((r-1))\simeq H^{d-r+1,d-r+1}(X,\mathbf{R}(d-r+1))^{\vee}$. Namely:

$$r_{r,1}(\xi)(\omega)=\frac{1}{(2\pi\mathrm{i})^{d-r+1}}\sum_j\int_{Z_j}\omega\log|f_j|.$$

Example 8.11. Suppose that X is a surface. Then we have

$$\mathrm{cl}_{2,1}:\mathrm{CH}^2_{\mathrm{hom}}(X,1)\to\frac{\{H^{2,0}(X)\oplus H^{1,1}(X)\}^{\vee}}{H_2(X,\mathbf{Z})}.$$

The corresponding transcendental regulator is defined to be

$$\Phi_{2,1}:\mathrm{CH}^2_{\mathrm{hom}}(X,1)\to\frac{H^{2,0}(X)^{\vee}}{H_2(X,\mathbf{Z})},$$

$$\Phi_{2,1}(\xi)(\omega)=\int_{\zeta}\omega.$$

and real regulator

$$r_{2,1}:\mathrm{CH}^2(X,1)\to H^{1,1}(X,\mathbf{R}(1))^{\vee}\simeq H^{1,1}(X,\mathbf{R}(1)),$$

$$r_{2,1}(\xi)(\omega)=\frac{1}{2\pi\mathrm{i}}\sum_j\int_{Z_j}\log|f_j|\omega.$$

There is an induced map

$$\underline{r}_{2,1}:\mathrm{CH}^2_{\mathrm{ind}}(X,1)\to H^{1,1}_{\mathrm{tr}}(X,\mathbf{R}(1)).$$

If X is a $K3$ surface, then $\mathrm{CH}^2_{\mathrm{hom}}(X,1)=\mathrm{CH}^2(X,1)$, hence there is an induced map

$$\underline{\Phi}_{2,1}:\mathrm{CH}^2_{\mathrm{ind}}(X,1)\to\frac{H^{2,0}(X)^{\vee}}{H_2(X,\mathbf{Z})}.$$

Theorem 8.12. (i) *([40]) Let $X \subset \mathbf{P}^3$ be a smooth surface of degree d. If $d \leq 3$, then $r_{2,1} : \mathrm{CH}^2(X, 1) \to H^{1,1}(X, \mathbf{R}(1))$ is surjective; moreover $\mathrm{CH}^2_{\mathrm{ind}}(X, 1; \mathbf{Q}) = 0$. Now assume that X is general. If $d \geq 5$, then $Im(r_{2,1})$ is "trivial", i.e. its image in the transcendental part of $H^{1,1}(X, \mathbf{R}(1))$ is zero.*

(ii) *[Hodge-$\mathscr{D}$-conjecture for K3 surfaces ([10])] Let X be a general member of a universal family of projective K3 surfaces, in the sense of the real analytic topology. Then*

$$r_{2,1} : \mathrm{CH}^2(X, 1) \otimes \mathbf{R} \to H^{1,1}(X, \mathbf{R}(1)),$$

is surjective.

(iii) *([12]) Let $X/\mathbf{C}$ be a general algebraic K3 surface. Then the transcendental regulator $\Phi_{2,1}$ is non-trivial. Quite generally, if X is a general member of a general subvariety of dimension $20 - \ell$, describing a family of K3 surfaces with general member of Picard rank ℓ, with $\ell < 20$, then $\Phi_{2,1}$ is non-trivial.*

Remark 8.13. (i) Regarding part (iii) of Theorem 8.12, one can ask whether $\Phi_{2,1}$ can be non-trivial for those K3 surfaces X with Picard rank 20, (which are rigid and therefore defined over $\overline{\mathbf{Q}}$)? In [12], some evidence is provided in support of this.

(ii) One of the key ingredients in the proof of the above theorem is the existence of plenty of nodal rational curves on a general K3 surface. Indeed, there is the following result:

Theorem 8.14 ([11]). *For a general K3 surface, the union of rational curves on X is a dense subset in the analytic topology.*

Remark 8.15. It is well known that for an elliptic curve E defined over an algebraically closed subfield $k \subset \mathbf{C}$, the torsion subgroup $E_{\mathrm{tor}}(\mathbf{C}) \subset E(k)$. An analogous result holds for rational curves on a K3 surface. Quite generally, the following result which may be common knowledge among experts, seems worthwhile mentioning:

Proposition 8.16. *Assume given $X/\mathbf{C}$ a smooth projective surface with* $\mathrm{Pg}(X) := \dim H^{2,0}(X) > 0$. *If we write $X/\mathbf{C} = X_k \times_k \mathbf{C}$, viz., $X/\mathbf{C}$ obtained by base change from a smooth projective surface X_k defined over an algebraically closed subfield $k \subset \mathbf{C}$, and if $C \subset X/\mathbf{C}$ is a rational curve, then C is likewise defined over k.*

Proof. By a standard spread argument, there is a smooth projective variety S/k of dimension ≥ 0, and a k-family $\mathscr{C} \twoheadrightarrow S$ of rational curves containing C as a general member, with embedding h:

$$\begin{array}{ccccc} \mathscr{C} & \xrightarrow{h} & S \times_k X & \xrightarrow{Pr_X} & X \\ & {\scriptstyle Pr_S} \searrow & & \swarrow & \\ & & S & & \end{array}$$

Since $\mathrm{Pg}(X) > 0$, there are only at most a countable number of rational curves on $X/\mathbf{C}$, and hence $Pr_X(h(\mathscr{C})) = Pr_X(h(Pr_S^{-1}(t)))$ for any $t \in S(\mathbf{C})$. Now use the fact that $S(k) \neq \emptyset$. $\square$

Now suppose that X is a $K3$ surface defined over $\overline{\mathbf{Q}}$. Let $\Sigma \subset X$ be the union of all rational curves on X. Then Σ is defined over $\overline{\mathbf{Q}}$. In discussions with Matt Kerr (personal communication), we naively raise the following:

Question 8.17. Is $X(\overline{\mathbf{Q}}) \subset \Sigma(\overline{\mathbf{Q}})$?

An affirmative answer to this question would not only imply that Σ is dense in $X(\mathbf{C})$ in the usual topology, but this would also provide a nontrivial instance of the Bloch–Beilinson conjecture on the injectivity of Abel–Jacobi maps for smooth projective varieties defined over $\overline{\mathbf{Q}}$. More specifically, by an application of the connectedness part of Bertini's theorem, Σ is connected, hence $\mathrm{CH}^2_{\mathrm{hom}}(X/\overline{\mathbf{Q}}) = 0$.

8.18 Torsion Indecomposables

The story about torsion indecomposable classes takes an interesting turn from the geometric story presented in Theorem 8.12(i). The situation is this, and for the moment let X be any projective algebraic manifold. An elementary consequence of the Merkurjev–Suslin theorem implies the following:

Theorem 8.19 (See [15]). *The kernel of the Abel–Jacobi map*

$$\underline{AJ}_X : \frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} \to J\left(\frac{H^2(X,\mathbf{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbf{Z}(2))}\right),$$

is uniquely divisible. This implies that $\underline{AJ}_X$ is injective on torsion indecomposables $\left\{\frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)}\right\}_{\mathrm{tor}}$.

(Here we remind the reader that since we are working integrally, we have an inclusion that for torsion reasons, need not be an equality:

$$\frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} \subseteq \frac{\mathrm{CH}^2(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)} =: \mathrm{CH}^2_{\mathrm{ind}}(X,1).)$$

On the other hand, one has the torsion subgroup $\{\mathrm{CH}^2_{\mathrm{ind}}(X,1)\}_{\mathrm{tor}}$. Put

$$H^2_{\mathrm{tr}}(X,\mathbf{Q}(2)/\mathbf{Z}(2)) = \mathrm{Cokernel}\big(H^2_{\mathrm{alg}}(X,\mathbf{Q}(2)/\mathbf{Z}(2)) \to H^2(X,\mathbf{Q}(2)/\mathbf{Z}(2))\big).$$

Theorem 8.20 ([29]). *There is an identification*

$$\{\mathrm{CH}^2_{\mathrm{ind}}(X,1)\}_{\mathrm{tor}} \xrightarrow{\sim} H^2_{\mathrm{tr}}(X,\mathbf{Q}(2)/\mathbf{Z}(2)).$$

In light of these two theorems, one expects that

$$\underline{AJ}_X : \left\{\frac{\mathrm{CH}^2_{\mathrm{hom}}(X,1)}{\mathrm{CH}^2_{\mathrm{dec}}(X,1)}\right\}_{\mathrm{tor}} \xrightarrow{\sim} \left\{J\left(\frac{H^2(X,\mathbf{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbf{Z}(2))}\right)\right\}_{\mathrm{tor}}.$$

For example, suppose that X is a $K3$ surface of Picard rank 20. Then $E := J\left(\frac{H^2(X,\mathbf{Z}(2))}{H^2_{\mathrm{alg}}(X,\mathbf{Z}(2))}\right)$ is an elliptic curve defined over a number field. In this case one expects the identification

$$\{\mathrm{CH}^2_{\mathrm{ind}}(X,1)\}_{\mathrm{tor}} \xrightarrow{\sim} \{E(\overline{\mathbf{Q}})\}_{\mathrm{tor}}.$$

8.21 Case m = 2 and Elliptic Curves

Regulator examples on $\mathrm{CH}^2(X,2)$. Let X be a compact Riemann surface. In [38] there is constructed a real regulator

$$r : \mathrm{CH}^2(X,2) \to H^1(X,\mathbf{R}(1)), \tag{13}$$

given by

$$\begin{aligned}\omega \in H^1(X,\mathbf{R}) \simeq H^1(X,\mathbf{R}(1))^\vee \mapsto & \int_X \Big[\log|f| d\log|g| - \log|g| d\log|f|\Big] \wedge \omega \\ &= 2\int_X \log|f| d\log|g| \wedge \omega, \text{ (by a Stokes}' \text{ theorem argument).}\end{aligned} \tag{14}$$

Alternatively, up to a twist, and real isomorphism, this is the same as the real part of the regulator $\mathrm{cl}_{2,2}$ in (4), viz.,

$$r_{2,2}(\omega) = \frac{1}{2\pi}\int_X [\log|f| d\arg g - \log|g| d\arg f)] \wedge \omega, \tag{15}$$

where the formula for:

$$\mathrm{cl}_{2,2} : \mathrm{CH}^2(X,2) \to H^2_{\mathscr{D}}(X,\mathbf{Z}(2)) \simeq H^1(X,\mathbf{C}/\mathbf{Z}(2)) \simeq \frac{H^1(X,\mathbf{C})}{H_1(X,\mathbf{Z}(-1))} = \frac{H^1(X,\mathbf{C})}{H_1(X,\mathbf{Z})(1)}, \tag{16}$$

(for $\omega \in H^1(X,\mathbf{C})$), which can be found for example in [32], is induced, up to a factor[10] of $(2\pi \mathrm{i})^{-1}$, by:

$$\{f,g\} \quad \mapsto \tag{17}$$

[10] The decision to consider the factor $(2\pi\mathrm{i})^{-1}$ is somewhat "political", as reflected in the remark on page 2 of [32]. From a cohomological point of view, one works with $\mathbf{Z}(2)$ coefficient periods, whereas homologically, is it with $\mathbf{Z}(1)$ coefficients. This is neatly illustrated via the Poincaré duality isomorphism in (16).

$$\int_{X\backslash f^{-1}[-\infty,0]} \log f d\log g \wedge \omega \; - 2\pi\mathrm{i} \int_{f^{-1}[-\infty,0]\backslash (f\times g)^{-1}[-\infty,0]^2} \log g \wedge \omega \; + \; (2\pi\mathrm{i})^2 \int_\zeta \omega,$$

where if we assume for the moment that $T\{f,g\}) = 0$, then ζ is a real membrane with $\partial\zeta = (f \times g)^{-1}[-\infty,0]^2$. Otherwise if $T\{f,g\} \neq 0$, we are then dealing with a situation where $\{f,g\}$ is replaced by a given $\prod_\alpha \{f_\alpha, g_\alpha\}$, where

$$T\Big(\prod_\alpha \{f_\alpha, g_\alpha\}\Big) = \sum_\alpha T\{f_\alpha, g_\alpha\} = 0,$$

and accordingly arrive at a corresponding ζ. Note that (17) is really the current written in the slang form:

$$[\,\log f d\log g - 2\pi\mathrm{i}\log g\delta_{f^{-1}(\mathbf{R}^-)} + (2\pi\mathrm{i})^2\delta_\zeta] =: \tilde{R}.$$

To connect formulas (15) and (17), one takes the imaginary part of $\tilde{R}$ (consistent with $\mathbf{C}/\mathbf{Z}(2) \twoheadrightarrow \mathbf{C}/\mathbf{R}(2) \simeq \mathbf{R}(1)$). This gives us

$$\mathrm{Im}(\tilde{R}) = [\,\log|f| d\arg g + \arg f d\log|g| - 2\pi\log|g|\delta_{f^{-1}(\mathbf{R}^-)}].$$

Now add the coboundary current $d[\,\log|g|\arg f]$ and apply a Stokes' theorem argument.[11]

8.22 *Constructing $K_2(X)$ Classes on Elliptic Curves X*

We consider the following trick due to Bloch [5]. Let X be an elliptic curve and assume given $f,\ g \in \mathbf{C}(X)^\times$ such that $\Sigma := |\mathrm{div}(f)| \cup |\mathrm{div}(g)|$ are points of order N in $\mathrm{Pic}(X)$. Then

$$T(\{f,g\}^N) \in \bigoplus \mathbf{C}^\times \quad \text{and} \ \mapsto 0 \in \mathrm{Pic}(X) \otimes \mathbf{C}^\times.$$

A clarification. This uses the Weil reciprocity theorem. Let X be a compact Riemann surface, $f,g \in \mathbf{C}(X)^\times$, and for $p \in X$, write

$$T_p\{f,g\} = (-1)^{\nu_p(g)\nu_p(f)}\bigg(\frac{f^{\nu_p(g)}}{g^{\nu_p(f)}}\bigg)\bigg|_p \in \mathbf{C}^\times.$$

Note that for $p \notin |\mathrm{div}(f)| \bigcup |\mathrm{div}(g)|$, we have $T_p\{f,g\} = 1$. Thus we can write $T\{f,g\} = \sum_{p\in X} T_p\{f,g\}$. Weil reciprocity says that $\prod_{p\in X} T_p\{f,g\} = 1$. Let us rewrite this as follows. If we write $T\{f,g\} = \sum_{j=1}^M (c_j, p_j)$, where $p_j \in X$ and $c_j \in \mathbf{C}^\times$, then $\prod_{j=1}^M c_j = 1$. Now fix $p \in X$ and let us suppose that $Np_j \sim_{\mathrm{rat}} Np$ for all j. Thus there exists $h_j \in \mathbf{C}(X)^\times$ such that $(h_j) = Np_j - Np$. Then $T\{h_j, c_j\} = (c_j^N, p) + (c_j^{-N}, p_j)$. The result is that

[11] Alternatively, taking $\mathrm{Re}((2\pi\mathrm{i})^{-1}\tilde{R})$ gives the formula in (15), viz., with the factor $(2\pi)^{-1}$, right on the nose.

$$T(\{f,g\}^N\{h_1,c_1\}\cdots\{h_M,c_M\}) = \prod_{j=1}^{M}(c_j^N,p) = (1,p) = 0.$$

Thus there exists $\{h_i\} \in \mathbf{C}(X)^\times$ and $\{c_i\} \in \mathbf{C}^\times$ such that $\{f,g\}^N \prod\{h_i,c_i\} \in \mathrm{CH}^2(X,2)$. Note that the terms $\{h_i,c_i\}$ do not contribute to the regulator value by the formula in (14) above. Clearly this construction takes advantage of the existence of a dense subset of torsion points on X. Bloch (*op. cit.*) shows that the real regulator is nontrivial for general elliptic curves, and indeed A. Collino [14] shows that the regulator image of $\mathrm{CH}^2(X,2)$ for a general elliptic curve X is infinite dimensional (over $\mathbf{Q}$). Actually it is pretty easy to see why $r_{2,2}$ is non-trivial for a general elliptic curve:

Theorem 8.23 (Hodge-$\mathcal{D}$-conjecture for elliptic curves). *If X is a general elliptic curve in the real analytic Zariski topology, then $r_{2,2}$ is surjective.*

Proof. Let X be an elliptic curve given in affine coordinates by the equation $y^2 = h(x)$, where $h(x)$ is a cubic polynomial with distinct roots. A basis for $H^1(X,\mathbf{R})$ is given by

$$\omega_1 := \frac{dx}{y} + \frac{d\overline{x}}{\overline{y}} \quad ; \quad \omega_2 := \mathrm{i}\Big(\frac{dx}{y} - \frac{d\overline{x}}{\overline{y}}\Big).$$

Next, we consider

$$f_1 := y + \mathrm{i}x \quad ; \quad f_2 = y + x \quad ; \quad g_1 = g_2 = x.$$

We claim that for general X,

$$\det\begin{bmatrix} \int_X \log|f_1| d\log|g_1| \wedge \omega_1 & \int_X \log|f_1| d\log|g_1| \wedge \omega_2 \\ \int_X \log|f_2| d\log|g_2| \wedge \omega_1 & \int_X \log|f_2| d\log|g_2| \wedge \omega_2 \end{bmatrix} \neq 0. \tag{18}$$

Now let us first assume that X is given for which (18) holds, and note that the rational functions f_1, f_2, g_1, g_2 can each be expressed in the form L_1/L_2, where L_j are homogeneous linear polynomials in the homogeneous coordinates of $\mathbf{P}^2$ (and where $X \subset \mathbf{P}^2$). Since X has a dense subset of torsion points X_{tor}, and by Abel's theorem, one can find $\tilde{L}_j$ "close" to L_j, $j = 1,2$, such that $\tilde{L}_j \cap X \subset X_{\mathrm{tor}}$. Thus up to $\mathbf{C}^\times$ multiple, $\tilde{L}_1/\tilde{L}_2$ is "close" to L_1/L_2. Hence one can find $\tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2$ for which

$$\Big\{|\mathrm{div}(\tilde{f}_1)| \bigcup |\mathrm{div}(\tilde{f}_2)| \bigcup |\mathrm{div}(\tilde{g}_1)| \bigcup |\mathrm{div}(\tilde{g}_2)|\Big\} \subset X_{\mathrm{tor}}, \tag{19}$$

and that by continuity considerations,

$$\det\begin{bmatrix} \int_X \log|\tilde{f}_1| d\log|\tilde{g}_1| \wedge \omega_1 & \int_X \log|\tilde{f}_1| d\log|\tilde{g}_1| \wedge \omega_2 \\ \int_X \log|\tilde{f}_2| d\log|\tilde{g}_2| \wedge \omega_1 & \int_X \log|\tilde{f}_2| d\log|\tilde{g}_2| \wedge \omega_2 \end{bmatrix} \neq 0. \tag{20}$$

Thus one can complete $\{\tilde{f}_1, \tilde{g}_1\}$, $\{\tilde{f}_2, \tilde{g}_2\}$ to classes $\xi_1,\ \xi_2 \in \mathrm{CH}^2(X, 2)$, for which

$$\det\begin{bmatrix} r_{2,2}(\xi_1)(\omega_1) & r_{2,2}(\xi_1)(\omega_2) \\ r_{2,2}(\xi_2)(\omega_1) & r_{2,2}(\xi_2)(\omega_2) \end{bmatrix} \neq 0, \tag{21}$$

and so modulo the claim in (18), we are done. We sketch a proof of the claim. With regard to $dV = d\mathrm{Re}(x) \wedge d\mathrm{Im}(x)$:

$$\frac{d\log|x| \wedge \omega_1}{2} = \frac{1}{4}\Big(\frac{1}{x\overline{y}} - \frac{1}{\overline{x}y}\Big)dx \wedge d\overline{x} = \frac{\mathrm{Im}(\overline{x}y)}{|x|^2|y|^2}dV \tag{22}$$

$$\frac{d\log|x| \wedge \omega_2}{2} = -\frac{\mathrm{i}}{4}\Big(\frac{1}{x\overline{y}} + \frac{1}{\overline{x}y}\Big)dx \wedge d\overline{x} = -\frac{\mathrm{Re}(\overline{x}y)}{|x|^2|y|^2}dV \tag{23}$$

Now let us degenerate X to the rational elliptic curve X_0 given by $y^2 = x^3$. Note that X_0 is given parametrically by $(x, y) = (z^2, z^3)$, $z \in \mathbf{C}$. Thus $\overline{x}y = |z|^4 z$, and up to a real positive multiplicative constant times the standard volume element on $\mathbf{C}$, which we will denote by dV_0, (22) and (23) become:

$$d\log|x| \wedge \omega_1 = \frac{\mathrm{Im}(z)}{|z|^4}dV_0 \quad ; \quad d\log|x| \wedge \omega_2 = -\frac{\mathrm{Re}(z)}{|z|^4}dV_0. \tag{24}$$

Let $\mathbf{H} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) \geq 0\}$ be the upper half plane. Now one has the following *formal* calculations after degenerating to X_0, and using symmetry arguments:

$$\int_{X_0} \log|f_1| d\log|g_1| \wedge \omega_1 = \int_{\mathbf{C}} \log|z^3 + \mathrm{i}z^2| \frac{\mathrm{Im}(z)}{|z|^4}dV_0 \tag{25}$$

$$= \int_{\mathbf{C}} \log|z + \mathrm{i}| \frac{\mathrm{Im}(z)}{|z|^4}dV_0 = \int_{\mathbf{H}} \log\Big|\frac{z + \mathrm{i}}{\overline{z} + \mathrm{i}}\Big| \frac{\mathrm{Im}(z)}{|z|^4}dV_0 \mapsto +\infty,$$

using the fact

$$\Big|\frac{z + \mathrm{i}}{\overline{z} + \mathrm{i}}\Big| > 1 \Leftrightarrow \mathrm{Im}(z) > 0.$$

$$\int_{X_0} \log|f_2| d\log|g_2| \wedge \omega_1 = \int_{\mathbf{C}} \log|z + 1| \frac{\mathrm{Im}(z)}{|z|^4}dV_0 = 0. \tag{26}$$

For the remaining two formal calculations, put $w = \mathrm{i}z$, and note that $\mathrm{Re}(z) = \mathrm{Im}(w)$, and that $|z + 1| = |w + \mathrm{i}|$. Then

$$\int_{X_0} \log|f_2| d\log|g_2| \wedge \omega_2 = -\int_{\mathbf{C}} \log|z + 1| \frac{\mathrm{Re}(z)}{|z|^4}dV_0 \tag{27}$$

$$= -\int_{\mathbf{C}} \log|w + \mathrm{i}| \frac{\mathrm{Im}(w)}{|w|^4}dV_0 = -\int_{\mathbf{H}} \log\Big|\frac{z + \mathrm{i}}{\overline{z} + \mathrm{i}}\Big| \frac{\mathrm{Im}(z)}{|z|^4}dV_0 \mapsto -\infty.$$

$$\int_{X_0} \log|f_1| d\log|g_1| \wedge \omega_2 = -\int_{\mathbf{C}} \log|z + \mathrm{i}| \frac{\mathrm{Re}(z)}{|z|^4}dV_0 = 0. \tag{28}$$

Note that two of these integrals blow up over the singular point $z = 0$ of the singular curve X_0, as expected. By using the Lebesgue theory of integration, we can make the calculations in (25)–(28) more precise. First, by using the projection $(x, y) \mapsto x$, we have a double covering $X \to \mathbf{P}^1$. Thus for $f, g \in \mathbf{C}(X)$, and $\omega = \omega_1$ or $\omega = \omega_2$, we can express $\int_X \log|f| d\log|g| \wedge \omega$ as the integral of some Lebesgue integrable function $H(x)$ over $\mathbf{P}^1$. Next, by converting to polar coordinates, viz. $x = \mathrm{e}^{\mathrm{i}t}$, we can Fubini integrate in $t \in [0, 2\pi]$ and $r \in [0, \infty]$. Let $h(r)$ be the result of integrating $H(x)$ with respect to t over $[0, 2\pi]$. As X degenerates to X_0, we can construct a sequence $\{h_n(r)\}$ which limits to $h_\infty(r)$ over X_0. In the cases of (25)–(28), we have that $h_\infty(r)$ is either zero, nonnegative, or nonpositive. By using the standard Lebesgue integral limit theorems, we arrive at the claim in (18), and hence the theorem. □

For curves X of genus $g > 1$, the problem of constructing classes in $\mathrm{CH}^2(X, 2)$ seems to be related to the fact that under the Abel–Jacobi mapping $\Phi : X \to J^1(X), \quad p \mapsto \{p - p_0\}$, the inverse image of the torsion subgroup, $\Phi^{-1}(J^1(X)_{\mathrm{tor}})$, is finite, this being the import of the Mumford–Manin theorem (see [43] for a proof). Indeed as explained in [38] (as well as in [39]), one can prove a weak version of the Mumford–Manin theorem based on the fact that for a general curve X of genus $g > 1$, the image of the regulator map $\mathrm{cl}_{2,2} : \mathrm{CH}^2(X, 2) \to H^2_{\mathscr{D}}(X, \mathbf{Z}(2))$ is torsion (A. Collino [14]). Collino's approach (*op. cit.*) uses infinitesimal methods. The reader should also consult [23] for similar refined results in this direction. For the benefit of the reader, we will provide an ad hoc explanation as to why this is the case (in "Observation 1" below). In order to do so, we must first digress and consider the following setting.

Assume given a dominant morphism $\overline{\rho} : \overline{X} \to \overline{C}$ of smooth complex projective varieties, where $\overline{X}$ is a surface and $\overline{C}$ is a curve. Let $C \subset \overline{C}$ be an affine open subset over which $\overline{\rho}$ is smooth, and $\Sigma := \overline{C} \backslash C$, $X = \overline{\rho}^{-1}(C)$ and $\rho = \overline{\rho}\big|_X : X \to C$. For $t \in \Sigma$, we will assume that the singular set of X_t is a single node. Next, we will assume given a class $\{\xi\} \in \mathrm{CH}^2(X, 2)$. In particular $\partial\xi = 0$ on X (here ∂ is the same thing as the Tame symbol). Note that ξ is given by a product of symbols of the form $\{f, g\}$, where $f, g \in \mathbf{C}(X)^\times$. However, since $\mathbf{C}(X) = \mathbf{C}(\overline{X})$, one can also think of ξ as defined on $\overline{X}$ (call it $\overline{\xi}$) with $\partial\overline{\xi}$ supported on $\overline{X}_\Sigma := \overline{\rho}^{-1}(\Sigma)$. Now for $t \in \Sigma$, the contribution ("residue") of $\partial\overline{\xi}$ gives rise to a class in $\mathrm{CH}^1(X_t, 1)$. If X_t were smooth, then $\mathrm{CH}^1(X_t, 1) = \mathbf{C}^\times$; but here we are assuming that X_t has a single node $P \in X_t$ as singularity. Under the desingularization $\sigma : \tilde{X}_t \to X_t$, let $\{Q, R\} = \sigma^{-1}(P)$. Next if $Q - R \in \mathrm{CH}^1_{\mathrm{tor}}(\tilde{X}_t)$, then for some integer N, $N \cdot (Q - R) = \mathrm{div}(f)$ for some $f \in \mathbf{C}(\tilde{X}_t)^\times$. But on X_t, $\mathrm{div}(f) = 0$, and hence $\mathbf{C}^\times \subsetneq \mathrm{CH}^1(X_t, 1)$. The upshot is that if $\partial\overline{\xi}$ contributes to a nonzero element of $\mathrm{CH}^1(X_t, 1)/\mathbf{C}^\times$ for some $t \in \Sigma$, then via a residue calculation and a calculation of the MHS $H^2(X, \mathbf{Q}(2))$, the current $d\log\xi$ (induced by $\{f, g\} \mapsto d\log f \wedge d\log g$) will contribute to a nonzero class in $[d\log\xi] \in \Gamma H^2(X, \mathbf{Q}(2))$. *The converse statement also holds:* if $Q - R \notin \mathrm{CH}^1_{\mathrm{tor}}(\tilde{X}_t)$, for all such $t \in \Sigma$, then $[d\log\xi] = 0 \in \Gamma H^2(X, \mathbf{Q}(2))$. Next, the Leray spectral sequence associated to ρ (which by Deligne, degenerates at E_2, see [25] (p. 466)), together with the fact that since C is an affine curve (hence $H^2(C, R^0\rho_*\mathbf{Q}(2)) = 0$), yields the short exact sequence of MHS:

$$0 \to H^1(C, R^1\rho_*\mathbf{Q}(2)) \to H^2(X, \mathbf{Q}(2)) \to H^0(C, R^2\rho_*\mathbf{Q}(2)) \to 0.$$

Note that $\Gamma H^0(C, R^2\rho_*\mathbf{Q}(2)) = 0$ as $H^0(C, R^2\rho_*\mathbf{Q}(2))$ is of pure weight -2. Hence $\Gamma H^2(X, \mathbf{Q}(2)) = \Gamma H^1(C, R^1\rho_*\mathbf{Q}(2))$. On the other hand, for $t \in C$, ξ restricts to a class $\xi_t \in \mathrm{CH}^2(X_t, 2)$, and hence we have a normal function

$$\nu_\xi : C \to \bigcup_{t \in C} J(H^1(X_t, \mathbf{Z}(2))),$$

whose topological invariant is the aforementioned class $[d\log\xi] \in \Gamma H^1(C, R^1\rho_*\mathbf{Q}(2))$, and which we will now denote it by $\delta(\nu_\xi) := [d\log\xi]$. It is a general fact that there is a short exact sequence:

$$0 \to J(H^0(C, R^1\rho_*\mathbf{Q}(2))) \to \begin{Bmatrix}\text{Normal}\\ \text{functions}\end{Bmatrix}_{\mathbf{Q}} \xrightarrow{\delta} \Gamma H^1(C, R^1\rho_*\mathbf{Q}(2)) \to 0, \qquad (29)$$

where $\{\cdots\}_{\mathbf{Q}}$ means with respect to $\mathbf{Q}$-periods. We will explain this in more detail below, but comment in passing that the technical details can be found in [31]. If $\delta(\nu_\xi) = 0$, then $\nu_\xi \in J(H^0(C, R^1\rho_*\mathbf{Q}(2)))$, i.e. belongs to the fixed part of a corresponding variation of Hodge structure. The situation is not unlike what occurs in the short exact sequence involving Deligne cohomology in (8) above, and the nature of this argument is completely analogous to that in Example 8.6. We can frame this discussion in more precise terms. One has a cycle class map $\mathrm{cl}_{2,2} : \mathrm{CH}^2(X, 2) \to H^2_{\mathcal{D}}(X, \mathbf{Z}(2))$, (Deligne–Beilinson cohomology); moreover by a weight argument, there is a short exact sequence:

$$0 \to J(H^1(X, \mathbf{Z}(2))) \to H^2_{\mathcal{D}}(X, \mathbf{Z}(2)) \to \Gamma H^2(X, \mathbf{Z}(2)) \to 0. \qquad (30)$$

For $t \in C$, X_t is a smooth curve. Then for such t, $H^2_{\mathcal{D}}(X_t, \mathbf{Z}(2)) = J(H^1(X_t, \mathbf{Z}(2)))$, and accordingly the map

$$t \in C \mapsto \mathrm{cl}_{2,2}(\xi_t) \in J(H^1(X_t, \mathbf{Z}(2))),$$

is our normal function ν_ξ; moreover the image of ξ via the composite

$$\mathrm{CH}^2(X, 2) \to H^2_{\mathcal{D}}(X, \mathbf{Q}(2)) \to \Gamma H^2(X, \mathbf{Q}(2)) = \Gamma H^1(C, R^1\rho_*\mathbf{Q}(2)),$$

is precisely $\delta(\nu_\xi)$. Finally to explain (29) more precisely, we observe that there is a short exact sequence:

$$0 \to H^1(C, R^0\rho_*\mathbf{Q}(2))) \to H^1(X, \mathbf{Q}(2)) \to H^0(C, R^1\rho_*\mathbf{Q}(2)) \to 0.$$

But $\Gamma H^0(C, R^1\rho_*\mathbf{Q}(2)) = 0$, hence we arrive at the short exact sequence:

$$0 \to J(H^1(C, R^0\rho_*\mathbf{Q}(2)))) \to J(H^1(X, \mathbf{Q}(2))) \to J(H^0(C, R^1\rho_*\mathbf{Q}(2))) \to 0.$$

This together with (29) and $(30)_{\mathbf{Q}}$ leads to the identification:

$$\begin{Bmatrix}\text{Normal}\\ \text{functions}\end{Bmatrix}_{\mathbf{Q}} \simeq \frac{H^2_{\mathscr{D}}(X,\mathbf{Q}(2))}{J(H^1(C,R^0\rho_*\mathbf{Q}(2)))}.$$

Now having discussed the relationship between a cycle class $\xi \in \mathrm{CH}^2(X,2)$, the associated normal function ν_ξ, and the topological invariant $\delta(\nu_\xi) \in \varGamma H^2(X,\mathbf{Q}(2))$ and how it is related to the "torsion" nature of the nodal singularities of the singular fibers $\{X_t\}_{t\in\varSigma}$, we are led to consider two divergent observations:

Observation 1. Suppose that X_0 is a general curve of genus $g > 1$. By general, we can assume that X_0 is a very general member of a pencil of curves $\{X_t\}_{t\in\mathbf{P}^1}$, defining a smooth surface $\overline{X}_{\mathbf{P}^1} := \bigcup_{t\in\mathbf{P}^1} X_t \to \mathbf{P}^1$, whose singular fibers are Lefschetz, i.e. admit a single ordinary node. Let $\xi_0 \in \mathrm{CH}^2(X_0,2)$. After a suitable base extension $\overline{C} \to \mathbf{P}^1$, for some smooth projective curve $\overline{C}$, and corresponding $\overline{X} := \overline{C} \times_{\mathbf{P}^1} \overline{X}_{\mathbf{P}^1}$, with setting as in the above discussion, ξ_0 will then spread to a class $\xi \in \mathrm{CH}^2(X,2)$, in a general family $\rho : X \to C$, where ρ is smooth and proper over an affine curve C. Granted that the singular fibers over $\varSigma \subset \overline{C}$ are not necessarily nodes (as $\overline{C} \to \mathbf{P}^1$ may ramify over the singular points), a similar line of reasoning as the nodal situation will occur, based on a parallel situation encountered in [9]. So for simplicity, let us assume that for each $t \in \varSigma$, that X_t is Lefschetz. Since $g(X_t) \geq 2$ for $t \in C$, it follows that for $t \in \varSigma$, $g(X_t) \geq 1$.

Proposition 8.24. *If X_0 is sufficiently general, then one can arrange for the following to hold:*

(i) $H^0(C,R^1\rho_*\mathbf{Q}(2)) = 0.$
(ii) *For every $t \in \varSigma$, the corresponding $Q - R$ is nontorsion in* $\mathrm{CH}^1(\tilde{X}_t)$.

Proof. Although we won't prove this, it goes without mentioning that (i) is a standard result in the deformation theory of curves and corresponding VHS. For (ii), one considers via deformation, a family of nodal curves of genus at least 1, together with an argument of Baire type using the fact that the torsion points on a curve of genus $g \geq 1$ is at most countable. $\square$

It follows that such a ξ would define a normal function for which $\delta(\nu_\xi) = 0 \in \varGamma H^1(C,R^1\rho_*\mathbf{Q}(2))$, and so $\nu_\xi \in J(H^0(C,R^1\rho_*\mathbf{Q}(2))) = 0$. This leads to $\mathrm{cl}_{2,2}(\xi_t) = 0 \in H^2_{\mathscr{D}}(X_t,\mathbf{Q}(2))$ for very general $t \in C$, and hence $\mathrm{cl}_{2,2}(\xi_0)$ is torsion as a class in $H^2_{\mathscr{D}}(X_0,\mathbf{Z}(2))$.

Observation 2. Consider an elliptic surface $\overline{\rho} : \overline{X} \to \overline{C}$. The singular fibers $X_\varSigma$ are unions of rational curves. If for some $t \in \varSigma$, X_t is nodal with node $P \in X_t$, then on $\tilde{X}_t$, $Q-R \sim_{\mathrm{rat}} 0$, hence $\mathrm{CH}^1(X_t,1)/\mathbf{C}^\times \neq 0$, and the possibility of a class $\xi \in \mathrm{CH}^2(X,2)$, with nontrivial value $[d\log\xi] \in \varGamma H^2(X,\mathbf{Q}(2))$ arises. Assuming this is the case, then ν_ξ is nontrivial, and hence for general X_t, $\mathrm{cl}_{2,2}(\xi_t)$ is a nontorsion class (using a Baire category argument). This will be illustrated in Theorem 8.26 below, but as a preliminary warm-up, consider the nodal curve $D = \overline{V(y^2 - x^3 - x^2)} \subset \mathbf{P}^2$, with singular point $P = (0,0)$. By making the substitution $(x,y) = (x,ux)$, we end up with the desingularization $\sigma : \tilde{D} := \overline{V(u^2 = x+1)} \to D$, and where $\sigma^{-1}(P) = \{Q = (0,1), R = (0,-1)\}$ in (x,u)-coordinates. Let

$$f = \frac{u+1}{u-1} = \frac{y+x}{y-x}.$$

Then viewing $f \in \mathbf{C}(\tilde{D})$, $\mathrm{div}_{\tilde{D}}(f) = R - Q$, and viewing $f \in \mathbf{C}(D) = \mathbf{C}(\tilde{D})$, $\mathrm{div}_D(f) = 0$. We apply this to the following.

Example 8.25. Let $\pi : X \to \mathbf{P}^1$ be the elliptic surface defined by

$$y^2 = x^3 + x^2 + t =: h(x),$$

and let $\Sigma \subset \mathbf{P}^1$ be the singular set of π. One shows that

$$\Sigma = \left\{0, \infty, \frac{-4}{27}\right\},$$

furthermore X_0, $X_{\frac{-4}{27}}$ are nodal curves, and X_∞ is a simply-connected tree of $\mathbf{P}^1$'s. We then have:

Theorem 8.26. *Let* $U = X \backslash \{X_0, X_{\frac{-4}{27}}, X_\infty\}$. *Then*

$$\Gamma(H^2(U, \mathbf{Q}(2))) \simeq \mathbf{Q}^2;$$

moreover it is generated by $[d\log(\xi_1)], [d\log(\xi_2)]$, *where*

$$\xi_1 = \left\{\frac{(y-x)^3}{8}, \frac{(y+x)^3}{8}\right\}\left\{\frac{y+x}{y-x}, t\right\}^3,$$

$$\xi_2 = \left\{\frac{(\mathrm{i}y+x+2/3)^3}{8}, \frac{(\mathrm{i}y-x-2/3)^3}{8}\right\}\left\{\frac{\mathrm{i}y+x+2/3}{\mathrm{i}y-x-2/3}, -t-4/27\right\}^3,$$

are classes in $\mathrm{CH}^2(U,2;\mathbf{Q})$.[12]

Now choose a class $\xi \in \mathrm{CH}^2(U,2)$ such that $[d\log\xi] \neq 0 \in \Gamma(H^2(U,\mathbf{Q}(2)))$. Thus for general $t \in \mathbf{P}^1$, $\mathrm{cl}_{2,2}(\xi_t)$ is nontorsion.

Remark 8.27. As pointed out by the referee, another class of examples pertaining to Observation 2 are the modular families of elliptic curves studied in [4], where every node P does give rise to such a class ξ, by Beilinson's Eisenstein symbol construction.

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[12] M. Asakura informed me of his work in [1], which includes this theorem as a special case. Further he provides an upper bound for the rank of the $d\log$ image for variants of the family in Example 8.25.

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