Preface

This volume is an outgrowth of the Simons Symposium "Geometry over Non-Closed Fields," which took place in a magnificent setting on the island of St. John in February 2012. It gathered in a tropical paradise a small group of experts working at the interface of algebraic geometry and number theory and experts from complementary branches of these fields. In addition to a series of very interesting and high-level lectures, the meeting was characterized by intense discussions and collaborative research. These lively discussions were greatly enhanced by the multitude of points of view and approaches advocated at the meeting.

This volume reflects the main topics of the symposium: interactions between birational geometry, the theory of rational curves, and arithmetic.

Among the actively studied problems in higher-dimensional birational geometry are the questions of rationality, unirationality, and birational geometry of varieties naturally appearing in geometric and arithmetic problems such as moduli spaces of various origins. An important role is played by the theory of birational automorphisms and birational models; these themes are taken up in several papers of the volume.

Specifically, the survey paper by I. Arzhantsev, H. Flenner, S. Kaliman, K. Kutzschebauch, and M. Zaidenberg compares the notions of *flexible* and *infinitely transitive* algebraic varieties. Their main result is that smooth varieties X of dimension ≥ 2 with a transitive action of the subgroup $\mathrm{SAut}(X)$ of algebraic automorphisms generated by additive one-parameter subgroups are flexible, and in fact the action of $\mathrm{SAut}(X)$ is infinitely transitive.

This theme is also treated in the article of F. Bogomolov, I. Karzhemanov, and K. Kuyumzhiyan. They formulate a provocative conjecture linking unirationality of the function field $K = k(X)$ of an algebraic variety X over an algebraically closed field k with the existence of an infinitely transitive model after stabilization, i.e., addition of sufficiently many algebraically independent elements to K. Furthermore, they prove new results on the existence of infinitely transitive models for unirational function fields with many cancellations and provide evidence for their conjecture for some interesting classes of unirational functions fields, e.g., function fields of cubic hypersurfaces and some complete intersections.

Birational models and fibration structures of a variety X are encoded, to a large extent, in the convex, linear geometry of the cones of ample and effective divisors of X. The article of Kovács supplies a characteristic independent proof of the basic result concerning the shape of the ample cone of a $K3$ surface: either it consists of all vectors with square more than zero, in the upper half-space of the second cohomology, or it is a cone whose faces are dual to smooth rational (-2)-curves.

The interplay between cones of curves and divisors and finite-generation questions is at the forefront of Lazić's contribution. He discusses implications of his method of proof of finite generation of canonical rings to other problems in higher-dimensional algebraic geometry, specifically concerning varieties of zero or negative Kodaira dimension. In particular, he discusses applications to the Mori program which conjecturally constitute the core in the class of Fano varieties and relations between abundance conjecture and the cone conjecture for Calabi–Yau manifolds. He also develops a new viewpoint on the birational properties of Mori Dream spaces, i.e., varieties for which the total coordinate ring (over all line bundles) is finitely generated.

The article of Bertram and Coskun discusses in detail cones of curves and ample divisors for Hilbert schemes of points on surfaces, with a special emphasis on del Pezzo surfaces. They show Hilbert schemes of del Pezzo and Hirzebruch surfaces are Mori Dream spaces. Moreover, in many cases, birational models can be interpreted as moduli spaces for certain choices of Bridgeland stability data, and this allows one to identify classes of extremal curves.

The paper by Cheltsov, Katzarkov, and Przhyjalkowski develops a new categorical approach to birational properties of algebraic varieties, combining ideas from homological mirror symmetry and techniques which evolved in the study of derived categories of coherent sheaves on projective varieties. Their expectation is that the presence of *phantom* subcategories, as almost direct summands of the derived category of a variety, is indicative of nonrationality. They verify this for many representative examples of threefolds.

An alternative obstruction to nonrationality, leading to many explicit examples in higher dimensions, relies on *stable* and *unramified stable* cohomology. Among these examples are quotient varieties, obtained from finite-dimensional linear representations of finite groups. In this case, the birational properties are tightly linked to group cohomology and its birational parts, i.e., stable cohomology. These are not easy to compute, in general. The article of Bogomolov and Böhning treats stable cohomology of finite groups obtained as iterated wreath products of cyclic groups of prime order. The main result is that all such elements are detected on elementary abelian subgroups. The article includes several results showing that stable cohomology of isoclinic groups exhibits similar properties.

The structure of the Brauer group is a prime motive for Lieblich's survey on *twisted sheaves*, i.e., sheaves indigenous to stacks associated with a variety. These shed new light on long-standing algebraic and arithmetic problems: the period-index problem, the validity of the Brauer–Manin formalism for rational varieties over global fields, and the classification of K3 surfaces over small fields. After the translation to stacks, geometric techniques may be brought to bear, e.g., the birational triviality of moduli spaces of stable vector bundle with fixed determinant over a curve and the theory of rationally connected varieties.

Várilly-Alvarado summarizes the current state of the art for the arithmetic of rational surfaces over global fields, with particular emphasis on counterexamples for small degree del Pezzo surfaces. Effective computations in the Brauer group play a central rôle.

Liedtke offers a broad survey of the geometry of surfaces over fields of positive characteristic. Here again, much of what distinguishes positive characteristic is the unexpected behavior of rational curves, e.g., quasi-elliptic fibrations, the existence of unirational surfaces among supersingular K3 surfaces, and the appearance of "new" rational curves when a K3 surface is reduced modulo p. Liedtke shows how this rich geometry can be exploited to establish theorems even in characteristic zero, e.g., complex K3 surfaces with odd Picard rank admit infinitely many rational curves.

Kebekus turns the emphasis of rational curves on its head: What can one say when there are very few rational curves? This has major implications for the rigidity of morphisms between such varieties. On the other hand, the criteria of Mori and Miyaoka for uniruledness, via positivity of the tangent bundle, suggest quantitative approaches to the structure of rational curves, expressed in the language of foliations.

Debarre investigates the structure of varieties parametrizing smooth curves of a given geometric genus and degree in smooth hypersurfaces and discusses how they can be used to study the geometry of the hypersurface. When are these spaces of curves themselves uniruled, and what does it tell us about the position of the underlying variety in the classification scheme?

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