Ramanujan–Sato-Like Series

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Abstract Using the theory of Calabi–Yau differential equations we obtain all the parameters of Ramanujan–Sato-like series for $1/\pi^2$ as q-functions valid in the complex plane. Then we use these q-functions together with a conjecture to find new examples of series of non-hypergeometric type. To motivate our theory we begin with the simpler case of Ramanujan–Sato series for $1/\pi$.

Key words Ramanujan–Sato-like series • Examples of complex series for $1/\pi$ • Calabi–Yau differential equations • Mirror map • Yukawa coupling • Examples of non-hypergeometric series for $1/\pi^2$

1 Introduction

In his famous paper in 1914 S. Ramanujan published 17 formulas for $1/\pi$ [18], all of hypergeometric form

$$\sum_{n=0}^{\infty} \frac{(1/2)_n (s)_n (1-s)_n}{n!^3} (a+bn) z^n = \frac{1}{\pi}.$$

Here $(c)_n = c(c+1)(c+2)\cdots(c+n-1)$ is the Pochhammer symbol, $s = 1/2$, $1/3$, $1/4$, or $1/6$ and z, b, a are algebraic numbers. The most impressive is

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$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{1}{99^{4n}} (26390n + 1103) = \frac{9801\sqrt{2}}{4\pi}, \tag{1.1}$$

which gives eight decimal digits of π per term. All the 17 series were rigorously proved in 1987 by the Borwein brothers [9]. Independently, the Borwein [9] and the Chudnovsky brothers [12] studied and proved Ramanujan series of the form

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3} (a + bn) z^n = \frac{1}{\pi}. \tag{1.2}$$

The value of z can be found in the following way: Let us take the Chudnovsky brothers series (of Ramanujan type with $s = 1/6$)

$$\sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3} (10177 + 261702n) \frac{1}{(-5280^3)^n} = \frac{880^2\sqrt{330}}{\pi}.$$

The series

$$w_0 = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!n!^3} z^n = \sum_{n=0}^{\infty} 12^{3n} \cdot \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^3} z^n$$

satisfies the differential equation

$$\Big(\theta^3 - 24z(2\theta + 1)(6\theta + 1)(6\theta + 5)\Big) w_0 = 0,$$

where $\theta = z d/dz$. A second solution is

$$w_1 = w_0 \ln z + 744z + 562932z^2 + 570443360z^3 + \cdots .$$

Define

$$q = \exp\left(\frac{w_1}{w_0}\right) = z + 744z^2 + 750420z^3 + 872769632z^4 + \cdots .$$

Then

$$J(q) = \frac{1}{z(q)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

is the famous modular invariant and

$$J\left(-e^{-\pi\sqrt{67}}\right) = -5280^3.$$

A similar construction, getting a different $J := 1/z$, can be made starting with any third order differential equation which is the symmetric square of a second-order differential equation. This kind of series are called Ramanujan–Sato series for $1/\pi$ because Sato discovered the first example of this type, one involving the Apéry numbers [10].

Similarly, the first formulas for $1/\pi^2$, found by the second author, were of hypergeometric type, using a function

$$w_0 = \sum_{n=0}^{\infty} \frac{(1/2)_n (s_1)_n (1-s_1)_n (s_2)_n (1-s_2)_n}{n!^5} z^n,$$

where the 14 possible pairs (s_1, s_2) are given in [14] or [6] and w_0 satisfies a fifth-order differential equation

$$\left(\theta^5 - z(\theta + \frac{1}{2})(\theta + s_1)(\theta + 1 - s_1)(\theta + s_2)(\theta + 1 - s_2)\right) w_0 = 0.$$

This differential equation is of a very special type. It is a Calabi–Yau equation with a fourth-order pullback with solutions y_0, y_1, y_2, y_3, where

$$w_0 = y_0 (\theta y_1) - (\theta y_0) y_1.$$

This was used in [6, 14], where one new hypergeometric formula was found. Unfortunately, fifth-order Calabi–Yau differential equations are quite rare. The simplest non-hypergeometric cases are Hadamard products of second- and third-order equations (labeled $A * \alpha$, etc., in [5]). Seven formulas of this kind have been found [6], like, for example,

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{i=0}^{n} \frac{(-1)^i 3^{n-3i} (3i)!}{i!^3} \binom{n}{3i} \binom{n+i}{i} \frac{(-1)^n}{3^{6n}} (803n^2 + 416n + 68) = \frac{486}{\pi^2}, \tag{1.3}$$

which involves the Almkvist–Zudilin numbers. Two of the others were proved by Zudilin [20]. In this paper we explore more complicated fifth-order equations, most of them found by the first author (#130 was found by Verrill).

To find q_0 in the $1/\pi$ case, we solve the equation $\alpha(q) = \alpha_0$, where α_0 is a rational and

$$\alpha(q) = \frac{\ln^2 |q|}{\pi^2}.$$

The real solutions are $q_0 = \pm e^{-\pi\sqrt{\alpha_0}}$. As there are many examples in the literature with q_0 real, in this paper we will show some series corresponding to $q_0 = e^{i\pi r_0} e^{-\pi\sqrt{\alpha_0}}$, where r_0 is a rational such that $e^{i\pi r_0}$ is complex. If we calculate $J_0 = J(q_0)$, then $z_0 = 1/J_0$. In the $1/\pi^2$ case we have two functions,

$$\alpha(q) = \frac{\frac{1}{6}\ln^3 |q| - T(q) - h\zeta(3)}{\pi^2 \ln |q|}, \quad \tau(q) = \frac{\frac{1}{2}\ln^2 |q| - (\theta_q T)(q)}{\pi^2} - \alpha(q), \tag{1.4}$$

where h is an invariant and $T(q)$ essentially is the Gromov–Witten potential in string theory. Solving the equation $\alpha(q) = \alpha_0$ numerically, where α_0 is rational, we get an approximation of q_0. Replacing q_0 in the second equation, we get τ_0. We conjecture that the corresponding series is of Ramanujan type for $1/\pi^2$ if, and only if, τ_0^2 is also rational. The success in finding the examples of this paper depends heavily on our experimental method to get the invariant h. It uses the critical value $z = z_c$, the radius of convergence for the power series w_0. From the conjecture $(dz/dq)(q_c) = 0$ we get an approximation of q_c and using the PSLQ algorithm to find an integer relation among the numbers

$$\frac{\ln^3|q_c|}{6} - T(q_c), \qquad \pi^2 \ln|q_c|, \qquad \zeta(3),$$

we obtain simultaneously α_c and the invariant h. Replacing α_c and q_c in the second equation we get τ_c.

In the $1/\pi$ case, the algebraic but nonrational z_0 dominate the rational solutions (see the tables in [4]). In the case $1/\pi^2$ the only known series with a nonrational z_0 is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^3 \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n^5} \left(\frac{15\sqrt{5}-33}{2}\right)^{3n} \times \left[(1220/3 - 180\sqrt{5})n^2 + (303 - 135\sqrt{5})n + (56 - 25\sqrt{5}) \right] = \frac{1}{\pi^2}, \quad (1.5)$$

which was discovered by the second author [15]. See also the corresponding mosaic supercongruences in [16].

We obtain the q-functions for all the parameters of general Ramanujan–Sato-like series for $1/\pi$ and $1/\pi^2$. Contrary to the series for $1/\pi$ in which everything can be proved rigorously by means of modular equations, in the case $1/\pi^2$, we can only evaluate the functions numerically and then guess the algebraic values when they exist. A modular-like theory which explains the rational and algebraic quantities observed is still not available [19]. For an excellent account of these topics, see [22].

2 Ramanujan–Sato-Type Series for $1/\pi$

Certain differential equations of order 3 are the symmetric square of a differential equation of second order. Suppose

$$\theta^3 w = e_2(z)\theta^2 w + e_1(z)\theta w + e_0(z)w, \qquad \theta = z\frac{d}{dz}, \quad (2.1)$$

is the symmetric square of the second-order equation

$$\theta^2 y = c_1(z)\theta y + c_0(z)y, \qquad 3c_1(z) = e_2(z). \quad (2.2)$$

We define the following function

$$P(z) = \exp \int \frac{-2c_1(z)}{z} dz,$$

with $P(0) = 1$, which plays an important role in the theory. In the examples of this paper $P(z)$ is a polynomial but we have also found examples for which $P(z)$ is a rational function.

The fundamental solutions w_0, w_1, w_2 of the third-order differential equation are connected to the fundamental solutions y_0, y_1 of the second-order equation by

$$w_0 = y_0^2, \quad w_1 = y_0 y_1, \quad w_2 = \frac{1}{2} y_1^2 \tag{2.3}$$

[7, Prop. 9]. We define the wronskians

$$W(w_i, w_j) = \begin{vmatrix} w_i & \theta w_i \\ w_j & \theta w_j \end{vmatrix}, \qquad W(y_0, y_1) = \begin{vmatrix} y_0 & \theta y_0 \\ y_1 & \theta y_1 \end{vmatrix}.$$

Observe that this notation is not the same as in [1], where in the definition of $W(y_0, y_1)$ we have y_0' and y_1' instead of θy_0 and θy_1.

Theorem 2.1. *We have*

$$W(w_0, w_1) = \frac{y_0^2}{\sqrt{P}}, \quad W(w_0, w_2) = \frac{y_0 y_1}{\sqrt{P}}, \quad W(w_1, w_2) = \frac{y_1^2}{2\sqrt{P}}. \tag{2.4}$$

Proof. Using (2.3), we get

$$W(w_0, w_1) = y_0^2 W(y_0, y_1), \quad W(w_0, w_2) = y_0 y_1 W(y_0, y_1)$$

and

$$W(w_1, w_2) = \frac{1}{2} y_1^2 W(y_0, y_1).$$

If we denote with f the wronskian $W(y_0, y_1)$, then from (2.2), we see that $\theta f = c_1(z) f$. This implies $f = 1/\sqrt{P}$. $\square$

2.1 Series for $1/\pi$

Let $q = e^{i\pi r} e^{-\pi\tau}$. If the function

$$w_0(z) = \sum_{n=0}^{\infty} A_n z^n$$

satisfies a differential equation of order 3 as above, then we will find two functions $b(q)$ and $a(q)$ with good arithmetical properties, such that

$$\sum_{n=0}^{\infty} A_n \Big(a(q) + b(q)n \Big) z^n(q) = \frac{1}{\pi}.$$

The interesting cases are those with z, b, a algebraic. They are called Ramanujan–Sato-type series for $1/\pi$.

The usual q-parametrization is

$$q = \exp\left(\frac{y_1}{y_0}\right) = \exp\left(\frac{w_1}{w_0}\right),$$

and we can invert it to get z as a series of powers of q. The function $z(q)$ is the mirror map, and for this kind of differential equations, it has been proved that it is a modular function. We also define $J(q) := 1/z(q)$.

Theorem 2.2. *The functions $\alpha(q)$, $b(q)$, $a(q)$ such that*

$$\sum_{j=0}^{2} \Big[(w_j)a + (\theta w_j)b \Big] x^j = e^{i\pi r x} \left(\frac{1}{\pi} - \frac{\pi}{2}\alpha x^2 \right) \text{ truncated at } x^3 \tag{2.5}$$

are given by

$$\alpha(q) = \tau^2(q), \quad b(q) = \tau(q)\sqrt{P(z)}, \quad a(q) = \frac{1}{\pi w_0}\left(1 + \frac{\ln|q|}{w_0} q \frac{dw_0}{dq}\right). \tag{2.6}$$

In addition, if r and τ_0^2 are rational, then $z(q_0)$, $b(q_0)$, $a(q_0)$ are algebraic.

Proof. First, we see that $q = e^{i\pi r} e^{-\pi\tau}$ implies that

$$\tau(q) = -\frac{\ln|q|}{\pi}.$$

We can write (2.5) in the following equivalent form:

$$\begin{aligned} (w_0)\,a + (\theta w_0)\,b &= \frac{1}{\pi}, \\ (w_1)\,a + (\theta w_1)\,b &= ir, \\ (w_2)\,a + (\theta w_2)\,b &= -\frac{\pi}{2}(\alpha + r^2). \end{aligned} \tag{2.7}$$

In what follows, we will use the wronskians (2.4). As we want this system to be compatible, we have

$$\begin{vmatrix} w_0 & \theta w_0 & \frac{1}{\pi} \\ w_1 & \theta w_1 & ir \\ w_2 & \theta w_2 & -\frac{\pi}{2}(\alpha + r^2) \end{vmatrix} = 0. \tag{2.8}$$

Expanding along the last column, we get

$$\frac{1}{2\pi}\left(\frac{y_1}{y_0}\right)^2 - ir\left(\frac{y_1}{y_0}\right) - \frac{\pi}{2}(\alpha + r^2) = 0.$$

Hence

$$\frac{1}{\pi}\frac{\ln^2 q}{2} - ir\ln q - \frac{\pi}{2}(\alpha + r^2) = 0.$$

As $\ln q = \ln|q| + i\pi r$, we obtain the function $\alpha(q)$. To obtain b we apply Cramer's method to the system formed by the two first equations of (2.7). We get

$$b = \left(ir - \frac{1}{\pi}\frac{w_1}{w_0}\right)\sqrt{P(z)} = \left(\frac{i\pi r - \ln q}{\pi}\right)\sqrt{P(z)} = -\frac{\ln|q|}{\pi}\sqrt{P(z)}.$$

Then, replacing w_1 with $w_0 \ln q$ in the second equation of (2.7) and solving the system formed by the two first equations, we obtain the identity

$$w_0 = \frac{q}{z\sqrt{P(z)}}\frac{dz}{dq}.$$

Finally, using the two last formulas and the first equation of (2.7), we derive the formula for $a(q)$ in (2.6). From $b = \tau\sqrt{1-z}$ we see that b takes algebraic values when r and τ^2 are rational. By an analogue to the argument given in [14, Sect. 2.4], we see that the same happens to $a(q)$. □

2.2 Examples of Series for $1/\pi$

There are many examples in the literature (see [8] and the references in it), but until very recently, all of them were with $r = 0$ (series of positive terms) or with $r = 1$ (alternating series). The first example of a complex series was found and proved, with a hypergeometric transformation, by the second author and Wadim Zudilin in [17, Eq. 44]. Other complex series, proved by modular equations or hypergeometric transformations, are in [11], like, for example,

$$\sum_{n=0}^{\infty} \frac{(4n)!}{n!^4}\left(\frac{10+2\sqrt{-3}}{28\sqrt{3}}\right)^{4n}\left((320 - 55\sqrt{-3})n + (52 - 12\sqrt{-3})\right) = \frac{98\sqrt{3}}{\pi}. \tag{2.9}$$

Tito Piezas (Ramanujan-type complex series available at Tito Piezas's web-site, personal communication)found numerically and then guessed the series

$$\sum_{n=0}^{\infty} \frac{(2n)!(3n)!}{n!^5} \frac{3+(17-i)n}{(2(7+i)(2+i)^4)^n} = \frac{33-6i}{4} \frac{1}{\pi}, \tag{2.10}$$

which involves only Gaussian rational numbers. It leads to taking $q = e^{\frac{2\pi i}{3}} e^{-\frac{4\sqrt{2}}{3}\pi}$, and of course, it is possible to prove it rigorously using modular equations. For our following examples, we have chosen the sequence of numbers

$$A_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k},$$

which is the Hadamard product $\binom{2n}{n} * (d)$ (see [7]). The differential equation is

$$\left(\theta^3 - 8z(2\theta+1)(3\theta^2+3\theta+1) + 128z^2(\theta+1)(2\theta+1)(2\theta+3)\right) w = 0.$$

The polynomial $P(z)$ is $P(z) = (1-16z)(1-32z)$ and

$$J(q) = \frac{1}{z(q)} = \frac{1}{q} + 16 + 52q + 834q^3 + 4760q^5 + 24703q^7 + \cdots.$$

For $q = ie^{-\pi\frac{\sqrt{13}}{2}}$, we find

$$\sum_{n=0}^{\infty} A_n \frac{(-1+6i)+(-9+33i)n}{(16+288i)^n} = \frac{52+91i}{\sqrt{13(1+18i)^3}} \cdot \frac{50}{\pi}. \tag{2.11}$$

For $q = ie^{-\pi\frac{\sqrt{37}}{2}}$, we get

$$\sum_{n=0}^{\infty} A_n \frac{(11842+11741i)+112665(1+i)n}{(16-14112i)^n} = \left(\frac{37}{1+882i}\right)^{\frac{3}{2}} \cdot \frac{2\cdot 5^3 \cdot 29^3}{\pi}. \tag{2.12}$$

Taking $q = e^{i\frac{\pi}{4}} e^{-\pi\frac{\sqrt{15}}{4}}$, we find

$$\sum_{n=0}^{\infty} A_n \left[4(-3\sqrt{3}+13i) + 3(29\sqrt{3}-44i)n\right] \left(\frac{2}{65-15\sqrt{3}i}\right)^n = \frac{14^2\sqrt{5}}{\pi}. \tag{2.13}$$

We give a final example with the sequence of Domb's numbers [10], called (α) in [7]

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}.$$

The differential equation is

$$\left(\theta^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + 64z^2(\theta+1)^3\right) w = 0.$$

We have $P(z) = (1-4z)(1-16z)$ and

$$J(q) = q^{-1} + 6 + 15q + 32q^2 + 87q^3 + 192q^4 + \cdots.$$

For $q = e^{\frac{i\pi}{3}} e^{-\frac{2\sqrt{2}}{3}\pi}$, we find

$$\sum_{n=0}^{\infty} A_n \frac{(1+i)+(4+2i)n}{(16+16i)^n} = \frac{6}{\pi}. \tag{2.14}$$

Taking the real and imaginary parts, we get

$$\sum_{n=0}^{\infty} A_n \frac{1+4n}{32^n} (\sqrt{2})^n \cos\frac{n\pi}{4} + \sum_{n=0}^{\infty} A_n \frac{1+2n}{32^n} (\sqrt{2})^n \sin\frac{n\pi}{4} = \frac{6}{\pi},$$

and

$$\sum_{n=0}^{\infty} A_n \frac{1+2n}{32^n} (\sqrt{2})^n \cos\frac{n\pi}{4} = \sum_{n=0}^{\infty} A_n \frac{1+4n}{32^n} (\sqrt{2})^n \sin\frac{n\pi}{4}.$$

The first author is preparing a collection of series for $1/\pi$ in [4]. Although we have guessed our examples from numerical approximations, the exact evaluations can be proved rigorously by using modular equations [11].

3 Ramanujan–Sato-Like Series for $1/\pi^2$

A Calabi–Yau differential equation is a fourth-order differential equation

$$\theta^4 y = c_3(z)\theta^3 y + c_2(z)\theta^2 y + c_1(z)\theta y + c_0(z) y, \qquad \theta = z\frac{d}{dz}, \tag{3.1}$$

where $c_i(z)$ are quotients of polynomials of z with rational coefficients, which satisfies several conditions [6]. There are two functions associated to these equations which play a very important role, namely, the mirror map and the Yukawa coupling. The mirror map $z(q)$ is defined as the functional inverse of

$$q = \exp(\frac{y_1}{y_0})$$

and the Yukawa coupling as

$$K(q) = \theta_q^2 \left(\frac{y_2}{y_0}\right), \qquad \theta_q = q\frac{d}{dq}.$$

We define $T(q)$ as the unique power series of q such that $T(0) = 0$ and

$$\theta_q^3 T(q) = 1 - K(q).$$

The function

$$\Phi = \frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) = \frac{1}{6}\ln^3 q - T(q) \tag{3.2}$$

is well known in string theory and is called the Gromov–Witten potential (see [13, p. 28]).

3.1 Pullback

The solution of some differential equations of fifth order can be recovered from the solutions of a fourth-order Calabi–Yau differential equation. We say that they admit a pullback. If (3.1) is the ordinary pullback of the differential equation

$$\theta^5 w = e_4(z)\theta^4 w + e_3(z)\theta^3 w + e_2(z)\theta^2 w + e_1(z)\theta w + e_0(z)w, \tag{3.3}$$

then we know that w_0, w_1, w_2, w_3, w_4 can be recovered from the four fundamental solutions y_0, y_1, y_2, y_3 of (3.1) in the following way:

$$w_0 = \begin{vmatrix} y_0 & y_1 \\ \theta y_0 & \theta y_1 \end{vmatrix}, \; w_1 = \begin{vmatrix} y_0 & y_2 \\ \theta y_0 & \theta y_2 \end{vmatrix}, \; w_3 = \frac{1}{2}\begin{vmatrix} y_1 & y_3 \\ \theta y_1 & \theta y_3 \end{vmatrix}, \; w_4 = \frac{1}{2}\begin{vmatrix} y_2 & y_3 \\ \theta y_2 & \theta y_3 \end{vmatrix}, \tag{3.4}$$

$$w_2 = \begin{vmatrix} y_0 & y_3 \\ \theta y_0 & \theta y_3 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ \theta y_1 & \theta y_2 \end{vmatrix}. \tag{3.5}$$

We define the following function

$$P(z) = \exp \int \frac{-2c_3(z)}{z} dz,$$

with $P(0) = 1$, which plays an important role in the theory. In the Yifan Yang's pullback the corresponding coefficient is $4c_3(z)$ instead of $c_3(z)$. In all the examples of this paper $P(z)$ is a polynomial.

We denote as $W(w_i, w_j, w_j)$ and $W(w_i, w_j)$ the following wronskians [1]:

$$W(w_i, w_j, w_k) = \begin{vmatrix} w_i & \theta w_i & \theta^2 w_i \\ w_j & \theta w_j & \theta^2 w_j \\ w_k & \theta w_k & \theta^2 w_k \end{vmatrix}, \qquad W(w_i, w_j) = \begin{vmatrix} w_i & \theta w_i \\ w_j & \theta w_j \end{vmatrix}. \tag{3.6}$$

Due to different definition and notation, f in [1] is $1/\sqrt[4]{P(z)}$ here and the powers of x (z here) do not appear now. We will need the following wronskians of order 3 (see [1]):

$$W(w_1,w_2,w_3)=\frac{1}{2}\frac{y_1y_2-y_0y_3}{\sqrt{P}},\quad W(w_0,w_2,w_3)=\frac{y_1^2}{\sqrt{P}},\quad W(w_0,w_1,w_3)=\frac{y_0y_1}{\sqrt{P}},$$

$$W(w_0,w_1,w_2)=\frac{y_0^2}{\sqrt{P}},\quad W(w_0,w_1,w_4)=\frac{y_0y_2}{\sqrt{P}},\quad W(w_1,w_2,w_4)=\frac{y_2^2}{2\sqrt{P}},$$

and

$$W(w_0,w_2,w_4)=\frac{y_0y_3+y_1y_2}{2\sqrt{P}}.$$

We will also need the following wronskians of order 2 (see [1]):

$$W(w_0,w_1)=\frac{y_0^2}{\sqrt[4]{P}},\quad W(w_0,w_2)=\frac{y_0y_1}{\sqrt[4]{P}},\quad W(w_1,w_2)=\frac{y_0y_2}{\sqrt[4]{P}}.$$

3.2 Series for $1/\pi^2$

Suppose that the function

$$w_0(z)=\sum_{n=0}^{\infty}A_nz^n, \tag{3.7}$$

is a solution of a fifth-order differential equation which has a pullback to a Calabi–Yau differential equation. We will determine functions $a(q)$, $b(q)$, $c(q)$ in terms of $\ln|q|$, $z(q)$ and $T(q)$, such that

$$\sum_{n=0}^{\infty}A_nz(q)^n(a(q)+b(q)n+c(q)n^2)=\frac{1}{\pi^2}.$$

The interesting cases are those for which z, c, b, a are algebraic numbers. We will call them Ramanujan–Sato-like series for $1/\pi^2$. In this paper we improve and generalize to the complex plane the theory developed in [6, 15]. We let $q=|q|\,e^{i\pi r}$ and consider an expansion of the form

$$\sum_{j=0}^{4}\Big[(w_j)a+(\theta w_j)b+(\theta^2w_j)c\Big]x^j$$
$$=e^{i\pi rx}\left(\frac{1}{\pi^2}-\alpha x^2+h\frac{\zeta(3)}{\pi^2}x^3+\frac{\pi^2}{2}(\tau^2-\alpha^2)x^4\right)\text{ truncated at }x^5. \tag{3.8}$$

The number h is a rational constant associated to the differential operator D such that $Dw_0 = 0$. The motivation of this expansion is due to the fact that in the case of Ramanujan–Sato-like series for $1/\pi^2$ (z, c, b, a algebraic), we have experimentally observed that r, α, and τ^2 are rational while h is a rational constant (see the remark at the end of this section). We have the equivalent system

$$\begin{aligned}
(w_0)\,a + (\theta w_0)\,b + (\theta^2 w_0)\,c &= \frac{1}{\pi^2}, \\
(w_1)\,a + (\theta w_1)\,b + (\theta^2 w_1)\,c &= \frac{i}{\pi} r, \\
(w_2)\,a + (\theta w_2)\,b + (\theta^2 w_2)\,c &= -\frac{r^2}{2} - \alpha, \\
(w_3)\,a + (\theta w_3)\,b + (\theta^2 w_3)\,c &= i\pi r\left(-\frac{r^2}{6} - \alpha\right) + h\frac{\zeta(3)}{\pi^2}, \\
(w_4)\,a + (\theta w_4)\,b + (\theta^2 w_4)\,c &= \pi^2\left(\frac{r^4}{24} + \frac{\tau^2 - \alpha^2}{2} + \frac{r^2}{2}\alpha\right) + \frac{i}{\pi} h\zeta(3) r.
\end{aligned} \tag{3.9}$$

This system allows us to develop the theory. In the next theorem we obtain α and τ as non-holomorphic functions of q.

Theorem 3.1. *We have*

$$\alpha(q) = \frac{\frac{1}{6}\ln^3|q| - T(q) - h\zeta(3)}{\pi^2 \ln|q|}, \tag{3.10}$$

and

$$\tau(q) = \frac{\frac{1}{2}\ln^2|q| - (\theta_q T)(q)}{\pi^2} - \alpha(q). \tag{3.11}$$

Proof. In the proof we use the wronskians above. As we want the system (3.9) to be compatible, we have

$$\begin{vmatrix} w_0 & \theta w_0 & \theta^2 w_0 & p_0 \\ w_1 & \theta w_1 & \theta^2 w_1 & p_1 \\ w_2 & \theta w_2 & \theta^2 w_2 & p_2 \\ w_3 & \theta w_3 & \theta^2 w_3 & p_3 \end{vmatrix} = 0, \tag{3.12}$$

where p_0, p_1, etc., stand for the independent terms. Expanding the determinant along the last column, we obtain

$$-p_0\left(\frac{y_1 y_2 - y_0 y_3}{2}\right) + p_1(y_1^2) - p_2(y_0 y_1) + p_3(y_0^2) = 0.$$

Then, dividing by y_0^2, we get

$$-p_0\frac{1}{2}\left(\frac{y_1}{y_0}\frac{y_2}{y_0} - \frac{y_3}{y_0}\right) + p_1\left(\frac{y_1}{y_0}\right)^2 - p_2\left(\frac{y_1}{y_0}\right) + p_3 = 0.$$

Hence

$$-p_0\left(\frac{1}{6}\ln^3 q - T(q)\right) + p_1 \ln^2 q - p_2 \ln q + p_3 = 0.$$

Using $\ln q = \ln|q| + i\pi r$ and replacing p_0, p_1, p_2, and p_3 with there values in the system, we arrive at (3.10). Then, from the first, second, third, and fifth equations and using the the function $\alpha(q)$ obtained already, we derive (3.11). □

In the next theorem we obtain c, b, a as non-holomorphic functions of q.

Theorem 3.2.

$$c(q) = \tau(q)\sqrt[4]{P(z)}, \tag{3.13}$$

$$b(q) = \frac{z(q)}{\theta_q z(q)}\left(\frac{1}{\pi^2}\left(\theta_q^2 T(q) - \ln|q|\right) - \tau(q)\frac{\theta_q L(q)}{L(q)}\right)\sqrt[4]{P(z)}, \tag{3.14}$$

$$a(q) = \frac{1}{w_0(q)}\left(\frac{1}{\pi^2} - (\theta w_0)b(q) - (\theta^2 w_0)c(q)\right), \tag{3.15}$$

with

$$L(q) = \frac{y_0^2}{\sqrt[4]{P(z)}} = \frac{w_0(q)}{\sqrt[4]{P(z)}}\frac{\theta_q z(q)}{z(q)} = \frac{1}{\sqrt{P(q)}K(q)}\left(\frac{\theta_q z(q)}{z(q)}\right)^3, \tag{3.16}$$

where y_0 is the ordinary pullback.

Proof. Solving for c by Cramer's rule from the three first equations of (3.9), we get

$$\frac{c}{\sqrt[4]{P(z)}} = \frac{1}{\pi^2}\left(\frac{y_2}{y_0}\right) - \frac{i}{\pi}r\left(\frac{y_1}{y_0}\right) - \frac{r^2}{2} - \alpha.$$

Hence

$$\frac{c}{\sqrt[4]{P(z)}} = \frac{1}{\pi^2}\left(\frac{1}{2}\ln^2 q - \theta_q T\right) - \frac{i}{\pi}r\ln q - \frac{r^2}{2} - \alpha.$$

Replacing $\ln q$ with $\ln|q| + i\pi r$, we obtain (3.13). Then, solving for b from the two first equations of (3.9), we obtain

$$b = \frac{1}{\pi^2}\frac{w_0}{L}\left(i\pi r - \frac{w_1}{w_0}\right) - c(z)\frac{\theta_z L}{L}, \tag{3.17}$$

where $L = w_0(\theta w_1) - w_1(\theta w_0)$. But, as $q = \exp(y_1/y_0)$, we obtain

$$\theta_q\left(\frac{y_2}{y_0}\right) = \frac{q}{\theta_z q}\theta_z\left(\frac{y_2}{y_0}\right) = \frac{y_0\,\theta y_2 - y_2\,\theta y_0}{y_0\,\theta y_1 - y_1\,\theta y_0} = \frac{w_1}{w_0}.$$

Applying θ_q to the two extremes of it, we get

$$\theta_q\left(\frac{w_1}{w_0}\right) = \frac{w_0(\theta w_1) - w_1(\theta w_0)}{w_0^2}\frac{\theta_q z}{z} = K(q) = 1 - \theta_q^3 T(q),$$

which implies

$$\frac{w_1}{w_0} = \ln q - \theta_q^2 T(q) \tag{3.18}$$

and

$$w_0 \theta w_1 - w_1 \theta w_0 = \frac{w_0^2 K(q) z(q)}{\theta_q z(q)} = L(q). \tag{3.19}$$

But

$$L = w_0(\theta w_1) - w_1(\theta w_0) = \frac{y_0^2}{\sqrt[4]{P(z)}}. \tag{3.20}$$

In [1] we have the formula

$$y_0^2 = \left(\frac{\theta_q z(q)}{z(q)}\right)^3 \frac{1}{\sqrt[4]{P(q)}\, K(q)}. \tag{3.21}$$

From (3.19)–(3.21), we obtain

$$w_0 = \left(\frac{\theta_q z(q)}{z(q)}\right)^2 \frac{1}{\sqrt[4]{P(q)}\, K(q)}. \tag{3.22}$$

From the three last identities we arrive at (3.16). From (3.17), (3.18), and (3.22) we deduce (3.14). The proof of (3.15) is trivial from the first equation of (3.9). □

The relevant fact is that the functions $\alpha(q)$, $\tau(q)$, $c(q)$, $b(q)$, $a(q)$ have good arithmetical properties. This is stated in the following conjecture which is crucial to discover Ramanujan–Sato-like series for $1/\pi^2$:

Conjecture

Let $\alpha_0 = \alpha(q_0)$, $\tau_0 = \tau(q_0)$, $z_0 = z(q_0)$, $a_0 = a(q_0)$, etc. If two of the quantities α_0, τ_0^2, z_0, a_0, b_0, c_0 are algebraic, so are all the others. Even more, in that case, α_0 and τ_0^2 are rational.

Remark

As one has

$$\sum_{n=0}^{\infty} \frac{A_{n+x}}{A_x} z^{n+x} = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4 + O(x^5),$$

we can write (3.8) in the following way:

$$\frac{1}{A_x}\sum_{n=0}^{\infty} z^{n+x}A_{n+x}(a+b(n+x)+c(n+x)^2)$$
$$= e^{i\pi r x}\left(\frac{1}{\pi^2}-\alpha x^2+h\frac{\zeta(3)}{\pi^2}x^3+\frac{\pi^2}{2}(\tau^2-\alpha^2)x^4\right)+\mathcal{O}(x^5).$$

The rational constant h appears (and can be defined) by the coefficient of x^3 in the expansion of A_x (analytic continuation of A_n) [6, Eq. 4]. In the hypergeometric cases we know how to extend A_n to A_x because the function Γ is the analytic continuation of the factorial. To determine h in the non-hypergeometric cases, we will not use this definition because it is not clear how to extend A_n to A_x in an analytic way. Instead, we will use the following conjecture:

Conjecture

The radius of convergence z_c of $w_0(z)$ is the smallest root of $P(z)=0$ and

$$\frac{dz}{dq}(q_c)=0.$$

In addition, α_c is rational. Hence there is a relation with integer coefficients among the numbers $\frac{1}{6}\ln^3|q_c|-T(q_c)$, $\pi^2\ln|q_c|$, and $\zeta(3)$, which we can discover with the PSLQ algorithm, and it determines the invariant h. This solution corresponds to the degenerated series $z=z_c$, $c(q_c)=b(q_c)=a(q_c)=0$.

3.3 New Series for $1/\pi^2$

To discover Ramanujan-like series for $1/\pi^2$, we first obtain the mirror map, the Yukawa coupling, and the function $T(q)$. Solving the equation

$$\frac{dz(q)}{dq}=0,$$

we get the value q_c which corresponds to z_c. Let $q=e^t e^{i\pi r}$, where $t<0$ is real. If we choose a value of r, then we can write (3.10) in the form

$$\alpha(t)=\frac{\frac{1}{6}t^3-T(q)-h\zeta(3)}{\pi^2 t}.$$

For $r = 0$ we get series of positive terms and for $r = 1$, we get alternating series. Solving numerically the equation $\alpha(t) = \alpha_0$, where α_0 is rational, we find an approximation of t_0 and hence also an approximation of q_0. Substituting this q_0 in (3.11), we get the value of τ_0. If τ_0^2 is also rational, then with the mirror map, we get the corresponding approximation of z_0. To discover the exact algebraic number z_0, we use the Maple function MinimalPolynomial which finds the minimal polynomial of a given degree, then we use the functions $c(q)$, $b(q)$, $a(q)$ to get the numerical values c_0, b_0, and a_0. To recognize the exact algebraic values of these parameters, we use minimal polynomial again. It is remarkable that in the "divergent" cases, we can compute c_0, b_0, a_0 with high precision by using formula (3.22) for w_0.

Big Table

In [5] there is a collection of many differential equations of Calabi–Yau type. We select some of those which are pullbacks of differential equations of fifth order. The ones not mentioned below gave no result. The symbol # stands as a reference of the equation in the Big Table. In [2] one can learn the art of finding Calabi–Yau differential equations. In Table 1, we show the invariants corresponding to the cases #60, #130, #189, #355, and #356. For all the cases cited above we have found examples of Ramanujan-like series for $1/\pi^2$. In Table 2 we show those examples, indicating the algebraic values of $\alpha - \alpha_c$, $z_c^{-1} \cdot z$, a, b, and c for which we have

$$\sum_{n=0}^{\infty} \tilde{A}_n \left(z_c^{-1} \cdot z\right)^n (a + bn + cn^2) = \frac{1}{\pi},$$

where $\widetilde{A_n} = A_n z_c^n$. If $|z_c^{-1} \cdot z| > 1$, then the series diverges, but we avoid the divergence considering the analytic continuation given by the parametrization with q. In all cases we have found congruences mod p^5 or mod p^3 (see [21]).

For #355 an explicit formula for A_n is not known but we can easily compute these numbers from the fifth-order differential equation $Dw = 0$, where D is the following operator:

$$\theta^5 - 2z(2\theta+1)(43\theta^4 + 86\theta^3 + 77\theta^2 + 34\theta + 6) \\ + 48z^2(\theta+1)(2\theta+1)(2\theta+3)(6\theta+5)(6\theta+7).$$

Table 1 Table of invariants

#60	$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n+k}{n}\binom{2n-k}{n}$
$P(z)=(1-16z)^2(1-108z)^2,\quad z_c=\frac{1}{2^2\cdot 3^3},\quad \alpha_c=\frac{1}{3},\quad \tau_c^2=\frac{2}{23},\quad h=\frac{50}{23}$	
#130	$A_n=\sum_{\substack{i+j+k+l\\+m+s=n}}\left(\frac{n!}{i!\,j!\,k!\,l!\,m!\,s!}\right)^2$
$P(z)=(1-4z)^2(1-16z)^2(1-36z)^2,\quad z_c=\frac{1}{36},\quad \alpha_c=\frac{1}{6},\quad \tau_c^2=\frac{2}{45},\quad h=\frac{2}{3}$	
#189	$A_n=\binom{2n}{n}\sum_{j,k}\binom{n}{j}^2\binom{n}{k}^2\binom{j+k}{n}^2$
$P(z)=(1-4z)^2(1-256z)^2,\quad z_c=\frac{1}{256},\quad \alpha_c=\frac{1}{2},\quad \tau_c^2=\frac{8}{21},\quad h=\frac{30}{7}$	
#355	Explicit formula for A_n not known
$P(z)=(1-64z)^2(1-108z)^2,\quad z_c=\frac{1}{108},\quad \alpha_c=\frac{1}{3},\quad \tau_c^2=\frac{4}{33},\quad h=\frac{30}{11}$	
#356	$A_0=1,\quad A_{n>0}=2\binom{2n}{n}\sum_{k=0}^{[n/4]}\frac{n-2k}{3n-4k}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}\binom{3n-4k}{2n}$
$P(z)=(1-108z)^2(1-128z)^2,\quad z_c=\frac{1}{128},\quad \alpha_c=\frac{1}{3},\quad \tau_c^2=\frac{1}{10},\quad h=\frac{14}{5}$	

Complex Series for $1/\pi^2$

Another method to obtain series for $1/\pi^2$ is by applying suitable transformations to the already known series for $1/\pi^2$; see [3, 6, 20]. Although we can use this technique to obtain other real Ramanujan-like series for $1/\pi^2$, our interest here is to find examples of Ramanujan-like complex series for $1/\pi^2$. For that purpose we will use the following very general transformation:

$$\sum_{n=0}^{\infty}A_n z^n=\frac{1}{1-z}\sum_{n=0}^{\infty}a_n\left[u\left(\frac{z}{1-z}\right)^m\right]^n,\quad A_n=\sum_{k=0}^{n}u^k\binom{n}{mk}a_k,$$

where $u=1$ or $u=-1$ and m is a positive integer (check that both sides satisfy the same Calabi–Yau differential equation). For example, translating the hypergeometric series

$$\sum_{n=0}^{\infty}\frac{(3n)!(4n)!}{n!^7}(252n^2+63n+5)(-1)^n\left(\frac{1}{24}\right)^{4n}=\frac{48}{\pi^2},\tag{3.23}$$

taking $u=-1$ and $m=4$, we find four series, one of them is the complex series

$$\sum_{n=0}^{\infty}A_n\left(9072n^2+(9072-756i)n+(2875-516i)\right)\left(\frac{1}{1-24i}\right)^n=\frac{27504+3454i}{\pi^2},\tag{3.24}$$

Table 2 Table of examples

#	$\alpha_0 - \alpha_c$	$z_c^{-1} \cdot z_0$	a_0	b_0	c_0
60	$\frac{4}{23}$	$\frac{1}{2}$	$\frac{3}{3 \cdot 23}$	$\frac{20}{3 \cdot 23}$	$\frac{40}{3 \cdot 23}$
	$\frac{8}{23}$	$\frac{3^3}{5^3}$	$\frac{40}{5^2 \cdot 23}$	$\frac{282}{5^2 \cdot 23}$	$\frac{616}{5^2 \cdot 23}$
	$\frac{43}{46}$	$-\frac{1}{48}$	$\frac{706}{2^5 \cdot 3^2 \cdot 23}$	$\frac{5895}{2^5 \cdot 3^2 \cdot 23}$	$\frac{16380}{2^5 \cdot 3^2 \cdot 23}$
	$\frac{3}{46}$	-2	$\frac{178}{2^5 \cdot 23}$	$\frac{719}{2^5 \cdot 23}$	$\frac{860}{2^5 \cdot 23}$
130	$\frac{1}{6}$	$-\frac{3^2}{4^2}$	$\frac{21}{96}$	$\frac{74}{96}$	$\frac{85}{96}$
	0	-4	$\frac{38}{54}$	$\frac{94}{54}$	$\frac{65}{54}$
189	$\frac{2}{21}$	$\frac{8^2}{9^2}$	$\frac{48}{3^5 \cdot 7}$	$\frac{328}{3^5 \cdot 7}$	$\frac{680}{3^5 \cdot 7}$
	$\frac{4}{7}$	$\frac{1}{3^2}$	$\frac{87}{2^4 \cdot 3^2 \cdot 7}$	$\frac{710}{2^4 \cdot 3^2 \cdot 7}$	$\frac{1840}{2^4 \cdot 3^2 \cdot 7}$
	$\frac{19}{42}$	$-\frac{2^4}{3^4}$	$\frac{843}{2^2 \cdot 3^5 \cdot 7}$	$\frac{5750}{2^2 \cdot 3^5 \cdot 7}$	$\frac{12610}{2^2 \cdot 3^5 \cdot 7}$
	$\frac{139}{42}$	$-\frac{2^4}{21^4}$	$\frac{1655799}{2^2 \cdot 3^5 \cdot 7^5}$	$\frac{24749870}{2^2 \cdot 3^5 \cdot 7^5}$	$\frac{122761930}{2^2 \cdot 3^5 \cdot 7^5}$
	$\frac{1}{14}$	$-\frac{4^2}{3^2}$	$\frac{51}{252}$	$\frac{254}{252}$	$\frac{370}{252}$
355	$\frac{1}{11}$	$\frac{3}{4}$	$\frac{1}{2^2 \cdot 3 \cdot 11}$	$\frac{12}{2^2 \cdot 3 \cdot 11}$	$\frac{30}{2^2 \cdot 3 \cdot 11}$
	$\frac{5}{22}$	$-\frac{3^2}{4^2}$	$\frac{9}{2^2 \cdot 11}$	$\frac{42}{2^2 \cdot 11}$	$\frac{60}{2^2 \cdot 11}$
	$\frac{5}{11}$	$\frac{27}{196}$	$\frac{21}{2^2 \cdot 7 \cdot 11}$	$\frac{164}{2^2 \cdot 7 \cdot 11}$	$\frac{390}{2^2 \cdot 7 \cdot 11}$
	$\frac{13}{11}$	$\frac{1}{108}$	$\frac{3119}{2^2 \cdot 3^6 \cdot 11}$	$\frac{29860}{2^2 \cdot 3^6 \cdot 11}$	$\frac{93090}{2^2 \cdot 3^6 \cdot 11}$
	$\frac{1}{22}$	-3	$\frac{16}{3 \cdot 11}$	$\frac{60}{3 \cdot 11}$	$\frac{60}{3 \cdot 11}$
356	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{2}{160}$	$\frac{27}{160}$	$\frac{74}{160}$
	1	$\frac{1}{50}$	$\frac{74}{800}$	$\frac{679}{800}$	$\frac{2002}{800}$
	$\frac{7}{10}$	$-\frac{1}{2^4}$	$\frac{158}{1280}$	$\frac{1113}{1280}$	$\frac{2618}{1280}$

where

$$A_n = \sum_{k=0}^{n} (-1)^k \binom{n}{4k} \frac{(3k)!(4k)!}{k!^7}.$$

Transformations preserve the value of the invariants h, α_c, and τ_c, and the series (3.24) has $2(\alpha - \alpha_c) = 3$ and $\tau = 3\sqrt{3}$ because it is a transformation of (3.23); see [6]. Looking at the transformation with $u = -1$ and $m = 4$, we see that the mirror maps z and z' corresponding to A_n and a_n, are related in the following way:

$$z = \frac{\sqrt[4]{z'}}{1 + \sqrt[4]{z'}}.$$

Write $z = z(q)$ and $z' = z'(q')$. Then, the first terms of $J(q)$ are

$$J(q) = \frac{1}{q} + 1 + 582q^3 + 277263q^7 + 167004122q^{11} + \cdots,$$

with $q = \sqrt[4]{q'}$. Writing, as usual, $q = e^{i\pi r}|q|$, we deduce that as the series (3.23) has $r = 1$, then the series (3.24) has $r = 1/4$.

4 Addendum

The method used in this paper, to find h, $\alpha_c = \alpha(q_c)$ and $\tau_c = \tau(q_c)$, is valid for those Calabi–Yau differential equations such that $K(q_c) = 0$, where q_c is a solution of $dz/dq = 0$. In these cases, we conjecture that $z(q_c)$ is the smallest root of $P(z)$ and that $a(q_c) = b(q_c) = c(q_c) = 0$. But from Theorem 3.2, we see that $b(q_c) = 0$ implies that $\tau_c = f(q_c)$, where

$$f(q) = \frac{1}{\pi^2} \left(\theta_q^2 T(q) - \ln|q| \right) \frac{L(q)}{\theta_q L(q)},$$

which allows us to obtain the critical value of τ. Then, replacing $q = q_c$ in (3.11), we can obtain α_c. Finally, replacing $q = q_c$ in (3.10), we obtain the value of h. As $q_c > 0$, the formula for h can be written in the form $h = h(q_c)$, where

$$h(q) = \frac{1}{\zeta(3)} \left(\Phi(q) - \ln(q)\, \theta_q \Phi(q) - \ln(q) \frac{L(q)}{\theta_q L(q)} \theta_q^2 \Phi(q) \right),$$

and $\Phi(q)$ is the Gromov–Witten potential (3.2). The advantage of this way of getting the invariants τ_c, α_c, and h is that we use explicit formulas instead of the PSLQ algorithm.

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