

Preface

This book is written as an introduction to analysis on the sphere and on the ball, and it provides a cohesive account of recent developments in approximation theory and harmonic analysis on these domains. Analysis on the unit sphere appears as part of Fourier analysis, in the study of homogeneous spaces, and in several fields in applied mathematics, from numerical analysis to geoscience, and it has seen increased activity in recent years. Its materials, however, are mostly scattered in papers and sections of books that cover more general topics. Our goals are twofold. The first is to provide a self-contained background for readers who are interested in analysis on the sphere. The second is to give a complete treatment of some recent advances in approximation theory and harmonic analysis on the sphere developed in the last fifteen years or so, of which both authors are among the earnest participants, and several chapters of the book are based on materials from their own research.

The book is loosely divided into four parts. The first part deals with analysis on the sphere with respect to the surface measure $d\sigma$, the only rotation-invariant measure on the sphere. We give a self-contained exposition on spherical harmonics, written with analysis in mind, in the first chapter, and present classical results of harmonic analysis on the sphere, including convolution structure, Cesàro summability of orthogonal expansions, the Littlewood–Paley theory, and the multiplier theorem due to Bonami–Clerc in the next two chapters. Approximation on the sphere is discussed in the fourth section, where a recent characterization of best approximation by polynomials on the sphere is given in terms of a modulus of smoothness and its equivalent K -functional. An introduction to cubature formulas, which are necessary for discretizing integrals to obtain discrete processes of approximation, is given in the sixth chapter. A recent proof of a conjecture on spherical design, synonym of equal-weight cubature formulas, by Bondarenko, Radchenko, and Viazovska, is included, for which the necessary ingredient of the Marcinkiewicz–Zygmund inequality is established in the fifth chapter, where the inequality and several others are established for the doubling weight on the sphere.

The second part discusses analysis in weighted spaces on the sphere. The background of this part is a far-reaching extension of spherical harmonics due to C. Dunkl, in which the role of the orthogonal group is replaced by a finite reflection

group, the measure $d\sigma$ is replaced by $h_\kappa^2(x)d\sigma$, where h_κ is a weight function invariant under a reflection group with κ being a parameter, and spherical harmonics are replaced by h -spherical harmonics associated with the Dunkl operators, a family of commuting differential–difference operators that replace partial derivatives. The study of h -spherical harmonic expansions started about fifteen years ago. Many deeper results in analysis were established only in the case of the group \mathbb{Z}_2^d , for which h_κ is given by $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$. In order to avoid heavy algebraic preparations, we give a self-contained exposition of Dunkl’s theory in the case of \mathbb{Z}_2^d , which is composed to highlight its parallel to the theory of spherical harmonics. Most results on ordinary spherical harmonic expansions can be extended to h -spherical harmonic expansions, including finer L^p results on projection operators and the Cesàro means, maximal functions, and multiplier theorem, as well as a characterization of best approximation that was developed by many authors. We give complete proofs of these results, which are more challenging than proofs for classical results for $d\sigma$, and in fact, in some cases, simplify those proofs for classical results when the parameters κ are set to zero.

The third part deals with analysis on the unit ball and on the simplex. There are close relations between analysis on spheres and that on balls of different dimensions, which enables us to utilize the results in the part two to develop a parallel theory for approximation theory and harmonic analysis on the unit ball. There is also a connection between analysis on the ball and that on the simplex, which carries much, but not all, of analysis on the ball over to the simplex. These results are composed in parallel to the development on the sphere.

The fourth part consists of one chapter, the last chapter of the book, which discusses five topics related to the main theme of the book: highly localized polynomial frames, distribution of nodes of positive cubature, positive and strictly positive definite functions, asymptotics of minimal discrete energy, and computerized tomography.

Analysis on the sphere has seen increased activity in the past two decades. There are other related topics that we decided not to include, for example scattered data interpolation, applications of spherical radial basis functions (zonal functions), and numerical or computational analysis on the sphere. These topics are more closely related to the applied and computational branches of approximation theory. Our choices, dictated by our own strengths and limitations, are those topics that are closely related to the main theme—approximation theory *and* harmonic analysis—of this book.

We keep the references in the text to a minimum and leave references and historical remarks to the last section of each chapter, entitled “Notes and Further Results,” where we also point out further results related to the materials in the chapter. Some common notation and terminology are given in the preamble at the front of the book, and there are two fairly detailed indexes: a subject index and a symbol index.

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