Chapter 2
Convolution Operator and Spherical Harmonic Expansion

The convergence of spherical harmonic expansions is studied through projection operators and various summability methods. We start with translation and convolution operators on the sphere in the first section, which are essential for the rest of the book. In particular, the projection operators and the Poisson integrals for the Fourier expansion in spherical harmonics, discussed in the second section, are convolution operators, which are also multiplier operators. The convolution and translation operators are used to define and study the Hardy–Littlewood maximal function on the sphere in the third section. As in the classical Fourier series, spherical harmonic series do not in general converge beyond the L^2 metric. It is then necessary to consider summation methods. One family of summation methods is that of Cesàro means, which will serve as an important tool in our later chapters. In the fourth section, we define the Cesàro means (C,δ) of the spherical harmonics and establish their convergence for δ above the critical index. Further results, in greater depth, on the convergence of these means are collected in the fifth section. A family of convolution operators that combine the polynomial-preserving property of the partial sum operator and the convergence of the Cesàro means is defined in terms of a smooth cutoff function in the sixth section. These operators provide near-optimal polynomial approximation, and their convolution kernels are proven to be highly localized in the sense that they decay faster than any polynomial order away from the diagonal. Such operators and their kernels will be instrumental for approximation on the sphere.

2.1 Convolution and Translation Operators on the Sphere

The distance between two points $x,y \in \mathbb{S}^{d-1}$ is defined as the geodesic distance

$$\mathtt{d}(x,y) := \arccos \langle x,y \rangle,$$

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and the reproducing kernel of $\mathscr{H}_n^d$ depends only on $\langle x,y\rangle$. This suggests a definition of a convolution operator on the sphere. Let

$$w_\lambda(x) = (1-x^2)^{\lambda-1/2}, \quad \lambda > -\frac{1}{2}, \quad x \in (-1,1).$$

Definition 2.1.1. For $f \in L^1(\mathbb{S}^{d-1})$ and $g \in L^1(w_\lambda;[-1,1])$ with $\lambda = \frac{d-2}{2}$,

$$(f*g)(x) := \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)g(\langle x,y\rangle)\mathrm{d}\sigma(y). \tag{2.1.1}$$

Denote the norm of the space $L^p(w_\lambda;[-1,1])$ by $\|\cdot\|_{\lambda,p}$; for $g \in L^p(w_\lambda;[-1,1])$,

$$\|g\|_{\lambda,p} := \left(c_\lambda \int_{-1}^{1} |g(x)|^p w_\lambda(x)\mathrm{d}x\right)^{1/p}, \quad 1 \le p \le \infty,$$

where c_λ is the normalization constant such that $c_\lambda \int_{-1}^{1} w_\lambda(t)\mathrm{d}t = 1$, and the norm is taken as the uniform norm when $p = \infty$. The convolution on the sphere satisfies Young's inequality:

Theorem 2.1.2. *Let $p,q,r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(\mathbb{S}^{d-1})$ and $g \in L^r(w_\lambda;[-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$\|f*g\|_p \le \|f\|_q \|g\|_{\lambda,r}. \tag{2.1.2}$$

In particular, for $1 \le p \le \infty$,

$$\|f*g\|_p \le \|f\|_p\|g\|_{\lambda,1} \quad \textit{and} \quad \|f*g\|_p \le \|f\|_1\|g\|_{\lambda,p}. \tag{2.1.3}$$

Proof. The standard proof (cf. [15, p. 6]) applies in this setting. By Minkowski's inequality,

$$\|f*g\|_q \le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |f(y)| \left(\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} |g(\langle x,y\rangle)|^q \mathrm{d}\sigma(y)\right)^{1/q} \mathrm{d}\sigma(x) = \|f\|_1\|g\|_{\lambda,q},$$

on using (A.5.1). And by Hölder's inequality and (A.5.1), it follows readily that

$$\|f*g\|_\infty \le \|f\|_{q'}\|g\|_{\lambda,q}, \quad \frac{1}{q'}+\frac{1}{q} = 1.$$

Applying the Riesz–Thorin theorem to interpolate the above two inequalities with $\theta = q(1-\frac{1}{p})$ gives the stated result. $\square$

In particular, (2.1.3) shows that $f*g$ is well defined. By (1.2.4) and (1.2.7), proj_n is a convolution operator:

$$\operatorname{proj}_n f = f * Z_n, \qquad Z_n(t) := \frac{n+\lambda}{\lambda} C_n^\lambda(t) \quad \text{with} \quad \lambda = \frac{d-2}{2}. \tag{2.1.4}$$

For $g \in L^1(w_\lambda; [-1,1])$, let $\hat{g}_n^\lambda$ denote the Fourier coefficient of g with respect to the Gegenbauer polynomials,

$$\hat{g}_n^\lambda := c_\lambda \int_{-1}^{1} g(t) \frac{C_n^\lambda(t)}{C_n^\lambda(1)} (1-t^2)^{\lambda - \frac{1}{2}} \mathrm{d}t.$$

Theorem 2.1.3. *For $f \in L^1(\mathbb{S}^{d-1})$ and $g \in L^1(w_\lambda; [-1,1])$ with $\lambda = \frac{d-2}{2}$,*

$$\operatorname{proj}_n(f * g) = \hat{g}_n^\lambda \operatorname{proj}_n f, \quad n = 0, 1, 2 \ldots. \tag{2.1.5}$$

Proof. By (1.2.4) and the Funk–Hecke formula in Theorem 1.2.9,

$$\begin{aligned} \operatorname{proj}_n(f*g)(x) &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} (f*g)(\xi) Z_n(x,\xi) \mathrm{d}\sigma(\xi) \\ &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) \left(\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} g(\langle \xi, y \rangle) Z_n(x,\xi) \mathrm{d}\sigma(\xi) \right) \mathrm{d}\sigma(y) \\ &= \hat{g}_n^\lambda \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) Z_n(x,y) \mathrm{d}\sigma(y) = \hat{g}_n^\lambda \operatorname{proj}_n f(x), \end{aligned}$$

where we have used the fact that $c_\lambda = \omega_{d-1}/\omega_d$ when $\lambda = \frac{d-2}{2}$. □

The identity (2.1.5) can be viewed as an analogue of the fact that the Fourier transform of $f * g$ is equal to the product of the Fourier transforms of f and g. It justifies our calling the right-hand side of (2.1.1) a convolution.

The translation operator $T_\theta f$ on the sphere can be interpreted in terms of the geodesic distance. It is defined as follows:

Definition 2.1.4. For $0 \le \theta \le \pi$ and $f \in L^1(\mathbb{S}^{d-1})$, define

$$T_\theta f(x) := \frac{1}{\omega_{d-1} (\sin\theta)^{d-1}} \int_{\langle x,y \rangle = \cos\theta} f(y) \mathrm{d}\ell_{x,\theta}(y), \tag{2.1.6}$$

where $\mathrm{d}\ell_{x,\theta}(y)$ denotes Lebesgue measure on the set $\{y \in \mathbb{S}^{d-1} : \langle x,y \rangle = \cos\theta\}$.

The basic properties of the translation operator are listed below:

Proposition 2.1.5. *Let $0 \le \theta \le \pi$ and $f \in L^2(\mathbb{S}^{d-1})$. Then*

1. Let $\mathbb{S}_x^\perp := \{y \in \mathbb{S}^{d-1} : \langle x,y \rangle = 0\}$, the equator in $\mathbb{S}^{d-1}$ with respect to x; then

$$T_\theta f(x) = \frac{1}{\omega_{d-1}} \int_{\mathbb{S}_x^\perp} f(x\cos\theta + u\sin\theta) \mathrm{d}\sigma(u). \tag{2.1.7}$$

In particular, if $f_0(x) := 1$, then $T_\theta f_0(x) = 1$.

2. *For a generic* $g : [-1,1] \mapsto \mathbb{R}$,

$$(f * g)(x) = \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) T_\theta f(x) (\sin\theta)^{d-2} \mathrm{d}\theta. \tag{2.1.8}$$

Proof. The first item follows from a change of variable $y \mapsto x\cos\theta + u\sin\theta$. For the second, we choose a coordinate system such that x becomes the north pole and set again $y = x\cos\theta + u\sin\theta$ to obtain

$$\begin{aligned}(f * g)(x) &= \frac{1}{\omega_d} \int_0^\pi g(\cos\theta) \int_{\mathbb{S}_x^\perp} f(x\cos\theta + u\sin\theta) \mathrm{d}\sigma(u) (\sin\theta)^{d-2} \mathrm{d}\theta \\ &= \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) T_\theta f(x) (\sin\theta)^{d-2} \mathrm{d}\theta,\end{aligned}$$

since $\mathbb{S}_x^\perp$ is isomorphic to the sphere $\mathbb{S}^{d-2}$. $\square$

The next proposition gives the interaction between T_θ and orthogonal expansions:

Lemma 2.1.6. *The operator $T_\theta f$ maps $\Pi_n(\mathbb{S}^{d-1})$ onto itself; for $f \in L^1(\mathbb{S}^{d-1})$,*

$$\operatorname{proj}_n T_\theta f = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} \operatorname{proj}_n f, \qquad \lambda = \frac{d-2}{2}. \tag{2.1.9}$$

Proof. Let $Y \in \mathscr{H}_n^d$. Denote by $\langle f, Y \rangle$ the Fourier coefficient of f with respect to Y. By Theorem 2.1.3,

$$\begin{aligned}\langle f * g, Y \rangle &= \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \operatorname{proj}_n (f * g)(x) Y(x) \mathrm{d}\sigma(x) \\ &= \langle f, Y \rangle \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} (\sin\theta)^{d-2} \mathrm{d}\theta.\end{aligned}$$

On the other hand, by (2.1.8),

$$\langle f * g, Y \rangle = \frac{\omega_{d-1}}{\omega_d} \int_0^\pi g(\cos\theta) \langle T_\theta f, Y \rangle (\sin\theta)^{d-2} \mathrm{d}\theta.$$

Since the above holds for a generic g whenever the integrals make sense, this shows that the Fourier coefficient of $T_\theta f$ with respect to Y satisfies $\langle T_\theta f, Y \rangle = \langle f, Y \rangle C_n^\lambda(\cos\theta)/C_n^\lambda(1)$, which proves the stated formula. $\square$

Lemma 2.1.7. *For $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$,*

$$\|T_\theta f\|_p \le \|f\|_p \qquad \text{and} \qquad \lim_{\theta \to 0^+} \|T_\theta f - f\|_p = 0.$$

Proof. For $f \in L^1(\mathbb{S}^{d-1})$ and $\lambda = \frac{d-2}{2}$, we have

$$\begin{aligned}\|T_\theta f\|_1 &\le \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} T_\theta(|f|)\,\mathrm{d}\sigma(x) = \mathrm{proj}_0(T_\theta|f|)\\ &= \frac{C_0^\lambda(\cos\theta)}{C_0^\lambda(1)}\mathrm{proj}_0(|f|) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}}|f(x)|\,\mathrm{d}\sigma(x) = \|f\|_1,\end{aligned}$$

where we have used the positivity of T_θ in the first step, and Lemma 2.1.6 in the third step. On the other hand, it follows directly from the definition that $\|T_\theta f\|_\infty \le \|f\|_\infty$. Thus, using the Riesz–Thorin interpolation theorem, we deduce that $\|T_\theta f\|_p \le \|f\|_p$ for all $1 \le p \le \infty$. This further implies $\|T_\theta f - f\|_p \le 2\|f-P\|_p + \|T_\theta P - P\|_p$ for every polynomial P. By Lemma 2.1.6,

$$T_\theta P - P = \sum_{j=0}^{n}\left(\frac{C_j^\lambda(\cos\theta)}{C_j^\lambda(1)} - 1\right)\mathrm{proj}_j P, \qquad P \in \Pi_n(\mathbb{S}^{d-1}),$$

so that $T_\theta P - P \to 0$ as $\theta \to 0^+$, from which the convergence of $\|T_\theta f - f\|_p$ follows from the density of polynomials. □

2.2 Fourier Orthogonal Expansions

With respect to an orthonormal basis $\{Y_\alpha\}$, say (1.5.6), a function f in $L^2(\mathbb{S}^{d-1})$ can be expanded in a Fourier series

$$f(x) = \sum c_\alpha Y_\alpha(x), \quad \text{where} \quad c_\alpha = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)Y_\alpha(y)\mathrm{d}\sigma.$$

It is often more convenient to consider the orthogonal expansions in terms of the spaces $\mathscr{H}_n^d$. Collecting terms of spherical harmonics of the same degree, the Fourier series takes the form, by (1.2.4) and (1.2.3),

$$f(x) = \sum_{n=0}^{\infty}\mathrm{proj}_n f(x), \tag{2.2.1}$$

where $\mathrm{proj}_n f$ is the orthogonal projection of f onto the space $\mathscr{H}_n^d$. The formulation of (2.2.1) is independent of a particular choice of orthogonal basis. In particular, the nth partial sum of (2.2.1) is defined by

$$S_n f := \sum_{k=0}^{n}\mathrm{proj}_k f. \tag{2.2.2}$$

By (1.2.4), $S_n f$ can be written as an integral operator whose kernel enjoys a closed form in terms of Jacobi polynomials.

Proposition 2.2.1. *For* $n = 0, 1, 2, \ldots$,

$$S_n f(x) = (f * K_n)(x), \qquad x \in \mathbb{S}^{d-1}, \tag{2.2.3}$$

where the kernel K_n *satisfies, with* $\lambda = \frac{d-2}{2}$,

$$K_n(t) := \sum_{k=0}^{n} \frac{k+\lambda}{\lambda} C_k^{\lambda}(t) = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(t). \tag{2.2.4}$$

Proof. The definition follows from the closed form of the zonal harmonics in (1.2.7). The closed form follows from the Eq. (B.1.9). □

Since the space of spherical polynomials is dense in $C(\mathbb{S}^{d-1})$ by Weierstrass's theorem and, as a consequence, dense in $L^2(\mathbb{S}^{d-1})$, the following theorem is a standard Hilbert space result for $L^2(\mathbb{S}^{d-1})$:

Theorem 2.2.2. *The family of spherical harmonics is dense in* $L^2(\mathbb{S}^{d-1})$, *and*

$$L^2(\mathbb{S}^{d-1}) = \sum_{n=0}^{\infty} \mathscr{H}_n^d \quad \text{i.e.} \quad f = \sum_{n=0}^{\infty} \operatorname{proj}_n f$$

in the sense that $\lim_{n\to\infty} \|f - S_n f\|_2 = 0$ *for every* $f \in L^2(\mathbb{S}^{d-1})$. *In particular, for* $f \in L^2(\mathbb{S}^{d-1})$, *Parseval's identity holds,*

$$\|f\|_2^2 = \sum_{n=0}^{\infty} \|\operatorname{proj}_n f\|_2^2.$$

Just as in the case of classical Fourier series in several variables, $S_n f$ does not in general converge either pointwise or in L^p for $p \neq 2$. The summability of Fourier series will be studied in the next chapter. Here we are content with one result.

Definition 2.2.3. For $f \in L^1(\mathbb{S}^{d-1})$, the Poisson integral of f is defined by

$$P_r f(\xi) := (f * P_r)(\xi), \quad \xi \in \mathbb{S}^{d-1}, \tag{2.2.5}$$

where the kernel $P_r(\langle x, \cdot\rangle)$ is given by, for $0 < r < 1$,

$$P_r(t) := \frac{1-r^2}{(1-2rt+r^2)^{d/2}}. \tag{2.2.6}$$

Lemma 2.2.4. *For* $0 < r < 1$, *the Poisson kernel satisfies the following properties:*

(1) For $x,y \in \mathbb{S}^{d-1}$, $P_r(\langle x,y\rangle) = \sum_{n=0}^{\infty} Z_n(x,y) r^n$.
(2) $P_r f = \sum_{n=0}^{\infty} r^n \operatorname{proj}_n f$.
(3) $P_r(\langle x,y\rangle) \ge 0$ *and* $\omega_d^{-1} \int_{\mathbb{S}^{d-1}} P_r(\langle x,y\rangle) \mathrm{d}\sigma(y) = 1$.

Proof. The first item follows from (1.2.7) and the Poisson kernel of the Gegenbauer polynomials in (B.2.8). The second item follows from the first. The infinite series converges uniformly, since $r < 1$. Integration term by term shows that $\omega_d^{-1} \int_{\mathbb{S}^{d-1}} P_r(\langle x,y\rangle) \mathrm{d}\sigma(y) = 1$. □

Theorem 2.2.5. *Let* f *be a continuous function on* $\mathbb{S}^{d-1}$. *For* $0 \le r < 1$, $u(r\xi) := P_r f(\xi)$ *is a harmonic function in* $x = r\xi$ *and* $\lim_{r\to 1^-} u(r\xi) = f(\xi)$, $\forall \xi \in \mathbb{S}^{d-1}$.

Proof. The proof is standard, and we shall be brief. By Lemma 2.2.4,

$$\begin{aligned} |u(r\xi) - f(\xi)| &= \frac{1}{\omega_d} \left| \int_{\mathbb{S}^{d-1}} [f(y) - f(\xi)] P_r(\langle \xi, y\rangle) \mathrm{d}\sigma(y) \right| \\ &\le \sup_{\|\xi - y\| \le \delta} |f(y) - f(\xi)| + 2\|f\|_\infty \int_{\|\xi - y\| \ge \delta} P_r(\langle \xi, y\rangle) \mathrm{d}\sigma(y) \end{aligned}$$

for every $\delta > 0$. If $\|\xi - y\| \ge \delta$, then $2(1 - \langle \xi, y\rangle) = \|\xi - y\|^2 \ge \delta^2$ for $y \in \mathbb{S}^{d-1}$, so that $P_r(\langle \xi, y\rangle) \le (1-r^2)/((1-r)^2 + r\delta^2) \to 0$, as $r \to 1^-$. Thus, taking $r \to 1^-$ and then $\delta \to 0$, the proof follows because f is continuous. □

In other words, u is the solution of the Dirichlet problem $\Delta u = 0$ inside the unit ball with the boundary condition $u = f$ on the unit sphere.

Corollary 2.2.6. *If* $f, g \in L^1(\mathbb{S}^{d-1})$ *and* $\operatorname{proj}_n f = \operatorname{proj}_n g$ *for all* $n = 0, 1, \ldots$, *then* $f = g$.

Proof. If $\operatorname{proj}_n f = \operatorname{proj}_n g$ for all $n \in \mathbb{N}_0$, then $P_r f = P_r g$ for $0 \le r < 1$. The desired conclusion $f = g$ then follows from Theorem 2.2.5 and the uniqueness of the limit in L^1. □

Next, we define multiplier operators of spherical harmonic expansions. The operator norm of an operator $T : L^p(\mathbb{S}^{d-1}) \to L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, is defined by

$$\|T\|_{(p,p)} := \sup_{\|f\|_p \le 1} \|Tf\|_p,$$

where in the case of $p = \infty$, we assume $f \in C(\mathbb{S}^{d-1})$.

Definition 2.2.7. A linear operator T on $L^p(\mathbb{S}^{d-1})$ for some $1 \le p \le \infty$ is called a multiplier operator if there exists a sequence $\{\mu_n\}$ of real numbers such that

$$\operatorname{proj}_n Tf = \mu_n \operatorname{proj}_n f, \qquad \forall f \in L^p(\mathbb{S}^{d-1}), \quad \forall n \in \mathbb{N}_0.$$

It is a bounded multiplier operator on $L^p(\mathbb{S}^{d-1})$ if $\|T\|_{(p,p)}$ is finite.

Because of Theorem 2.1.3, the convolution operator $f \mapsto f * g$ is a multiplier operator for every $g : [-1,1] \mapsto \mathbb{R}$. In particular, the translation operator T_θ in (2.1.6) is an example of a multiplier operator, as is the Cesàro operator defined later. We are interested in the L^p bounded multiplier operators. It is clear that a multiplier operator is bounded on $L^2(\mathbb{S}^{d-1})$ if and only if its associated sequence $\{\mu_j\}$ is bounded.

Proposition 2.2.8. *If T is a bounded multiplier operator on $L^p(\mathbb{S}^{d-1})$ for some p and $1 \le p \le \infty$, then it extends to a bounded operator on $L^q(\mathbb{S}^{d-1})$ with $\|T\|_{(q,q)} = \|T\|_{(q',q')}$ for all $|\frac{1}{q} - \frac{1}{2}| \le |\frac{1}{p} - \frac{1}{2}|$, where $\frac{1}{q} + \frac{1}{q'} = 1$.*

Proof. Recall the inner product $\langle f,g \rangle = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(x) g(x)\, \mathrm{d}\sigma(x)$, $f,g \in L^2(\mathbb{S}^{d-1})$. By the orthogonality of spherical harmonics, if f and g are polynomials, then it is easy to see that $\langle Tf, g \rangle = \langle f, Tg \rangle$. By the density of spherical polynomials and the Riesz representation theorem, we deduce that

$$\begin{aligned}
\|T\|_{(q',q')} &= \sup_{\|f\|_{q'} \le 1} \sup_{\|g\|_q \le 1} |\langle Tf, g \rangle| = \sup_{\|f\|_{q'} \le 1} \sup_{\|g\|_q \le 1} |\langle f, Tg \rangle| \\
&= \sup_{\|g\|_q \le 1} \|Tg\|_q = \|T\|_{(q,q)}.
\end{aligned}$$

Thus, if T is a bounded multiplier operator on L^p, then it is bounded on $L^{p'}$ as well. Using the Riesz–Thorin interpolation theorem, we further conclude that T is bounded on $L^q(\mathbb{S}^{d-1})$ for all $|\frac{1}{q} - \frac{1}{2}| \le |\frac{1}{p} - \frac{1}{2}|$. This completes the proof. □

Our next proposition gives a characterization of multiplier operators on the sphere. Recall the operator T_Q defined by $T_Q f(x) = f(Q^{-1}x)$ for $Q \in O(d)$.

Proposition 2.2.9. *A bounded linear operator T on $L^2(\mathbb{S}^{d-1})$ is a multiplier operator if and only if it is invariant under the group of rotations, that is, if and only if $TT_Q = T_Q T$ for all $Q \in O(d)$.*

Proof. We begin with the proof of the necessity. Let T be a multiplier operator associated with a bounded sequence $\{\mu_j\}$. By (2.1.4) and the rotation invariance of the Lebesgue measure $\mathrm{d}\sigma(x)$, $T_Q \operatorname{proj}_n = \operatorname{proj}_n T_Q$ for all $Q \in O(d)$. Thus, for each $f \in L^2(\mathbb{S}^{d-1})$ and $n \in \mathbb{N}_0$, □

$$\operatorname{proj}_n(T_Q T f) = T_Q \operatorname{proj}_n(Tf) = \mu_n T_Q \operatorname{proj}_n f = \mu_n \operatorname{proj}_n(T_Q f) = \operatorname{proj}_n(T T_Q f).$$

It then follows by Corollary 2.2.6 that $TT_Q = T_Q T$. Next, we prove the sufficiency. Assume that T is bounded on L^2 and that $TT_Q = T_Q T$ for all $Q \in O(d)$. By definition, it suffices to show that there exists a sequence of real numbers μ_n such that $\operatorname{proj}_n(Tf) = \mu_n \operatorname{proj}_n f$ for each $f \in \mathscr{H}_n^d$ and all $n \in \mathbb{N}_0$. However, by (2.1.4), it suffices to show that

$$T[Z_n(\langle \cdot, y \rangle)](x) = \mu_n Z_n(\langle x, y \rangle), \quad x, y \in \mathbb{S}^{d-1}.$$

Since the reproducing kernel $Z_n(\langle x,y\rangle)$ is invariant under simultaneous rotation in both variables, the rotation invariance of T shows that for all $Q \in O(d)$,

$$T\left[Z_n\left(\langle \cdot, Qy\rangle\right)\right](Qx) = T\left[Z_n\left(\langle Q(\cdot), Qy\rangle\right)\right](x) = T\left[Z_n\left(\langle \cdot, y\rangle\right)\right](x),$$

which implies, as in the proof of Lemma 1.2.5, that $T\left[Z_n\left(\langle \cdot, y\rangle\right)\right](x)$ is a zonal function $F_n(\langle x,y\rangle)$ of $\langle x,y\rangle$. On the other hand, for each fixed $x \in \mathbb{S}^{d-1}$, it follows directly by (1.2.3) that the function $y \mapsto T\left[Z_n\left(\langle \cdot, y\rangle\right)\right](x) = F_n\left(\langle x,y\rangle\right)$ is a spherical harmonic of degree n. Thus, using (1.2.2) and the Funk–Hecke formula (1.2.11), we conclude that for some real number μ_n,

$$F_n(\langle x,y\rangle) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} F_n(\langle x,z\rangle) Z_n\big(\langle y,z\rangle\big)\,\mathrm{d}\sigma(z) = \mu_n Z_n\big(\langle x,y\rangle\big),$$

which completes the proof. □

2.3 The Hardy–Littlewood Maximal Function

For $x \in \mathbb{S}^{d-1}$ and $\theta > 0$, we define a spherical cap $\mathsf{c}(x,\theta)$ centered at x by

$$\mathsf{c}(x,\theta) := \{y \in \mathbb{S}^{d-1} : \langle x,y\rangle \ge \cos\theta\}. \tag{2.3.1}$$

Let $|\mathsf{c}(x,\theta)|$ denote the surface area of $\mathsf{c}(x,\theta)$, that is,

$$|\mathsf{c}(x,\theta)| := \int_{\mathsf{c}(x,\theta)} \mathrm{d}\sigma(y) = \omega_{d-1}\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi, \tag{2.3.2}$$

where the second equation follows from (A.5.1), which is independent of x.

Definition 2.3.1. For $f \in L^1(\mathbb{S}^{d-1})$, we define the Hardy-Littlewood maximal function

$$Mf(x) := \sup_{0<\theta\le\pi} \frac{1}{|\mathsf{c}(x,\theta)|}\int_{\mathsf{c}(x,\theta)} |f(y)|\mathrm{d}\sigma(y).$$

An alternative definition of Mf is given in the following lemma.

Lemma 2.3.2. *For a nonnegative function $f \in L^2(\mathbb{S}^{d-1})$,*

$$Mf(x) = \sup_{0<\theta\le\pi} \frac{\omega_d}{|\mathsf{c}(x,\theta)|}(f * \chi_{[\cos\theta,1]})(x) \tag{2.3.3}$$

$$= \sup_{0<\theta\le\pi} \frac{\int_0^\theta T_\phi f(x)(\sin\phi)^{d-2}\mathrm{d}\phi}{\int_0^\theta(\sin\phi)^{d-2}\mathrm{d}\phi}, \tag{2.3.4}$$

where $\chi_{[a,1]}$ denotes the characteristic function of the interval $[a,1]$.

Proof. The first equation follows from (2.3.2) and

$$\frac{1}{\omega_d}\int_{\mathsf{c}(x,\theta)} f(y)\mathrm{d}\sigma(y) = \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y)\chi_{[\cos\theta,1]}(\langle x,y\rangle)\mathrm{d}\sigma(y) = (f*\chi_{[\cos\theta,1]})(x),$$

whereas the second equation follows from the above equation and (2.1.8). □

Given $E\subset\mathbb{S}^{d-1}$, we denote by $\mathrm{meas}(E)$ the Lebesgue measure $\int_E \mathrm{d}\sigma(x)$ of E. The maximal function satisfies a weak estimate for L^1 functions given by the following weak type-$(1,1)$ inequality:

Theorem 2.3.3. *For $f \in L^1(\mathbb{S}^{d-1})$ and $\alpha > 0$,*

$$\mathrm{meas}\left\{x\in\mathbb{S}^{d-1} : Mf(x)\ge\alpha\right\}\le c\frac{\|f\|_1}{\alpha}.$$

Proof. As in the case of the maximal function defined for functions in $\mathbb{R}^d$, the proof relies on a covering lemma. Let $E_\alpha := \{x\in\mathbb{S}^{d-1} : Mf(x) > \alpha\}$. For every $x\in E_\alpha$, there exists a spherical cap $\mathsf{c}(x,\theta)$ such that $\int_{\mathsf{c}(x,\theta)}|f(y)|d\sigma > \alpha|\mathsf{c}(x,\theta)|$ by the definition of Mf. The collection of such $\mathsf{c}(x,\theta)$ for $x\in E_\alpha$ clearly covers E_α. The covering lemma, which can be proved exactly as in the classical case of $\mathbb{R}^d$ (cf. [159]), states that given an arbitrary compact subset E of E_α, there exists a sequence of spherical caps $\mathsf{c}(x_k,\theta_k)$ that are mutually disjoint such that $\sum_k|\mathsf{c}(x_k,\theta_k)|\ge c\,\mathrm{meas}\,E$, where c is a constant depending only on the dimension. Hence, it follows that

$$\|f\|_1 \ge \int_{\cup_k\mathsf{c}(x_\kappa,\theta_k)}|f(y)|\mathrm{d}\sigma(y)\ge\alpha\sum_k|\mathsf{c}(x_k,\theta_k)|\ge c\,\alpha\,\mathrm{meas}\,E.$$

Taking the supremum over all compact subsets E of E_α, we deduce the desired inequality, which completes the proof. □

Corollary 2.3.4. *Let $f\in L^p(\mathbb{S}^{d-1})$, $1<p<\infty$, or $f\in C(\mathbb{S}^{d-1})$ for $p=\infty$. Then the maximal function is a strong type-(p,p) operator for $1<p\le\infty$; that is,*

$$\|Mf\|_p\le c\|f\|_p,\qquad 1<p\le\infty.$$

Proof. The definition of Mf shows immediately that $\|Mf\|_\infty\le\|f\|_\infty$. Thus, M is of weak type $(1,1)$ and strong type (∞,∞), from which the result for $1<p<\infty$ follows from the Marcinkiewicz interpolation theorem [159]. □

As an application of the boundedness of maximal functions, let us consider

$$f_\theta(x) := \frac{1}{|\mathsf{c}(x,\theta)|}\int_{\mathsf{c}(x,\theta)}|f(y)|\mathrm{d}\sigma(y),\quad 0<\theta\le1,\quad x\in\mathbb{S}^{d-1}.$$

Lemma 2.3.5. *If $f \in L^1(\mathbb{S}^{d-1})$, then $\lim_{\theta \to 0^+} f_\theta(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$.*

Proof. By (2.3.4), we can write

$$f_\theta(x) - f(x) = \frac{\int_0^\theta (T_\phi f(x) - f(x))(\sin\phi)^{d-2}\mathrm{d}\phi}{\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi},$$

which implies $\|f_\theta - f\|_1 \le \sup_{0\le\phi\le\theta} \|T_\phi f - f\|_1$, and by Lemma 2.1.7, f_θ converges to f in $L^1(\mathbb{S}^{d-1})$. To prove the almost-everywhere convergence, we show that

$$\Omega f(x) := \left| \limsup_{\theta\to 0^+} f_\theta(x) - \liminf_{\theta\to 0^+} f_\theta(x) \right| = 0$$

for almost every $x \in \mathbb{S}^{d-1}$. Since $C(\mathbb{S}^{d-1})$ is dense in $L^1(\mathbb{S}^{d-1})$, we can write $f = h + g$ with $h \in C(\mathbb{S}^{d-1})$ and $\|g\|_1$ arbitrarily small. Since $\Omega g(x) \le 2Mg(x)$, the maximal inequality implies

$$\operatorname{meas}\{x : \Omega g(x) \ge \alpha\} \le \operatorname{meas}\{x : 2Mg(x) \ge \alpha\} \le c\frac{\|g\|_1}{\alpha}.$$

Since $\Omega h = 0$, this shows that $\Omega f(x) = 0$ almost everywhere. □

Theorem 2.3.6. *Assume that $g \in L^1([-1,1]; w_\lambda)$, $\lambda = \frac{d-2}{2}$, and that $k(\theta) := g(\cos\theta)$ is a continuous, nonnegative, and decreasing function on $[0,\pi]$. Then for $f \in L^1(\mathbb{S}^{d-1})$,*

$$|(f * g)(x)| \le cM(|f|)(x), \quad x \in \mathbb{S}^{d-1},$$

where $c = \int_0^\pi k(\theta)(\sin\theta)^{d-2}\mathrm{d}\theta$.

Proof. For fixed x, define $F_x(\theta) = \int_0^\theta T_\phi f(x)(\sin\phi)^{d-2}\mathrm{d}\phi$. By (2.1.8) and (2.3.3), an integration by parts shows that

$$\begin{aligned}
|f * g(x)| &= \frac{\omega_{d-1}}{\omega_d} \left| \int_0^\pi g(\cos\theta) T_\theta f(x)(\sin\theta)^{d-2}\mathrm{d}\theta \right| \\
&= \frac{\omega_{d-1}}{\omega_d} \left| F_x(\pi)k(\pi) - \int_0^\pi k'(\phi)F_x(\phi)\mathrm{d}\phi \right| \\
&\le Mf(x) \left[k(\pi)\int_0^\pi (\sin\phi)^{d-2}\mathrm{d}\phi - \int_0^\pi k'(\theta)\int_0^\theta (\sin\phi)^{d-2}\mathrm{d}\phi\mathrm{d}\theta \right],
\end{aligned}$$

where we have used the fact that $k(\theta)$ is nonnegative and $k'(\theta) \le 0$. Consequently, integrating by parts again, we see that $|f * g(x)| \le cMf(x)$ follows from the fact that $\int_0^\pi k(\theta)(\sin\theta)^{d-2}\mathrm{d}\theta$ is bounded. □

2.4 Spherical Harmonic Series and Cesàro Means

By Theorem 2.2.2, the spherical harmonic series converges in $L^2(\mathbb{S}^{d-1})$. In particular, for $f \in L^2(\mathbb{S}^{d-1})$, the partial sum operator $S_n f$ converges to f in the $\|\cdot\|_2$ norm, and the operator norm $\|S_n\|_2$ is uniformly bounded. The operator norm of S_n in $L^p(\mathbb{S}^{d-1})$ is defined by

$$\|S_n\|_p := \sup_{\|f\|_p=1} \|S_n f\|_p.$$

Theorem 2.4.1. *Let $d > 2$. Then $\|S_n\|_\infty = \|S_n\|_1 = \Lambda_n$, where*

$$\Lambda_n := \max_{x\in\mathbb{S}^{d-1}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |K_n(x,y)| \mathrm{d}\sigma(y) \sim n^{\frac{d-2}{2}}.$$

Proof. That $\|S_n\|_\infty = \|S_n\|_1 = \Lambda_n$ follows from a standard argument for linear integral operators. By the closed form of $K_n(x,y)$ in (2.2.4) and the integral relation (A.5.1), we have

$$\Lambda_n = \frac{(2\lambda+1)_n}{(\lambda+\frac{1}{2})_n} c_\lambda \int_{-1}^{1} |P_n^{(\lambda+\frac{1}{2},\lambda-\frac{1}{2})}(t)|(1-t^2)^{\lambda-\frac{1}{2}} \mathrm{d}t, \quad \lambda = \frac{d-2}{2},$$

from which the asymptotic relation follows from (B.1.8). $\square$

In the case of $d = 2$, we have $\|S_n\|_\infty \sim \log n$, as shown in classical Fourier analysis.

The constant Λ_n is often called the Lebesgue constant. Since it is unbounded as $n \to \infty$, the uniform boundedness principle implies that there is a function $f \in C(\mathbb{S}^{d-1})$ for which $S_n f$ does not converge to f in the uniform norm. We then look for summability methods for the spherical harmonic series that will ensure convergence. One important class of such methods is that of Cesàro means.

Definition 2.4.2. For $\delta \in \mathbb{R}$, the Cesàro, or (C,δ), means of the sequence $\{a_k\}_{k=0}^\infty$ are defined by

$$s_n^\delta := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta a_k, \quad n = 0,1,\ldots, \tag{2.4.1}$$

where

$$A_k^\delta = \binom{k+\delta}{k} = \frac{(\delta+k)(\delta+k-1)\ldots(\delta+1)}{k!}. \tag{2.4.2}$$

The sequence is said to be (C,δ) summable if s_n^δ converges as $n \to \infty$.

The properties of A_k^δ and s_k^d are collected in Appendix A. By (A.4.4), a simple exercise shows that if s_n^δ converges to s, then $s_n^{\delta+\tau}$ converges to s for all $\tau > 0$.

Denote by $S_n^\delta f$ the (C,δ) means of the spherical harmonic series,

$$S_n^\delta f := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \operatorname{proj}_k f. \tag{2.4.3}$$

If $\delta = 0$, then $S_n^\delta f = S_n f$. By (2.1.4), $S_n^\delta f$ can be written as a convolution operator

$$S_n^\delta f = f * K_n^\delta, \qquad K_n^\delta(t) := \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^\delta \frac{k+\lambda}{\lambda} C_k^\lambda(t), \tag{2.4.4}$$

where $\lambda = (d-2)/2$. This kernel is closely connected to the Cesàro means $s_n^\delta(w_\lambda; f)$ of the Fourier orthogonal series in the Gegenbauer polynomials. Indeed, let $k_n^\delta(w_\lambda; \cdot, \cdot)$ denote the kernel of $s_n^\delta(w_\lambda; f)$,

$$s_n^\delta(w_\lambda; f, x) = c_\lambda \int_{-1}^{1} f(y) k_n^\delta(w_\lambda; x, y) w_\lambda(y) \mathrm{d}y.$$

Then it is easy to verify that

$$K_n^\delta(t) = k_n^\delta(w_\lambda; 1, t), \qquad \lambda = \frac{d-2}{2}. \tag{2.4.5}$$

As a consequence, some of the convergence results of spherical harmonic series can be deduced from those of the Gegenbauer series. Here is one example:

Theorem 2.4.3. *For $\delta \ge d-1$, S_n^δ is a nonnegative operator; that is, $S_n^\delta f(x) \ge 0$ if $f(x) \ge 0$ for all $x \in \mathbb{S}^{d-1}$.*

Proof. By (2.4.4), we need only show that $K_n^{d-1}(t) \ge 0$ for $t \in [-1, 1]$, which follows, by (2.4.5), from the classical result of Kogbetiliantz that $k_n^\delta(w_\lambda; 1, t) \ge 0$ if $\delta \ge 2\lambda + 1$ ([5, p. 389]). $\square$

Moreover, the relation (2.4.5) shows that the Lebesgue constant of S_n^δ can be deduced from the Lebesgue function of $s_n^\delta(w_\lambda)$ evaluated at the point $x = 1$.

Theorem 2.4.4. *Let $\lambda = \frac{d-2}{2}$. Then $\|S_n^\delta\|_\infty = \|S_n^\delta\|_1 = \Lambda_n^\delta$, where*

$$\Lambda_n^\delta := \max_{x \in \mathbb{S}^{d-1}} \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} |K_n^\delta(\langle x, y \rangle)| \mathrm{d}\sigma(y) \sim \begin{cases} n^{\lambda-\delta}, & 0 < \delta < \lambda, \\ \log n, & \delta = \lambda, \\ 1 & \delta > \lambda. \end{cases}$$

In particular, $\|S_n^\delta\|_\infty$ and $\|S_n^\delta\|_1$ are bounded if and only if $\delta > \frac{d-2}{2}$.

Proof. Again, that $\|S_n^\delta\|_\infty = \|S_n^\delta\|_1 = \Lambda_n^\delta$ follows from a standard argument. By the integral relation (A.5.1),

$$\Lambda_n^\delta = c_\lambda \int_{-1}^{1} \left| k_n^\delta(w_\lambda; 1, t) \right| (1-t^2)^{\lambda - \frac{1}{2}} \mathrm{d}t.$$

The asymptotic behavior of this integral, as $n \to \infty$, is given in [162, Sect. 9.4]. $\square$

Corollary 2.4.5. *If $\delta > \frac{d-2}{2}$, then for $f \in L^p(\mathbb{S}^{d-1})$ and $1 \le p \le \infty$, or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$,*

$$\sup_n \|S_n^\delta f\|_p \le c\|f\|_p \quad \textit{and} \quad \lim_{n\to\infty} \|S_n^\delta f - f\|_p = 0.$$

Furthermore, for $p = 1$ or ∞, the convergence fails in general if $\delta = \frac{d-2}{2}$.

Proof. The boundedness of $\|S_n^\delta f\|_p$ for $\delta > \frac{d-2}{2}$ follows from Theorem 2.4.4 and the Riesz–Thorin interpolation theorem. Since $S_n P = P$ for every polynomial P of fixed degree m and $S_n = S_n^0$, it follows that $S_n^\delta P$ converges to P as $n \to \infty$ for all $\delta > 0$. Hence, the norm convergence of $S_n^\delta f$ follows from the boundedness of the norm. If $\delta = \frac{d-2}{2}$, then $\|S_n^\delta\|_p$ is unbounded for $p = 1$ and $p = \infty$, and the convergence fails in general by the uniform boundedness principle. □

The index $\lambda = \frac{d-2}{2}$ is often called the critical index for the (C,δ) means of the spherical harmonic series on $\mathbb{S}^{d-1}$.

A pointwise estimate for the kernel of the Cesàro means of the Jacobi series is given in Lemma B.1.2, from which the relation (2.4.5) gives the following:

Lemma 2.4.6. *Let $\lambda = \frac{d-2}{2} \ge 0$. If $0 \le \delta \le \lambda + 1$, then*

$$|K_n^\delta(t)| \le cn^{\lambda-\delta}\left[(1-t+n^{-2})^{-(\delta+\lambda+1)/2} + (1+t+n^{-2})^{-\lambda/2}\right].$$

If $\lambda + 1 \le \delta \le 2\lambda + 1$, then

$$|K_n^\delta(t)| \le cn^{-1}\left[(1-t+n^{-2})^{-(\lambda+1)} + (1+t+n^{-2})^{-(2\lambda+1-\delta)/2}\right].$$

If $\delta \ge 2\lambda + 1$, then

$$|K_n^\delta(t)| \le cn^{-1}(1-t+n^{-2})^{-(\lambda+1)}.$$

These estimates of the kernel functions can be used to establish the upper bound of Λ_n^δ in Theorem 2.4.4. We use them to study the almost-everywhere convergence of the Cesàro means. For $\delta \ge 0$, we define the maximal Cesàro (C,δ) operator by

$$S_*^\delta f(x) := \sup_{N\ge 0} |S_N^\delta f(x)|, \qquad x \in \mathbb{S}^{d-1}.$$

It turns out that the maximal Cesàro operator S_*^δ can be controlled pointwise by the Hardy–Littlewood maximal function whenever $\delta > \frac{d-2}{2}$.

Theorem 2.4.7. *If $\delta > \frac{d-2}{2}$ and $f \in L^1(\mathbb{S}^{d-1})$, then for every $x \in \mathbb{S}^{d-1}$,*

$$S_*^\delta f(x) \le c\,[Mf(x) + Mf(-x)]\,. \tag{2.4.6}$$

If, in addition, $\delta \ge d-1$, then the term $Mf(-x)$ in (2.4.6) can be dropped.

Proof. For the proof of (2.4.6), it suffices to consider the case $\lambda < \delta \le \lambda + 1$, where $\lambda = \frac{d-2}{2}$, since by (A.4.4), $S_*^{\delta+\tau} f(x) \le S_*^\delta (f)(x)$ for every $\tau > 0$. Setting

$$G_{n,1}^\delta(\cos\theta) :=\ n^{\lambda-\delta}(n^{-1}+\theta)^{-(\delta+\lambda+1)}\chi_{[0,\frac{\pi}{2}]}(\theta),$$
$$G_{n,2}^\delta(\cos\theta) :=\ n^{\lambda-\delta}(n^{-1}+\theta)^{-\lambda}\chi_{[0,\frac{\pi}{2}]}(\theta),$$

we obtain from Lemma 2.4.6 that for $\lambda < \delta \le \lambda + 1$,

$$K_n^\delta(\cos\theta) \le c\left[G_{n,1}^\delta(\cos\theta) + G_{n,2}^\delta(\cos(\pi-\theta))\right].$$

It is easy to see that $g(t) = G_{n,i}^\delta(t)$ satisfies the conditions of Theorem 2.3.6, so that by $T_\theta(-x) = T_{\pi-\theta}(x)$, (2.4.4), and Theorem 2.3.6, we have then

$$\begin{aligned}|S_n^\delta f(x)| &\le \left[(|f| * G_{n,1}^\delta)(x) + (|f| * G_{n,2}^\delta)(-x)\right]\\ &\le c\left[Mf(x) + Mf(-x)\right].\end{aligned}$$

Furthermore, if $\delta > 2\lambda + 1$, then Lemma 2.4.6 shows that $|K_n^\delta(\cos\theta)|$ is bounded by a single term, and the same proof yields $|S_n^\delta f(x)| \le cMf(x)$. □

From Theorem 2.4.7, Theorem 2.3.3, and the density argument in the proof of Lemma 2.3.5, we deduce the following corollary.

Corollary 2.4.8. *If $\delta > \frac{d-2}{2}$ and $f \in L^1(\mathbb{S}^{d-1})$, then $\lim_{N\to\infty} S_N^\delta f(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$, and moreover,*

$$\mathrm{meas}\{x \in \mathbb{S}^{d-1} : S_*^\delta f(x) > \alpha\} \le c\frac{\|f\|_1}{\alpha}, \ \ \forall \alpha > 0.$$

2.5 Convergence of Cesàro Means: Further Results

According to Corollary 2.4.5, the (C,δ) means $S_n^\delta f$ converge to f in the $L^1(\mathbb{S}^{d-1})$ norm or in the uniform norm if and only if $\delta > \frac{d-2}{2}$. We also know, since $S_n^0 f = S_n f$, that convergence holds for $\delta \ge 0$ in the $L^2(\mathbb{S}^{d-1})$ norm. The case $1 < p < \infty$ is more delicate and far more difficult to resolve.

Below, we shall state several results for $1 < p < \infty$ without proofs. Some of these results will be proved in the more general setting of weighted approximation in Chap. 9. Throughout this section, we set, for $1 \le p \le \infty$,

$$\delta(p) := \max\left\{0, (d-1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}\right\}. \tag{2.5.1}$$

We start with a negative result of Bonami and Clerc [18, Theorem 5.1].

Theorem 2.5.1. *If $1 \le p \le \infty$ and $0 \le \delta \le \delta(p)$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ such that $S_n^\delta f$ does not converge in $L^p(\mathbb{S}^{d-1})$. In particular, if $\delta = 0$ and $p \ne 2$, then there exists a function $f \in L^p(\mathbb{S}^{d-1})$ such that $S_n f$ does not converge in $L^p(\mathbb{S}^{d-1})$.*

Theorem 2.5.1 also implies that if $1 \le p \le \infty$ and $0 \le \delta \le \delta(p)$, then $\{\|S_n^\delta\|_p\}_{n=1}^\infty$ is unbounded.

In the positive direction, the convergence of $S_n^\delta f$ depends on a sharp bound of the projection operator. Such a bound was established by Sogge [155].

Theorem 2.5.2. *If $1 \le p \le \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{d}$, then*

$$\|\operatorname{proj}_n f\|_2 \le c_d n^{\delta(p)} \|f\|_p. \tag{2.5.2}$$

The connection between (2.5.2) and the convergence of S_n^δ, revealed in the proof of Theorem 5.2 in [18], leads to the following theorem in [155].

Theorem 2.5.3. *If $1 \le p < \infty$ and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{d}$, then $\lim_{n\to\infty} \|S_n^\delta f - f\|_p = 0$ holds for all $f \in L^p(\mathbb{S}^{d-1})$ if and only if $\delta > \delta(p)$.*

Theorems 2.5.2 and 2.5.3 will be proved in Chap. 9 in the more general setting of weighted approximation on the sphere.

In the case of $d = 3$, Sogge [155] further proved that the conclusion of Theorem 2.5.3 remains true without the assumption $|1/2 - 1/p| \ge 1/3$.

For the maximal Cesàro operator S_*^δ, the following result was proved, using Stein's interpolation theorem for analytic families of operators, in [18].

Theorem 2.5.4. *If $1 < p \le 2$, $\delta > (d-2)(\frac{1}{p} - \frac{1}{2})$, and $f \in L^p(\mathbb{S}^{d-1})$, then*

$$\|S_*^\delta f\|_p \le C_p \|f\|_p.$$

Together with Corollary 2.4.8, Theorem 2.5.4 implies the following corollary.

Corollary 2.5.5. *If $1 \le p \le 2$, $f \in L^p(\mathbb{S}^{d-1})$, and $\delta > (d-2)(\frac{1}{p} - \frac{1}{2})$, then*

$$\lim_{n\to\infty} S_n^\delta f(x) = f(x)$$

for almost every $x \in \mathbb{S}^{d-1}$.

2.6 Near-Best Approximation Operators and Highly Localized Kernels

For a given function $f \in L^p(\mathbb{S}^{d-1})$, its Cesàro means $S_n^\delta f$ provide a sequence of polynomials that approximate f. These means are useful for our further study, but they are not ideal for quantitative results in approximation theory.

Definition 2.6.1. Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $n \ge 0$, the error of the best approximation to f by polynomials of degree at most n is defined by

$$E_n(f)_p := \inf_{g \in \Pi_n(\mathbb{S}^{d-1})} \|f - g\|_p, \qquad 1 \le p \le \infty. \tag{2.6.1}$$

The best-approximation element exists, since $\Pi_n(\mathbb{S}^{d-1})$ is a finite-dimensional space, by a general theorem on Banach spaces [54, p. 59]. Finding such a polynomial, however, is not easy. For most applications, it fortunately is sufficient to find a polynomial that is a near-best approximation.

Definition 2.6.2. Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge 2$. Define

$$L_n f(x) := f * L_n(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(y) L_n(\langle x, y \rangle)\, \mathrm{d}\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{2.6.2}$$

for $n = 0, 1, 2, \ldots$, where

$$L_n(t) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \frac{k+\lambda}{\lambda} C_k^\lambda(t), \quad \lambda = \frac{d-2}{2}, \quad t \in [-1,1]. \tag{2.6.3}$$

In the following, if a function satisfies the properties of the η function in the theorem, we shall call it a C^∞ cutoff function, or simply a cutoff function.

Since η is supported on $[0,2]$, the summation in $L_n f$ can be terminated at $k = 2n-1$, so that $L_n f$ is a polynomial of degree at most $2n-1$. It approximates f as well as the best-approximation polynomial of degree n.

Theorem 2.6.3. *Let $f \in L^p$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*

(1) *$L_n f \in \Pi_{2n-1}(\mathbb{S}^{d-1})$ and $L_n f = f$ for $f \in \Pi_n(\mathbb{S}^{d-1})$.*
(2) *For $n \in \mathbb{N}$, $\|L_n f\|_p \le c\|f\|_p$.*
(3) *For $n \in \mathbb{N}$,*

$$\|f - L_n f\|_p \le (1+c) E_n(f)_p. \tag{2.6.4}$$

Proof. We have already shown that $L_n f$ is a polynomial of degree at most $2n-1$. Using the projection operator proj_n of $\mathscr{H}_n^d$, we can write

$$L_n f = \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \mathrm{proj}_k f.$$

Since the definition of η shows that $\eta(\frac{k}{n}) = 1$ for $0 \le k \le n$, it follows readily that $L_n f = \sum_{k=0}^{n} \mathrm{proj}_k f = f$ if $f \in \Pi_n(\mathbb{S}^{d-1})$. This proves (1).

By Young's inequality, Theorem 2.1.2, $\|L_n f\|_p \le \|f\|_p \|L_n\|_{\lambda,1}$, where $\lambda = \frac{d-2}{2}$. The proof of (2) reduces to showing that $\|L_n\|_{\lambda,1}$ is bounded. Let σ be a positive integer such that $\sigma \ge d-1$ so that the (C,σ) means $K_n^\sigma(t)$ of the sequence $\frac{k+\lambda}{\lambda}C_k^\lambda(t)$ are nonnegative on $[-1,1]$ (see Theorem 2.4.3). Let $\triangle$ denote the difference operator defined in (A.3.1). Using summation by parts repeatedly, we can write

$$L_n(t) = \sum_{k=1}^{\infty} \triangle^{\sigma+1}\eta\left(\frac{k}{n}\right)\binom{k+\sigma}{k}K_k^\sigma(t),$$

where $\triangle^m$ acts on the function $t \mapsto \eta(\frac{t}{n})$. Since $\eta \in C^\infty[0,+\infty)$ implies that $|\triangle^{\sigma+1}\eta(k/n)| \le cn^{-\sigma-1}$ and $\binom{k+\sigma}{k} \le ck^\sigma$, it follows that

$$\|L_n\|_{\lambda,1} \le cn^{-\sigma-1}\sum_{k=1}^{2n} k^\sigma \|K_k^\sigma\|_{\lambda,1} \le cn^{-\sigma-1}\sum_{k=1}^{2n} k^\sigma \le c,$$

since the support of η is on $[0,2]$. This completes the proof of (2).

The proof of (3) is an easy consequence of (1) and (2). Indeed, let p_n be the best-approximation polynomial of degree n. Then (1) shows that $L_n p_n = p_n$, so that

$$\|f - L_n f\|_p \le \|f-p_n\| + \|L_n(f-p_n)\| \le (1+c)\|f-p_n\|_p = (1+c)E_n(f)_p,$$

by the triangle inequality and (2). □

Recall that the Laplace–Beltrami operator Δ_0 can be decomposed, by (1.8.3), in terms of the angular derivative $D_{i,j}$ defined in (1.8.1).

Proposition 2.6.4. *The operator $D_{i,j}$ commutes with convolution. More precisely, for $f \in C^1(\mathbb{S}^{d-1})$ and $g \in C^1[-1,1]$, $D_{i,j}(f*g) = (D_{i,j}f)*g$. In particular, the operator $D_{i,j}$, hence Δ_0, commutes with the operator L_n.*

Proof. Directly from the definition,

$$D_{i,j}^{(x)} g(\langle x,y\rangle) = g'(\langle x,y\rangle)(x_i y_j - y_i x_j) = -D_{i,j}^{(y)} g(\langle x,y\rangle),$$

where $D_{i,j}^{(x)}$ means that $D_{i,j}$ acts on the x variable. Hence,

$$\begin{aligned} D_{i,j}(f*g)(x) &= \frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y) D_{i,j}^{(x)} g(\langle x,y\rangle)\mathrm{d}\sigma(y) \\ &= -\frac{1}{\omega_d}\int_{\mathbb{S}^{d-1}} f(y) D_{i,j}^{(y)} g(\langle x,y\rangle)\mathrm{d}\sigma(y) = (D_{i,j}f)*g(x), \end{aligned}$$

where the last step follows from (1.8.7). □

The fact that $L_n f$ reproduces polynomials of degree up to n and that it is a polynomial of degree at most $2n-1$ itself makes it a fundamental tool in polynomial approximation. Even more, its kernel, $L_n(t)$, possesses the remarkable property that

L_n and its derivatives $L_n^{(j)}$ are highly localized at $t = 1$. More precisely, we state the following theorem.

Theorem 2.6.5. *Let ℓ be a positive integer. For $n \ge 1$ and $\theta \in [0,\pi]$,*

$$|L_n^{(j)}(\cos\theta)| \le c_{\ell,j} \left\| \eta^{(3\ell-1)} \right\|_\infty n^{d-1+2j} (1+n\theta)^{-\ell}, \quad j = 0,1,\ldots. \tag{2.6.5}$$

By choosing ℓ large but fixed, the theorem shows that L_n and its derivatives decay faster than any polynomial of a fixed degree. This desirable property will be used in a number of occasions in this book. As an application, we prove the following corollary first.

Corollary 2.6.6. *Let ℓ be a positive integer and let $\delta > 0$. Then*

$$\sup_{z \in \mathtt{c}(y,\frac{\delta}{n})} |L_n(\langle x,y\rangle) - L_n(\langle x,z\rangle)| \le c\delta n^{d-1}(1+n\mathtt{d}(x,y))^{-\ell} \tag{2.6.6}$$

for all $x, y \in \mathbb{S}^{d-1}$ that satisfy $\mathtt{d}(x,y) \ge 4\delta/n$.

Proof. If $z \in \mathtt{c}(y,\frac{\delta}{n})$ and $\mathtt{d}(x,y) \ge 4\delta/n$, then $1+n\mathtt{d}(x,z) \sim 1+n\mathtt{d}(x,y)$ by the triangle inequality. Applying the estimate (2.6.5) with $j = 1$ and $\ell+1$ instead of ℓ, we obtain, by the mean value theorem,

$$|L_n(\langle x,y\rangle) - L_n(\langle x,z\rangle)| \le c|\langle x,y\rangle - \langle x,z\rangle| n^{d+1}(1+n\mathtt{d}(x,y))^{-\ell-1}.$$

Since $\langle x,y\rangle = \cos d(x,y)$, it follows by the triangle inequality that

$$\begin{aligned} |\langle x,y\rangle - \langle x,z\rangle| &= 2\sin\frac{\mathtt{d}(x,z)-\mathtt{d}(x,y)}{2}\sin\frac{\mathtt{d}(x,z)+\mathtt{d}(x,y)}{2} \\ &\le c\mathtt{d}(z,y)(\mathtt{d}(x,y)+n^{-1}) \le c\delta n^{-2}(1+n\mathtt{d}(x,y)). \end{aligned}$$

Putting the two inequalities together proves the stated result. $\square$

Recall that the Gegenbauer polynomials C_n^λ are special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}$. For further uses, we shall prove a more general result than Theorem 2.6.5. For this, we define, for $\alpha \ge \beta \ge -\frac{1}{2}$,

$$G_n^{(\alpha,\beta)}(t) := \sum_{k=0}^\infty \varphi\left(\frac{k}{n}\right) \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)} P_k^{(\alpha,\beta)}(t) \tag{2.6.7}$$

for some smooth cutoff function φ. If $\varphi = \eta$, then by (B.2.1) in Appendix B,

$$L_n(t) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-1)} G_n^{\left(\frac{d-3}{2},\frac{d-3}{2}\right)}(t),$$

so that Theorem 2.6.5 follows directly from the estimates of the kernels $G_n^{(\alpha,\beta)}(t)$. Furthermore, the assumption about the cutoff function in the theorem below is considerably weaker than that of Theorem 2.6.5.

Theorem 2.6.7. *Let ℓ be a positive integer and let $\varphi \in C^{3\ell-1}[0,\infty)$ satisfy supp $\varphi \subset [0,2]$ and $\varphi^{(j)}(0) = 0$ for $j = 1,2,\ldots,3\ell-2$. Then the kernel function $G_n \equiv G_n^{(\alpha,\beta)}$ defined in (2.6.7), with $\alpha \ge \beta \ge -1/2$, satisfies, for $\theta \in [0,\pi]$ and $n \in \mathbb{N}$,*

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j,\alpha}\left\|\varphi^{(3\ell-1)}\right\|_\infty n^{2\alpha+2j+2}(1+n\theta)^{-\ell}, \quad j = 0,1,\ldots. \tag{2.6.8}$$

Proof. Taking derivatives and using (B.1.5) in Appendix B, it follows that

$$G_n^{(j)}(t) = \sum_{k=0}^{\infty} \varphi\left(\frac{k+j}{n}\right) \frac{(2k+\alpha+\beta+2j+1)\Gamma(k+\alpha+\beta+2j+1)}{2^j\Gamma(k+\beta+j+1)} P_k^{(\alpha+j,\beta+j)}(t).$$

Because φ is supported on $[0,2)$, the summation terminates at $2n-j$. Summing by parts ℓ times and using (B.1.10) with $(\alpha+j+i,\beta+j)$ in place of (α,β) at the ith time for $i = 1,\ldots,\ell$, we obtain

$$G_n^{(j)}(t) = 2^{-j}\sum_{k=0}^{\infty} a_{n,\ell}(k)\frac{\Gamma(k+\alpha+\beta+2j+\ell+1)}{\Gamma(k+\beta+j+1)} P_k^{(\alpha+j+\ell,\beta+j)}(t), \tag{2.6.9}$$

where $\{a_{n,\ell}\}_{\ell=0}^{\infty}$ is a sequence of functions on $[0,\infty)$ defined recursively by

$$a_{n,0}(s) := (2s+\alpha+\beta+2j+1)\varphi\left(\frac{s+j}{n}\right),$$

$$a_{n,\ell+1}(s) := \frac{a_{n,\ell}(s)}{2s+\alpha+\beta+2j+\ell+1} - \frac{a_{n,\ell}(s+1)}{2s+\alpha+\beta+2j+\ell+3}.$$

We claim that if $m+j \le \ell$ and $\ell \ge 1$, then

$$\left|a_{n,\ell}^{(m)}(s)\right| \le c_{\ell,j}(s+1)^{-m-2j+1}\left(\frac{s+1}{n}\right)^{2\ell-1}\left\|\varphi^{(2\ell+m+j-1)}\right\|_{L^\infty\left[0,\frac{s+j+\ell}{n}\right]}, \tag{2.6.10}$$

which implies, in particular, with $m = 0$ and $j = \ell$,

$$|a_{n,\ell}(k)| \le c_{\ell,\ell}\left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1}. \tag{2.6.11}$$

For the moment, we take (2.6.10) for granted and proceed with the proof of (2.6.8). Using (2.6.11) and (2.6.9), we obtain

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j}\left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1}\sum_{k=0}^{2n}(k+1)^{\alpha+j+\ell}\left|P_k^{(\alpha+j+\ell,\beta+j)}(\cos\theta)\right|,$$

which implies, by (B.1.7), that for $\theta \in [0, \pi/2]$,

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1} \left[\sum_{0\le k\le \max\{\theta^{-1},2n\}} (k+1)^{2\alpha+2j+2\ell} \right.$$

$$\left. + \sum_{\max\{\theta^{-1},2n\}<k\le 2n} (k+1)^{\alpha+j+\ell-\frac{1}{2}} \theta^{-\alpha-j-\ell-\frac{1}{2}} \right]$$

$$\le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{2\alpha+2j+2} (1+n\theta)^{-(\alpha+j+\ell+\frac{1}{2})},$$

and that for $\theta \in [\frac{\pi}{2}, \pi]$,

$$\left|G_n^{(j)}(\cos\theta)\right| \le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{-2\ell+1} \sum_{k=0}^{2n} (k+1)^{\alpha+j+\ell} (k+1)^{\beta+j}$$

$$\le c_{\ell,j} \left\|\varphi^{(3\ell-1)}\right\|_\infty n^{2\alpha+2j+2} n^{-\ell},$$

where the last step uses the assumption $\alpha \ge \beta$. Putting the above together and recalling that $\alpha \ge -\frac{1}{2}$, we have proved the desired estimate (2.6.8).

It remains to prove the claim (2.6.10). We first observe that by Taylor's theorem,

$$\left\|\varphi^{(m)}\right\|_{L^\infty[0,t]} \le \frac{t^k}{k!} \left\|\varphi^{(m+k)}\right\|_{L^\infty[0,t]}, \quad t \ge 0, \tag{2.6.12}$$

whenever $m \ge 1$ and $m+k \le 3\ell-1$. Next we prove the case $j=1$ of (2.6.10). Since $a_{n,1}$ is supported on $[0, 2n]$, we may assume, without loss of generality, that $0 \le s \le 2n$. Directly from the definition, we have

$$a_{n,1}(s) = \varphi\left(\frac{s+j}{n}\right) - \varphi\left(\frac{s+j+1}{n}\right) = -\int_{\frac{j}{n}}^{\frac{j+1}{n}} \varphi'\left(\frac{s}{n}+t\right) \mathrm{d}t,$$

which implies, if we combine (2.6.12) with $k = 2\ell-1$, that

$$\left|a_{n,1}^{(m)}(s)\right| \le n^{-m} \left| \int_{\frac{j}{n}}^{\frac{j+1}{n}} \varphi^{(m+1)}\left(\frac{s}{n}+t\right) \mathrm{d}t \right|$$

$$\le n^{-m-1} \frac{1}{(2\ell-1)!} \left(\frac{s+j+1}{n}\right)^{2\ell-1} \left\|\varphi^{(m+2\ell)}\right\|_{L^\infty[0,\frac{s+j+1}{n}]}$$

$$\le c_{\ell,j} (s+1)^{-m-1} \left(\frac{s+1}{n}\right)^{2\ell-1} \left\|\varphi^{(m+2\ell)}\right\|_{L^\infty[0,\frac{s+j+1}{N}]},$$

where the last step uses the assumption $0 \le s \le 2n$. This proves (2.6.10) when $\ell = 1$. We now proceed by induction. Assuming that (2.6.10) has been proven for some $\ell \ge 1$ and observing that

$$a_{n,\ell+1}^{(m)}(s) = -\int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \left(\frac{a_{n,\ell}(s+t)}{2s+2t+\alpha+\beta+2j+\ell+1}\right) \mathrm{d}t,$$

we obtain, by the product formula of derivatives, that for $m+j+1 \le \ell$,

$$\begin{aligned}\left|a_{n,\ell+1}^{(m)}(s)\right| &\le \int_0^1 \max_{k_1+k_2=m+1} \left|a_{n,j}^{(k_1)}(s+t)\right| (s+1)^{-k_2-1}\,\mathrm{d}t \\ &\le c_{\ell,j}(s+1)^{-m-2j-1}\left(\frac{s+1}{n}\right)^{2\ell-1} \left\|\varphi^{(2\ell+m+j)}\right\|_{L^\infty[0,\frac{s+j+\ell+1}{n}]}.\end{aligned}$$

This proves (2.6.10) in the case of $\ell+1$ and completes the induction. $\square$

2.7 Notes and Further Results

The translation operator T_θ is also called the spherical mean in the literature. Most of the properties of T_θ stated in the first section can be found in [14, 135, 148]; see also [133, 174]. The following useful representation of T_θ using the Haar measure on $SO(d)$ was proved in [42] in the case of even d:

$$T_\theta f(x) = \int_{SO(d)} f(Q^{-1}M_\theta Q x)\,dQ, \quad x \in \mathbb{S}^{d-1},\ f \in L(\mathbb{S}^{d-1}),$$

where dQ denotes the Haar measure on $SO(d)$ normalized by $\int_{SO(d)} dQ = 1$, and M_θ is a $d\times d$ orthogonal matrix given by

$$M_\theta := \begin{pmatrix} \cos\theta & \sin\theta & \cdots & 0 & 0 \\ -\sin\theta & \cos\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos\theta & \sin\theta \\ 0 & 0 & \cdots & -\sin\theta & \cos\theta \end{pmatrix}.$$

A general theory of convolutions and multipliers on $\mathbb{S}^{d-1}$ can be found in [63].

Sharp estimates of the Cesáro kernels and their derivatives are often needed in analysis. The kernels K_n^δ are the (C,δ) means of the Gegenbauer polynomials, a special case of the Jacobi polynomials. For the estimates of these kernels and their derivatives, we refer to [18,30,34]. A more refined result is the following asymptotic expression for the Cesàro kernels in [109, 155]: for $\lambda = \frac{d-2}{2}$ and $\frac{\pi}{2n+2} \le \theta \le \pi - \frac{\pi}{2n+2}$,

$$K_n^\delta(\cos\theta) = \frac{2A_n^\lambda}{A_n^\delta} \frac{\sin\left(\left(n+\lambda+\frac{\delta+1}{2}\right)\theta - \frac{\lambda+\delta}{2}\pi\right)}{(2\sin\theta)^\lambda (2\sin\frac{\theta}{2})^{1+\delta}}$$
$$+ \frac{(n+1)^{\lambda-\delta-1}\eta_n^{\lambda,\delta}(\theta)}{(\sin\theta)^{\lambda+1}(\sin\frac{\theta}{2})^{1+\delta}} + \frac{\xi_n^{\lambda,\delta}(\theta)}{(n+1)(\sin\frac{\theta}{2})^{2+2\lambda}},$$

where $|\eta_n^{\lambda,\delta}(\theta)| + |\xi_n^{\lambda,\delta}(\theta)| \le c$. Precise constants in the main term of the asyptotics of the Lebesgue constants of the Cesàro means were obtained in [109]. For a detailed discussion of Cesàro summability in L^p and H^p spaces, the reader is referred to [18, 34, 155].

A complement of Corollary 2.4.8 is a counterexample of [163]: there exists a function $f \in L(\mathbb{S}^{d-1})$ for which $\lim_{n\to\infty} \sup_{j\ge n} |S_j^{\frac{d-2}{2}}(f)(x)| = \infty$ for almost every $x \in \mathbb{S}^{d-1}$. Furthermore, a complement of Corollary 2.4.5 is the following result in [24] on the convergence of the Cesàro means at the critical index $\delta = \frac{d-2}{2}$: if $\int_{\mathbb{S}^{d-1}} |f(x)|\log^2(1+|f(x)|)\,d\sigma(x) < \infty$, then $\lim_{n\to\infty} S_n^{\frac{d-2}{2}}(f)(x) = f(x)$ for almost every $x \in \mathbb{S}^{d-1}$.

The operator $L_n f$ was used in [148], but the use of such an operator on the sphere appeared already in [95]. The fast decay of the kernel was established in [23, 120, 129, 138]. Under additional assumptions on the cutoff function, the rate of decay can be improved to the subexponential estimate [91]

$$\left|L_n^{(j)}(\cos\theta)\right| \le c_1 n^{2\alpha+2j+2} \exp\left\{-\frac{c_2 n\theta}{[\ln(e+n\theta)]^{1+\varepsilon}}\right\}, \quad 0 \le \theta \le \pi, \qquad (2.7.1)$$

where $c_2 = c'\varepsilon$ with $c' > 0$ an absolute constant and $c_1 = c''8^j$ with $c'' > 0$ depending only on α, β, and ε.

Fast Fourier spherical transforms are available for numerical computation of the spherical harmonic expansions; see [62, 101, 143].

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