

Preface

There has been a great deal of interest in optimal control systems described by stochastic and partial differential equations. These optimal control problems lead to stochastic and partial differential inclusions. The aim of this book is to present a unified theory of stochastic differential inclusions written in integral form with both types of stochastic set-valued integrals defined as subsets of the space $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and as multifunctions with closed values in the space \mathbb{R}^n . Such defined inclusions are therefore divided into two types: stochastic functional inclusions ($SFI(F, G)$) and stochastic differential inclusions ($SDI(F, G)$), respectively. The main results of the book deal with properties of solution sets of stochastic functional inclusions and some of their applications in stochastic optimal control theory and in the theory of partial differential inclusions. In particular, apart from the existence of weak solutions for initial value problems of stochastic functional inclusions, the existence of their strong and weak viable solutions is also investigated. An important role in applications is played by theorems on weak compactness of solution sets of weak and viable weak solutions for the above initial value problems. As a result of these properties, some optimal control problems for dynamical systems described by stochastic and partial differential inclusions are obtained. Let us remark that for a given pair (F, G) of multifunctions, the sets $\mathcal{X}(F, G)$ and $\mathcal{S}(F, G)$ of all weak solutions of $SFI(F, G)$ and $SDI(F, G)$, respectively, are defined as families of systems $(\mathcal{P}_{\mathbb{F}}, x, B)$ consisting of a filtered probability space $\mathcal{P}_{\mathbb{F}}$, a continuous process $x = (x_t)_{t \geq 0}$, and an \mathbb{F} -Brownian motion $B = (B_t)_{t \geq 0}$ satisfying these inclusions. Immediately from the definitions of $SFI(F, G)$ and $SDI(F, G)$, it follows that $\mathcal{X}(F, G) \subset \mathcal{S}(F, G)$. It is natural to extend the results of this book to the set $\mathcal{S}(F, G)$ and consider weak solutions with x a càdlàg process instead of a continuous one. These problems are quite complicated and need new methods. Therefore, in this book, they are left as open problems.

The first papers dealing with stochastic functional inclusions written in integral form are due to Hiai [38] and Kisielewicz [50–56, 58, 60–62]. Independently, Ahmed [2], Da Prato and Frankowska [23], Aubin and Da Prato [9], and Aubin et al. [10] have considered stochastic differential inclusions symbolically written in the differential form $dx_t \in F(t, x_t)dt + G(t, x_t)dB_t$ and understood as a problem

consisting in finding a system $(\mathcal{P}_{\mathbb{F}}, x, B)$ consisting of a filtered probability space $\mathcal{P}_{\mathbb{F}}$, a continuous process $x = (x_t)_{t \geq 0}$, and an \mathbb{F} -Brownian motion such that $x_t = x_0 + \int_0^t f_\tau d\tau + \int_0^t g_\tau dB_\tau$ with $f_t \in (F \circ x)_t =: F(t, x_t)$ and $g_t \in (G \circ x)_t =: G(t, x_t)$ a.s. for $t \geq 0$. Stochastic functional inclusions defined by Hiai [38] and Kisielewicz [51] are in the general case understood as a problem consisting in finding a system $(\mathcal{P}_{\mathbb{F}}, x, B)$ such that $x_t - x_s \in \text{cl}_{\mathbb{L}}\{J_{st}(F \circ x) + \mathcal{J}_{st}(G \circ x)\}$ for every $0 \leq s \leq t < \infty$, where $J_{st}(F \circ x)$ and $\mathcal{J}_{st}(G \circ x)$ denote set-valued functional integrals on the interval $[s, t]$ of $F \circ x$ and $G \circ x$, respectively. It is evident that some properties of stochastic functional inclusions written in integral form follow from properties of set-valued stochastic integrals. Such properties are difficult to obtain for stochastic differential inclusions written in differential form.

The first results dealing with set-valued stochastic integrals with respect to the Wiener process with application to some set-valued stochastic differential equations are due to Bocşan [22]. More general definitions and properties of set-valued stochastic integrals were given in the above-cited papers of Hiai and Kisielewicz, where set-valued stochastic integrals are defined as certain subsets of the spaces $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$ of all square integrable random variables with values at \mathbb{R}^n and \mathcal{X} , respectively, where \mathcal{X} is a Hilbert space. In this book, such integrals are called stochastic functional set-valued integrals. Unfortunately, such integrals do not admit a representation by set-valued random variables with values in \mathbb{R}^n and \mathcal{X} , because they are not decomposable subsets of $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$, respectively. Later, Jung and Kim [46] (see also [98]) defined a set-valued stochastic integral as a set-valued random variable determined by a closed decomposable hull of the above-mentioned set-valued stochastic functional integral. Unfortunately, the authors did not obtain any properties of such integrals. In Chap. 3, we apply the above approach to the theory of set-valued stochastic integrals of \mathbb{F} -nonanticipative multiprocesses and obtain some properties of such integrals.

The first results dealing with partial differential inclusions were in fact simple generalizations of ordinary differential inclusions. They dealt with hyperbolic partial differential inclusions of the form $z''_{x,y} \in F(x, y, z)$. Later on, partial differential inclusions $z''_{x,y} \in F(x, y, z, z'_x, z'_y)$ were also investigated. Such partial differential inclusions have been considered by Kubiacyk [65], Dawidowski and Kubiacyk [24], Dawidowski et al. [25], and Sosulski [92, 93], among others. Some hyperbolic partial differential inclusions were considered in Aubin and Frankowska [11]. A new idea dealing with partial differential inclusions was given by Bartuzel and Fryszkowski in their papers [15–17], where partial differential inclusions of the form $Du \in F(u)$ with a lower semicontinuous multifunction F and a partial differential operator D are considered. The existence and properties of solutions of initial and boundary value problems of such inclusions follow from classical results dealing with abstract differential inclusions. As usual, certain types of continuous selection theorems for set-valued mappings play an important role in investigations of such inclusions.

The partial differential inclusions considered in this book have the forms $u'_t(t, x) \in (\mathbb{L}_{FG}u)(t, x) + c(t, x)u(t, x)$ and $\psi(t, x) \in (\mathbb{L}_{FG}u)(t, x) + c(t, x)u(t, x)$,

where c and ψ are given functions and \mathbb{L}_{FG} denotes the set-valued diffusion generator defined by given multifunctions F and G . The first results dealing with such partial differential inclusions are due to Kisielewicz [60, 61]. The initial and boundary value problems of such inclusions are investigated by stochastic methods. Their solutions are characterized by weak solutions of stochastic functional inclusions $SFI(F, G)$. Such an approach leads to natural methods of solving some optimal control problems for systems described by the above type of partial differential inclusions. It is a consequence of weak compactness with respect to the convergence in distribution of sets of all weak solutions of considered stochastic functional inclusions.

The content of the book is divided into seven parts. Chapter 1 covers basic notions and theorems of the theory of stochastic processes. Chapter 2 contains the fundamental notions of the theory of set-valued mappings and the theory of set-valued stochastic processes. Chapter 3 is devoted to the theory of set-valued stochastic integrals. Apart from their properties, it contains some important selection theorems. The main results of Chap. 4 deal with properties of stochastic functional and differential inclusions. In particular, it contains theorems dealing with weak compactness with respect to convergence in distribution of solution sets of weak solutions of initial value problems for stochastic functional inclusions. Chapter 5 contains some results dealing with viability theory for forward and backward stochastic functional and differential inclusions, whereas Chaps. 6 and 7 are devoted to some applications of the above-mentioned results to partial differential inclusions and to some optimal control problems for systems described by stochastic functional and partial differential inclusions.

The present book is intended for students, professionals in mathematics, and those interested in applications of the theory. Selected probabilistic methods and the theory of set-valued mappings are needed for understanding the text. Formulas, theorems, lemmas, remarks, and corollaries are numbered separately in each chapter and denoted by pairs of numbers. The first stands for the section number, the second for the number of the formula, theorem, etc. If we need to quote some formula or theorem given in the same chapter, we always write only this pair. In other cases, we will use this pair with information indicated the chapter number. The ends of proofs, theorems, remarks, and corollaries are denoted by \square .

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