Chapter 2
Set-Valued Stochastic Processes

This chapter is devoted to basic notions of the theory of set-valued mappings and set-valued stochastic processes. We begin with the notions and basic properties of the space of subsets of a given metric space. Selected properties of set-valued mappings, Aumann integrals, and set-valued stochastic processes are presented. The last two parts of this chapter discuss properties of a set-valued conditional expectation and selection properties of set-valued integrals depending on random parameters.

1 Spaces of Subsets of a Metric Space

Let (X,ρ) be a metric space and $(A_n)_{n=1}^{\infty}$ a sequence of subsets of X. The sets $\bigcap_{n=1}^{\infty}\bigcup_{k=0}^{\infty} A_{n+k}$ and $\bigcup_{n=1}^{\infty}\bigcap_{k=0}^{\infty} A_{n+k}$ are denoted by $\mathrm{Lim\,sup}\, A_n$ and $\mathrm{Lim\,inf}\, A_n$, respectively and said to be a limit superior and a limit inferior, respectively of a sequence $(A_n)_{n=1}^{\infty}$. Immediately from the above definitions, the following properties of $\mathrm{Lim\,sup}\, A_n$ and $\mathrm{Lim\,inf}\, A_n$ follow.

Lemma 1.1. *Let $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be sequences of subsets of X and let $C \subset X$. Then*

(i) $\mathrm{Lim\,inf}\, A_n = (\mathrm{Lim\,sup}\, A_n^{\sim})^{\sim}$, *where* $D^{\sim} = X \setminus D$ *for* $D \subset X$,
(ii) $\mathrm{Lim\,inf}(A_n \cap B_n) = \mathrm{Lim\,inf}\, A_n \cap \mathrm{Lim\,inf}\, B_n$,
(iii) $\mathrm{Lim\,inf}(A_n \cap C) = (\mathrm{Lim\,inf}\, A_n) \cap C$,
(iv) $\bigcap_{n=1}^{\infty} A_n \subset \mathrm{Lim\,inf}\, A_n \subset \mathrm{Lim\,sup}\, A_n \subset \bigcup_{n=1}^{\infty} A_n$.

Corollary 1.1. *For every family $\{A_n^i : i,n = 1,2,\ldots\}$ of subsets of X, one has* $\bigcap_{i=1}^{\infty}[\mathrm{Lim\,inf}\ A_n^i) = \mathrm{Lim\,inf}[\bigcap_{i=1}^{\infty} A_n^i]$. □

Apart from the limits $\mathrm{Lim\,sup}\, A_n$ and $\mathrm{Lim\,inf}\, A_n$, we can also define the Kuratowski limits $\mathrm{Li}\, A_n$ and $\mathrm{Ls}\, A_n$. The set $\mathrm{Li}\, A_n$ is defined by the property $x \in \mathrm{Li}\, A_n$ if and only if for every neighborhood $\mathcal{U}$ of x, there is an integer $N \geq 1$

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such that $\mathcal{U} \cap A_n \neq \emptyset$ for every $n \geq N$. It is said to be the Kuratowski limit inferior of a sequence $(A_n)_{n=1}^{\infty}$. Similarly, the Kuratowski limit superior $\mathrm{Ls}\, A_n$ of a sequence $(A_n)_{n=1}^{\infty}$ is defined by the property: $x \in \mathrm{Ls}\, A_n$ if and only if for every neighborhood $\mathcal{U}$ of x, there are infinitely many n with $\mathcal{U} \cap A_n \neq \emptyset$.

Corollary 1.2. *For every sequence $(A_n)_{n=1}^{\infty}$ of subsets of X, one has*

(i) $\mathrm{Li}\, A_n \subset \mathrm{Ls}\, A_n$,
(ii) $x \in \mathrm{Li}\, A_n$ *if and only if there exist an integer $N \geq 1$ and a sequence $(x_n)_{n=1}^{\infty}$ of X with $x_n \in A_n$ for $n \geq N$ such that $x = \lim_{n\to\infty} x_n$,*
(iii) $x \in \mathrm{Ls}\, A_n$ *if and only if there exist an increasing subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(x_{n_k})_{k=1}^{\infty}$ of X such that $x_{n_k} \in A_{n_k}$ for $k = 1, 2, \ldots$ and $x = \lim_{k\to\infty} x_{n_k}$.* □

The following properties of the Kuratowski limits follow immediately from the above definitions.

Lemma 1.2. *Let $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ be sequences of subsets of X. Then*

(i) if $A_n \subset B_n$ for $n \geq 1$, then $\mathrm{Li}\, A_n \subset \mathrm{Li}\, B_n$ *and* $\mathrm{Ls}\, A_n \subset \mathrm{Ls}\, B_n$,
(ii) $\mathrm{Lim\,inf}\, A_n \subset \mathrm{Li}\, A_n$,
(iii) $\mathrm{Li}(A_n \cap B_n) \subset (\mathrm{Li}\, A_n) \cap (\mathrm{Li}\, B_n)$,
(iv) $\mathrm{Ls}(A_n \cap B_n) \subset (\mathrm{Ls}\, A_n) \cap (\mathrm{Ls}\, B_n)$,
(v) $\mathrm{Ls}\, A_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=0}^{\infty} A_{k+n}}$,
(vi) if $A_n = A$ for $n \geq 1$, then $\mathrm{Li}\, A_n = \bar{A} = \mathrm{Ls}\, A_n$.

Let $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X. For every $A, B \in \mathrm{Cl}(X)$, we define the Hausdorff distance $h(A, B)$ with respect to the metric ρ on X by setting $h(A, B) = \inf\{\varepsilon : A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\}$, where $V_\varepsilon(C)$ denotes the ε-neighborhood of $C \in \mathrm{Cl}(X)$, i.e., $V_\varepsilon(C) = \{x \in X : \mathrm{dist}(x, C) \leq \varepsilon\}$.

Lemma 1.3. *The function $h : \mathrm{Cl}(X) \times \mathrm{Cl}(X) \to [0, \infty]$ has the following properties:*

(i) $h(A, B) = 0$ *if and only if* $A = B$ *for* $A, B \in \mathrm{Cl}(X)$,
(ii) $h(A, B) = h(B, A)$ *for every* $A, B \in \mathrm{Cl}(X)$,
(iii) $h(A, B) \leq h(A, C) + h(C, B)$ *for every* $A, B, C \in \mathrm{Cl}(X)$.

Proof. To prove (i), let us observe that $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$, where $\bar{h}(C, D) = \sup_{x\in C} \mathrm{dist}(x, D)$ for $C, D \in \mathrm{Cl}(X)$. Hence it follows that $h(A, B) = 0$ implies that $A \subset B$ and $B \subset A$, because $A, B \in \mathrm{Cl}(X)$. Then $A = B$. Statement (ii) is evident. To prove (iii), if $A \subset V_\varepsilon(C)$ and $C \subset V_\eta(B)$, then $A \subset V_{\varepsilon+\eta}(B)$. Consequently, we get $\bar{h}(A, B) \leq \bar{h}(A, C) + \bar{h}(C, B)$. Thus $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\} \leq \max\{\bar{h}(A, C) + \bar{h}(C, B), \bar{h}(B, C) + \bar{h}(C, A)\} \leq \max\{h(A, C) + h(C, B), h(B, C) + h(C, A)\} = h(A, C) + h(C, B)$. □

Theorem 1.1. *Let* (X, ρ) *be a compact metric space. Then* $(\mathrm{Cl}(X), h)$ *is a compact metric space, too. In such a case, a sequence* $(A_n)_{n=1}^{\infty}$ *of* $\mathrm{Cl}(X)$ *converges to* $A \in \mathrm{Cl}(X)$ *in the* h*-metric topology if and only if* $\mathrm{Li}\, A_n = A = \mathrm{Ls}\, A_n$.

Proof. By virtue of Lemma 1.3, the mapping h is a metric on $\mathrm{Cl}(X)$. The proof of compactness of $(\mathrm{Cl}(X), h)$ can be found in [49]. If a sequence $(A_n)_{n=1}^{\infty}$ of $\mathrm{Cl}(X)$ converges to $A \in \mathrm{Cl}(X)$ in the h-metric topology, then by the definitions of the metric h and the Kuratowski limits $\mathrm{Li}\, A_n$ and $\mathrm{Ls}\, A_n$, we get $A \subset \mathrm{Li}\, A_n$ and $\mathrm{Ls}\, A_n \subset A$. Then $\mathrm{Li}\, A_n = A = \mathrm{Ls}\, A_n$. Conversely, let $A \subset X$ be such that $\mathrm{Li}\, A_n = A = \mathrm{Ls}\, A_n$. By the compactness of the metric space (X, ρ), we have $A \neq \emptyset$. Then $A \in \mathrm{Cl}(X)$. We have to show that for every $\varepsilon > 0$ and sufficiently large $n \geq 1$, one has $A_n \subset V_\varepsilon(A)$ and $A \subset V_\varepsilon(A_n)$. If the first inclusion were false, we would obtain a contradiction to $A = \mathrm{Ls}\, A_n$. If the second inclusion were false, we would obtain a contradiction to $\mathrm{Li}\, A_n = A$. □

Remark 1.1. The above results can be extended to the case of a locally compact separable metric space (X, ρ), because it possesses a one-point compactification, denoted by $X \cup \{\infty\}$. □

We can extend the definition of Hausdorff distance on the family $\mathcal{P}_b(X)$ of all nonempty bounded subsets of a metric space (X, ρ). Similarly as above, for every $A, B \in \mathcal{P}_b(X)$, we define $\bar{h}(A, B) = \inf\{\varepsilon > 0 : A \subset V_\varepsilon\}$, and then the Hausdorff pseudometric h on $\mathcal{P}_b(X)$ is defined by $h(A, B) = \max\{\bar{h}(A, B), \bar{h}(B, A)\}$ for every $A, B \in \mathcal{P}_b(X)$. It can be verified that $h(A, B) = 0$ if and only if $\bar{A} = \bar{B}$.

Corollary 1.3. *For every* $A, B \in \mathcal{P}_b(X)$*, one has* $\bar{h}(A, B) = \sup\{\mathrm{dist}(a, B) : a \in A\}$*, where* $\mathrm{dist}(a, B) = \inf\{\rho(a, b) : b \in B\}$.

Proof. For every $A, B \in \mathcal{P}_b(X)$, we have $A \subset V_\varepsilon(B)$ if for every $a \in A$, we have $\mathrm{dist}(a, B) \leq \varepsilon$. Then $A \subset V_\varepsilon(B)$ implies $\bar{h}(A, B) \leq \varepsilon$. Similarly, we can verify that $\bar{h}(A, B) \leq \varepsilon$ implies $A \subset V_\varepsilon(B)$. Hence it follows that $\inf\{\varepsilon > 0 : A \subset V_\varepsilon(B)\} = \inf\{\varepsilon > 0 : \varepsilon \geq \bar{h}(A, B)\} = \bar{h}(A, B)$. □

Lemma 1.4. *For every* $A, B \in \mathcal{P}_b(X)$*, one has* $h(\bar{A}, \bar{B}) \leq h(A, B)$.

Proof. For every $a \in \bar{A}$ and $\varepsilon > 0$, there is $a_\varepsilon \in A$ such that $\rho(a, a_\varepsilon) \leq \varepsilon$. Therefore, $\mathrm{dist}(a, \bar{B}) \leq \rho(a, a_\varepsilon) + \mathrm{dist}(a_\varepsilon, \bar{B}) \leq \varepsilon + \inf\{\rho(a_\varepsilon, b) : b \in \bar{B}\} \leq \varepsilon + \inf\{\rho(a_\varepsilon, b) : b \in B\} \leq \varepsilon + \bar{h}(A, B)$. Thus $\sup\{\mathrm{dist}(a, \bar{B}) : a \in \bar{A}\} \leq \varepsilon + \bar{h}(A, B)$, i.e., $\bar{h}(\bar{A}, \bar{B}) \leq \varepsilon + \bar{h}(A, B)$ for every $\varepsilon > 0$. Then $\bar{h}(\bar{A}, \bar{B}) \leq \bar{h}(A, B)$. Similarly, we get $\bar{h}(\bar{B}, \bar{A}) \leq \bar{h}(B, A)$. □

Remark 1.2. It can be verified that for every $A, B \in \mathcal{P}_b(X)$, one has $h(\bar{A}, \bar{B}) = h(A, B)$. □

If X is a linear normed space and $A, B \in \mathcal{P}_b(X)$, then we define $A + B = \{x \in X : x = a + b,\ a \in A,\ b \in B\}$. Similarly, for $A \in \mathcal{P}_b(X)$ and $\mu \in \mathbb{R}$, we define $\mu \cdot A = \{x \in X : x = \mu a,\ a \in A\}$. Immediately from the last definition, it follows that we can define a set $A + (-1)B$, which is often called the Minkowski difference of sets $A, B \in \mathcal{P}_b(X)$. In the general case, we have $A + (-1)A \neq \{0\}$.

For some nonempty compact convex sets $A, B \subset X$, a difference $A - B$, known as a Hukuhara difference, can be defined such that $A - A = \{0\}$. It is easy to verify that for all compact convex sets $A, B \in \mathcal{P}_b$ and $\lambda, \mu \in \mathbb{R}^+$, one has (i) $A + \{0\} = \{0\} + A = A$, (ii) $(A + B) + C = A + (B + C)$ (iii) $A + B = B + A$, (iv) $A + C = B + C$ implies $A = B$, (v) $1 \cdot A = A$, (vi) $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$, and (vii) $(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$.

Lemma 1.5. *Let $(X, ||\cdot||)$ be a linear normed space. For every $A, B, C, D \in \mathcal{P}_b(X)$ and $\mu \in \mathbb{R}^+$, one has (i) $\bar{h}(\mu A, \mu B) = \mu \bar{h}(A, B)$ and (ii) $\bar{h}(A + B, C + D) \leq \bar{h}(A, C) + \bar{h}(B, D)$.*

Proof. (i) If $A \subset V_\varepsilon(B)$, then $\mu A \subset V_{\mu\varepsilon}(\mu B)$. Hence it follows that $\inf\{\eta > 0 : \mu A \subset V_\eta(\mu B)\} = \mu \inf\{\eta > 0 : A \subset V_\eta(B)\} = \mu \bar{h}(A, B)$. (ii) If $A \subset V_\varepsilon(C)$ and $B \subset V_\eta(D)$, then $A + B \subset V_{\varepsilon+\eta}(C + D)$. Therefore, $\inf\{\varepsilon + \eta : A + B \subset V_{\varepsilon+\eta}(C + D)\} \leq \inf\{\varepsilon : A \subset V_\varepsilon(C)\} + \inf\{\eta : B \subset V_\eta(D)\} = \bar{h}(A, C) + \bar{h}(B, D)$. □

Corollary 1.4. *For every $\mu \in [0, 1]$ and $A, B, C, D \in \mathcal{P}_b(X)$, one has $\bar{h}(\mu A + (1 - \mu)B, \mu C + (1 - \mu)D) \leq \mu \bar{h}(A, C) + (1 - \mu)\bar{h}(B, D)$.* □

Corollary 1.5. *For every $A, B, C, D \in \mathcal{P}_b(X)$, one has $\bar{h}(\overline{A + B}, \overline{C + D}) \leq \bar{h}(A, C) + \bar{h}(B, D)$.* □

Corollary 1.6. *For every $A, B, C, D \in \mathcal{P}_b(X)$, one has $h(\overline{A + B}, \overline{C + D}) \leq h(A, C) + h(B, D)$.* □

2 Set-Valued Mappings

Let X and Y be nonempty sets and let $\mathcal{P}(Y)$ denote the family of all nonempty subsets of Y. By a set-valued mapping defined on X with values in $\mathcal{P}(Y)$ we mean a mapping $F : X \to \mathcal{P}(Y)$. It is clear that a set-valued mapping F can be defined as a relation contained in $X \times Y$ with the domain $\text{Dom}(F) = X$. It is defined by its graph: $\text{Graph}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. In applications, we need set-valued mappings having some special regularities, such as continuity and measurability. To define such set-valued mappings, we have to consider X and $\mathcal{P}(Y)$ as topological or measurable spaces. It can be verified that if $(Y, \mathcal{T})$ is a topological space, then we can define on $\mathcal{P}(Y)$ the upper topology $\mathcal{T}_u$ generated by the base $\mathcal{U} = \{[\cdot, G] : G \in \mathcal{T}\}$, where $[\cdot, G] = \{A \in \mathcal{P}(Y) : A \subset G\}$. Similarly, the lower topology $\mathcal{T}_l$ on $\mathcal{P}(Y)$ is generated by the subbase $\mathcal{L}$ defined by $\mathcal{L} = \{I_G : G \in \mathcal{T}\}$, where $I_G = \{U \in \mathcal{P}(Y) : U \cap G \neq \emptyset\}$. If (Y, d) is a separable metric space, then the Borel σ-algebra of the metric space $(\text{Comp}(Y), h)$ is generated by sets $\{K \in \text{Comp}(Y) : K \cap V \neq \emptyset\}$ for every open set $V \subset Y$, where Comp(Y) $\subset \mathcal{P}(Y)$ contains all compact subsets of Y, and h is the Hausdorff metric on $\text{Comp}(Y)$. These observations lead to the following definitions.

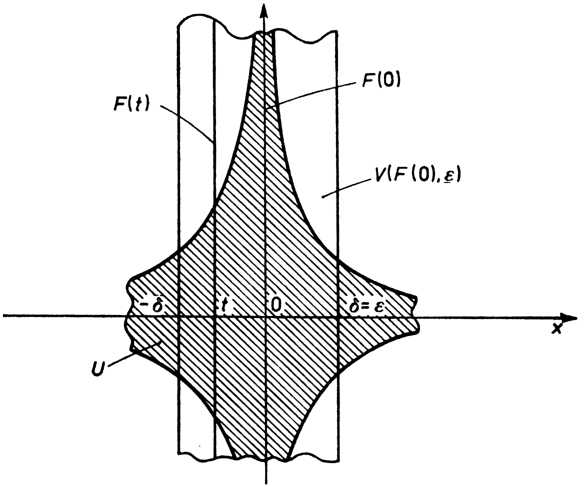

Fig. 2.1 A mapping that is H-u.s.c. but not u.s.c. at $t = 0$

If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are given topological spaces, then $F : X \to \mathcal{P}(Y)$ is said to be lower semicontinuous (l.s.c.) at $\bar{x} \in X$ if for every $U \in \mathcal{T}_Y$ with $F(\bar{x}) \cap U \neq \emptyset$, there is $V \in \mathcal{T}_X$ such that $\bar{x} \in V$ and $F(x) \cap U \neq \emptyset$ for every $x \in V$. We call $F : X \to \mathcal{P}(Y)$ upper semicontinuous (u.s.c.) at $\bar{x} \in X$ if for every $U \in \mathcal{T}_Y$ such that $F(\bar{x}) \subset U$, there is $V \in \mathcal{T}_X$ such that $\bar{x} \in V$ and $F(x) \subset U$ for every $x \in V$. If (X, ρ) and (Y, d) are given metric spaces, then a set-valued mapping $F : X \to \mathcal{P}(Y)$ is said to be H-l.s.c. at $\bar{x} \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(\bar{x}) \subset V(F(x), \varepsilon)$ for every $x \in B(\bar{x}, \delta)$, where $V(F(x), \varepsilon) = \{z \in X : \mathrm{dist}(z, F(x)) \leq \varepsilon\}$ and $B(\bar{x}, \delta)$ is an open ball of X centered at $\bar{x}$ with radius δ. It is clear that if F is H-l.s.c. at $\bar{x} \in X$, then it is also l.s.c. If $F(\bar{x}) \in \mathrm{Comp}(Y)$, then F is H-l.s.c. at $\bar{x} \in X$ if and only if it is l.s.c. at $\bar{x} \in X$. We say that F is l.s.c. (H-l.s.c.) on X if it is l.s.c. (H-l.s.c.) at every point $\bar{x} \in X$. In a similar manner, we can define H-u.s.c. set-valued mappings on X. There are some H-u.s.c. set-valued mappings that are not u.s.c. This is is illustrated in Fig. 2.1, where $F(t) = \{(y, z) \in \mathbb{R}^2 : y = t\}$ for $t \in \mathbb{R}$.

Let us observe that for a given l.s.c. set-valued mapping, we can change its values at finite points in such a way that it remains l.s.c. This follows from the following result.

Remark 2.1. If $F : X \to \mathcal{P}(Y)$ is l.s.c. on X and $(x_0, y_0) \in \mathrm{Graph}(F)$, then the set-valued mapping $G : X \to \mathcal{P}(Y)$ defined by taking $G(x) = F(x)$ for $x \in X \setminus \{x_0\}$ and $G(x) = \{y_0\}$ for $x = x_0$, is also l.s.c. on X.

Proof. It is clear that G is l.s.c. at every point $x \in X \setminus \{x_0\}$. By the lower semicontinuity of F at x_0 and the property of the point (x_0, y_0), for every neighborhood $\mathcal{U}$ of y_0 we have $F(x_0) \cap \mathcal{U} \neq \emptyset$, and there is a neighborhood V of x_0 such that $F(x) \cap \mathcal{U} \neq \emptyset$ for every $x \in V$. Therefore, for every $\mathcal{U} \in \mathcal{T}_Y$

such that $G(x_0) \cap \mathcal{U} \neq \emptyset$, there is $V \in \mathcal{T}_X$, a neighborhood of x_0, such that $G(x) \cap \mathcal{U} \neq \emptyset$ for every $x \in V$. Then G is l.s.c. at x_0. □

A set-valued mapping $F : X \to \mathcal{P}(Y)$ is said to be continuous (H-continuous) on X if it is l.s.c. (H-l.s.c.) and u.s.c. (H-u.s.c.) on X. It can be verified that a multifunction $F : X \to \mathrm{Comp}(Y)$ is continuous if and only if it is H-continuous. If $Y = \mathbb{R}^d$ and $F : X \to \mathrm{Comp}(Y)$ takes convex values, then F is continuous if and only if a function $X \ni x \to \sigma(p, F(x)) \in \mathbb{R}$ is continuous for every $p \in \mathbb{R}^d$, where $\sigma(\cdot, A)$ denotes the support function of $A \subset \mathbb{R}^d$. In optimal control theory, we have to deal with parameterized set-valued functions of the form $F(x) = \{f(x,u) : u \in U\}$, where $f : X \times U \to Y$ is a given function. We shall show that if $f(\cdot, u)$ is continuous, then the multifunction F is l.s.c. Some other properties of such multifunctions are given in Chap. 7.

Lemma 2.1. *Assume that X and Y are topological Hausdorff spaces and let $f : X \times U \to Y$, where $U \neq \emptyset$. If $f(\cdot, u)$ is continuous on X for every $u \in U$, then the set-valued mapping $F : X \to \mathcal{P}(Y)$ defined by $F(x) = f(x, U)$ is l.s.c. on X.*

Proof. Let $\bar{x} \in X$ be fixed and let $\mathcal{N}$ be an open set of Y. Suppose $\bar{u} \in U$ is such that $f(\bar{x}, \bar{u}) \in \mathcal{N}$. By the continuity of $f(\cdot, \bar{u})$ at $\bar{x}$, there is a neighborhood V of $\bar{x}$ such that $f(x, \bar{u}) \in \mathcal{N}$ for every $x \in V$. Therefore, for every $x \in V$, we get $F(x) \cap \mathcal{N} \neq \emptyset$. □

Let $(T, \mathcal{F})$ be a measurable space and (Y, d) a separable metric space. A set-valued mapping $F : T \to \mathcal{P}(Y)$ is said to be measurable (weakly measurable) if for every closed (open) set $E \subset Y$, we have $\{t \in T : F(t) \cap E \neq \emptyset\} \in \mathcal{F}$. It is clear that if F is measurable, then it is weakly measurable. The converse statement is not true in general.

Remark 2.2. Let $(T, \mathcal{F})$ be a measurable space and $(Y, ||\cdot||)$ a separable Banach space. For $F : T \to \mathcal{P}(Y)$, we denote by $\overline{\mathrm{co}}\, F$ the set-valued mapping $\overline{\mathrm{co}}\, F : T \to \mathcal{P}(Y)$ defined by $(\overline{\mathrm{co}}\, F)(t) = \overline{\mathrm{co}}\, F(t)$ for every $t \in T$, where $\overline{\mathrm{co}}\, F(t)$ denotes the closed convex hull of $F(t)$. It is clear that $\overline{\mathrm{co}}\, F$ is measurable whenever F is measurable. □

Remark 2.3. If $(T, \mathcal{F})$ is a measurable space, $Y = \mathbb{R}^d$, and $F : T \to \mathrm{Cl}(Y)$ is measurable, then the function $T \ni t \to \sigma(p, F(t)) \in \mathbb{R}$ is measurable for every $p \in \mathbb{R}^d$. If $F : T \to \mathrm{Cl}(\mathbb{R}^d)$ is convex-valued, then F is measurable if and only if $\sigma(p, F(\cdot))$ is measurable for every $p \in \mathbb{R}^d$. □

Remark 2.4. It can be proved that if X is a metric space, $Y = \mathbb{R}^d$, and $F : X \to \mathrm{Comp}(Y)$ is continuous, then $\sigma(p, F(\cdot))$ is continuous for every $p \in \mathbb{R}^d$. □

It is natural to expect that for a given multifunction $F : X \to \mathcal{P}(Y)$, there exists a function $f : X \to Y$ such that $f(x) \in F(x)$ for $x \in X$. The existence of such a function f, called a selector or a selection for F, follows immediately from Zermelo's axiom of choice. We recall first the Kuratowski–Zorn lemma, and then we will verify how from this principle, the axiom of choice can be deduced.

Lemma (Kuratowski–Zorn lemma). Let P be a nonempty partially ordered set with the property that every completely ordered subset of P has an upper bound in P. Then P contains at least one maximal element.

Lemma (Axiom of choice). Let $\mathcal{E}$ be a nonempty family of nonempty subsets of a set X. Then there exists a function $f : \mathcal{E} \to X$ such that $f(E) \in E$ for each E in $\mathcal{E}$.

Proof. Consider the class P of all functions $p : \mathcal{D}(p) \to X$ such that the domain $\mathcal{D}(p)$ of p belongs to $\mathcal{E}$ and $p(E) \in E$ for each E in $\mathcal{D}(p)$. This is a nonempty class, because $\mathcal{E}$ contains a nonempty set E, and if $x \in E$, the function with domain $\{E\}$ and range $\{x\}$ is a member of P. We order P by the inclusion relation in $\mathcal{E} \times X$. It can be verified that P satisfies the conditions of the Kuratowski–Zorn lemma. Therefore, we infer that there exists a function $f : \mathcal{E} \to X$ such that $f(E) \in E$ for each $E \in \mathcal{E}$. $\square$

Corollary 2.1. *For nonempty sets X and Y, every set-valued mapping $F : X \to \mathcal{P}(Y)$ possesses at least one selector.*

Proof. Let $\mathcal{E} = \{F(x)\}_{x \in X}$. The family $\mathcal{E}$ satisfies the conditions of Zermelo's axiom of choice. Therefore, there exists a function $g : \mathcal{E} \to Y$ such that $g(F(x)) \in F(x)$ for every $x \in X$. Thus the function $f : X \to Y$ defined by $f(x) = g(F(x))$ for $x \in X$ is a selector for F. $\square$

In applications of the theory of set-valued mappings, the existence of special selectors for given multifunctions plays a crucial role. The most difficult part is to deduce the existence of selectors with prescribed properties. In what follows, we shall present some results dealing with the existence of continuous, measurable, and Lipschitz continuous selectors. The fundamental problem deals with the existence of continuous selections. The following example shows that continuous set-valued mappings need not have, in general, continuous selections.

Example 2.1. Let F be the set-valued mapping defined on the interval $(-1, 1)$ by setting

$$F(x) = \begin{cases} \{(v_1, v_2) : v_1 = \cos\theta,\ v_2 = t\sin\theta \text{ and } \frac{1}{t} \le \theta \le \frac{1}{t} + 2\pi - |t|\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t \in (-1, 2) \setminus \{0\}, \\ \{(v_1, v_2) : -1 \le v_1 \le 1,\ \ v_2 = 0\} \qquad \text{for} \quad t = 0\,. \end{cases}$$

For $t \neq 0$ and $t \in (-1, 1)$, $F(t)$ is a subset of an ellipse in $\mathbb{R}^2$ (see Fig. 2.2), whose minor axis shrinks to zero as $t \to 0$, so that the ellipse collapses to a segment $F(0)$.

The subset of the ellipse given by $F(t)$ is obtained by removing a section, from the angle $(1/t) - |t|$ to the angle $(1/t)$. As t gets smaller, the arc length of this hole decreases, while the initial angle increases like $(1/t)$, i.e., it spins around the origin with increasing angular velocity. However, F is continuous at the origin, while no selection $f : (-1, 0) \to \mathbb{R}^2$ or $g : (0, 1) \to \mathbb{R}^2$, for example

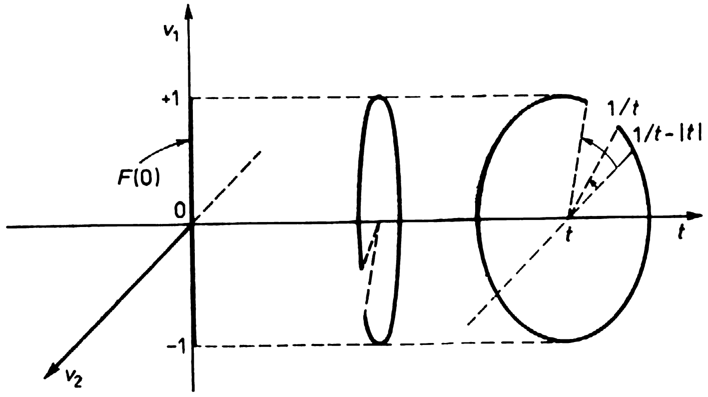

Fig. 2.2 The mapping F

$f(t) = (\cos(1/t), t\sin(1/t))$, can be continuously extended to the whole interval $(-1, 1)$. In fact, the hole in the ellipse would force this selection to rotate around the origin with an angle $\rho(t)$ between $(1/t)$ and $(1/t) + 2\pi - |t|$, and $\lim_{t\to 0} f(t)$ cannot exist.

We shall show that in some special cases, l.s.c. multifunctions possess continuous selections. This follows from the famous Michael continuous selection theorem. We precede it by the following lemmas.

Lemma 2.2. *Let (X, ρ) and $(Y, ||\cdot||)$ be a metric and a Banach space, respectively, and let $\Phi : X \to \mathcal{P}(Y)$ be a convex-valued and l.s.c. multifunction. For every $\varepsilon > 0$, there is a continuous function $\varphi : X \to Y$ such that* $\text{dist}(\varphi(x), \Phi(x)) \le \varepsilon$ *for $x \in X$.*

Proof. Let $x \in X$ be fixed and select $y_x \in \Phi(x)$ and $\delta_x > 0$ such that $(y_x + \varepsilon K_0) \cap \Phi(x') \ne \emptyset$ for every $x' \in B_x$, where $B_x = B(x, \delta_x)$ denotes the open ball of X centered at x with radius $\delta_x > 0$, and K_0 is the unit open ball of Y centered at $0 \in Y$. Since X is paracompact, there exists a locally finite refinement $\{U_z\}_{z\in\Lambda}$ of $\{B_z\}_{z\in X}$. Let $\{p_x\}_{x\in\Lambda}$ be a partition of unity subordinated to it and define a function $\varphi : X \to Y$ by setting $\varphi(x) = \sum_{z\in\Lambda} p_z(x) y_z$ for $x \in X$. It is clear that φ is a continuous function on X. Furthermore, we have $x \in U_z \subset B_z$ whenever $p_z(x) > 0$. Hence it follows that $y_z \in \Phi(x) + \varepsilon K_0$. Since this set is convex, every convex combination of such y_z, in particular $\varphi(x)$, belongs to it, too. Therefore, $\text{dist}(\varphi(x), \Phi(x)) \le \varepsilon$ for $x \in X$. □

Lemma 2.3. *Let (X, d) and (Y, ρ) be metric spaces, let $G : X \to \mathcal{P}(Y)$ be l.s.c., and let $g : X \to Y$ be continuous on X. If a real-valued function $X \ni x \to \varepsilon(x) \in \mathbb{R}^+$ is l.s.c. on X, then the set-valued mapping $\Phi : X \to \mathcal{P}(Y)$ defined by $\Phi(x) = B(g(x), \varepsilon(x)) \cap G(x)$ is l.s.c. at every $x \in X$ such that $\Phi(x) \ne \emptyset$.*

Proof. Let $\bar{x} \in X$ be such that $\Phi(\bar{x}) \neq \emptyset$. Select $\bar{y} \in \Phi(\bar{x})$ and let $\eta > 0$. Assume $\varepsilon(\bar{x}) > \rho(\bar{y}, g(\bar{x}))$ and let $\sigma > 0$ be such that $\rho(\bar{y}, g(\bar{x})) = \varepsilon(\bar{x}) - \sigma$. There exists $\sigma_1 > 0$ such that to every $x \in X$ with $d(x, \bar{x}) < \sigma_1$ we can associate $y_x \in G(x)$ such that $\rho(y_x, \bar{y}) < \min(\eta, (1/3)\sigma)$, $\sigma_2 > 0$ such that $d(x, \bar{x}) < \sigma_2$ implies $\varepsilon(x) > \varepsilon(\bar{x}) - (1/3)\sigma$, and $\sigma_3 > 0$ such that $d(x, \bar{x}) < \sigma_3$ implies $\rho(g(\bar{x}), g(x)) < (1/3)\sigma$. Thus

$$\begin{aligned} \rho(y_x, g(x)) &\leq \rho(y_x, \bar{y}) + \rho(\bar{y}, g(\bar{x})) + \rho(g(\bar{x}), g(x)) \\ &< (1/3)\sigma + \varepsilon(\bar{x}) - \sigma + (1/3)\sigma = \varepsilon(\bar{x}) - (1/3)\sigma < \varepsilon(x), \end{aligned}$$

whenever $d(x, \bar{x}) < \min\{\sigma_1, \sigma_2, \sigma_3\}$. Then $y_x \in \Phi(x)$ and $\rho(y_x, y) < \eta$. $\square$

Now we can prove Michael's continuous selection theorem.

Theorem 2.1 (Michael). *Let (X, ρ) and $(Y, |\cdot|)$ be a metric and a Banach space, respectively, and let $F : X \to \mathcal{P}(Y)$ be l.s.c. with closed convex values. Then there exists a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for $x \in X$.*

Proof. By virtue of Lemma 2.2, for $\varepsilon_1 = 1/2$ and $\Phi = F$, there exists a continuous function $f_1 : X \to Y$ such that $\operatorname{dist}(f_1(x), F(x)) \leq \varepsilon_1$ for $x \in X$. Let $\Phi_1(x) = (f_1(x) + \varepsilon_1 K_0) \cap F(x)$ for $x \in X$. We have $\Phi_1(x) \neq \emptyset$ for $x \in X$. By Lemma 2.3, the multifunction Φ_1 is l.s.c. Then by Lemma 2.2, for $\varepsilon_2 = (1/2)^2$, there exists a continuous function $f_2 : X \to Y$ such that $\operatorname{dist}(f_2(x), \Phi_1(x)) \leq \varepsilon_2$ for $x \in X$. Thus $\operatorname{dist}(f_2(x), F(x)) \leq \varepsilon_2$ and $\operatorname{dist}(f_2(x), (f_1(x) + \varepsilon_1 K_0)) \leq \varepsilon_2$, i.e., $f_2(x) - f_1(x) \in (\varepsilon_1 + \varepsilon_2)K_0$ for $x \in X$. Continuing the above procedure, we can deduce that for every $\varepsilon_n = (1/2)^n$ with $n = 0, 1, 2, \ldots$, there exists a continuous function $f_n : X \to Y$ such that $\operatorname{dist}(f_n(x), F(x)) \leq \varepsilon_n$ and $f_n(x) - f_{n-1}(x) \in (\varepsilon_{n-1} + \varepsilon_n)K_0$ for $x \in X$. Hence in particular, it follows that $\sup_{x \in X} \|f_n(x) - f_{n-1}(x)\| \leq \varepsilon_{n-1} + \varepsilon_n$ for $n \geq 1$, which implies that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space $C(X, Y)$ of all continuous bounded functions $g : X \to Y$ with the supremum norm. Thus there exists a continuous function $f : X \to Y$ such that $\sup_{x \in X} \|f_n(x) - f(x)\| \to 0$ as $n \to \infty$. Hence it follows that $f(x) \in F(x)$ for $x \in X$, because $F(x)$ is a closed subset of Y and $\operatorname{dist}(f_n(x), F(x)) \leq \varepsilon_n$ for $x \in X$ and $n = 1, 2, \ldots$. $\square$

Remark 2.5. There are closed convex-valued u.s.c. multifunctions that do not possess continuous selections. A simple example is the set-valued mapping F defined by the formula

$$F(x) = \begin{cases} \{-1\} & \text{for} \quad x < 0, \\ [-1, 1] & \text{for} \quad x = 0, \\ \{+1\} & \text{for} \quad x > 0, \end{cases}$$

with the graph presented in Fig. 2.3. $\square$

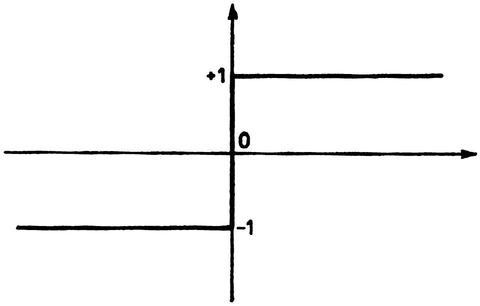

Fig. 2.3 The mapping F

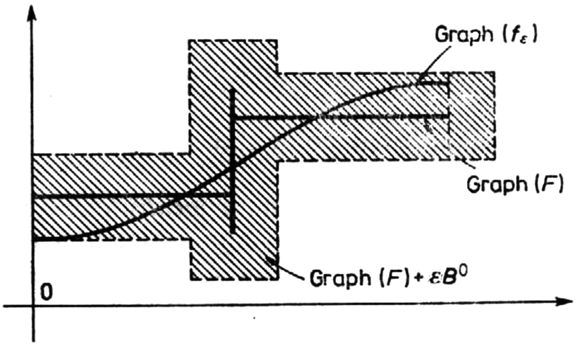

Fig. 2.4 Approximation continuous selection of u.s.c mutifunction

It can be proved that the above set-valued mapping possesses an approximation continuous selection of u.s.c mutifunction

Immediately from Michael's continuous selection theorem we obtain the existence of continuous approximation selections for some special multifunctions. The proof of such a theorem is based on the following lemma.

Lemma 2.4. *Let* (X, ρ), $(Y, |\cdot|)$ *and* $(Z, \|\cdot\|)$ *be Polish and Banach spaces, respectively. If* $\lambda : X \times Y \to Z$ *and* $v : X \to Z$ *are continuous and* $F : X \to \mathcal{P}(Y)$ *is l.s.c. such that* $v(x) \in \lambda(\{x\} \times F(x))$ *for* $x \in X$, *then for every l.s.c. function* $\varepsilon : X \to (0, \infty)$, *the set-valued mapping* $\Phi : X \to \mathcal{P}(Y)$ *defined by* $\Phi(x) = F(x) \cap \{u \in Y : \|\lambda(x,u) - v(x)\| < \varepsilon(x)\}$ *for* $x \in X$ *is l.s.c. on* X.

Proof. Let $\bar{x} \in X$. For every open set $\mathcal{U} \subset Y$ such that $\mathcal{U} \cap \Phi(\bar{x}) \neq \emptyset$, there are $\bar{u} \in \Phi(\bar{x})$ and $\eta > 0$ such that $(\bar{x}, \bar{u}) \in \text{Graph}(\Phi)$ and $(\bar{u} + \eta K_0) \subset \mathcal{U}$, where K_0 is the unit ball of Y. There is $\sigma > 0$ such that $\|\lambda(\bar{x}, \bar{u}) - v(\bar{x}))\| = \varepsilon(\bar{x}) - \sigma$. Let $\delta > 0$ be such that $\|\lambda(x,u) - \lambda(\bar{x}, \bar{u})\| < (1/3)\sigma$ for every $(x,u) \in X \times Y$ satisfying $\max\{\rho(x, \bar{x}), |u - \bar{u}|\} < \delta$. By the lower semicontinuity of F, there is $\sigma_1 > 0$ such that for every $x \in X$ satisfying $\rho(x, \bar{x}) < \sigma_1$, there exists $y_x \in F(x)$ such that $|y_x - \bar{u}| < \min\{\eta, (1/3)\sigma, \delta\}$. By the continuity of v, there is $\sigma_2 > 0$ such that $\|v(x) - v(\bar{x})\| < (1/3)\sigma$ for $x \in X$ satisfying $\rho(x, \bar{x}) < \sigma_2$. Furthermore, by the lower semicontinuity of ε, there is $\sigma_3 > 0$ such that $\varepsilon(x) > \varepsilon(\bar{x}) - (1/3)\sigma$

for every $x \in X$ satisfying $\rho(x, \bar{x}) < \sigma_3$. Then for every $x \in X$ satisfying $\rho(x, \bar{x}) < \min\{\delta, \sigma_1, \sigma_2, \sigma_3\}$, we get

$$\begin{aligned} \|\lambda(x, y_x) - v(x)\| &\leq \|\lambda(x, y_x) - \lambda(\bar{x}, \bar{u})\| \\ &\quad + \|\lambda(\bar{x}, \bar{u}) - v(\bar{x})\| + \|v(\bar{x}) - v(x)\| \\ &< (1/3)\sigma + \varepsilon(\bar{x}) - \sigma + (1/3)\sigma < \varepsilon(x) . \end{aligned}$$

Thus $y_x \in \Phi(x)$ and $\|y_x - \bar{u}\| < \eta$. For $\bar{u} \in \Phi(\bar{x})$ and $\eta > 0$ chosen above, we can choose $\bar{\varepsilon} = \min\{\delta, \sigma_1, \sigma_2, \sigma_3\}$ such that $(\bar{u} + \eta K_0) \cap \Phi(x) \neq \emptyset$ for every $x \in B(\bar{x}, \bar{\varepsilon})$. Therefore, for every open set $\mathcal{U} \subset Y$ such that $\mathcal{U} \cap \Phi(\bar{x}) \neq \emptyset$, there is $\bar{\varepsilon} > 0$ such that $(\bar{u} + \eta K_0) \cap \Phi(x) \neq \emptyset$ and $(\bar{u} + \eta K_0) \cap \Phi(x) \subset \mathcal{U} \cap \Phi(x)$ for every $x \in B(\bar{x}, \bar{\varepsilon})$. □

Theorem 2.2. *Let (X, ρ), $(Y, |\cdot|)$ and $(Z, \|\cdot\|)$ be Polish and Banach spaces, respectively. Assume that $\lambda : X \times Y \to Z$ and $v : X \to Z$ are continuous and $F : X \to \mathcal{P}(Y)$ is l.s.c. with closed convex values. If $\lambda(x, \cdot)$ is affine and $v(x) \in \lambda(x, F(x))$ for $x \in X$, then for every $\varepsilon > 0$, there exists a continuous function $f_\varepsilon : X \to Y$ such that $f_\varepsilon(x) \in F(x)$ and $\|\lambda(x, f_\varepsilon(x)) - v(x)\| \leq \varepsilon$ for $x \in X$.*

Proof. By virtue of Lemma 2.4, for every $\varepsilon > 0$, the set-valued mapping $\Phi_\varepsilon : X \to \mathcal{P}(Y)$ defined by $\Phi_\varepsilon(x) = F(x) \cap \{u \in Y : \|\lambda(x, u) - v(x)\| < \varepsilon\}$ for $x \in X$ is l.s.c. on X. Therefore, $\mathrm{cl}(\Phi_\varepsilon)$ is also l.s.c. on X. By the convexity of $F(x)$ and the property of $\lambda(x, \cdot)$ for fixed $x \in X$, it follows that $\Phi_\varepsilon(x)$ and $\mathrm{cl}(\Phi_\varepsilon)(x)$ are convex for $x \in X$. Therefore, by Michael's theorem, for every $\varepsilon > 0$, there exists a continuous selector f_ε for $\mathrm{cl}(\Phi_\varepsilon)$. It is clear that f_ε is a selector of F and satisfies $\|\lambda(x, f_\varepsilon(x)) - v(x)\| < \varepsilon$ for $x \in X$. □

Now we consider the problem of the existence of more regular selections of multifunctions. Such selections are connected with special properties of the "Steiner point map" $s : \mathrm{Conv}(\mathbb{R}^d) \to \mathbb{R}^d$ defined by

$$s(A) = \begin{cases} (d/2)\,[\sigma(1, A) + \sigma(-1, A)] & \text{for } d = 1, \\ d \int_{\Gamma_1} y\, \sigma(y, A) dr & \text{for } d \geq 1, \end{cases} \tag{2.1}$$

for $A \in \mathrm{Conv}(\mathbb{R}^d)$, where Γ_1 is the boundary of an open unit ball of $\mathbb{R}^d$ and $\mathrm{d}r$ denotes a differential of the surface measure r on Γ_1 proportional to the Lebesgue measure such that $r(\Gamma_1) = 1$. As usual, $\sigma(\cdot, A)$ denotes the support functions of $A \in \mathrm{Conv}(\mathbb{R}^d)$, and $\mathrm{Conv}(\mathbb{R}^d)$ is the family of all nonempty convex compact subsets of $\mathbb{R}^d$.

Immediately from the above definition, it follows that (i) $s(\{x\}) = x$ for every $x \in \mathbb{R}^d$. Furthermore, (ii) $s(A + B) = s(A) + s(B)$ and (iii) $s(\lambda A) = \lambda s(A)$ for $A, B \in \mathrm{Conv}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$. Indeed, for every $A, B \in \mathrm{Conv}(\mathbb{R}^d)$, one obtains

$$
\begin{aligned}
s(A+B) &= d\int_{\Gamma_1} \sigma(y, A+B)y\,\mathrm{d}r \\
&= d\int_{\Gamma_1} \sigma(y, A)\,\mathrm{d}r + d\int_{\Gamma_1} \sigma(y, B)\,\mathrm{d}r \\
&= s(A) + s(B).
\end{aligned}
$$

Quite similarly, we also get $s(\lambda A) = \lambda s(A)$ for $\lambda \in \mathbb{R}$ and $A \in \mathrm{Conv}(\mathbb{R}^d)$. Then conditions (ii) and (iii) are also satisfied.

We shall show that for every $A \in \mathrm{Conv}(\mathbb{R}^d)$, one has $s(A) \in A$. To prove this, let us recall some properties of the group $\mathcal{O}(\mathbb{R}^d)$ of all orthogonal linear transformations on $\mathbb{R}^d$. It can be verified that $s(l[A]) = l[s(A)]$ for every $l \in \mathcal{O}(\mathbb{R}^d)$ and $A \in \mathrm{Conv}(\mathbb{R}^d)$. It is also known that the surface measure $r(\cdot)$ on Γ_1 is invariant under the action of elements in $\mathcal{O}(\mathbb{R}^d)$.

Lemma 2.5. *For every* $A \in \mathrm{Conv}(\mathbb{R}^d)$, *one has* $s(A) \in A$.

Proof. Suppose there is $A \in \mathrm{Conv}(\mathbb{R}^d)$ such that $s(A) \notin A$. Define $C = A - s(A)$. Then $0 \notin C$, and by (i)–(iii), we get $s(C) = 0$. Let $0 \neq \hat{c}$ be such that $\langle c - \hat{c}, \hat{x}\rangle > 0$ for every $c \in C$, where $\hat{x} = \hat{c}\|\hat{c}\|^{-1}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^d$. But $\langle c, \hat{x}\rangle = \langle \hat{c} + (c - \hat{c}), \hat{x}\rangle = \langle \hat{c}, \hat{x}\rangle + \langle c - \hat{c}, \hat{x}\rangle$ and $\langle \hat{c}, \hat{x}\rangle = \|\hat{c}\|$. Then for every $c \in C$, one has $\|\hat{c}\| \leq \langle c, \hat{x}\rangle$.

Let $l : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation defined by $l(\hat{x}) = \hat{x}$ and $l(x) = -x$ for $x \in \mathbb{R}^d$ orthogonal to $\hat{x}$. It can be verified that l belongs to the group $\mathcal{O}(\mathbb{R}^d)$ of orthogonal linear transformations on $\mathbb{R}^d$ and $l^2 = I$, the identity map. So $l = l^*$. Let $D = C + l(C)$. Then $l(D) = D$, and so $s(D) = 0$. In addition, for every $d \in D$, we have $\langle d, \hat{x}\rangle \geq 2\|\hat{c}\| > 0$, and so $0 \notin D$. Now let

$$
\begin{aligned}
&\Gamma_1^0 = \{y \in \Gamma_1 : \langle y, \hat{x}\rangle = 0\}, \Gamma_1^+ = \{y \in \Gamma_1 : \langle y, \hat{x}\rangle > 0\} \text{ and} \\
&\Gamma_1^- = \{y \in \Gamma_1 : \langle y, \hat{x}\rangle < 0\}.
\end{aligned}
$$

Then $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+ \cup \Gamma_1^-$, and these three sets Γ_1^0, Γ_1^+, Γ_1^- are disjoint. Also, $r(\Gamma_1^0) = 0$. So we have

$$
\begin{aligned}
s(D) &= d\int_{\Gamma_1^+} \sigma(y, D)\,\mathrm{d}r + d\int_{\Gamma_1^-} \sigma(y, D)\,\mathrm{d}r \\
&= d\int_{\Gamma_1} [\sigma(y, D) - \sigma(-y, D)]\,\mathrm{d}r.
\end{aligned}
$$

Let $y \in \Gamma_1^+$ and $e \in D$ be such that $\sigma(-y, D) = \langle -y, e\rangle$. Then

$$
\begin{aligned}
\sigma(y, D) - \sigma(-y, D) &= \sigma(y, l(D)) - \sigma(-y, D) \\
&= \sigma(l(y), D) - \sigma(-y, D) \\
&\geq \langle l(y), e\rangle + \langle y, e\rangle = \langle (l + I)(y), e\rangle\,.
\end{aligned}
$$

But $(l + I)(y) = 2\langle y, \hat{x}\rangle \hat{x}$. Then $\sigma(y, D) - \sigma(-y, D) \geq 2\langle y, \hat{x}\rangle \cdot \langle \hat{x}, e\rangle > 0$, since $y \in \Gamma_1^+$ and $\langle \hat{x}, e\rangle > 0$. Therefore,

$$\langle s(D), \hat{x}\rangle = d \int_{\Gamma_1} [\sigma(y, D) - \sigma(-y, D)] \cdot \langle y, \hat{x}\rangle \, \mathrm{d}r > 0,$$

which contradicts $s(D) = 0$. Then $s(A) \in A$ for $A \in \mathrm{Conv}(\mathbb{R}^d)$. □

Corollary 2.2. *There is* $K > 0$ *such that for every* $A, B \in \mathrm{Conv}(\mathbb{R}^d)$, *one has* $|s(A) - s(B)| \leq K \cdot h(A, B)$.

Proof. Let us observe that for $A, B \in \mathrm{Conv}(\mathbb{R}^d)$, we have $h(A, B) = \max\{|\sigma(x, A) - \sigma(x, B)| : \|x\| = 1\}$. Then $|s(A) - s(B)| \leq d \int_{\Gamma_1} y\, |\sigma(y, A) - \sigma(y, B)| \, \mathrm{d}r \leq K \cdot h(A, B)$ for every $K \geq d$. □

Remark 2.6. In the above inequality we can compute the optimal Lipschitz constant $K(d) > 0$. It is equal to $d!!/(d-1)!!$ if d is odd, and $K(d) = d!!/[\pi(d-1)!!]$ if d is even. □

Theorem 2.3. *If* (X, ρ) *is a metric space and* $F : X \to \mathrm{Conv}(\mathbb{R}^d)$ *is Lipschitz continuous, then* F *admits a Lipschitz continuous selection.*

Proof. Let $h(F(x_1), F(x_2)) \leq L\rho(x_1, x_2)$ for some $L > 0$ and every $x_1, x_2 \in X$. Put $f(x) = s(F(x))$ for $x \in X$. By Corollary 2.2, we get $|f(x_1) - f(x_2)| = |s(F(x_1)) - s(F(x_2))| \leq K(d) \cdot h(F(x_1), F(x_2)) \leq K(d) \cdot L\, \rho(x_1, x_2)$, where $K(d)$ is as in Remark 2.6. By Lemma 2.5, for every $x \in X$, we have $f(x) \in F(x)$. □

Remark 2.7. Theorem 2.3 cannot be extended to multifunctions with values in an infinite-dimensional Banach space $(Y, ||\cdot||)$. It can be proved that if a Lipschitz continuous multifunction $F : X \to \mathrm{Conv}(Y)$ admits a Lipschitz continuous selection, then Y is finite-dimensional. □

Remark 2.8. It can be proved that if $F : X \to \mathcal{P}(\mathbb{R}^d)$ with $X \in \mathrm{Conv}(\mathbb{R}^m)$ is convex-valued such that $F^-(\{y\}) = \{x \in X : y \in F(x)\}$ is an open set in X for every $y \in \mathbb{R}^d$, then F admits an C^∞- selection. □

We shall now show that some measurable multifunctions admit measurable selections. We begin with the following lemma.

Lemma 2.6. *Let* (X, ρ) *be a separable metric space and* $(T, \mathcal{F})$ *a measurable space. Then a multifunction* $F : T \to \mathcal{P}(X)$ *is weakly measurable if and only if the function* $T \ni t \to \mathrm{dist}(x, F(t)) \in \mathbb{R}^+$ *is measurable for each* $x \in X$.

Proof. Let us observe that F is weakly measurable if and only if $F^-(B(x, \varepsilon)) \in \mathcal{F}$ for every $x \in X$ and $\varepsilon > 0$. On the other hand, a function $T \ni t \to \mathrm{dist}(x, F(t)) \in \mathbb{R}^+$ is measurable for fixed $x \in X$ if and only if $\{t \in T : \mathrm{dist}(x, F(t)) < \varepsilon\} \in \mathcal{F}$ for every $\varepsilon > 0$. But $F^-(B(x, \varepsilon)) = \{t \in T : F(t) \cap B(x, \varepsilon) \neq \emptyset\} = \{t \in T : \mathrm{dist}(x, F(t)) < \varepsilon\}$. This completes the proof. □

Theorem 2.4 (Kuratowski and Ryll-Nardzewski). *Let (X, ρ) be a Polish space and $(T, \mathcal{F})$ a measurable space. If $F : T \to \mathrm{Cl}(X)$ is measurable, then F admits a measurable selector.*

Proof. Let $\{x_1, x_2, \ldots\}$ be a countable dense subset in X and let $B_n(i) = \{x \in X : \rho(x, x_i) \le 1/n\}$ for $i, n \ge 1$. Without any loss of generality, we may assume that $\mathrm{diam}(X) < 1$, where $\mathrm{diam}(X) = \sup\{\rho(x, y) : x, y \in X\}$. We will construct a sequence $(f_n)_{n=1}^{\infty}$ of measurable functions $f_n : T \to X$ such that

$$\text{(i)} \quad \mathrm{dist}(f_n(t), F(t)) \le \varepsilon_n \quad \text{and} \quad \text{(ii)} \quad \rho(f_n(t), f_{n-1}(t)) \le \varepsilon_{n-1}$$

for $n \ge 0$ and $t \in T$, where $\varepsilon_n = (1/2)^n$ for $n = 0, 1, 2, \ldots$. Let $f_0(t) = x_1$ for $t \in T$. Then $\mathrm{dist}(f_0(t), F(t)) < 1$. Suppose $f_0, \ldots, f_{n-1}$ have been constructed and let $A_k^n = \{t \in T : \mathrm{dist}(f_k(t), F(t)) < \varepsilon_n\}$ and $C_k^n = \{t \in T : \rho(x_k, f_{n-1}(t)) < \varepsilon_{n-1}\}$. Put $D_k^n = A_k^n \cap C_k^n$. We claim that $T = \bigcup_{k \ge 1} D_k^n$ for $n \ge 1$. Fix $t \in T$. By the inductive hypothesis, we can find $z \in F(t)$ such that $\rho(f_{n-1}(t), z) < \varepsilon_{n-1}$. On the other hand, there is $k \ge 1$ such that $\rho(x_k, z) < \varepsilon_n$ and $\rho(x_k, z) + \rho(z, f_{n-1}(t)) < \varepsilon_n + \varepsilon_{n-1} < 2\varepsilon_{n-2} = \varepsilon_{n-1}$. Therefore, $t \in D_k^n$ and $T \subset \bigcup_{k \ge 1} D_k^n$. By virtue of Lemma 2.6 and the continuity of the function $\mathrm{dist}(\cdot, F(t))$ for fixed $t \in T$, we obtain that $A_k^n \in \mathcal{F}$. The inductive hypothesis gives that $C_k^n \in \mathcal{F}$. Then $D_k^n \in \mathcal{F}$. Now define $f_n : T \to X$ by setting $f_n(t) = x_k$ for $t \in D_k^n \setminus \bigcup_{i=1}^{k-1} D_i^n$. Clearly, f_n is measurable. Moreover, by (ii), we see that $(f_n(t))_{n=1}^{\infty}$ is a Cauchy sequence in X for every fixed $t \in T$. Then there exists a function $f : T \to X$ such that $f_n(t) \to f(t)$ for every $t \in T$ as $n \to \infty$. We also have $\mathrm{dist}(f(t), F(t)) = 0$ for every $t \in T$. Hence it follows that f is measurable such that $f(t) \in F(t)$ for every $t \in T$. $\square$

In what follows, we shall consider "complete" measurable spaces defined in the following way. For a given measurable space $(T, \mathcal{F})$ and every probability measure μ on $\mathcal{F}$, we denote by $\mathcal{F}_\mu$ the μ-completion of $\mathcal{F}$ and define $\tilde{\mathcal{F}} = \bigcap_\mu \mathcal{F}_\mu$. The space $(T, \mathcal{F})$ is said to be complete if $\mathcal{F} = \tilde{\mathcal{F}}$.

Remark 2.9. It can be proved that for a given complete measure space $(T, \mathcal{F}, \mu)$, a multifunction $F : T \to \mathcal{P}(\mathbb{R}^n)$ such that $\mathrm{Graph}(F) \in \mathcal{F} \otimes \beta(\mathbb{R}^m)$ is measurable and admits a measurable selection. $\square$

A consequence of the above measurable selection theorem is the following implicit function theorem.

Theorem 2.5. *Assume that (X, ρ) is a Polish space, $(T, \mathcal{F})$ a measurable space, and (Y, d) a metric space. Suppose $f : T \times X \to Y$ is a function measurable in $t \in T$ and continuous in $x \in X$, and let $\Gamma : T \to \mathrm{Comp}(X)$ be a measurable multifunction and $g : T \to Y$ a measurable function such that $g(t) \in f(t, \Gamma(t))$ for $t \in T$. Then there exists a measurable function $\gamma : T \to X$ such that $\gamma(t) \in \Gamma(t)$ and $g(t) = f(t, \gamma(t))$ for $t \in T$.*

Proof. Let us observe that the set-valued function $F : T \to \mathcal{P}(X)$ defined by $F(t) = \{x \in X : f(t, x) \in \mathcal{U}\}$ for $t \in T$ is measurable for every open set $\mathcal{U} \subset Y$.

Indeed, let B be a closed subset of X and let A be a countable dense subset of B. We have

$$\begin{aligned} F^-(B) &= \{t \in T : F(t) \cap B \neq \emptyset\} \\ &= \{t \in T : f(t,x) \in \mathcal{U} \ \text{ for some } \ x \in B\} \\ &= \{t \in T : f(t,a) \in \mathcal{U} \ \text{ for some } \ a \in A\} \\ &= \bigcup_{a \in A} \{t \in T : f(t,a) \in \mathcal{U}\}. \end{aligned}$$

Therefore, $F^-(B) \in \mathcal{F}$, because we have $\{t \in T : f(t,a) \in \mathcal{U}\} \in \mathcal{F}$ for every fixed $a \in A$. Define multifunctions $H(t) = \Gamma(t) \cap \{x \in X : d(f(t,x), g(t)) = 0\}$ for $t \in T$ and $F_n(t) = \{x \in X : d(f(t,x), g(t)) < 1/n\}$ for $t \in T$ and $n \geq 1$. For every $n = 1, 2, \ldots$, a multifunction F_n is measurable and also weakly measurable. Hence it follows that its closure $\bar{F_n}$ is weakly measurable, because $F_n^-(B) = \bar{F_n}^-(B)$ for every open set $B \subset X$. Clearly, $\{x \in X : d(f(t,x), g(t)) = 0\} = \bigcap_{n=1}^{\infty} \bar{F_n}(t)$ for $t \in T$, because $\bar{F_n}(t) \subset \{x \in X : d(f(t,x), g(t)) \leq 1/n\}$ for $t \in T$ and $n \geq 1$. Hence it follows that the multifunction H defined above can be also defined by $H(t) = \Gamma(t) \cap \left[\bigcap_{n=1}^{\infty} \bar{F_n}(t)\right]$ for $t \in T$, which implies that H is measurable. Therefore, by Theorem 2.4, there is a measurable selector γ for H that in particular is a selector for Γ satisfying $d(f(t,\gamma(t)), g(t)) = 0$ for $t \in T$. □

Corollary 2.3. *If (X,ρ) is a Polish space, $(T,\mathcal{F})$ a measurable space, and $\Gamma : T \to \mathrm{Comp}(X)$ and $g : T \to X$ are measurable, then there exists a measurable selector γ for Γ such that $\mathrm{dist}(g(t), \Gamma(t)) = \rho(g(t), \gamma(t))$ for $t \in T$.* □

The following important result follows immediately from the Kuratowski and Ryll-Nardzewski measurable selection theorem.

Theorem 2.6. *Let (X,ρ) be a Polish space, $(T,\mathcal{F})$ a measurable space, and let $F : T \to \mathrm{Cl}(X)$. The following conditions are equivalent:*

(i) F is measurable;
(ii) there exists a sequence $(f_n)_{n=1}^{\infty}$ of measurable selectors of F such that $F(t) = \mathrm{cl}\{f_1(t), f_2(t), \ldots\}$ for every $t \in T$.

Proof. Let F be measurable and $(x_n)_{n=1}^{\infty}$ a dense sequence of X. For every $n, k \geq 1$, we define

$$F_{n,k}(t) = \begin{cases} F(t) \cap B(x_n, \varepsilon_k) & \text{if } t \in F^-(B(x_n,\varepsilon_k)), \\ F(t) & \text{otherwise,} \end{cases}$$

where $\varepsilon_k = (1/2)^k$ and $F^-(B(x_n,\varepsilon_k)) = \{t \in T : F(t) \cap B(x_n,\varepsilon_k) \neq \emptyset\}$. Note that $F^-(B(x_n,\varepsilon_k)) \in \mathcal{F}$ and that the set-valued function $T \ni t \to F(t) \cap B(x_n,\varepsilon_k) \subset X$ is measurable. So $F_{n,k}$ is measurable, which implies that $\mathrm{cl}\,[F_{n,k}]$

is also measurable. Therefore, by Theorem 2.4, there exist measurable functions $f_{n,k} : T \to X$ such that $f_{n,k}(t) \in \mathrm{cl}\,[F_{n,k}]\,(t)$ for every $t \in T$. We shall show that $F(t) = \mathrm{cl}\{f_{n,k}(t) : n,k \geq 1\}$ for $t \in T$. Indeed, fix $t \in T$ and let $x \in F(t)$ and $\varepsilon > 0$. Let $k \geq 1$ and $n \geq 1$ be such that $\varepsilon_{k-1} \leq \varepsilon$ and $x \in B(x_n, \varepsilon_k)$. Then $t \in F^-(B(x_n, \varepsilon_k))$ and $f_{n,k}(t) \in \overline{B(x_n, \varepsilon_k)}$. So $\rho(f_{n,k}(t), x) \leq \rho(f_{n,k}(t), x_n) + \rho(x_n, x) \leq \varepsilon$, which proves that $F(t) = \mathrm{cl}\{f_{n,k}(t) : n,k \geq 1\}$. Then (i) $\Rightarrow$ (ii). Assume that (ii) is satisfied. Then for every open set $\mathcal{U} \subset X$, we have

$$F^-(\mathcal{U}) = \{t \in T : F(t) \cap \mathcal{U} \neq \emptyset\} = \bigcup_{n \geq 1} \{t \in T : f_n(t) \in \mathcal{U}\} \in \mathcal{F}.$$

Then F is weakly measurable and therefore measurable. Thus (ii)$\Rightarrow$(i). □

Remark 2.10. It can be proved that if $(T, \mathcal{F})$ is a complete measurable space, $(G, \mathcal{G})$ is a measurable space, X is a Suslin space, $g : T \times G \to X$ is jointly measurable, $\Gamma : T \to \mathcal{P}(G)$ is a multifunction such that $\mathrm{Graph}(\Gamma) \in \mathcal{F} \otimes \mathcal{G}$, and $h : T \to X$ is a measurable map such that $h(t) \in g(t, \Gamma(t))$ for $t \in T$, then there exists a measurable selector $\gamma : T \to G$ of Γ such that $h(t) = g(t, \gamma(t))$ for $t \in T$. □

We shall consider now the existence of Carathéodory-type selections of measurable multifunctions depending on two variables. More precisely, let $(T, \mathcal{F})$ be a measurable space, (X, ρ) a Polish space, and $(Y, \|\cdot\|)$ a separable Banach space. Consider the set-valued mapping $F : T \times X \to \mathrm{Cl}(Y)$, which is assumed to be measurable, i.e., for every closed set $A \subset Y$, we have $F^-(A) = \{(t,x) \in T \times X : F(t,x) \cap A \neq \emptyset\} \in \mathcal{F} \otimes \beta(X)$. We are interested in the existence of a function $f : T \times X \to Y$, a selector of F, such that $f(\cdot, x)$ is measurable for fixed $x \in X$, and $f(t, \cdot)$ is continuous for fixed $t \in T$. Such selectors of F are said to be of Carathéodory type or simply to be Carathéodory selectors for F.

Theorem 2.7. *Let $(T, \mathcal{F})$ be a complete measurable space, (X, ρ) a Polish space, $(Y, \|\cdot\|)$ a separable Banach space, and $F : T \times X \to \mathrm{Cl}(Y)$ a convex-valued measurable set-valued mapping. If furthermore, $F(t, \cdot)$ is l.s.c. for fixed $t \in T$, then F admits a Carathéodory selection.*

Proof. Let $(y_n)_{n=1}^{\infty}$ be a dense sequence of Y. For $t \in T$, $n \geq 1$, and $\varepsilon > 0$, define $G_n^\varepsilon(t) = \{x \in X : y_n \in (F(t,x) + \varepsilon B)\}$, where B is an open unit ball in Y. By the lower semicontinuity of $F(t, \cdot)$, a set $G_n^\varepsilon(t)$ is open for every $t \in T$, $\varepsilon > 0$, and $n \geq 1$. Also, the family $\{G_n^\varepsilon(t) : n \geq 1\}$ is an open covering of X. Moreover,

$$\mathrm{Graph}(G_n^\varepsilon) = \{(t,x) \in T \times X : \mathrm{dist}(y_n, F(t,x)) < \varepsilon\} \in \mathcal{F} \otimes \beta(X),$$

because of the measurability of F. Let $\varepsilon_m = (1/2)^m$ and

$$G_{n,m}^\varepsilon(t) = \{x \in G_n^\varepsilon(t) : \mathrm{dist}(x, X \setminus G_n^\varepsilon) \geq \varepsilon_m\} \text{ and } \mathcal{U}_n^\varepsilon(t) = G_n^\varepsilon(t) \setminus \bigcup_{1 \leq k < n} G_{n,k}^\varepsilon(t)$$

for $n, m \geq 1$. It can be verified that the family $\{\mathcal{U}_n^\varepsilon(t) : n \geq 1\}$ is a locally finite covering of X and every multifunction $\mathcal{U}_n^\varepsilon : T \to \mathcal{P}(X)$ has a measurable graph. Hence it follows that the set-valued mapping $T \ni t \to X \setminus \mathcal{U}_n^\varepsilon(t) \subset X$ is measurable with closed values. Let

$$p_n^\varepsilon(t,x) = \frac{\operatorname{dist}(x, X \setminus \mathcal{U}_n^\varepsilon(t))}{\sum_{n\geq 1} \operatorname{dist}(x, X \setminus \mathcal{U}_n^\varepsilon(t))}.$$

By virtue of Lemma 2.6, the function $p_n^\varepsilon(\cdot, x)$ is measurable for every $n \geq 1$ and fixed $x \in X$. By the above definition, $p_n^\varepsilon(t, \cdot)$ is continuous for fixed $t \in T$. Then p_n^ε is a Carathéodory function for every $\varepsilon > 0$ and $n \geq 1$. Furthermore, $\sum_{n\geq 1} p_n^\varepsilon(t,x) = 1$. Let $f^\varepsilon(t,x) = \sum_{n\geq 1} p_n^\varepsilon(t,x) \cdot y_n$. It is clear that f^ε is a Carathéodory function. By the convexity of $F(t,x)$, for every $(t,x) \in T \times X$ we get $f^\varepsilon(t,x) \in F(t,x) + \varepsilon B$ for $(t,x) \in T \times X$ and every $\varepsilon > 0$.

Let $\varepsilon_n = (1/2)^n$ for $n = 1, 2, \ldots$. We define now a sequence $(f_n)_{n=1}^\infty$ of Carathéodory functions $f_n : T \times X \to Y$ such that $f_n(t,x) \in F(t,x) + \varepsilon_n B$ and $\|f_n(t,x) - f_{n-1}\| < \varepsilon_{n-1}$ for $(t,x) \in T \times X$ and $n \geq 2$. We start with $f_1 = f^{\varepsilon_1}$ and then we put $F_2(t,x) = F(t,x) \cap \{f_1(t,x) + \varepsilon_1 B\}$ for $(t,x) \in T \times X$. By virtue of Lemma 2.3, a multifunction $F_2(t,\cdot)$ is l.s.c. for fixed $t \in T$. It is easy to see that F_2 is measurable. Consequently, its closure $\mathrm{cl}\,[F_2]$ is measurable and $\mathrm{cl}\,[F_2]\,(t,\cdot)$ is l.s.c. for fixed $t \in T$. From this and the first part of the proof, it follows that for $\varepsilon = \varepsilon_2$, there exists a Carathéodory function f_2 such that $f_2(t,x) \in \mathrm{cl}\,[F_2]\,(t,x) + \varepsilon_2 B$ for $(t,x) \in T \times X$. It is clear that $f_2(t,x) \in F(t,x) + \varepsilon_2 B$ and $\|f_2(t,x) - f_1(t,x)\| < \varepsilon_1$ for $(t,x) \in T \times X$. By the inductive procedure, we can define a sequence $(f_n)_{n=1}^\infty$ of Carathéodory functions $f_n : T \times X \to Y$ such that $f_n(t,x) \in F(t,x) + \varepsilon_n B$ and $\|f_n(t,x) - f_{n-1}(t,x)\| < \varepsilon_{n-1}$ for $(t,x) \in T \times X$. Hence it follows that there exists a Carathéodory function $f : T \times X \to Y$ such that $f_n(t,x) \to f(t,x)$ as $n \to \infty$ for $(t,x) \in T \times X$. By the closedness of $F(t,x)$, this implies that $f(t,x) \in F(t,x)$ for $(t,x) \in T \times X$. □

Remark 2.11. It can be proved that if T is a locally compact metric space furnished with a Radon measure μ, X is a Polish space, Y is a separable reflexive Banach space, and $F : T \times X \to \mathrm{Cl}(Y)$ is as in Theorem 2.7, then there exists a sequence $(f_m)_{m=1}^\infty$ of Carathéodory selectors $f_m : T \times X \to Y$ of F such that $F(t,x) = \mathrm{cl}\{f_m(t,x) : m \geq 1\}$ for every $(t,x) \in T \times X$. □

There are quite a number of set-valued fixed-point theorems. We present below one of them that generalizes the classical Banach fixed-point theorem.

Theorem 2.8 (Covitz–Nadler). *Let* (X, ρ) *be a complete metric space and let* $F : X \to \mathrm{Cl}(X)$ *be such that* $h(F(x), F(\bar{x})) \leq K\rho(x, \bar{x})$ *for every* $x, \bar{x} \in X$ *with* $K \in (0,1)$. *Then there exists* $x \in X$ *such that* $x \in F(x)$.

Proof. Let $L \in (K, 1)$ and $\lambda = K^{-1}L$. For some $x \in X$, we have $B(x, \lambda \cdot \operatorname{dist}(x, F(x))) \cap F(x) \neq \emptyset$, because $\lambda > 1$. Then we can select $x_1 \in F(x)$ such that $\rho(x, x_1) \leq \lambda \cdot \operatorname{dist}(x, F(x))$. For such $x_1 \in X$, we can select $x_2 \in F(x_1)$ such that $\rho(x_1, x_2) \leq \lambda \cdot \operatorname{dist}(x_1, F(x_1))$. Continuing this procedure, we can find a sequence $(x_n)_{n=1}^{\infty}$ of X such that $\rho(x_n, x_{n+1}) \le \lambda \cdot \mathrm{dist}(x_n, F(x_n))$ for $n \ge 1$. Hence it follows that $\rho(x_n, x_{n+1}) \le \lambda \cdot \mathrm{dist}(x_n, F(x_n)) \le \lambda \cdot h(F(x_{n-1}, F(x_n)) \le L\rho(x_{n-1}, x_n) \le L^n \mathrm{dist}(x, F(x))$. Now, similarly as in the proof of the Banach fixed-point theorem, we can verify that the above defined sequence $(x_n)_{n=1}^{\infty}$ has a limit, say x,, belonging to X. Since F is H-continuous and $\mathrm{dist}(x, F(x)) \le \rho(x, x_n) + \mathrm{dist}(x_n, F(x_{n+1})) + h(F((x_{n+1}), F(x))$ for $n \ge 1$, it follows that $x \in F(x)$. □

3 The Aumann Integral

Let $(T, \mathcal{F}, \mu)$ be a σ-finite measure space that is not necessarily complete. For $p \ge 1$, by $\mathbb{L}^p(T, \mathbb{R}^d)$ we denote the Banach space $\mathbb{L}^p(T, \mathcal{F}, \mu, \mathbb{R}^d)$ with the norm $\|\cdot\|$ defined in the usual way, i.e., by $\|f\|^p = \int_T |f(t)|^p d\mu$ for $f \in \mathbb{L}^p(T, \mathbb{R}^d)$. In what follows, we shall consider $\mathbb{L}^p(T, \mathbb{R}^d)$ with $p = 1$ and $p = 2$. Instead of $\mathbb{L}^1(T, \mathbb{R}^d)$, we shall write $\mathbb{L}(T, \mathbb{R}^d)$. Let us recall that if $\mu(T) < \infty$, then a set $K \subset \mathbb{L}^p(T, \mathbb{R}^d)$ is relatively sequentially weakly compact if K is bounded and uniformly integrable, i.e., if $\lim_{\mu(E)\to 0} \int_E f(t) d\mu = 0$ uniformly for $f \in K$. By the reflexivity of $\mathbb{L}^2(T, \mathbb{R}^d)$, a set $K \subset \mathbb{L}^2(T, \mathbb{R}^d)$ is relatively sequentially weakly compact if and only if it is bounded. By the Eberlein–Šmulian theorem, it follows that for a bounded set $K \subset \mathbb{L}^2(T, \mathbb{R}^d)$, its closure $\mathrm{cl}_w K$ with respect to the weak topology of $\mathbb{L}^2(T, \mathbb{R}^d)$ is weakly compact. In particular, if K is also closed and convex, then it is weakly compact, because in such a case, we have $K = \mathrm{cl}_w K$.

Given a measurable set-valued mapping $F : T \to \mathrm{Cl}(\mathbb{R}^d)$, we define subtrajectory integrals $S(F)$ of F as the subset of the space $\mathbb{L}^p(T, \mathbb{R}^d)$ defined by $S(F) = \{f \in \mathbb{L}^p(T, \mathbb{R}^d) : f(t) \in F(t) \; a.e.\}$. It can be verified that $S(F)$ is a closed subset of $\mathbb{L}^p(T, \mathbb{R}^d)$. In what follows we shall consider only the cases $p = 1$ and $p = 2$. Immediately from properties of multifunction F it will be easily seen if $S(F)$ is a subset of $\mathbb{L}(T, \mathbb{R}^d)$ or $\mathbb{L}^2(T, \mathbb{R}^d)$, respectively. In what follows, we shall denote by $\mathcal{M}(T, \mathbb{R}^d)$ the space of all measurable set-valued mappings $F : T \to \mathrm{Cl}(\mathbb{R}^d)$ and by $\mathcal{A}(T, \mathbb{R}^d)$ the subspace of $\mathcal{M}(T, \mathbb{R}^d)$ containing all $F \in \mathcal{M}(T, \mathbb{R}^d)$ such that $S(F) \ne \emptyset$. It can be proved that every $F \in \mathcal{M}(T, \mathbb{R}^d)$ belongs to $\mathcal{A}(T, \mathbb{R}^d)$ if and only if there exists $k \in \mathbb{L}^p(T, \mathbb{R}^+)$ such that $\mathrm{dist}(0, F(t)) \le k(t)$ for a.e. $t \in T$. We have the following simple results.

Lemma 3.1. *If $F \in \mathcal{A}(T, \mathbb{R}^d)$, then there exists a sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n \in S(F)$ such that $F(t) = \mathrm{cl}\{f_1(t), f_2(t), \ldots\}$ for $t \in T$.*

Proof. By virtue of Theorem 2.6, there exists a sequence $(g_n)_{n=1}^{\infty}$ of measurable functions $g_n : T \to \mathbb{R}^d$ such that $F(t) = \mathrm{cl}\{g_1(t), g_2(t), \ldots\}$ for $t \in T$. Taking a countable measurable partition $\{A_1, A_2, \ldots\}$ of T with $\mu(A_k) < \infty$ and a function $f \in \mathbb{L}^p(T, \mathbb{R}^d)$ such that $f(t) \in F(t)$ for $t \in T$, we define $B_{j,m,k} = \{t \in T : m - 1 \le |g_j(t)| < m\} \cap A_k$ and $f_{j,m,k} = \mathbb{1}_{B_{j,m,k}} g_j + \mathbb{1}_{T \setminus B_{j,m,k}} f$ for $j, m, k \ge 1$. It is easy to see that $f_{j,m,k} \in S(F)$ and $F(t) = \overline{\{f_{j,m,k}(t) : j, m, k \ge 1\}}$ for $t \in T$. □

Corollary 3.1. *If* $F, G \in \mathcal{A}(T, \mathbb{R}^d)$*, then* $S(F) = S(G)$ *if and only if* $F(t) = G(t)$ *for a.e.* $t \in T$. □

Lemma 3.2. *Let* $F \in \mathcal{A}(T, \mathbb{R}^d)$ *and let* $(f_n)_{n=1}^{\infty}$ *be a sequence of* $S(F)$ *such that* $F(t) = \mathrm{cl}\{f_1(t), f_2(t), \ldots\}$ *for* $t \in T$*. Then for every* $f \in S(F)$ *and* $\varepsilon > 0$*, there exists a finite measurable partition* $\{A_1, \ldots, A_m\}$ *of* T *such that* $\|f - \sum_{i=1}^{m} \mathbb{1}_{A_i} f_i\| < \varepsilon$.

Proof. Assume $f(t) \in F(t)$ for every $t \in T$ and let $\rho \in \mathbb{L}^p(T, \mathbb{R})$ be strictly positive such that $\int_T \rho \mathrm{d}\mu < \varepsilon/3$. Then there exists a countable measurable partition $\{B_1, B_2, \ldots\}$ of T such that $|f(t) - f_i(t)| < \rho(t)$ for $t \in B_i$ and $i \geq 1$. Take an integer m such that $\sum_{i=m+1}^{\infty} \int_{B_i} |f(t)| \mathrm{d}\mu < \varepsilon/6$ and $\sum_{i=m+1}^{\infty} \int_{B_i} |f_i(t)| \mathrm{d}\mu < \varepsilon/6$ and define a finite measurable partition $\{A_1, \ldots, A_m\}$ as follows: $A_1 = B_1 \cup (\bigcup_{i=m+1}^{\infty} B_i)$ and $A_j = B_j$ for $2 \leq j \leq m$. Then we have

$$\left\| f - \sum_{i=1}^{m} \mathbb{1}_{A_i} f_i \right\| = \sum_{i=1}^{m} \int_{B_i} |f(t) - f_i(t)| \mathrm{d}\mu + \sum_{i=m+1}^{\infty} \int_{B_i} |f(t) - f_i(t)| \mathrm{d}\mu$$

$$\leq \int_T \rho \mathrm{d}\mu + \sum_{i=m+1}^{\infty} \int_{B_i} (|f(t)| + |f_i(t)|) \mathrm{d}\mu < \varepsilon. \qquad \square$$

Lemma 3.3. *Let* $(T, \mathcal{F}, \mu)$ *be a measure space with a* σ*-finite measure* μ*. If* $F \in \mathcal{A}(T, \mathbb{R}^d)$*, then* $\overline{\mathrm{co}}\, S(F) = S(\overline{\mathrm{co}}\, F)$.

Proof. We have $\overline{\mathrm{co}}\, S(F) \subset S(\overline{\mathrm{co}}\, F)$. Assume that there exists $f \in S(\overline{\mathrm{co}}\, F)$ such that $f \notin \overline{\mathrm{co}}\, S(F)$. By the strong separation theorem, we can find $h \in \mathbb{L}^{\infty}(T, \mathbb{R}^d)$ such that $\sup\{(h, g) : g \in S(F)\} < (h, f)$, where $(\cdot, \cdot)$ denotes the duality bracket. Hence it follows that $\int_T \sigma(h(t), \overline{\mathrm{co}}\, F(t)) \mathrm{d}\mu < \int_T \langle h(t), f(t) \rangle \, \mathrm{d}\mu$. On the other hand, $f(t) \in \overline{\mathrm{co}}\, F(t)$ a.e. Then $\int_T \langle h(t), f(t) \rangle \, \mathrm{d}\mu \leq \int_T \sigma(h(t), \overline{\mathrm{co}}\, F(t)) \mathrm{d}\mu$, a contradiction. Therefore, $\overline{\mathrm{co}}\, S(F) = S(\overline{\mathrm{co}} F)$. □

A multifunction $F : T \to \mathcal{P}(\mathbb{R}^n)$ is said to be p-integrably bounded if there is $k \in \mathbb{L}^p(T, \mathbb{R}^+)$ such that $\|F(t)\| =: h(\{0\}, F(t)) \leq k(t)$ for a.e. $t \in T$. In particular, for $p = 1$, we say simply integrably bounded instead of 1-integrably bounded. Similarly, if $p = 2$, then instead of 2-integrably bounded, we say square integrably bounded. It is clear that F is p-integrably bounded if and only if the function $T \ni t \to \|F(t)\| \in \mathbb{R}^+$ belongs to $\mathbb{L}^p(T, \mathbb{R}^+)$. For every p-integrably bounded multifunction $F \in \mathcal{M}(T, \mathbb{R}^n)$, we have $S(F) \neq \emptyset$.

Remark 3.1. Immediately from the definition of subtrajectory integrals, it follows that for every measurable and p-integrably bounded multifunction $F : T \to \mathrm{Conv}(\mathbb{R}^d)$, its subtrajectory integral $S(F)$ is a nonempty convex weakly sequentially compact subset of $\mathbb{L}^p(T, \mathbb{R}^d)$. In particular, it is a weakly compact convex subset of this space for $p > 1$. □

Lemma 3.4. *If* $F, G \in \mathcal{A}(T, \mathbb{R}^d$ *then* $S(\overline{F + G}) = \overline{S(F) + S(G)}$.

Proof. Immediately from Theorem 2.6, it follows that $H = \overline{F+G}$ is measurable. It is clear that $S(H)$ is closed, and therefore, $\overline{S(F)+S(G)} \subset S(H)$. On the other hand, we may find sequences $(f_n)_{n=1}^{\infty} \subset S(F)$ and $(g_m)_{m=1}^{\infty} \subset S(G)$ such that $F(t) = \text{cl}\{f_n(t) : n \geq 1\}$ and $G(t) = \text{cl}\{g_n(t) : n \geq 1\}$ a.e. Evidently, $H(t) = \overline{\{f_n(t) + g_m(t) : n, m \geq 1\}}$, which, by Lemma 3.2, implies that for given $h \in S(H)$ and $\varepsilon > 0$, we can select a finite $\mathcal{F}$-measurable partition $(A_k)_{k=1}^{N}$ of T and positive integers $n_1, \ldots, n_N$ and $m_1, \ldots, m_N$ such that $\|h - \sum_{k=1}^{N} \mathbb{1}_{A_k}(f_{n_k} + g_{m_k})\| < \varepsilon$. Hence it follows that $h \in \overline{S(F)+S(G)}$. Then $S(H) \subset \overline{S(F)+S(G)}$. □

Let $(T, \mathcal{F}, \mu)$ be a measure space, $\tilde{\mathbb{R}} = [-\infty, +\infty]$ and let $\phi : T \times X \to \tilde{\mathbb{R}}$ be an $\mathcal{F} \otimes \beta(\mathbb{R}^d)$-measurable function. The functional $\mathcal{T}_\phi$ defined on the space $\mathbb{L}^0(T, \mathbb{R}^d)$ of measurable functions $f : T \to \mathbb{R}^d$ by setting $\mathcal{T}_\phi(f) = \int_T \phi(t, f(t))\mathrm{d}\mu$ if the integral exists, permitting $+\infty$ or $-\infty$, is called the integral functional.

Lemma 3.5. *Let $F \in \mathcal{M}(T, \mathbb{R}^d)$ and let $\phi : T \times \mathbb{R}^d \to \tilde{\mathbb{R}}$ be $\mathcal{F} \otimes \beta(\mathbb{R}^d)$-measurable. Assume either that (i) $\phi(t,x)$ is u.s.c. in x for every fixed $t \in T$ or that (ii) $(T, \mathcal{F}, \mu)$ is complete and $\phi(t,x)$ is l.s.c. in x for every fixed $t \in T$. Then the function $T \ni t \to \inf\{\phi(t,x) : x \in F(t)\} \subset \tilde{\mathbb{R}}$ is measurable.*

Proof. Let $\xi(t) = \inf\{\phi(t,x) : x \in F(t)\}$ and assume that (i) is satisfied. By Theorem 2.6, there exists a sequence $(f_n)_{n=1}^{\infty}$ of measurable selectors of F such that $F(t) = \text{cl}(\{f_1(t), f_2(t), \ldots\})$ for $t \in T$. Then we have $\xi(t) = \inf_{n \geq 1} \phi(t, f_n(t))$ for $t \in T$, which implies that ξ is measurable. Let (ii) be satisfied and let $H : T \to \mathcal{P}(\mathbb{R}^d \times \mathbb{R})$ be defined by $H(t) = \{(x, \alpha) \in \mathbb{R}^d \times \mathbb{R} : x \in F(t),\ \phi(t,x) \leq \alpha\}$ for $t \in T$. Then $H(t)$ is closed in $\mathbb{R}^d \times \mathbb{R}$ for every $t \in T$, and $\text{Graph}(H) = [\text{Graph}(F) \cap \mathbb{R}] \cap \{(t, x, \alpha) : \Phi(t,x) - \alpha \leq 0\}$ belongs to $\mathcal{F} \otimes \beta(\mathbb{R}^d) \otimes \beta(\mathbb{R}) = \mathcal{F} \otimes \beta(\mathbb{R}^d \otimes \mathbb{R})$. Therefore, by virtue of Remark 2.9 and Theorem 2.6, there exists a sequence $(g_n, \xi_n)_{n=1}^{\infty}$ of measurable functions $g_n : T \to \mathbb{R}^d$ and $\xi_n : T \to \mathbb{R}$ such that $H(t) = \text{cl}(\{(g_1, \xi_1)(t), (g_2, \xi_2)(t), \ldots\})$ for $t \in \text{Dom}(H)$. Hence we have $\xi(t) = \inf_{n \geq 1} \xi_n(t)$ for $t \in \text{Dom}(H)$ and $\xi(t) = \infty$ for $t \in T \setminus \text{Dom}(H)$. This shows that ξ is measurable. □

Theorem 3.1. *Let $F \in \mathcal{A}(T, \mathbb{R}^d)$ and let $\phi : T \times X \to \tilde{\mathbb{R}}$ be $\mathcal{F} \otimes \beta(\mathbb{R}^d)$-measurable. Assume either that (i) $\phi(t,x)$ is u.s.c. in x for every fixed $t \in T$, or that (ii) $(T, \mathcal{F}, \mu)$ is complete and $\phi(t,x)$ is l.s.c. in x for every fixed $t \in T$. If the integral functional $\mathcal{T}_\phi$ is defined for all $f \in S(F)$ and $\mathcal{T}_\phi(f_0) < \infty$ for some $f_0 \in S(F)$, then $\inf\{\mathcal{T}_\phi(f) : f \in S(F)\} = \int_T \inf\{\phi(t,x) : x \in F(t)\}\mathrm{d}\mu$.*

Proof. Let $\xi(t) = \inf\{\phi(t,x) : x \in F(t)\}$. By virtue of Lemma 3.4, ξ is measurable and $\xi(t) \leq \phi(t, f(t))$ a.e. for every $f \in S(F)$. Taking $f = f_0$, we can see that the integral of ξ exists and $\int_T \xi \mathrm{d}\mu \leq \inf\{\mathcal{T}_\phi(f) : f \in S(F)\}$. If $\mathcal{T}_\phi(f_0) = -\infty$, then the proof is complete. Thus assume $\mathcal{T}_\phi(f_0)$ to be finite, so that the function $T \ni t \to \phi(t, f_0(t)) \in \tilde{\mathbb{R}}$ is in $\mathbb{L}(T, \mathbb{R})$. Let $\beta > \int_T \xi \mathrm{d}\mu$ be given. We shall show that $\mathcal{T}_\phi(f) < \beta$ for some $f \in S(F)$. Take a sequence

$(A_n)_{n=1}^{\infty}$ of measurable sets $A_n \in \mathcal{F}$ such that $\mu(A_n) < \infty$ and such that $A_n \uparrow T$ and a strictly positive function $\rho \in \mathbb{L}(T, \mathbb{R})$. For $n \geq 1$, define $B_n = A_n \cap \{t \in T : \phi(t, f_0(t)) \geq -n\}$ and

$$\xi_n(t) = \begin{cases} \xi(t) + \rho(t)/n \quad textif \;\; t \in B_n \;\; \text{and} \;\; \xi(t) \geq -n, \\ -n + \rho(t)/n \;\; \text{if} \;\; t \in B_n \;\; \text{and} \;\; \xi(t) < -n, \\ \phi(t, f_0(t)) + \rho(t)/n \;\; \text{if} \;\; t \in T \setminus B_n \, . \end{cases}$$

It is easy to see that $\xi_n \in \mathbb{L}(T, \mathbb{R})$ for $n \geq 1$ and $\xi_n(t) \downarrow \xi(t)$ a.e., so that $\int_T \xi_{n_0} \mathrm{d}\mu < \beta$ for some n_0. Setting $\zeta = \xi_{n_0}$, we have $\int_T \zeta \mathrm{d}\mu < \beta$ and $\xi(t) < \zeta(t)$ a.e. We claim now that there exists a measurable function $g : T \to \mathbb{R}^d$ satisfying $g(t) \in F(t)$ a.e. and $\phi(t, g(t)) \leq \zeta(t)$ a.e. For case (i), take a sequence $(g_i)_{i=1}^{\infty}$ of measurable functions such that $F(t) = \mathrm{cl}(\{g_1(t), g_2(t), \ldots\}$ for all $t \in T$. Since $\inf_{i \geq 1} \phi(t, g_i(t)) = \xi(t)$ a.e., there exists a measurable function g satisfying the conditions desired above. For case (ii), define $F_1(t) = F(t) \cap \{x \in \mathbb{R}^d : \phi(t, x) \leq \zeta(t)\}$ for $t \in T$. Since $F_1(t)$ is closed for every $t \in T$ and $\mathrm{Graph}(F_1) \in \mathcal{F} \otimes \beta(\mathbb{R}^d)$ it follows by Remark 2.9 that F_1 has a measurable selection on $\mathrm{Dom}(F_1) \in \mathcal{F}$. Thus the desired g is obtained from the condition $\mu(T \setminus \mathrm{Dom}(F_1)) = 0$. Using the function g defined above, we define $C_n = A_n \cap \{t \in T : |g(t)| \leq n\}$ and $f_n = \mathbb{1}_{C_n} g + \mathbb{1}_{T \setminus C_n} f_0$ for $n \geq 1$ such that $f_n \in S(F)$ for $n \geq 1$ and

$$\begin{aligned} \mathcal{T}_{\phi}(f_n) &= \int_{C_n} \phi(t, g(t)) \mathrm{d}\mu + \int_{T \setminus C_n} \phi(t, f_0(t)) \mathrm{d}\mu \\ &\leq \int_T \zeta \mathrm{d}\mu + \int_{T \setminus C_n} [\phi(t, f_0(t)) - \zeta] \mathrm{d}\mu. \end{aligned}$$

Since $\int_T \zeta \mathrm{d}\mu < \beta$ and $C_n \uparrow T$, we have $\mathcal{T}_{\phi}(f_n) < \beta$. □

Corollary 3.2. *If* $F \in \mathcal{A}(T, \mathbb{R}^d)$ *if* $\phi : T \times X \to \tilde{\mathbb{R}}$ *is* $\mathcal{F} \otimes \beta(\mathbb{R}^d)$*-measurable and satisfies (i) or (ii) of Theorem 3.1, and if* $\mathcal{T}_{\phi}$ *is defined for all* $f \in S(F)$ *and* $\mathcal{T}_{\phi}(f_0) > -\infty$ *for some* $f_0 \in S(F)$*, then* $\sup\{\mathcal{T}_{\phi}(f) : f \in S(F)\} = \int_T \sup\{\phi(t, x) : x \in F(t)\} \mathrm{d}\mu$. □

Corollary 3.3. *For every* $F \in \mathcal{A}(T, \mathbb{R}^d)$*, one has* $\sup\{\|f\|^p : f \in S(F)\} = \int_T \sup\{|x|^p : x \in F(t)\} \mathrm{d}\mu = \int_T \|F(t)\|^p \mathrm{d}\mu$. *Then* F *is* p*-integrably bounded if and only if* $S(F)$ *is a bounded subset of* $\mathbb{L}^p(T, \mathbb{R}^d)$. □

Let $M \subset \mathbb{L}^0(T, \mathbb{R}^d)$ be a set of measurable functions $f : T \to \mathbb{R}^d$. We call M decomposable with respect to $\mathcal{F}$ if $f_1, f_2 \in M$ and $A \in \mathcal{F}$ imply $\mathbb{1}_A f_1 + \mathbb{1}_{T \setminus A} f_2 \in M$. It is clear that if M is decomposable, then $\sum_{i=1}^m \mathbb{1}_{A_i} f_i \in M$ for each finite $\mathcal{F}$-measurable partition $\{A_1, \ldots, A_m\}$ of T and $\{f_1, \ldots, f_m\} \subset M$. The following theorem is a characterization of decomposable subsets of the space $\mathbb{L}^p(T, \mathbb{R}^d)$.

Theorem 3.2. *Let M be a nonempty closed subset of $\mathbb{L}^p(T, \mathbb{R}^d)$ with $p \geq 1$. Then there exists an $F \in \mathcal{A}(T, \mathbb{R}^d)$ such that $M = S(F)$ if and only if M is decomposable.*

Proof. Let us observe that $S(F)$ is decomposable for every $F \in \mathcal{A}(T, \mathbb{R}^d)$. If $M \subset \mathbb{L}^p(T, \mathbb{R}^d)$ is such that there exists $F \in \mathcal{A}(T, \mathbb{R}^d)$ such that $M = S(F)$, then it is decomposable. To prove the converse, assume that M is a nonempty closed decomposable subset of $\mathbb{L}^p(T, \mathbb{R}^d)$. Let us observe that a multifunction G defined by $G(t) = \mathbb{R}^d$ for every $t \in T$ belongs to $\mathcal{A}(T, \mathbb{R}^d)$. Therefore, by virtue of Lemma 3.1, there exists a sequence $(f_i)_{i=1}^{\infty}$ of $\mathbb{L}^p(T, \mathbb{R}^n)$ such that $\mathbb{R}^d = \text{cl}(f_i(t) : i \geq 1\}$ for every $t \in T$. Let $\alpha_i = \inf\{\|f_i - g\| : g \in M\}$ for $i \geq 1$ and choose a sequence $\{g_{ij} : j \geq 1\} \subset M$ such that $\|f_i - g_{ij}| \to \alpha_i$ as $j \to \infty$. Define $F \in \mathcal{A}(T, \mathbb{R}^d)$ by $F(t) = \text{cl}\{g_{ij}(t) : i, j \geq 1\}$. We shall prove that $M = S(F)$. By Lemma 3.2, for each $f \in S(F)$ and $\varepsilon > 0$, we can select a finite measurable partition $\{A_1, \ldots, A_m\}$ of T and $\{h_1, \ldots, h_m\} \subset \{g_{ij}(t) : i, j \geq 1\}$ such that $\|f - \sum_{k=1}^m \mathbb{1}_{A_k} h_k\| < \varepsilon$. Since $\sum_{k=1}^m \mathbb{1}_{A_k} h_k \in M$, this implies that $f \in M$. Then $S(F) \subset M$. Now suppose that $S(F) \neq M$. Then there exist an $f \in M$, an $A \in \mathcal{F}$ with $\mu(A) > 0$, and a $\delta > 0$ such that $\inf_{i,j \geq 1} |f(t) - g_{ij}(t)| \geq \delta$ for $t \in A$. Take an integer i, fixed in the rest of the proof, such that the set $B = A \cap \{t \in T : |f(t) - f_i(t)| < \delta/3\}$ has positive measure, and let $g'_j = \mathbb{1}_B f + \mathbb{1}_{T \setminus B} g_{ij}$, for $j \geq 1$. Since $g'_j \in M$ for $j \geq 1$ and $|f_i(t) - g_{ij}(t)| \geq |f(t) - g_{ij}(t)| - |f(t) - f_i(t)| > 2\delta/3$ it follows that

$$
\begin{aligned}
\|f_i - g_{ij}\|^p - \alpha_i &\geq \|f_i - g_{ij}\|^p - \|f_i - g'_j\|^p \\
&= \int_B \left(|f_i(t) - g_{ij}(t)|^p - |f_i(t) - f(t)|^p\right) \mathrm{d}\mu \\
&\geq [(2\delta/3)^p - (\delta/3)^p] \cdot \mu(B) > 0
\end{aligned}
$$

for $j \geq 1$. If j tends to infinity, we get $\lim_{j \to \infty} \|f_i - g_{ij}\| > \alpha_i$, a contradiction. Thus $M = S(F)$. □

Remark 3.2. The above result is also true for nonempty closed subsets of $\mathbb{L}^p(T, X)$, where X is a separable Banach space. □

Remark 3.3. Similarly as in the proof of Michael's continuous selection theorem, it can be proved that if (X, ρ) is a separable metric space and $(T, \mathcal{F}, \mu)$ is a measure space, then every l.s.c. multifunction $F : X \to \text{Cl}(\mathbb{L}^p(T, \mathbb{R}^d))$ with decomposable values admits a continuous selection $f : X \to \mathbb{L}^p(T, \mathbb{R}^d)$.

Proof (Sketch of proof). The proof follows from the following construction procedure. For every $\varepsilon > 0$, we define continuous mappings $f_\varepsilon : X \to \mathbb{L}^p(T, \mathbb{R}^d)$ and $\varphi_\varepsilon : X \to \mathbb{L}^p(T, \mathbb{R}^+)$ such that $F_\varepsilon(x) = \{u \in F(x) : |u(t) - f_\varepsilon(t)| < \varphi_\varepsilon(t) \text{ a.e.}\}$ is nonempty and $\|\varphi_\varepsilon\|_p < \varepsilon$. Now, by the inductive procedure, we can define sequences $(f_n)_{n \geq 0}$, $(\varphi_n)_{n \geq 0}$, and $(F_n)_{n \geq 0}$ such that $\|\varphi_n(x)\| < 1/2^n$, $|f_n(x)(t) - f_{n-1}(x)(t)| \leq \varphi_n(x)(t) + \varphi_{n-1}(x)(t)$ a.e., and $F_n(x) \neq \emptyset$ for $x \in X$.

Hence the existence of a continuous selector f for F follows similarly as in the proof of Michael's theorem. $\square$

Given $C \subset \mathbb{L}^p(T, \mathbb{R}^d)$, by $\operatorname{dec}\{C\}$ we denote the decomposable hull of C, i.e., the smallest decomposable set of $\mathbb{L}^p(T, \mathbb{R}^d)$ containing C. The closed decomposable hull $\overline{\operatorname{dec}}\{C\}$ of C is defined by $\overline{\operatorname{dec}}\{C\} = \operatorname{cl}_{\mathbb{L}}[\operatorname{dec}\{C\}]$. It is easy to see that

$$\operatorname{dec}\{C\} = \left\{ \sum_{i=1}^{m} \mathbb{1}_{A_i} f_i \ : (A_i)_{i=1}^m \in \Pi(T, \mathcal{F}) \text{ and } (f_i)_{i=1}^m \subset C \right\},$$

where $\Pi(T, \mathcal{F})$ denotes the family of all finite $\mathcal{F}$-measurable partitions of T. Immediately from the above definition, it follows that the decomposable hull of the unit ball $\mathcal{B}$ of $\mathbb{L}^p(T, \mathbb{R}^d)$ is equal to the whole space, i.e., $\operatorname{dec}\{\mathcal{B}\} = \mathbb{L}^p(T, \mathbb{R}^d)$. We have the following results dealing with decomposable hulls.

Lemma 3.6. *Let (X, ρ) be a metric space. If $\Gamma : X \to \mathcal{P}(\mathbb{L}^p(T, \mathbb{R}^d))$ is l.s.c., then the multifunction $X \ni x \to \operatorname{dec}\{\Gamma(x)\} \subset \mathbb{L}^p(T, \mathbb{R}^d)$ is also l.s.c.*

Proof. By virtue of ([49], Theorem II.2.8), one has to verify that $\operatorname{dec}(\Gamma)_-(C) := \{x \in X : \operatorname{dec}\{\Gamma(x)\} \subset C\}$ is a closed subset of X for every closed set $C \subset \mathbb{L}^p(T, \mathbb{R}^d)$. Let C be a closed subset of $\mathbb{L}^p(T, \mathbb{R}^d)$ and $(x_n)_{n=1}^\infty$ a sequence of $\operatorname{dec}(\Gamma)_-(C)$ converging to $x \in X$. For every $u \in \operatorname{dec}\{\Gamma(x)\} \subset \overline{\operatorname{dec}}\{\Gamma(x)\}$ and $\varepsilon > 0$, there exist a measurable partition $(A_k^\varepsilon)_{k=1}^{N_\varepsilon}$ of T and a family $(v_k^\varepsilon)_{i=1}^{N_\varepsilon} \subset \mathbb{L}^p(T, \mathbb{R}^d)$ such that $\|u - \sum_{i=1}^{N_\varepsilon} \mathbb{1}_{A_k^\varepsilon} v_k^\varepsilon\| < \varepsilon$ and $v_k^\varepsilon \in \Gamma(x)$ for every $k = 1, \ldots, N_\varepsilon$. But Γ is l.s.c. at $x \in X$. Therefore, by virtue of ([49], Theorem II.2.9), for every $k = 1, \ldots, N_\varepsilon$ and $\varepsilon > 0$ there exists a sequence $(v_k^{n,\varepsilon})_{n=1}^\infty$ converging to v_k^ε such that $v_k^{n,\varepsilon} \in \Gamma(x_n)$ for every $n \geq 1$, $k = 1, \ldots, N_\varepsilon$ and $\varepsilon > 0$. Hence it follows that $\| \sum_{k=1}^{N_\varepsilon} \mathbb{1}_{A_k^\varepsilon} v_k^{n,\varepsilon} - \sum_{k=1}^{N_\varepsilon} \mathbb{1}_{A_k^\varepsilon} v_k^\varepsilon \| \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. Therefore, $\lim_{n\to\infty} \|u - \sum_{k=1}^{N_\varepsilon} \mathbb{1}_{A_k^\varepsilon} v_k^{n,\varepsilon}\| \leq \varepsilon$ for every $\varepsilon > 0$. But $\sum_{i=1}^{N_\varepsilon} \mathbb{1}_{A_k^\varepsilon} v_k^{n,\varepsilon} \in \operatorname{dec}\{\Gamma(x_n)\} \subset C$ for every $n \geq 1$ and $\varepsilon > 0$. Then $u \in C + \varepsilon B$, where B denotes the closed unit ball of $\mathbb{L}^p(T, \mathbb{R}^d)$. Therefore, for every $u \in \operatorname{dec}\{\Gamma(x)\}$, one has $u \in \overline{C} = C$. Thus $\operatorname{dec}\{\Gamma(x)\} \subset C$, which implies that $x \in \operatorname{dec}(\Gamma)_-(C)$. Therefore, $\operatorname{dec}(\Gamma)_-(C)$ is a closed subset of X for every closed set $C \subset \mathbb{L}^p(T, \mathbb{R}^d)$. $\square$

Remark 3.4. Immediately from Lemma 3.6, it follows that by the assumption of Lemma 3.6, the multifunction $X \ni x \to \overline{\operatorname{dec}}\{\Gamma(x)\} \subset \mathbb{L}^p(T, \mathbb{R}^d)$ is l.s.c.

Proof. By virtue of ([49], Theorem II.2.9) one has to verify that for every $x \in X$, every sequence $(x_n)_{n=1}^\infty$ of X converging to x, and $u \in \overline{\operatorname{dec}}\{\Gamma(x)\}$, there exists a sequence $(y_n)_{n=1}^\infty$ of $\mathbb{L}^p(T, \mathbb{R}^d)$ converging to u such that $y_n \in \overline{\operatorname{dec}}\{\Gamma(x_n)\}$ for every $n \geq 1$. Let $x \in X$ be fixed, let $(x_n)_{n=1}^\infty$ be a sequence of X converging to x, and let $u \in \overline{\operatorname{dec}}\{\Gamma(x)\}$. For every $\varepsilon > 0$, one has $\operatorname{dec}\{\Gamma(x)\} \cap B(u, \varepsilon) \neq \emptyset$. By virtue of ([49], Proposition II.2.4) and Lemma 3.6, a multifunction $\Phi(x) = \operatorname{dec}\{\Gamma(x)\} \cap B(u, \varepsilon)$ is l.s.c. Then there exists a sequence $(y_n)_{n=1}^\infty$ of $\mathbb{L}^p(T, \mathbb{R}^d)$

converging to u such that $y_n \in \text{dec}\{\Gamma(x_n)\} \cap B(u,\varepsilon)$, which implies that $y_n \in \overline{\text{dec}}\{\Gamma(x_n)\}$. $\square$

Theorem 3.3. *The decomposable hull of a convex set $K \subset \mathbb{L}^p(T,\mathbb{R}^d)$ is itself convex, and its closure is convex and sequentially weakly closed. If $(\Omega,\mathcal{F},\mu)$ is a σ-fini te nonatomic space and K is a nonempty subset of $\mathbb{L}^p(\Omega,\mathcal{F},\mu,\mathbb{R}^d)$, then $\overline{\text{dec}_w}\{K\} = \overline{\text{co}}[\text{dec}\{K\}]$, where $\overline{\text{dec}_w}\{K\}$ denotes the closure of* dec$\{K\}$ *with respect to a weak topology of* $\mathbb{L}^p(T,\mathbb{R}^d)$.

Proof. Let K be a convex subset of $\mathbb{L}^p(T,\mathbb{R}^d)$ and $u,v \in \text{dec}\{K\}$. There are partitions $(A_n)_{n=1}^N, (B_m)_{m=1}^M \in \Pi(T,\mathcal{F})$, and $(u_n)_{n=1}^N, (v_m)_{m=1}^M \subset K$ such that $u = \sum_{n=1}^N \mathbb{1}_{A_n} u_n$ and $v = \sum_{m=1}^M \mathbb{1}_{B_m} v_m$. Let $(D_k)_{k=1}^K \in \Pi(T,\mathcal{F})$ be such that $u = \sum_{k=1}^K \mathbb{1}_{D_k}\bar{u}_k$ and $v = \sum_{k=1}^K \mathbb{1}_{D_k}\bar{v}_k$, where $\bar{u}_k = u_{n_k}$ and $\bar{v}_k = v_{m_k}$ for $n_k \in \{1,\dots,N\}$ and $m_k \in \{1,\dots,M\}$ for every $k = 1,\dots,K$. For every $\lambda \in [0,1]$ and $1 \le k \le K$, one has $\lambda\bar{u}_k + (1-\lambda)\bar{v}_k \in K$. Therefore, $\lambda u + (1-\lambda) v = \sum_{k=1}^K \mathbb{1}_{D_k}[\lambda\bar{u}_k + (1-\lambda)\bar{v}_k] \in \text{dec}\{K\}$. Thus $\text{dec}\{K\}$ is a convex subset of $\mathbb{L}^p(\Omega,\mathcal{F},\mathbb{R}^r)$. Hence the convexity of $\overline{\text{dec}_w}\{K\}$ follows. Now, immediately from Mazur's theorem ([4], Theorem 9.11), it follows that $\overline{\text{dec}}\{K\}$ is sequentially weakly closed. Finally, immediately from ([41], Theorem 2.3.17), the equality $\overline{\text{dec}_w}\{K\} = \overline{\text{co}}[\text{dec}\{K\}]$ follows. $\square$

Remark 3.5. If $K \subset \mathbb{L}^2(T,\mathbb{R}^d)$ is convex and square integrably bounded, then $\overline{\text{dec}}\{K\}$ is convex and weakly compact.

Proof. If $K \subset \mathbb{L}^2(T,\mathbb{R}^d)$ is square integrably bounded, then $\overline{\text{dec}}\{K\}$ is square integrably bounded, too. Therefore, $\overline{\text{dec}}\{K\}$ is relatively weakly compact, which by virtue of Theorem 3.3, implies that it is convex and weakly compact. $\square$

Remark 3.6. If $F : T \to \mathbb{R}^d$ is measurable and p-integrably bounded, then the interior $\text{Int}[S(F)]$ of $S(F)$ is the empty set and $S(F) = \overline{\text{dec}}\{f_n : n \ge 1\}$, where $f_n \in S(F)$ for $n \ge 1$ are such that $F(t) = \text{cl}\{f_n(t) : n \ge 1\}$ for $t \in T$.

Proof. Suppose $\text{Int}[S(F)] \ne \emptyset$. For every $f \in \text{Int}[S(F)])$, there exists an open ball $\mathcal{B}(f)$ containing f such that $\mathcal{B}(f) \subset \text{Int}[S(F)] \subset S(F)$. Hence it follows that $\text{dec}\{\mathcal{B}(f)\} \subset \text{dec}\{S(F)\}$. But $S(F)$ is a decomposable subset of $\mathbb{L}^p(T,\mathbb{R}^d)$. Therefore, $\text{dec}\{\mathcal{B}(f)\} \subset S(F)$, which is a contradiction, because $S(F)$ is bounded and $\text{dec}\{\mathcal{B}(f)\} = \mathbb{L}^p(T,\mathbb{R}^d)$. Then $\text{Int}[S(F)] = \emptyset$. Let us observe that by the properties of $S(F)$, we have $\overline{\text{dec}}\{f_n : n \ge 1\} \subset S(F)$. On the other hand, by virtue of Lemma 3.2, for every $f \in S(F)$ and $\varepsilon > 0$ there exist a partition $(A_k)_{k=1}^N \in \Pi(T,\mathcal{F})$ and a family $(f_{n_k})_{k=1}^N \subset \{f_n : n \ge 1\}$ such that $\|f - \sum_{k=1}^N \mathbb{1}_{A_k} f_{n_k}\| \le \varepsilon$, which implies that $f \in \overline{\text{dec}}\{f_n : n \ge 1\}$. Thus $S(F) = \overline{\text{dec}}\{f_n : n \ge 1\}$. $\square$

Lemma 3.7. *Assume that $(T,\mathcal{F},\mu)$ and (X,ρ) are measure and metric spaces, respectively. Let $F : T \times X \to \text{Cl}(\mathbb{R}^d)$ be such that $F(\cdot,x)$ is measurable for fixed $x \in X$ and there exist $m,k \in \mathbb{L}^2(T,\mathbb{R}^+)$ such that $\|F(t,x)\| \le m(t)$ and $h(F(t,x),F(t,\bar{x})) \le k(t)\rho(x,\bar{x})$ for μ-a.e. $t \in T$ and $x,\bar{x} \in X$.*

Then $H(S(F(\cdot,x)),S(F(\cdot,\bar{x}))) \le K\rho(x,\bar{x})$ *for every* $x,\bar{x} \in X$*, where* $K = (\int_T k^2(t)\mathrm{d}\mu)^{1/2}$ *and* H *is the Hausdorff metric on* $\mathrm{Cl}(\mathbb{L}^2(T,\mathbb{R}^d))$.

Proof. Assume $x,\bar{x} \in X$ and select arbitrarily $f^x \in S(F(\cdot,x))$. By virtue of Theorem 3.1, one has

$$\begin{aligned}\mathrm{dist}(f^x,S(F(\cdot,\bar{x})) &= \inf\left\{\left(\int_T |f_t^x - f_t|^2\mathrm{d}\mu\right)^{1/2} : f \in S(F(\cdot,\bar{x}))\right\}\\ &= \left(\int_T \mathrm{dist}^2(f_t^x, F(t,\bar{x}))\mathrm{d}\mu\right)^{1/2}\\ &\le \left(\int_T k^2(t)\rho^2(x,\bar{x})\mathrm{d}\mu\right)^{1/2} \le K\rho(x,\bar{x}),\end{aligned}$$

where $K = (\int_0^T k^2(t)\mathrm{d}t)^{1/2}$. Then $\bar{H}(S(F(\cdot,x)),S((F(\cdot,\bar{x}))) \le K\rho(x,\bar{x})$. In a similar way, we obtain $\bar{H}(S(F(\cdot,\bar{x})),S(F(\cdot,x))) \le K\rho(x,\bar{x})$. □

Remark 3.7. Similarly as above, one can prove that if $(T,\mathcal{F},\mu)$ and (X,ρ) are as above and $F : T \times X \to \mathrm{Cl}(\mathbb{R}^d)$ is measurable and uniformly square integrably bounded such that $F(t,\cdot)$ is l.s.c. for a.e. fixed $t \in T$, then a set-valued mapping $X \ni x \to S(F(\cdot,x)) \in \mathrm{Cl}(\mathbb{L}^2(T,\mathbb{R}^d))$ is l.s.c.

Proof. Let us observe first that for given metric spaces X and Y, a multifunction $\Phi : X \to \mathcal{P}(Y)$ is l.s.c. at $\bar{x} \in X$ if it is H-l.s.c., i.e., if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in X$ satisfying $\rho(x,\bar{x}) < \delta$, one has $\bar{h}(\Phi(\bar{x}),\Phi(x)) \le \varepsilon$. Indeed, suppose the above condition is satisfied and Φ is not l.s.c. at $\bar{x}$. There exists an open set $U \subset Y$ with $\Phi(\bar{x}) \cap U \ne \emptyset$ such that in every neighborhood V of $\bar{x}$, there exists $\tilde{x} \in V$ such that $\Phi(\tilde{x}) \cap U = \emptyset$. Therefore, we can select a sequence $(x_n)_{n=1}^{\infty}$ of X converging to $\tilde{x}$ such that $\Phi(x_n) \cap U = \emptyset$ for every $n = 1,2,\ldots$. On the other hand, for every $\varepsilon > 0$, there exists $N_\varepsilon \ge 1$ such that for every $n \ge N_\varepsilon$, we have $\Phi(\bar{x}) \subset V^0[\Phi(x_n),\varepsilon]$. Hence in particular, it follows that $\Phi(\bar{x}) \cap U \subset V^0[\Phi(x_n),\varepsilon]$ for $n \ge N_\varepsilon$. Let $y \in \Phi(\bar{x}) \cap U$, $n_k = N_{1/k}$ for every $k = 1,2,\ldots$ and select $y_k \in \Phi(x_{n_k})$ such that $d(y_k,y) < 1/k$. For k sufficiently large, we have $y_k \in U$ and therefore $\Phi(x_{n_k}) \cap U \ne \emptyset$, a contradiction.

Let us observe now that if $\Phi(\bar{x})$ is a compact subset of Y, then Φ is l.s.c. at $\bar{x} \in X$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in X$ satisfying $\rho(x,\bar{x}) < \delta$, one has $\bar{h}(\Phi(\bar{x}),\Phi(x)) \le \varepsilon$. Indeed, for $i = 1,\ldots,m$, let y_i be such that $\{B^0(y_i,(1/2)\varepsilon) : i = 1,\ldots,m\}$ covers $\Phi(\bar{x})$ and for $i = 1,\ldots,m$, let $\delta_i > 0$ be such that $\rho(x,\bar{x}) < \delta_i$ implies $\Phi(x) \cap B^0(y_i,(1/2)\varepsilon) \ne \emptyset$. Let $\delta = \min\{\delta_i : i = 1,\ldots,m\}$. Then $\rho(x,\bar{x}) < \delta$ implies that $y_i \in V^0(\Phi(x),(1/2)\varepsilon)$ for $i = 1,\ldots,m$, i.e., $B^0(y_i,(1/2)\varepsilon) \subset V^0(\Phi(x),(1/2)\varepsilon)$ for all $i = 1,\ldots,m$. Therefore, $\Phi(\bar{x}) \subset \bigcap_{i=1}^m B^0(y_i,(1/2)\varepsilon) \subset V^0(\Phi(x),(1/2)\varepsilon)$ for $x \in B^0(\bar{x},\delta)$, which is equivalent to $\bar{h}[\Phi(\bar{x}),\Phi(x)] \le \varepsilon$ for $x \in B^0(\bar{x},\delta)$.

Let $m \in \mathbb{L}^2(T, \mathbb{R}^+)$ be such that $\|F(t,x)\| \le m(t)$ for every $x \in X$ and a.e. $t \in T$. Therefore, $F(t,x)$ is a compact subset of $\mathbb{R}^d$ for every $x \in X$ and a.e. $t \in T$. Similarly as in the proof of Lemma 3.7, we can verify that for every $\bar{x}, x \in X$, one has

$$\bar{H}[S(F(\cdot,\bar{x})), S(F(\cdot,x))] \le \left(\int_T \bar{h}^2[F(t,\bar{x}), F(t,x)]\mathrm{d}t\right)^{\frac{1}{2}}.$$

Thus for every $\bar{x} \in X$ and every sequence $(x_n)_{n=1}^{\infty}$ of X converging to $\bar{x}$, we obtain $\int_T \bar{h}^2[F(t,\bar{x}), F(t,x_n)]\mathrm{d}t \to 0$ as $n \to \infty$, which implies that $\bar{H}[S(F(\cdot,\bar{x})), S(F(\cdot,x_n))] \to 0$ as $n \to \infty$. Then the set-valued mapping $X \ni x \to S(F(\cdot,x)) \in \mathrm{Cl}(\mathbb{L}^2(T,\mathbb{R}^d))$ is l.s.c. at $\bar{x}$. □

Lemma 3.8. *Assume that T is an interval of the real line and let $F : T \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : T \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ be measurable uniformly p-integrably bounded and such that $F(t,\cdot)$ and $G(t,\cdot)$ are l.s.c. for fixed $t \in T$. There are continuous functions $u : \mathbb{R}^d \to \mathbb{L}^p(T,\mathbb{R}^d)$ and $v : \mathbb{R}^d \to \mathbb{L}^p(T,\mathbb{R}^{d\times m})$ such that*

(i) $u(x) \in S(F(\cdot,x))$ and $v(x) \in S(G(\cdot,x))$ for $x \in \mathbb{R}^d$;
(ii) mappings $f : T \times \mathbb{R}^d \ni (t,x) \to u(x)(t) \in \mathbb{R}^d$ and $g : T \times \mathbb{R}^d \ni (t,x) \to v(x)(t) \in \mathbb{R}^{d\times m}$ are $\beta_T \otimes \beta(\mathbb{R}^d)$-measurable such that $f(t,x) \in F(t,x)$ and $g(t,x) \in G(t,x)$ for a.e. $t \in T$ and $x \in \mathbb{R}^d$.

Proof. The existence of continuous functions u and v satisfying (i) follows immediately from Remarks 3.3 and 3.7. Let $\mathcal{I}$ be the identity mapping on T and define $(\mathcal{I} \times u) : T \times \mathbb{R}^d \to T \times \mathbb{L}^p(T,\mathbb{R}^d)$ by setting $(\mathcal{I} \times u)(t,x) = (t, u(x))$ for $(t,x) \in T \times \mathbb{R}^d$. The function $\mathcal{I} \times u$ is continuous on $T \times \mathbb{R}^d$ and therefore $(\beta_T \otimes \beta(\mathbb{R}^d), \beta_T \otimes \beta(\mathbb{L}^p))$-measurable, where β_T, $\beta(\mathbb{R}^d)$ and $\beta(\mathbb{L}^p)$ denote the Borel σ-fields on T, $\mathbb{R}^d$ and $\mathbb{L}^p(T,\mathbb{R}^d)$, respectively. Let $\rho : T \times \mathbb{L}^p(T,\mathbb{R}^d) \to \mathbb{R}^d$ be defined by $\rho(t,z) = z(t)$ for $(t,z) \in T \times \mathbb{L}^p(T,\mathbb{R}^d)$. The mapping ρ is $(\beta_T \otimes \beta(\mathbb{L}^p), \beta(\mathbb{R}^d))$-measurable because ρ is such that $\rho(t,\cdot)$ is continuous and $\rho(\cdot,z)$ is measurable for fixed $t \in T$ and $z \in \mathbb{L}^p(T,\mathbb{R}^d)$, respectively. Hence it follows that a mapping $f : T \times \mathbb{R}^d \ni (t,x) \to u(x)(t) \in \mathbb{R}^d$ is measurable on $T \times \mathbb{R}^d$, i.e., is $(\beta_T \otimes \beta(\mathbb{R}^d), \beta(\mathbb{R}^d))$-measurable because $f(t,x) = [\rho \circ (\mathcal{I} \times u)](t,x) = \rho(t, u(x))$ for $(t,x) \in T \times \mathbb{R}^d$. Measurability of a mapping g can be verified in a similar way. It is clear that $f(t,x) \in F(t,x)$ and $g(t,x) \in G(t,x)$ for a.e. $t \in T$ and $x \in \mathbb{R}^d$. □

Similarly as above, let T be an interval of the real line. Denote by J the linear mapping defined on $\mathbb{L}^p(T,\mathbb{R}^d)$ by setting $J(f) = \int_T f(t)\mathrm{d}t$ for $f \in \mathbb{L}^p(T,\mathbb{R}^d)$. For a nonempty set $K \subset \mathbb{L}^p(T,\mathbb{R}^d)$, by $J(K)$ we denote its image by the mapping J, i.e., a set of the form $\{\int_T f(t)\mathrm{d}t : f \in K\}$.

Lemma 3.9. *If $K \subset \mathbb{L}^p(T,\mathbb{R}^d)$ is nonempty decomposable, then $J(K)$ is a nonempty convex subset of $\mathbb{R}^d$.*

Proof. Let $z_1, z_2 \in J(K)$ and $\lambda \in [0,1]$. There exist $f_1, f_2 \in K$ such that $z_1 = \int_T f_1(t)\mathrm{d}t$ and $z_2 = \int_T f_2(t)\mathrm{d}t$. Let $\mathcal{L}_T$ be the family of all Lebesgue measurable subsets of T and put $\mu(E) = (\int_E f_1(t)\mathrm{d}t, \int_E f_2(t)\mathrm{d}t)$ for $E \in \mathcal{L}_T$. By Lyapunov's theorem, $\mu(\mathcal{L}_T)$ is a convex compact subset of $\mathbb{R}^{2d}$. Since $(0,0)$ and (z_1, z_2) belong to $\mu(\mathcal{L}_T)$, then we have also $(\lambda z_1, \lambda z_2) \in \mu(\mathcal{L}_T)$. Therefore, there exists $H \in \mathcal{L}_T$ such that $(\lambda z_1, \lambda z_2) = \mu(H)$, which by the definition of the measure μ implies that $\lambda z_1 = \int_T \mathbb{1}_H f_1(t)\mathrm{d}t$ and $\lambda z_2 = \int_T \mathbb{1}_H f_2(t)\mathrm{d}t$. Let $f = \mathbb{1}_H f_1 + \mathbb{1}_{T\setminus H} f_2$. By the decomposability of K, we have $f \in K$. Therefore, $\int_T f(t)\mathrm{d}t \in J(K)$. But $\int_T f(t)\mathrm{d}t = \int_T (\mathbb{1}_H f_1 + \mathbb{1}_{T\setminus H} f_2)(t)\mathrm{d}t = \int_T \mathbb{1}_H (f_1 - f_2)(t)\mathrm{d}t + \int_T f_2(t)\mathrm{d}t = \lambda z_1 - \lambda z_2 + z_2 = \lambda z_1 + (1-\lambda) z_2$. Then $\lambda z_1 + (1-\lambda) z_2 \in J(K)$. $\square$

For $F \in \mathcal{A}(T, \mathbb{R}^d)$, the set $J(S(F))$ is denoted by $\int_T F(t)\mathrm{d}t$ and is said to be the Aumann integral of F on the interval T.

Corollary 3.4. *For every $F \in \mathcal{A}(T, \mathbb{R}^d)$, the Aumann integral $\int_T F(t)\mathrm{d}t$ is a nonempty convex subset of $\mathbb{R}^d$. If furthermore, F is p-integrably bounded, then $\int_T F(t)\mathrm{d}t$ is a bounded subset of $\mathbb{R}^d$.* $\square$

Denote by $V(\sigma^r)$ the set of $r+1$ vertices of the $(r+1)$-dimensional simplex $\sigma^r = \{(\xi_0, \ldots, \xi_r) \in \mathbb{R}^{r+1} : 0 \le \xi_i \le 1,\ \sum_{i=0}^r \xi_i = 1\}$. It is clear that if $u_i \in \mathbb{L}^\infty(T, \mathbb{R}^1)$ for $i = 0, 1, \ldots, r$, then $(u_0, \ldots, u_r) \in \mathbb{L}^\infty(T, \mathbb{R}^{r+1})$, where $\mathbb{L}^\infty(T, \mathbb{R}^1)$ consists of all μ-essentially bounded measurable scalar functions defined on T.

Lemma 3.10. *Let $Y(t)$ be an $n \times (r+1)$-matrix-valued function with components in $\mathbb{L}^\infty(T, \mathbb{R}^1)$, $\Psi = \{u \in \mathbb{L}^\infty(T, \mathbb{R}^{r+1}) : u(t) \in \sigma^r$ for $t \in T\}$, and $\Psi_0 = \{u \in \mathbb{L}^\infty(T, \mathbb{R}^{r+1}) : u(t) \in V(\sigma^r)$ for $t \in T\}$. Then $\{\int_T Y(t)\cdot u(t)\mathrm{d}t : u \in \Psi\} = \{\int_T Y(t)\cdot u(t)\mathrm{d}t : u \in \Psi_0\}$, and both of these sets are compact and convex.*

Proof. Let $J(u) = \int_T Y(t)\cdot u(t)\mathrm{d}t$ for $u \in \mathbb{L}^\infty(T, \mathbb{R}^{r+1})$. Clearly, Ψ is convex and bounded in the $\mathbb{L}^\infty(T, \mathbb{R}^{r+1})$-norm topology. Hence if we can show that Ψ is weakly*-closed, it will imply that Ψ is weakly*-compact. Suppose u^0 is a weak*-limit of a sequence of Ψ that does not belong to Ψ. Then there is a set $E \subset T$ of positive measure such that $u^0(t) \in \sigma^r$ for $t \in E$ and $u^0 \in \Psi$. One may readily establish the existence of an $\varepsilon > 0$ and $\eta \in \mathbb{R}^{r+1}$ such that the inner product satisfies $\langle \eta, \xi \rangle \ge C$ if $\xi \in \sigma^r$ and $\langle \eta, u^0(t) \rangle < C - \varepsilon$ for t in a subset E_1 of E having a positive measure $\mu(E_1)$. Define a function $w(t) = (w_0(t), \ldots, w_r(t))$ by setting

$$w_i(t) = \begin{cases} \eta_i/\mu(E_1) & \text{for } t \in E_1, \\ 0 \quad \text{for} & t \notin E_1, \end{cases}$$

for $i = 1, \ldots, r$. It is clear that $w \in \mathbb{L}^\infty(T, \mathbb{R}^{r+1})$. From the properties of $\eta \in \mathbb{R}^{r+1}$, it follows that w separates u^0 and Ψ, contradicting u^0 being a weak*-limit of a sequence of Ψ. Thus Ψ is closed, convex, and weak*-compact. It is easily seen that J is weak*-continuous, because the weak topology was defined so that the linear functionals that were continuous on a given normed space X with its norm topology are still continuous when X has its weak topology. In particular, $J = (J_1, \ldots, J_n)$ is a continuous linear mapping from X^* taken with its norm topology to $\mathbb{R}^d$ such that components J_i of J are representable as elements of X. Then J is continuous as a mapping of X^* with the weak*-topology to $\mathbb{R}^d$. Therefore, $J\Psi = \{Ju : u \in \Psi\}$ is a compac,t convex subset of $\mathbb{R}^d$. Clearly, $J\Psi_0 \subset J\Psi$. Similarly as in the proof of Lyapunov's theorem, we can also show that $J\Psi \subset J\Psi_0$. □

Lemma 3.11. *Let* $F : T \to \mathrm{Cl}(\mathbb{R}^d)$ *be measurable and integrably bounded. Then* $\int_T F(t)\mathrm{d}t = \int_T \mathrm{co}\, F(t)\mathrm{d}t$, *and both sets are nonempty and convex in* $\mathbb{R}^d$.

Proof. The nonemptiness and convexity of $\int_T F(t)\mathrm{d}t$ follow from Corollary 3.4. By the definition of the Aumann integral, it follows that $\int_T F(t)\mathrm{d}t \subset \int_T \mathrm{co}\, F(t)\mathrm{d}t$. Suppose $y \in \int_T \mathrm{co}\, F(t)\mathrm{d}t$, and let $f \in S(\mathrm{co}\, F)$ be such that $y = \int_T f(t)\mathrm{d}t$. By Carathéodory's theorem, for every $t \in T$, the point $f(t) \in \mathrm{co}\, F(t)$ may be expressed as a convex combination $f(t) = \sum_{i=0}^d \xi_i(t) f^i(t)$ with $f^i(t) \in F(t)$, $0 \le \xi_i(t) \le 1$, and $\sum_{i=0}^d \xi_i(t) = 1$. Let σ^d denote the simplex in the space $\mathbb{R}^{d+1}$, i.e., $\sigma^d = \{(\xi_0, \ldots, \xi_d) \in \mathbb{R}^{n+1} : 0 \le \xi_i \le 1, \sum_{i=0}^d \xi_i = 1\}$. Denote by $\xi(t)$ the vector $(\xi_0(t), \ldots, \xi_d(t)) \in \sigma^d$. Let us observe that the functions ξ_i and f^i can be chosen to be measurable. Indeed, let $g(t, \xi, \beta^0, \ldots, \beta^d) = \sum_{i=0}^{\mathrm{d}} \xi_i(t)\beta^i$ for $t \in T$ and $\beta^0, \ldots, \beta^d \in \mathbb{R}$ and let $\Gamma(t) = \sigma^{d+1} \times F(t) \times \cdots \times F(t)$ with $F(t)$ appearing $n+1$ times in the product. Since f is measurable and $f(t) \in g(t, \Gamma(t))$ for a.e. $t \in T$, then by Theorem 2.5, there exists a measurable function $T \ni t \to (\xi_0(t), \ldots, \xi_n(t), f^0(t), \ldots, f^{\mathrm{d}}(t)) \in \Gamma(t)$ such that $f(t) = g(t, (\xi_0(t), \ldots, \xi_n(t), f^0(t), \ldots, f^{\mathrm{d}}(t))$ for a.e. $t \in T$. Let the vectors $f^i(t)$ be the columns of an $d \times (d+1)$-matrix Y. By virtue of Lemma 3.10 there exists a measurable vector function $\xi^* = (\xi_0^*, \ldots, \xi_d^*)$ on T taking values in the vertices of the simplex σ^d such that $\int_T f(t)\mathrm{d}t = \int_T Y(t) \cdot \xi(t)\mathrm{d}t = \int_T Y(t) \cdot \xi^*(t)\mathrm{d}t$. Now $\xi_i^*(T) \subset \{0, 1\}$ for all $i = 0, 1, \ldots, d$ and $\sum_{i=0}^d \xi_i^*(t) = 1$. Let $T_i = \{t \in T : \xi_i^*(t) = 1\}$. Then T_i is measurable and $\bigcup_{i=0}^d T_i = T$ and $T_i \cap T_j = \emptyset$ for $i \ne j$. Define $f^*(t) = f^i(t)$ for $t \in T_i$ for $i = 0, 1, \ldots, d$. It is clear that f^* is measurable and such that $f^*(t) \in F(t)$ and $\int_T f^*(t)\mathrm{d}t = \int_T f(t)\mathrm{d}t$. Then $\int_T F(t)\mathrm{d}t = \int_T \mathrm{co}\,\mathrm{F}(t)\mathrm{d}t$. □

Theorem 3.4 (Aumann). *If* $F : T \to \mathrm{Cl}(\mathbb{R}^d)$ *is measurable and integrably bounded, then* $\int_T F(t)\mathrm{d}t = \int_T \mathrm{co}\, F(t)\mathrm{d}t$, *and both integrals are nonempty convex, compact subsets of* $\mathbb{R}^d$.

Proof. By virtue of Lemma 3.11, we have $\int_T F(t)\mathrm{d}t = \int_T \mathrm{co}\, F(t)\mathrm{d}t$, and both integrals are nonempty convex subsets of $\mathbb{R}^d$. By virtue of Remark 3.1, a set $S(\mathrm{co}\, F)$ is a weakly sequentially compact subset of $\mathbb{L}(T, \mathbb{R}^d)$. By the definition of the Aumann integral, we have $\int_T \mathrm{co}\, F(t)\mathrm{d}t = J(S(\mathrm{co}\, F))$, where J is a linear and continuous mapping defined on $\mathbb{L}(T, \mathbb{R}^d)$. By the linearity of J, it follows that J is also continuous on $\mathbb{L}(T, \mathbb{R}^d)$ with respect to its weak topology. Therefore, $J(S(\mathrm{co}\, F))$ is a compact subset of $\mathbb{R}^d$. □

Remark 3.8. It can be proved that if $(X, \|\cdot\|)$ is a separable Banach space, T is an interval of the real line, and $F : T \to \mathrm{Cl}(X)$ is measurable and integrably bounded, then $\mathrm{cl}(\int_T F(t)\mathrm{d}t) = \mathrm{cl}(\int_T \mathrm{co}\, F(t)\mathrm{d}t)$, where the closure is taken in the norm topology of X. □

Theorem 3.5. *If $F : T \to \mathrm{Cl}(\mathbb{R}^d)$ is measurable and integrably bounded, then for every $p \in \mathbb{R}^d$ and $A \in \mathcal{L}_T$, one has $\int_A \sigma(p, F(t))\mathrm{d}t = \sigma(p, \int_A F(t)\mathrm{d}t)$.*

Proof. Let us observe that $\sigma(p, F(\cdot))$ is measurable and integrably bounded for every fixed $p \in \mathbb{R}^d$. Then it is integrable and $\int_A \sigma(p, F(t))\mathrm{d}t < \infty$ for every $p \in \mathbb{R}^d$ and $A \in \mathcal{L}_T$. For every $f \in S(F)$ and $p \in \mathbb{R}^d$, we have $\left\langle p, \int_A f(t)\mathrm{d}t \right\rangle = \int_A \langle p, f(t)\rangle\, \mathrm{d}t \le \int_A \sigma(p, F(t))\mathrm{d}t$. Therefore, for every $p \in \mathbb{R}^d$, one has $\sigma(p, \int_A F(t)\mathrm{d}t) \le \int_A \sigma(p, F(t))\mathrm{d}t$. We shall show now that for every $\alpha \in \mathbb{R}$ and $p \in \mathbb{R}^d$ such that $\alpha < \int_A \sigma(p, F(t))d$, there is $f \in S(F)$ such that $\alpha < \sigma(p, \int_A f(t)\mathrm{d}t)$. Indeed, let us take arbitrarily $g \in S(F)$ and define for every $n \ge 1$ a multifunction F_n by setting $F_n(t) = \{x \in F(t) : |x - g(t)| < n\}$. Similarly as in the proof of Theorem 2.5, we can verify that F_n, and hence also $\mathrm{cl}(F_n)$, is measurable. Then $\sigma(p, F_n(\cdot))$ is measurable for every $p \in \mathbb{R}^d$ and $n \ge 1$. It is also integrably bounded. Furthermore, $\sigma(p, F_n(t)) \to \sigma(p, F(t))$ for $t \in T$ as $n \to \infty$. Then $\int_A \sigma(p, F_n(t))\mathrm{d}t \to \int_A \sigma(p, F(t))\mathrm{d}t$ for every $p \in \mathbb{R}^d$ as $n \to \infty$. Thus we have $\alpha < \int_A \sigma(p, F_n(t))\mathrm{d}t$ for n large enough. Then there exists an integrable function $\varphi : T \to \mathbb{R}$ such that $\alpha < \int_A \varphi(t)\mathrm{d}t$ and $\varphi(t) < \sigma(p, F_n(t))$ for a.e. $t \in T$. Let $G(t) = \{x \in F(t) : \langle p, x\rangle > \varphi(t)\}$ for $t \in T$. It is clear that $G(t) \ne \emptyset$ and that G has a measurable graph. Therefore, by virtue of Remark 2.9, there exists a measurable selector f of G, and hence also of F, such that $\varphi(t) < \langle p, f(t)\rangle$. Thus $\int_A \varphi(t)\mathrm{d}t < \left\langle p, \int_A f(t)\mathrm{d}t \right\rangle$. Hence it follows that $\alpha < \left\langle p, \int_A f(t)\mathrm{d}t \right\rangle$. Now taking in particular $\alpha_n = \int_A \sigma(p, F(t))\mathrm{d}t - 1/n$ for $n \ge 1$, we can select $f_n \in S(F)$ such that $\alpha_n < \sigma(p, \int_A f_n(t)\mathrm{d}t) \le \sigma(p, \int_A F(t)\mathrm{d}t)$ for every $p \in \mathbb{R}^d$ and $n \ge 1$, which implies that $\int_A \sigma(p, F(t))\mathrm{d}t \le \sigma(p, \int_A F(t)\mathrm{d}t)$ for every $p \in \mathbb{R}^d$ and $A \in \mathcal{L}_T$. □

Remark 3.9. The above results are also true for measurable and p-integrably bounded multifunctions with $p \ge 1$. □

4 Set-Valued Stochastic Processes

Similarly as in Chap. 1, we assume that we are given a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ satisfying the usual conditions. By a set-valued random variable, we mean an $\mathcal{F}$-measurable multifunction $\mathcal{Z} : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$. If $\mathcal{Z} \in \mathcal{A}(\Omega, \mathbb{R}^d)$, then the Aumann integral $\int_\Omega \mathcal{Z}\mathrm{d}P$ is denoted by $E[\mathcal{Z}]$ and is said to be the mean value of the set-valued random variable $\mathcal{Z}$. A set-valued random variable $\mathcal{Z} \in \mathcal{A}(\Omega, \mathbb{R}^d)$ is said to be Aumann integrable. Immediately from properties of measurable set-valued mappings, the following results, dealing with set-valued random variables, follow.

Lemma 4.1. *Let $\mathcal{Z} : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ be an Aumann integrable set-valued random variable. Then*

(i) $S(\mathcal{Z})$ *is a closed decomposable subset of* $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{R}^d)$ *and* $S(\overline{\mathrm{co}}\, \mathcal{Z}) = \overline{\mathrm{co}}\, S(\mathcal{Z})$.

(ii) $\mathcal{Z}$ *is p-integrably bounded if and only if* $S(\mathcal{Z})$ *is a bounded subset of* $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{R}^d)$.

(iii) If $\mathcal{Z}$ *is p-integrably bounded, then* $\mathrm{Int}[S(\mathcal{Z})] = \emptyset$ *and* $S(\mathcal{Z}) \neq \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{R}^r)$.

(iv) There exists a sequence $(z_n)_{n=1}^{\infty}$ *of d-dimensional random variables such that* $z_n(\omega) \in \mathcal{Z}(\omega)$ *and* $\mathcal{Z}(\omega) = \mathrm{cl}\{z_n(\omega) : n \geq 1\}$ *for* $n \geq 1$ *and* $\omega \in \Omega$. *If* $\{z_n : n \geq 1\} \subset S(\mathcal{Z})$, *then* $S(\mathcal{Z}) = \overline{\mathrm{dec}}\{z_n(\omega) : n \geq 1\}$.

(v) If $(z_n)_{n=1}^{\infty} \subset S(\mathcal{Z})$ *is such that* $\mathcal{Z}(\omega) = \mathrm{cl}\{z_n(\omega) : n \geq 1\}$ *for* $\omega \in \Omega$, *then for every* $z \in S(\mathcal{Z})$ *and every* $\varepsilon > 0$, *there exist a partition* $(A_k)_{k=1}^N \in \Pi(\Omega, \mathcal{F})$ *and a family* $(z_{n_k})_{k=1}^N \subset \{z_n : n \geq 1\}$ *such that* $E|z - \sum_{k=1}^N \mathbb{1}_{A_k} z_{n_k}| \leq \varepsilon$.

(vi) If F *and* G *are Aumann integrable set-valued random variables such that* $S(F) = S(G)$, *then* $F(\omega) = G(\omega)$ *for a.e.* $\omega \in \Omega$.

(vii) If $\mathcal{Z}$ *is convex-valued and square integrably bounded, then* $S(\mathcal{Z})$ *is a decomposable, convex, and weakly compact subset of* $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$.

(viii) If F *and* G *are convex-valued and integrably bounded set-valued random variables, then* $S(F + G) = S(F) + S(G)$.

A family $\Phi = (\Phi_t)_{t \geq 0}$ of set-valued random variables $\Phi_t : \Omega \to \mathrm{Cl}(\mathbb{R}^q)$ is called a set-valued stochastic process. Similarly as in the case of point-valued stochastic processes, a set-valued process $\Phi = (\Phi_t)_{t \geq 0}$ can also be defined as a set-valued mapping $\Phi : \mathbb{R}^+ \times \Omega \ni (t, \omega) \to \Phi_t(\omega) \in \mathrm{Cl}(\mathbb{R}^q)$ such that $\Phi(t, \cdot)$ is a set-valued random variable for every $t \geq 0$. If such a multifunction Φ is $\beta(\mathbb{R}^+) \otimes \mathcal{F}$-measurable, then a set-valued process Φ is said to be measurable. If furthermore, for every $t \geq 0$, the set-valued mapping Φ_t is $\mathcal{F}_t$-measurable, then Φ is said to be $\mathbb{F}$-nonanticipative. It is easy to see that Φ is $\mathbb{F}$-nonanticipative if and only if it is $\Sigma_{\mathbb{F}}$-measurable, where $\Sigma_{\mathbb{F}} = \{A \in \beta_T \otimes \mathcal{F} : A^t \in \mathcal{F}_t \text{ for } t \in T\}$, and A^t denotes the t-section of a set $A \subset T \times \Omega$. Given $p \geq 1$, we call a set-valued process $\Phi = (\Phi_t)_{t \geq 0}$ p-integrably bounded if there exists $m \in \mathbb{L}^p(\mathbb{R}^+ \times \Omega, \mathbb{R}^+)$ such that $\|\Phi_t(\omega)\| \leq m(t, \omega)$ for a.e. $(t, \omega) \in \mathbb{R}^+ \times \Omega$. A set-valued process $\Phi = (\Phi_t)_{t \geq 0}$ is said to be bounded if there exists a number $M > 0$ such that $\|\Phi_t(\omega)\| \leq M$ for a.e. $(t, \omega) \in \mathbb{R}^+ \times \Omega$. It is clear that every bounded set-valued process is p-integrably bounded for every $p \geq 1$. Similarly as above, by $S(\Phi)$ we denote the subtrajectory integrals of a set-valued stochastic process $\Phi : \mathbb{R}^+ \times \Omega \to \mathrm{Cl}(\mathbb{R}^q)$, i.e., the set of all measurable and $\mathrm{d}t \times P$-integrable selectors of Φ. By $S_{\mathbb{F}}(\Phi)$ we denote the subset of $S(\Phi)$ containing all $\mathbb{F}$-nonanticipative elements of $S(\Phi)$. If Φ is an p-integrably bounded set-valued process defined on $[0, T] \times \Omega$, its subtrajectory integrals will be denoted by $S(\Phi)$ for every $p \geq 1$. In this case, $S(\Phi) \subset \mathbb{L}^p([0, T] \times \Omega, \beta_T \otimes \mathcal{F}_T, \mathbb{R}^q)$. Similarly, if $\Phi : [0, T] \times \Omega \to \mathrm{Cl}(\mathbb{R}^q)$ is $\mathbb{F}$-nonanticipative and square integrably bounded, then $S_{\mathbb{F}}(\Phi) \subset \mathbb{L}^2([0, T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^q)$. Similarly as above, Φ is said to be Aumann

(Itô) integrable if $S(\Phi) \neq \emptyset$ ($S_{\mathbb{F}}(\Phi) \neq \emptyset$). We shall consider set-valued stochastic processes with $q = d$ and $q = d \times m$.

Let us denote by $\mathcal{M}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ and $\mathcal{M}_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ the spaces of all measurable and $\mathbb{F}$-nonanticipative, respectively set-valued stochastic, processes on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with values in $\mathrm{Cl}(\mathbb{R}^d)$. Similarly, the space of all $\mathbb{F}$-nonanticipative processes on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with values in $\mathrm{Cl}(\mathbb{R}^{d\times m})$ will be denoted by $\mathcal{M}_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^{d\times m})$. We denote by $\mathcal{L}^2(\Omega, \mathbb{R}^d)$ the space of all (equivalence classes of) set-valued random variables $\mathcal{Z} : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ such that $E\|\mathcal{Z}\|^2 < \infty$, where $\|\mathcal{Z}\|(\omega) = \sup\{|x| : x \in \mathcal{Z}(\omega)\}$ for a.e. $\omega \in \Omega$. Elements of the space $\mathcal{L}^2(\Omega, \mathbb{R}^d)$ are called $\mathbb{R}^d$-set-valued square integrably bounded random variables. We shall consider $\mathcal{L}^2(\Omega, \mathbb{R}^d)$ as a metric space with a metric H defined by $H(\mathcal{Z}_1, \mathcal{Z}_2) = [Eh^2(\mathcal{Z}_1(\cdot), \mathcal{Z}_2(\cdot))]^{1/2}$ for $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{L}^2(\Omega, \mathbb{R}^d)$. Similarly as in the case of $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{R}^d)$, it can be verified that $(\mathcal{L}^2(\Omega, \mathbb{R}^d), H)$ is a complete metric space. By $\mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ and $\mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^{d\times m})$ we shall denote the spaces of all square integrably bounded elements of spaces $\mathcal{M}_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ and $\mathcal{M}_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^{d\times m})$, respectively. Similarly as above, the spaces $\mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ and $\mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^{d\times m})$ will be considered metric spaces with metric d_H defined by $d_H(\Phi, \Psi) = [E\int_0^\infty h^2(\Phi_t, \Psi_t)\mathrm{d}t]^{1/2}$ for every $\Phi = (\Phi_t)_{t\geq 0}, \Psi = (\Psi_t)_{t\geq 0} \in \mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$ or $\Phi = (\Phi_t)_{t\geq 0}, \Psi = (\Psi_t)_{t\geq 0} \in \mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^{d\times m})$. It can be verified that $(\mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d), d_H)$ is a complete metric space. For fixed $T > 0$, we define $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d) = \{(\mathbb{1}_{[0,T]}\Phi_t)_{t\geq 0} : (\Phi_t)_{t\geq 0} \in \mathcal{L}^2_{\mathbb{F}}(\mathbb{R}^+ \times \Omega, \mathbb{R}^d)$. The space $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^{d\times m})$ is defined similarly. We shall consider $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ with the metric d_H, which in this case, is defined by $d_H(\Phi, \Psi) = [E\int_0^T h^2(\Phi_t, \Psi_t)\mathrm{d}t]^{1/2}$ for $\Phi, \Psi \in \mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$. We shall also consider spaces $\mathcal{L}^4_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ and $\mathcal{L}^4_{\mathbb{F}}(T, \Omega, \mathbb{R}^{d\times m})$, defined in a similar way. In what follows, stochastic processes Φ and Ψ belonging to $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ and $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^{d\times m})$ will be written as families $\Phi = (\Phi_t)_{0\leq t\leq T}$ and $\Psi = (\Psi_t)_{0\leq t\leq T}$, respectively. We shall also consider metric spaces $\mathrm{Cl}[\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)]$ and $\mathrm{Cl}[\mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})]$ with Hausdorff metrics denoted in both cases by D. Given a sequence $(F^n)_{n=1}^\infty$ of set-valued stochastic processes, $F^n = (F^n_t)_{0\leq t\leq T} \in \mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ is said to be uniformly integrably bounded if there exists $m \in \mathbb{L}^2([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^+)$ such that $\|F^n_t(\omega)\| \leq m_t(\omega)$ for $n \geq 1$ and a.e. $(t,\omega) \in [0,T] \times \Omega$. It is said to be uniformly integrable if

$$\lim_{C\to\infty} \sup_{n\geq 1} \int\int_{\{(t,\omega):\|F^n_t(\omega)\|>C\}} \|F^n_t(\omega)\|^2 \mathrm{d}t\mathrm{d}P = 0.$$

It is clear that every uniformly integrably bounded sequence $(F^n)_{n=1}^\infty$ of set-valued stochastic processes of $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ is also uniformly integrable. It is easy to see that every sequence $(\varphi^n)_{n=1}^\infty$ of $\mathbb{F}$-nonanticipative selectors φ^n of a uniformly integrable sequence $(F^n)_{n=1}^\infty \subset \mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ is uniformly integrable. Finally, let us observe that every sequence $(F^n)_{n=1}^\infty$ of set-valued stochastic processes of $\mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ converging in the d_H-metric topology to $F \in \mathcal{L}^2_{\mathbb{F}}(T, \Omega, \mathbb{R}^d)$ is uniformly integrable.

Lemma 4.2. *Let J and $\mathcal{J}$ be linear continuous mappings defined on $\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^d)$ and $\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^{d\times m})$, respectively, with values at $\mathbb{L}^2(\Omega,\mathcal{F}_T,\mathbb{R}^d)$. If $(\Phi^n)_{n=1}^\infty$ and $(\Psi^n)_{n=1}^\infty$ are sequences of $\mathcal{L}^2_{\mathbb{F}}(T,\Omega,\mathbb{R}^d)$ and $\mathcal{L}^2_{\mathbb{F}}(T,\Omega,\mathbb{R}^{d\times m})$ converging in the d_H-metric topology to $\Phi\in\mathcal{L}^2_{\mathbb{F}}(T,\Omega,\mathbb{R}^d)$, and $\Psi\in\mathcal{L}^2_{\mathbb{F}}(T,\Omega,\mathbb{R}^{d\times m})$, respectively, then*

(i) $\lim_{n\to\infty}[\max\{D(S_{\mathbb{F}}(\Phi^n),S_{\mathbb{F}}(\Phi)),D(S_{\mathbb{F}}(\Psi^n),S_{\mathbb{F}}(\Psi))\}]=0$*;*
(ii) $\lim_{n\to\infty}[\max\{H(J(S_{\mathbb{F}}(\Phi^n)),J(S_{\mathbb{F}}(\Phi)),H(\mathcal{J}(S_{\mathbb{F}}(\Psi^n)),\mathcal{J}(S_{\mathbb{F}}(\Psi)))\}]=0.$

Proof. By Theorem 3.1, for every $\varphi\in S_{\mathbb{F}}(\Phi^n))$, one has $E[\int_0^T\inf\{\|\varphi_t(\omega)-x\|^2 : x\in\Phi(t,\omega)\}\mathrm{d}t]=\inf\{E\int_0^T\|\varphi_t-f_t\|^2\mathrm{d}t : f\in S_{\mathbb{F}}(\Phi)\}=\mathrm{Dist}^2(\varphi,S_{\mathbb{F}}(\Phi))$. Similarly, for every $f\in S_{\mathbb{F}}(\Phi)$, we get $\mathrm{Dist}^2(f,S_{\mathbb{F}}(\Phi^n))=E\int_0^T\inf\{\|f_t(\omega)-x\|^2 : x\in\Phi_t^n(\omega)\}\mathrm{d}t$. Hence it follows that $D(S_{\mathbb{F}}(\Phi^n),S_{\mathbb{F}}(\Phi))\le d_H(\Phi^n,\Phi)$ for every $n\ge 1$, which implies $D(S_{\mathbb{F}}(\Phi^n),S_{\mathbb{F}}(\Phi))\to 0$ as $n\to\infty$. In a similar way, we also get $D(S_{\mathbb{F}}(\Psi^n),S_{\mathbb{F}}(\Psi))\to 0$ as $n\to\infty$.

It is easy to see that (ii) follows immediately from (i) and the properties of the mappings J and $\mathcal{J}$. Indeed, let us observe first that by (i), continuity of J and boundedness of $S_{\mathbb{F}}(\Phi)$ and $S_{\mathbb{F}}(\Phi_n))$, there exists $M>0$ such that $(E|J(\varphi)-J(\psi)|^2)^{1/2}\le M(\int_0^T E|\varphi-\psi|^2\mathrm{d}t)^{1/2}$ for $n\ge 1$, $\varphi\in S_{\mathbb{F}}(\Phi)$ and $\psi\in S_{\mathbb{F}}(\Phi_n))$. Suppose now that (ii) is not satisfied and let $A=J[S_{\mathbb{F}}(\Phi)]$ and $A_n=J[S_{\mathbb{F}}(\Phi_n)]$ for $n\ge 1$. There exist $\bar{\varepsilon}>0$ and an increasing subsequence $(n_k)_{k=1}^\infty$ of $(n)_{n=1}^\infty$ such that $\overline{H}(A_{n_k},A)>\bar{\varepsilon}$ for every $k\ge 1$. Hence it follows that for every $k\ge 1$, there exists $g^k\in A_{n_k}$ such that $\bar{\varepsilon}/2<(E|g^k-f|^2)^{1/2}$ for every $f\in A$. Let $\varphi^k\in S_{\mathbb{F}}(\Phi_{n_k})$ and $\phi\in S_{\mathbb{F}}(\Phi)$ be such that $g^k=J(\varphi^k)$ for $k\ge 1$ and $f=J(\phi)$. For every $k\ge 1$, one has

$$\bar{\varepsilon}/2<(E|g^k-f|^2)^{1/2}\le M\left(\int_0^T E|\varphi_t^k-\phi_t|^2\mathrm{d}t\right)^{1/2}.$$

By (i), it follows that for every $\varphi^k\in S(\Phi_{n_k})$, with $k\ge 1$ sufficiently large, there exists $\xi^k\in S_{\mathbb{F}}(\Phi)$ such that $(E\int_0^T|\varphi_t^k-\xi_t^k|^2\mathrm{d}t)^{1/2}\le\bar{\varepsilon}/2M$. Taking in particular $\phi=\xi^k$ with sufficiently large $k\ge 1$, we obtain

$$\bar{\varepsilon}/2<(E|g^k-f|^2)^{1/2}\le M\left(\int_0^T E|\varphi_t^k-\xi_t^k|^2\mathrm{d}t\right)^{1/2} M\cdot\bar{\varepsilon}/2M=\bar{\varepsilon}/2,$$

a contradiction. Then $H[J(S_{\mathbb{F}}(\Phi^n),J(S_{\mathbb{F}}(\Phi))]\to 0$ as $n\to\infty$. In a similar way, we also get $H[\mathcal{J}(S_{\mathbb{F}}(\Psi^n),\mathcal{J}(S_{\mathbb{F}}(\Psi))]\to 0$ as $n\to\infty$. □

Remark 4.1. If $J(\varphi)=\int_0^T\varphi_t\mathrm{d}t$ and $\mathcal{J}(\psi)=\int_0^T\psi_t\mathrm{d}B_t$ for $\varphi\in\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^d)$ and $\psi\in\mathbb{L}^2([0,T]\times\Omega,\Sigma_{\mathbb{F}},\mathbb{R}^{d\times m})$, then $\overline{H}(J[S_{\mathbb{F}}(\Phi^n)],J[S_{\mathbb{F}}(\Phi)])\le\sqrt{T}d_H(\Phi^n,\Phi)$ and $\overline{H}(\mathcal{J}[S_{\mathbb{F}}(\Psi^n)],\mathcal{J}[S_{\mathbb{F}}(\Psi)])\le d_H(\Psi^n,\Psi)$ for every $n\ge 1$.

Proof. For every $u\in J[S_{\mathbb{F}}(\Phi^n)]$, one has $\mathrm{dist}^2(u,J[S_{\mathbb{F}}(\Phi)])\le E|u-v|^2$ for every $v\in J[S_{\mathbb{F}}(\Phi)]$. But $u=\int_0^T\varphi_t\mathrm{d}t$ and $v=\int_0^T\psi_t\mathrm{d}t$ for some $\varphi\in S_{\mathbb{F}}(\Phi^n)$

and $\psi \in S_{\mathbb{F}}(\Phi)$. Therefore, $\mathrm{dist}^2(u, J[S_{\mathbb{F}}(\Phi)]) \le E|\int_0^T(\varphi - \psi)\mathrm{d}t|^2$ for every $\psi \in S_{\mathbb{F}}(\Phi)$. By Theorem 3.1, we have

$$\inf\left\{E\left|\int_0^T \varphi_t \mathrm{d}t - \int_0^T f t \mathrm{d}t\right|^2 : f \in S_{\mathbb{F}}(\Phi)\right\}$$
$$\le T\inf\left\{E\int_0^T |\varphi_t - f_t|^2 \mathrm{d}t : f \in S_{\mathbb{F}}(\Phi)\right\}$$
$$= TE\int_0^T \mathrm{dist}^2(\varphi_t, \Phi_t)\mathrm{d}t \le T d_H^2(\Phi^n, \Phi).$$

□

Thus $\mathrm{dist}^2(u, J[S_{\mathbb{F}}(\Phi)]) \le T d_H^2(\Phi^n, \Phi)$ for every $u \in J[S_{\mathbb{F}}(\Phi^n)]$, which implies that $\overline{H}(J[S_{\mathbb{F}}(\Phi^n)], J[S_{\mathbb{F}}(\Phi)]) \le \sqrt{T} d_H(\Phi^n, \Phi)$ for $n \ge 1$. In a similar way, we also get $\overline{H}(\mathcal{J}[S_{\mathbb{F}}(\Psi^n)], \mathcal{J}[S_{\mathbb{F}}(\Psi)]) \le d_H(\Psi^n, \Psi)$ for every $n \ge 1$.

In what follows, we shall deal with a conditional expectation of set-valued integrals depending on a random parameter. We begin with the general definition of set-valued conditional expectation and its basic properties. Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a sub-$\sigma$-algebra $\mathcal{G}$ of $\mathcal{F}$, and a set-valued random variable $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ the following result follows immediately from Theorem 3.2.

Lemma 4.3. *If $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ is a set-valued random variable such that $S(\Phi) \ne \emptyset$, then there exists a unique in the a.s. sense $\mathcal{G}$-measurable set-valued random variable $\Psi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ such that $S(\Psi) = \mathrm{cl}_{\mathbb{L}}\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$.*

Proof. Let $A \in \mathcal{G} \subset \mathcal{F}$ and $\mathcal{H} = \{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$. For every $\psi_1, \psi_2 \in \mathcal{H}$, there exist $\varphi_1, \varphi_2 \in S(\Phi)$ such that $\psi_1 = E[\varphi_1|\mathcal{G}]$ and $\psi_2 = E[\varphi_2|\mathcal{G}]$. By the decomposability of $S(\Phi)$, it follows that $E[\mathbb{1}_A\varphi_1 + \mathbb{1}_{\Omega\setminus A}\varphi_2|\mathcal{G}] \in \mathcal{H}$. Then $\mathcal{H}$ is decomposable, because $E[\mathbb{1}_A\varphi_1 + \mathbb{1}_{\Omega\setminus A}\varphi_2|\mathcal{G}] = \mathbb{1}_A\psi_1 + \mathbb{1}_{\Omega\setminus A}\psi_2$. Therefore, $\mathrm{cl}_{\mathbb{L}}(\mathcal{H})$ is a decomposable subset of $\mathbb{L}^p(\Omega, \mathcal{G}, \mathbb{R}^d)$. By virtue of Theorem 3.2, there exists a $\mathcal{G}$-measurable set-valued mapping $\Psi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ such that $S(\Psi) = \mathrm{cl}_{\mathbb{L}}(\mathcal{H})$. Suppose there are two $\mathcal{G}$-measurable mappings $\Psi_1, \Psi_2 : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ such that $S(\Psi_1) = S(\Psi_2) = \mathrm{cl}_{\mathbb{L}}(\mathcal{H})$. By Corollary 3.1, it follows that $\Psi_1 = \Psi_2$ a.s. □

A $\mathcal{G}$-measurable set-valued mapping $\Psi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ such that $S(\Psi) = \mathrm{cl}_{\mathbb{L}}\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$ is denoted by $E[\Phi|\mathcal{G}]$ and is said to be a $\mathcal{G}$-conditional expectation of a set-valued mapping of $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$. Let us observe that for every square integrably bounded convex-valued set-valued random variable $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$, the set $S(\Phi)$ is a convex and weakly compact subset of $\mathbb{L}^2(\Omega, \mathbb{R}^d)$. Then $\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$ is a closed subset of this space. Indeed, for every $u \in \mathrm{cl}_{\mathbb{L}}\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$, there is a sequence $(\varphi_n)_{n=1}^{\infty} \subset S(\Phi)$ such that $E[\varphi_n|\mathcal{G}] \to u$ as $n \to \infty$. Let $(\varphi_{n_k})_{k=1}^{\infty}$ be a subsequence of $(\varphi_n)_{n=1}^{\infty}$ weakly converging to $\varphi \in S(\Phi)$. Therefore, for every $A \in \mathcal{G}$, one has $\int_A E[\varphi_{n_k}|\mathcal{G}]\mathrm{d}P = \int_A \varphi_{n_k}\mathrm{d}P \to \int_A \varphi\mathrm{d}P = \int_A E[\varphi|\mathcal{G}]\mathrm{d}P$ as $k \to \infty$. Then $E[\varphi_{n_k}|\mathcal{G}]$ converges weakly to $E[\varphi|\mathcal{G}]$ as $k \to \infty$, which implies that $u = E[\varphi|\mathcal{G}] \in \{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$.

Corollary 4.1. *If $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ is a square integrably bounded convex-valued set-valued random variable, then $S(E[\Phi|\mathcal{G}]) = \{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}$.* □

Theorem 4.1. *Let $\Phi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ and $\Psi : \Omega \to \mathrm{Cl}(\mathbb{R}^d)$ be $\mathcal{F}$-measurable integrably bounded and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then*

(i) $E[\mathbb{1}_A E[\Phi|\mathcal{G}]] = E[\mathbb{1}_A \Phi]$ *for every* $A \in \mathcal{G}$.
(ii) $E[\xi\Phi|\mathcal{G}] = \xi E[\Phi|\mathcal{G}]$ *for every* $\xi \in \mathbb{L}^\infty(\Omega, \mathcal{G}, \mathbb{R})$.
(iii) $E[\overline{\mathrm{co}}\,\Phi|\mathcal{G}] = \overline{\mathrm{co}}\,E[\Phi|\mathcal{G}]$.
(iv) $H(E[\Phi|\mathcal{G}], E[\Psi|\mathcal{G}]) \le H(\Phi, \Psi)$, *where* $H(\Phi, \Psi) = E[h(\Phi, \Psi)]$.
(v) $E[\overline{\Phi + \Psi}|\mathcal{G}] = \overline{E[\Phi|\mathcal{G}] + E[\Psi|\mathcal{G}]}$ *a.s.*

Proof. (i) Let $A \in \mathcal{G}$ be fixed. If $u \in S(E[\Phi|\mathcal{G}])$, then there exists a sequence $(\varphi_n)_{n=1}^\infty$ in $S(\Phi)$ such that $\|u - E[\varphi_n|\mathcal{G}]\| \to 0$ as $n \to \infty$. Then $E[\mathbb{1}_A u] = \lim_{n\to\infty} E[\mathbb{1}_A E[\varphi_n|\mathcal{G}]] = \lim_{n\to\infty} E[\mathbb{1}_A \varphi_n]$. Hence by the compactness of the Aumann integral $E[\mathbb{1}_A \Phi]$, it follows that $E[\mathbb{1}_A u] \in E[\mathbb{1}_A \Phi]$. Thus $E[\mathbb{1}_A E[\Phi|\mathcal{G}]] \subset E[\mathbb{1}_A \Phi]$. Let $\mathcal{H} = \{E[\varphi|\mathcal{G}] : \varphi \in S(\phi)\}$. Then $E[\mathbb{1}_A \mathcal{H}] = \{E[\mathbb{1}_A E[\varphi|\mathcal{G}] : \varphi \in S(\phi)\} = E[\mathbb{1}_A \Phi]$. Hence it follows that $E[\mathbb{1}_A \Phi] \subset E[\mathbb{1}_A \mathrm{cl}_{\mathbb{L}}(\mathcal{H})] = E[\mathbb{1}_A E[\Phi|\mathcal{G}]]$. Therefore, $E[\mathbb{1}_A E[\Phi|\mathcal{G}]] = E[\mathbb{1}_A \Phi]$ for every $A \in \mathcal{G}$.

(ii) Let $\xi \in \mathbb{L}^\infty(\Omega, \mathcal{G}, \mathbb{R})$. We have to show that $S(E[\xi\Phi|\mathcal{G}]) = S(\xi E[\Phi|\mathcal{G}])$. By the definition of a set-valued conditional expectation, we have $S(E[\xi\Phi|\mathcal{G}]) = \mathrm{cl}_{\mathbb{L}}(\{E[f|\mathcal{G}] : f \in S(\xi\Phi)\})$ and $S(\xi E[\Phi|\mathcal{G}]) = \xi S(E[\Phi|\mathcal{G}]) = \xi \mathrm{cl}_{\mathbb{L}}(\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\})$. Let $u \in \xi\mathrm{cl}_{\mathbb{L}}(\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\})$ and $(\varphi_n)_{n=1}^\infty$ be a sequence of $S(\Phi)$ such that $\|\xi E[\varphi_n|\mathcal{G}] - u\| \to 0$ as $n \to \infty$. But $\xi E[\varphi_n|\mathcal{G}] = E[\xi\varphi_n|\mathcal{G}]$ for $n \ge 1$. Then $\|E[\xi\varphi|\mathcal{G}] - u\| \to 0$ as $n \to \infty$. We also have $\xi\varphi_n \in S(\xi\Phi)$ for $n \ge 1$. Therefore, $E[\xi\varphi_n|\mathcal{G}] \in \{E[f|\mathcal{G}] : f \in S(\xi\Phi)\}$ for $n \ge 1$, which implies that $u \in \mathrm{cl}_{\mathbb{L}}\{E[f|\mathcal{G}] : f \in S(\xi\Phi)\}$. Thus

$$\xi\mathrm{cl}_{\mathbb{L}}(\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\}) \subset \mathrm{cl}_{\mathbb{L}}(\{E[f|\mathcal{G}] : f \in S(\xi\Phi)\}).$$

Let $v \in \mathrm{cl}_{\mathbb{L}}\{E[f|\mathcal{G}] : f \in S(\xi\Phi)\}$ and $(\varphi_n)_{n=1}^\infty \subset S(\Phi)$ be such that $\|E[\xi\varphi_n|\mathcal{G}] - v\| \to 0$ as $n \to \infty$. Hence it follows that $\|\xi E[\varphi_n|\mathcal{G}] - v\| \to 0$ as $n \to \infty$. Similarly as above, we get $\xi E[\varphi_n|\mathcal{G}] \in \xi\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\} \subset \xi\mathrm{cl}_{\mathbb{L}}(\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\})$ for every $n \ge 1$. Therefore, $v \in \xi\mathrm{cl}_{\mathbb{L}}(\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\})$. Then $\mathrm{cl}_{\mathbb{L}}(\{E[f|\mathcal{G}] : f \in S(\xi\Phi)\}) \subset \mathrm{cl}_{\mathbb{L}}(\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\})$, which implies that $S(E[\xi\Phi|\mathcal{G}]) = S(\xi E[\Phi|\mathcal{G}])$.

(iii) Let $G = E[\Phi|\mathcal{G}]$. By Lemma 3.3, we obtain $S(E[\overline{\mathrm{co}}\,\Phi|\mathcal{G}]) = \mathrm{cl}_{\mathbb{L}}\{E[\varphi|\mathcal{G}] : \varphi \in \overline{\mathrm{co}}\,S(\Phi)\} = \overline{\mathrm{co}}\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\} = \overline{\mathrm{co}}\,S(G) = S(\overline{\mathrm{co}}\,G)$. Hence, by Corollary 3.1, it follows $E[\overline{\mathrm{co}}\,\Phi|\mathcal{G}] = \overline{\mathrm{co}}\,E[\Phi|\mathcal{G}]$.

(iv) Let $A = \{\omega \in \Omega : \sup[\text{dist}(y, E[\Psi|\mathcal{G}](\omega)) : y \in E[\Phi|\mathcal{G}](\omega)] \geq \sup[\text{dist}(y, E[\Phi|\mathcal{G}](\omega)) : y \in E[\Psi|\mathcal{G}](\omega)]\}$. We have $A \in \mathcal{G}$ and

$$
\begin{aligned}
H(E[\Phi|\mathcal{G}], E[\Psi|\mathcal{G}]) &= E[h(E[\Phi|\mathcal{G}], E[\Psi|\mathcal{G}])] \\
&= E[\mathbb{1}_A \sup[\text{dist}(y, E[\Psi|\mathcal{G}](\omega)) : y \in E[\Phi|\mathcal{G}](\omega)] \\
&\quad + E[\mathbb{1}_{\Omega\setminus A} \sup[\text{dist}(E[y, E[\Phi|\mathcal{G}](\omega)) : y \in E[\Psi|\mathcal{G}](\omega)] \\
&= \sup E[\mathbb{1}_A \sup[\text{dist}(E[\varphi|\mathcal{G}], E[\Psi|\mathcal{G}]) : \varphi \in S(\Phi)] \\
&\quad + E[\mathbb{1}_{\Omega\setminus A} \sup[\text{dist}(E[\psi|\mathcal{G}], E[\Phi|\mathcal{G}]) : \psi \in S(\Psi)] \\
&\leq \sup_{\varphi\in S(\Phi)} \inf_{\psi\in S(\Psi)} E[\mathbb{1}_A E[|\varphi - \psi||\mathcal{G}]] \\
&\quad + \sup_{\psi\in S(\Psi)} \inf_{\varphi\in S(\Phi)} E[\mathbb{1}_{\Omega\setminus A} E[|\varphi - \psi||\mathcal{G}]] \\
&= \sup_{\varphi\in S(\Phi)} \inf_{\psi\in S(\Psi)} E[\mathbb{1}_A |\varphi - \psi|] \\
&\quad + \sup_{\psi\in S(\Psi)} \inf_{\varphi\in S(\Phi)} E[\mathbb{1}_{\Omega\setminus A} |\varphi - \psi|] \\
&= \int_A \sup[\text{dist}(x, \Psi(\omega)) : x \in \Phi(\omega)] d\mathbb{P} \\
&\quad + \int_{\Omega\setminus A} \sup[\text{dist}(x, \Phi(\omega)) : x \in \Psi(\omega)] d\mathbb{P} \\
&= \int_\Omega h(\Phi(\omega), \Psi(\omega)) d\mathbb{P} = H(\Phi, \Psi).
\end{aligned}
$$

(v) By the definition of a multivalued conditional expectation, we have $S(E[\overline{\Phi + \Psi}|\mathcal{G}]) = \text{cl}_{\mathbb{L}}\{E[g|\mathcal{G}] : g \in S(\overline{\Phi + \Psi})$. By virtue of Lemma 3.4, we have

$$
\begin{aligned}
S(E[\overline{\Phi + \Psi}|\mathcal{G}]) &= \text{cl}_{\mathbb{L}}(\{E[\phi|\mathcal{G}] + E[\psi|\mathcal{G}] : \phi \in S(\Phi), \psi \in S(\Psi)\}) \\
&= \overline{S(E[\Phi|\mathcal{G}]) + S(E[\Psi|\mathcal{G}])} = S(\overline{E[\Phi|\mathcal{G}] + E[\Psi|\mathcal{G}]}),
\end{aligned}
$$

which by Corollary 3.1, implies that $E[\overline{\Phi + \Psi}|\mathcal{G}] = \overline{E[\Phi|\mathcal{G}] + E[\Psi|\mathcal{G}]}$ a.s. □

Remark 4.2. It can be proved that if $\Phi \in \mathcal{A}(\Omega, \mathcal{F}, \mathbb{R}^d)$ is convex-valued and $\mathcal{T}$ is sub-σ-algebra of $\mathcal{G} \subset \mathcal{F}$, then $E[\Phi|\mathcal{T}]$ taken on the base space $(\Omega, \mathcal{F}, P)$ is equal to $E[\Phi|\mathcal{T}]$ taken on the base space $(\Omega, \mathcal{G}, P)$ and $E[E[\Phi|\mathcal{G}]|\mathcal{T}] = E[\Phi|\mathcal{T}]$, P-a.s. □

5 Notes and Remarks

The definitions and results of the first two sections of this chapter are mainly based on Aubin and Frankowska [12], Hu and Papageorgiou [41], Aubin and Cellina [5], Kisielewicz [49], Kuratowski [69], Hildenbrand [40] and Klein, and Thomson [63]. In particular, Michael's continuous selection theorem is taken from Aubin and Cellina [5] and Kisielewicz [49], whereas Theorem 2.2 comes from Kisielewicz [57]. The proofs of the Kuratowski and Ryll-Nardzewski measurable selection theorem and the Carathèodory selection theorem are taken from Hu and Papageorgiou [41]. The existence of measurable selectors for measurable multifunctions has been considered first by Kuratowski and Ryll-Nardzewski in [70]. The existence of Carathéodory selections has been considered by Rybiński in [91], Fryszkowski in [32], and Kucia and Nowak in [66]. The proof of Theorem 2.3, dealing with the existence of Lipschitz-type selectors, is taken from Hu and Papageorgiou [41]. The idea of this proof is due to Przesławski [90]. The proofs of Lemmas 1.1 and 1.2, Remark 1.1, and Corollary 1.2 can be found in Kuratowski [69] and Hildenbrand [40], respectively. Figures 2.1–2.4 are taken from Aubin and Cellina [5] and Kisielewicz [49]. The proof of Remark 2.9 can be found in Hu and Papageorgiou [41]. The definition and properties of Aumann integrals are taken from Hiai and Umegaki [39] and Kisielewicz [49]. The first results dealing with Aumann integrals are due to Aumann [14]. The existence of continuous selections of multifunctions with decomposable values was proved by Fryszkowski [32]. The sketch of the proof of this theorem given in Sect. 2 is taken from Hu and Papageorgiou [41]. The definition and properties of conditional expectation of set-valued mappings are taken from Hiai and Umegaki [39]. More information on the Hukuhara difference can be found in Hukuhara [42].

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