Chapter 2
Fractional Part Integrals

If I had the theorems! Then I should find the proofs easily enough.

Bernhard Riemann (1826–1866)

2.1 Single Integrals

I have had my results for a long time: but I do not yet know how I am to arrive at them.

Carl Friedrich Gauss (1777–1855)

2.1. A de la Vallée Poussin integral. Calculate

$$\int_0^1 \left\{\frac{1}{x}\right\} \mathrm{d}x.$$

2.2. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 \mathrm{d}x = \ln(2\pi) - \gamma - 1.$$

Let A denote the Glaisher–Kinkelin constant defined by the limit

$$A = \lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.28242\,71291\,00622\,63687\ldots.$$

(continued)

O. Furdui, *Limits, Series, and Fractional Part Integrals: Problems in Mathematical Analysis*, Problem Books in Mathematics, DOI 10.1007/978-1-4614-6762-5_2,

(continued)

2.3. A cubic integral. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\}^3 \mathrm{d}x = -\frac{1}{2} - \gamma + \frac{3}{2}\ln(2\pi) - 6\ln A.$$

2.4. Let $k \geq 0$ be an integer. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\}^k \mathrm{d}x = \sum_{p=1}^{\infty} \frac{\zeta(p+1)-1}{\binom{k+p}{p}}.$$

2.5. (a) Let $k \geq 1$ be an integer. Prove that

$$\int_0^1 \left\{\frac{k}{x}\right\} \mathrm{d}x = k\left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln k - \gamma\right),$$

where γ denotes the Euler–Mascheroni constant.
(b) More generally, if q is a positive real number, then

$$\int_0^1 \left\{\frac{q}{x}\right\} \mathrm{d}x = \begin{cases} q(1-\gamma-\ln q) & \text{if } q \leq 1, \\ q\left(1 + \frac{1}{2} + \cdots + \frac{1}{1+\lfloor q \rfloor} - \gamma - \ln q + \frac{\lfloor q \rfloor(\{q\}-1)}{q(1+\lfloor q \rfloor)}\right) & \text{if } q > 1. \end{cases}$$

2.6. Let $k \geq 1$ be an integer. Prove that

$$\int_0^1 \left\{\frac{k}{x}\right\}^2 \mathrm{d}x = k\left(\ln(2\pi) - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + 2k\ln k - 2k - 2\ln k!\right).$$

2.7. Let $k \geq 2$ be an integer. Prove that

$$\int_0^1 \left\{\frac{1}{\sqrt[k]{x}}\right\} \mathrm{d}x = \frac{k}{k-1} - \zeta(k),$$

where ζ denotes the Riemann zeta function.

2.8. Let $k \geq 2$ be an integer. Prove that

$$\int_0^1 \left\{ \frac{k}{\sqrt[k]{x}} \right\} \mathrm{d}x = \frac{k}{k-1} - k^k \left(\zeta(k) - \frac{1}{1^k} - \frac{1}{2^k} - \cdots - \frac{1}{k^k} \right).$$

2.9. Let $k \geq 2$ be an integer. Prove that

$$\int_0^1 \left\{ \frac{1}{k\sqrt[k]{x}} \right\} \mathrm{d}x = \frac{1}{k-1} - \frac{\zeta(k)}{k^k}.$$

2.10. A Euler's constant integral. Prove that

$$\int_0^1 \left\{ \frac{1}{x} \right\} \left\{ \frac{1}{1-x} \right\} \mathrm{d}x = 2\gamma - 1.$$

2.11. (a) **A quadratic integral and Euler's constant.** Prove that

$$\int_0^1 \left\{ \frac{1}{x} \right\}^2 \left\{ \frac{1}{1-x} \right\}^2 \mathrm{d}x = 4\ln(2\pi) - 4\gamma - 5.$$

(b) **A class of fractional part integrals.** Let $n \geq 2$ be an integer. Then

$$\int_0^1 \left\{ \frac{1}{x} \right\}^n \left\{ \frac{1}{1-x} \right\}^n \mathrm{d}x = 2\sum_{j=2}^{n-1} (-1)^{n+j-1}(\zeta(j) - 1) + (-1)^n$$
$$- 2n \sum_{m=0}^{\infty} \frac{\zeta(2m+n) - \zeta(2m+n+1)}{n+m},$$

where ζ is the Riemann zeta function and the first sum is missing when $n = 2$.

2.12. Prove that

$$\int_0^1 \left\{ \frac{1}{x} \right\}^2 \left\{ \frac{1}{1-x} \right\} \mathrm{d}x = \frac{5}{2} - \gamma - \ln(2\pi).$$

2.13. Prove that

$$\int_0^1 \left\{ (-1)^{\lfloor \frac{1}{x} \rfloor} \frac{1}{x} \right\} \mathrm{d}x = 1 + \ln\frac{2}{\pi},$$

where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a.

2.14. (a) Prove that

$$\int_0^1 x\left\{\frac{1}{x}\right\}\left\lfloor\frac{1}{x}\right\rfloor \mathrm{d}x = \frac{\pi^2}{12} - \frac{1}{2}.$$

(b) More generally, let α, β, and γ be positive real numbers. Study the convergence of

$$\int_0^1 x^\alpha \left\{\frac{1}{x}\right\}^\beta \left\lfloor\frac{1}{x}\right\rfloor^\gamma \mathrm{d}x,$$

where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a.

2.15. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\} \frac{x}{1-x} \mathrm{d}x = \gamma.$$

2.16. Let $m > -1$ be a real number. Prove that

$$\int_0^1 \{\ln x\} x^m \mathrm{d}x = \frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2}.$$

2.17. The first Stieltjes constant, γ_1, is the special constant defined by

$$\gamma_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln k}{k} - \frac{\ln^2 n}{2}\right).$$

Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\} \ln x \mathrm{d}x = \gamma + \gamma_1 - 1.$$

2.18. Let k be a positive real number. Find the value of

$$\lim_{n\to\infty}\int_0^1 \left\{\frac{n}{x}\right\}^k \mathrm{d}x.$$

2.19. Calculate

$$L = \lim_{n\to\infty}\int_0^1 \left\{\frac{n}{x}\right\}^n \mathrm{d}x \quad \text{and} \quad \lim_{n\to\infty} n\left(\int_0^1 \left\{\frac{n}{x}\right\}^n \mathrm{d}x - L\right).$$

2.20. Let $m > 0$ be a real number. Calculate

$$\int_0^1 t^m \left\{\frac{1}{t}\right\} \mathrm{d}t.$$

2.21. Let $m \ge 1$ be an integer. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\}^m x^m \mathrm{d}x = 1 - \frac{\zeta(2)+\zeta(3)+\cdots+\zeta(m+1)}{m+1}.$$

2.22. (a) Let m and k be positive real numbers. Prove that

$$\int_0^1 \left\{\frac{1}{x}\right\}^k x^m \mathrm{d}x = \frac{k!}{(m+1)!}\sum_{j=1}^{\infty}\frac{(m+j)!}{(k+j)!}\left(\zeta(m+j+1)-1\right).$$

(b) **A new integral formula for Euler's constant.** If $m \ge 1$ is an integer, then

$$\int_0^1 x^m \left\{\frac{1}{x}\right\}^{m+1} \mathrm{d}x = H_{m+1} - \gamma - \sum_{j=2}^{m+1}\frac{\zeta(j)}{j},$$

where H_{m+1} denotes the $(m+1)$th harmonic number.

2.2 Double Integrals

Nature laughs at the difficulties of integration.

Pierre-Simon de Laplace (1749–1827)

2.23. (a) Calculate

$$\int_0^1\int_0^1 x\left\{\frac{1}{1-xy}\right\}\mathrm{d}x\mathrm{d}y.$$

(b) More generally, if $k \ge 1$ is an integer, calculate

$$\int_0^1\int_0^1 x^k\left\{\frac{1}{1-xy}\right\}\mathrm{d}x\mathrm{d}y.$$

2.24. Let $m \ge 1$ be an integer. Calculate

$$\int_0^1\int_0^1 \left\{\frac{1}{x+y}\right\}^m \mathrm{d}x\mathrm{d}y.$$

2.25. Let $k \geq 1$ be an integer. Calculate

$$\int_0^1 \int_0^{1-x} \frac{\mathrm{d}x\mathrm{d}y}{\left(\left\lfloor \frac{x}{y} \right\rfloor + 1\right)^k},$$

where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a.

2.26. Let $k \geq 1$ be an integer. Calculate

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{\left(\left\lfloor \frac{x}{y} \right\rfloor + 1\right)^k},$$

where $\lfloor a \rfloor$ denotes the greatest integer not exceeding a.

2.27. Calculate

$$\int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\} \mathrm{d}x\mathrm{d}y.$$

2.28. Let $k \geq 1$ be an integer. Prove that

$$\int_0^1 \int_0^1 \left\{ k\frac{x}{y} \right\} \mathrm{d}x\mathrm{d}y = \frac{k}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln k - \gamma\right) + \frac{1}{4}.$$

2.29. Let m and n be positive integers such that $m \leq n$. Prove that

$$\int_0^1 \int_0^1 \left\{ \frac{mx}{ny} \right\} \mathrm{d}x\mathrm{d}y = \frac{m}{2n}\left(\ln \frac{n}{m} + \frac{3}{2} - \gamma\right).$$

2.30. Let $k \geq 0$ be a real number. Prove that

$$\int_0^1 \int_0^1 \left\{ \frac{x^k}{y} \right\} \mathrm{d}x\mathrm{d}y = \frac{2k+1}{(k+1)^2} - \frac{\gamma}{k+1}.$$

2.31. Prove that

$$\int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\}^2 \mathrm{d}x\mathrm{d}y = -\frac{1}{3} - \frac{\gamma}{2} + \frac{\ln(2\pi)}{2}.$$

2.32. Let $k \geq 0$ be an integer. Prove that

$$\int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\}^k \mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 \left\{ \frac{1}{x} \right\}^k \mathrm{d}x + \frac{1}{2(k+1)} = \frac{1}{2}\sum_{p=1}^{\infty} \frac{\zeta(p+1)-1}{\binom{k+p}{p}} + \frac{1}{2(k+1)}.$$

2.33. Let $k \geq 1$ be an integer. Prove that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \left(\frac{y}{x}\right)^k \mathrm{d}x\mathrm{d}y = 1 - \frac{\zeta(2)+\zeta(3)+\cdots+\zeta(k+1)}{2(k+1)}.$$

2.34. Let $k \geq 1$ be an integer and let p be a real number such that $k-p > -1$. Prove that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \frac{y^k}{x^p} \mathrm{d}x\mathrm{d}y = \frac{1}{k-p+1} - \frac{\zeta(2)+\zeta(3)+\cdots+\zeta(k+1)}{(k+2-p)(k+1)}.$$

2.35. Let $k \geq 1$ and let m, p be nonnegative integers such that $m-p > -2$ and $k-p > -1$. Prove that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \frac{y^m}{x^p} \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{m-p+2}\left(\frac{k!}{(m+1)!}\sum_{j=1}^{\infty}\frac{(m+j)!}{(k+j)!}(\zeta(m+j+1)-1)+\frac{1}{k-p+1}\right).$$

2.36. A surprising appearance of $\zeta(2)$. Prove that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} \mathrm{d}x\mathrm{d}y = 1 - \frac{\pi^2}{12}.$$

2.37. Let $n, m > -1$ be real numbers. Prove that

$$\int_0^1 \int_0^1 x^m y^n \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{m+n+2}\left(\frac{1}{n+1}+\frac{1}{m+1}-\frac{\zeta(n+2)}{n+2}-\frac{\zeta(m+2)}{m+2}\right).$$

2.38. Let $n > -1$ be a real number. Prove that

$$\int_0^1 \int_0^1 x^n y^n \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} \mathrm{d}x\mathrm{d}y = \frac{1}{(n+1)^2} - \frac{\zeta(n+2)}{(n+1)(n+2)}.$$

2.39. Let $k \geq 1$ be an integer and let

$$I_k = \int_0^1 \int_0^1 \left\{k\frac{x}{y}\right\}\left\{k\frac{y}{x}\right\} \mathrm{d}x\mathrm{d}y.$$

Prove that

(a) $I_k = \int_0^1 \{kx\}\{k/x\}\,\mathrm{d}x$.
(b) $I_2 = 49/6 - 2\pi^2/3 - 2\ln 2$.
(c) **Open problem.** Find an explicit formula for I_k when $k \geq 3$.

2.40. Let $m \geq 1$ be an integer. Prove that

$$\int_0^1\int_0^1 \left\{\frac{x}{y}\right\}^m \left\{\frac{y}{x}\right\}^m \mathrm{d}x\mathrm{d}y = 1 - \frac{\zeta(2)+\zeta(3)+\cdots+\zeta(m+1)}{m+1}.$$

2.41. Let $m, k \geq 1$ be two integers. Prove that

$$\int_0^1\int_0^1 \left\{\frac{x}{y}\right\}^m \left\{\frac{y}{x}\right\}^k \mathrm{d}x\mathrm{d}y = \frac{k!}{2(m+1)!}\sum_{p=1}^{\infty}\frac{(m+p)!}{(k+p)!}(\zeta(m+p+1)-1)$$
$$+\frac{m!}{2(k+1)!}\sum_{p=1}^{\infty}\frac{(k+p)!}{(m+p)!}(\zeta(k+p+1)-1).$$

2.42. 1. Let $a > 0$ be a real number and let k be a positive integer. Prove that

$$\int_a^{a+k}\{x\}\,\mathrm{d}x = \frac{k}{2}.$$

2. Let $n \geq 1$ be an integer and let $a_1, \ldots, a_n$ be positive integers. Calculate

$$\int_0^{a_1}\cdots\int_0^{a_n}\{k(x_1+x_2+\cdots+x_n)\}\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

The Stieltjes constants, γ_m, are the special constants defined by

$$\gamma_m = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{(\ln k)^m}{k} - \frac{(\ln n)^{m+1}}{m+1}\right).$$

2.43. A multiple integral in terms of Stieltjes constants. Let $m \geq 1$ be an integer. Prove that

$$\int_0^1\cdots\int_0^1\left\{\frac{1}{x_1x_2\cdots x_m}\right\}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_m = 1 - \sum_{k=0}^{m-1}\frac{\gamma_k}{k!}.$$

2.3 Quickies

In my opinion, a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy, an opinion, moreover, which has been expressed by many philosophers.

Henri Lebesgue (1875–1941)

2.44. Let $k > -1$ be a real number and let $n \geq 1$ be an integer. Prove that

$$\int_0^1 \{nx\}^k \,\mathrm{d}x = \frac{1}{k+1}.$$

2.45. Calculate

$$\int_0^1 \left\{\frac{1}{x} - \frac{1}{1-x}\right\} \mathrm{d}x.$$

2.46. Calculate

$$\int_0^1 \left\{\frac{1}{x} - \frac{1}{1-x}\right\} \left\{\frac{1}{x}\right\} \left\{\frac{1}{1-x}\right\} \mathrm{d}x.$$

2.47. Let $k > 0$ and let $m \geq 0$ be real numbers. Calculate

$$\int_0^1 \left\{\left(\frac{1}{x}\right)^k - \left(\frac{1}{1-x}\right)^k\right\} x^m(1-x)^m \mathrm{d}x.$$

2.48. Let n be a positive integer and let k be a natural number. Prove that

$$\int_0^1 (x-x^2)^k \{nx\} \,\mathrm{d}x = \frac{(k!)^2}{2(2k+1)!}.$$

2.49. Let f and g be functions on $[0,1]$ with g integrable and $g(x) = g(1-x)$. Prove that

$$\int_0^1 \{f(x) - f(1-x)\} \, g(x) \mathrm{d}x = \frac{1}{2} \int_0^1 g(x) \mathrm{d}x,$$

where $\{a\}$ denotes the fractional part of a.

2.50. Calculate the double integral

$$\int_0^1 \int_0^1 \{x-y\} \,\mathrm{d}x\mathrm{d}y.$$

2.51. Prove that

$$\int_0^1 \int_0^1 \left\{\frac{x-y}{x+y}\right\} \mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \left\{\frac{x+y}{x-y}\right\} \mathrm{d}x\mathrm{d}y = \frac{1}{2}.$$

2.52. Let $k > 0$ be a real number. Prove that

$$\int_0^1 \int_0^1 \left\{\frac{k}{x-y}\right\} \left\{\frac{1}{x}\right\} \left\{\frac{1}{y}\right\} \mathrm{d}x\mathrm{d}y = \frac{1}{2}(1-\gamma)^2.$$

2.53. Let $k > 0$ be a real number. Find

$$\int_0^1 \int_0^1 \left\{ \left(\frac{1}{x}\right)^k - \left(\frac{1}{y}\right)^k \right\} \mathrm{d}x\mathrm{d}y.$$

2.54. Let k and m be positive real numbers. Calculate

$$\int_0^1 \int_0^1 \left\{ \left(\frac{x}{y}\right)^k - \left(\frac{y}{x}\right)^k \right\} x^m y^m \mathrm{d}x\mathrm{d}y.$$

2.55. Let $n \geq 1$ be an integer and let $m > -1$ be a real number. Calculate

$$\int_0^1 \int_0^1 \left\{\frac{nx}{x+y}\right\} x^m y^m \mathrm{d}x\mathrm{d}y.$$

2.56. Let $n \geq 1$ be an integer and let $m > -1$ be a real number. Calculate

$$\int_0^1 \int_0^1 \left\{\frac{nx}{x-y}\right\} x^m y^m \mathrm{d}x\mathrm{d}y.$$

2.57. Let $a > 0$, let $k \neq 0$ be a real number, and let $g : [0,a] \times [0,a] \to \mathbb{R}$ be an integrable and symmetric function in x and y. Prove that

$$\int_0^a \int_0^a \left\{x^k - y^k\right\} g(x,y)\mathrm{d}x\mathrm{d}y = \frac{1}{2} \int_0^a \int_0^a g(x,y)\mathrm{d}x\mathrm{d}y,$$

where $\{x\}$ denotes the fractional part of x.

2.4 Open Problems

I could never resist an integral.

G. H. Hardy (1877–1947)

2.58. Integrating a product of fractional parts. Let $n \geq 3$ be an integer. Calculate

$$\int_0^1 \cdots \int_0^1 \left\{\frac{x_1}{x_2}\right\} \left\{\frac{x_2}{x_3}\right\} \cdots \left\{\frac{x_n}{x_1}\right\} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

2.59. A power integral. Let $n \geq 3$ and $m \geq 1$ be integers. Calculate, in closed form, the integral

$$\int_0^1 \cdots \int_0^1 \left\{ \frac{1}{x_1 + x_2 + \cdots + x_n} \right\}^m \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

2.5 Hints

To myself I am only a child playing on the beach, while vast oceans of truth lie undiscovered before me.

Sir Isaac Newton (1642–1727)

2.5.1 Single Integrals

It is not certain that everything is uncertain.

Blaise Pascal (1642–1662)

2.1. Make the substitution $1/x = t$.

2.3. Make the substitution $1/x = t$ and the integral reduces to the calculation of the series $\sum_{k=1}^{\infty} \left(3k^2 \ln(k+1)/k + 3/2 - 3k - 1/(k+1)\right)$. Then, calculate the nth partial sum of the series by using the definition of Glaisher–Kinkelin constant.

2.5. and **2.6.** Make the substitution $k/x = t$.

2.7. Make the substitution $x = 1/y^k$.

2.8. Calculate the integral by using the substitution $x = k^k/y^k$.

2.9. Make the substitution $x = 1/(k^k y^k)$.

2.10. Make the substitution $x = 1/t$ and show the integral reduces to the calculation of the series $2\sum_{k=1}^{\infty} \left(k \ln \frac{k}{k+1} + k \ln \frac{k+2}{k+1} + \frac{1}{k+2}\right)$.

2.13. Make the substitution $1/x = y$ and observe that if y is a positive real number, which is not an integer, one has $\{-y\} = 1 - \{y\}$.

2.14., **2.15.**, and **2.17.** Make the substitution $x = 1/y$.

2.18. Substitute $n/x = t$ and apply Stolz–Cesàro lemma (the $0/0$ case).

2.19. Make the substitution $n/x = t$ and prove that $x_n = \int_0^1 \{n/x\}^n \mathrm{d}x$ verifies the inequalities

$$\frac{n}{n+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}\right) < x_n < \frac{n}{n+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}\right)$$

and

$$\frac{n^2}{n+1}\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right) < nx_n < \frac{n^2}{n+1}\left(\frac{\pi^2}{6}-\sum_{k=1}^{n-1}\frac{1}{k^2}\right).$$

2.21. and **2.22.** Use the substitution $1/x = y$ and the improper integral formula $1/a^n = 1/\Gamma(n)\int_0^\infty e^{-ax}x^{n-1}\mathrm{d}x$ where $a > 0$.

2.5.2 *Double Integrals*

As for everything else, so for a mathematical theory; beauty can be perceived but not explained.

Arthur Cayley (1821–1895)

In general, these double integrals can be calculated by writing the double integral $\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y$ as an iterated integral $\int_0^1(\int_0^1 f(x,y)\mathrm{d}y)\mathrm{d}x$, making a particular change of variables in the inner integral, and then integrating by parts.

2.23. Write the integral in the form $\int_0^1 x\left(\int_0^1\left\{\frac{1}{1-xy}\right\}\mathrm{d}y\right)\mathrm{d}x$; make the substitution $xy = t$, in the inner integral; and integrate by parts.

2.24. Write the integral as $\int_0^1\left(\int_x^{x+1}\left\{\frac{1}{t}\right\}^m\mathrm{d}t\right)\mathrm{d}x$ and integrate by parts.

2.26. Use that $\int_0^1\left(\int_0^1\frac{\mathrm{d}y}{(\lfloor x/y\rfloor+1)^k}\right)\mathrm{d}x = \int_0^1 x\left(\int_0^{1/x}\frac{\mathrm{d}t}{(\lfloor 1/t\rfloor+1)^k}\right)\mathrm{d}x$ and integrate by parts.

2.28. Substitute $kx/y = t$, in the inner integral, to get that $\int_0^1\left(\int_0^1\{kx/y\}\mathrm{d}y\right)\mathrm{d}x = k\int_0^1 x\left(\int_{kx}^\infty\{t\}/t^2\mathrm{d}t\right)\mathrm{d}x$ and integrate by parts.

2.29. Write $\int_0^1\left(\int_0^1\left\{\frac{mx}{ny}\right\}\mathrm{d}y\right)\mathrm{d}x \overset{t=\frac{mx}{ny}}{=} \frac{m}{n}\int_0^1 x\left(\int_{\frac{mx}{n}}^\infty\frac{\{t\}}{t^2}\mathrm{d}t\right)\mathrm{d}x$ and integrate by parts.

2.30. $\int_0^1\left(\int_0^1\left\{\frac{x^k}{y}\right\}\mathrm{d}y\right)\mathrm{d}x \overset{t=x^k/y}{=} \int_0^1 x^k\left(\int_{x^k}^\infty\frac{\{t\}}{t^2}\mathrm{d}t\right)\mathrm{d}x$ and integrate by parts.

2.31.–2.35. Use the substitution $x/y = t$ and integrate by parts.

2.38. Write $\int_0^1 x^n\left(\int_0^1 y^n\left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\}\mathrm{d}y\right)\mathrm{d}x = \int_0^1 x^{2n+1}\left(\int_0^{1/x}t^n\{t\}\left\{\frac{1}{t}\right\}\mathrm{d}t\right)\mathrm{d}x$ and integrate by parts.

2.40. and **2.41.** Use the substitution $x/y = t$, in the inner integral, to get that $\int_0^1\left(\int_0^1\{x/y\}^m\{y/x\}^k\mathrm{d}y\right)\mathrm{d}x = \int_0^1 x\left(\int_x^\infty\{t\}^m\left\{\frac{1}{t}\right\}^k\frac{\mathrm{d}t}{t^2}\right)\mathrm{d}x$ and integrate by parts.

2.42. 1. Let $m = \lfloor a\rfloor$ be the floor of a and calculate the integral on intervals of the form $[j, j+1]$.
2. Write the multiple integral as an iterated integral and use part 1 of the problem.

2.5.3 *Quickies*

Nature not only suggests to us problems, she suggests their solution.

Henri Poincaré (1854–1912)

These integrals are solved by using either symmetry or the following identity involving the fractional part function.

A fractional part identity. If $x \in \mathbb{R} \setminus \mathbb{Z}$, then $\{x\} + \{-x\} = 1$.

2.44. Make the substitution $nx = y$.

2.45.–2.47. and **2.49.** Make the substitution $x = 1 - y$ and use the fractional part identity.

2.48. Observe that if n is a positive integer, then $\{n(1-y)\} = 1 - \{ny\}$, for all $y \in [0,1]$ except for 0, $1/n, 2/n, \ldots, (n-1)/n, 1$. Then, use the substitution $x = 1 - y$ and the definition of the Beta function.

2.50.–2.54. and **2.57.** Use symmetry and the fractional part identity.

2.55. and **2.56.** Observe that if n is a positive integer, then $\{n(1-y)\} = 1 - \{ny\}$, for all $y \in [0,1]$ except for 0, $1/n$, $2/n$, $\ldots$, $(n-1)/n$, 1. Then, use symmetry.

2.6 Solutions

Everything you add to the truth subtracts from the truth.

Alexander Solzhenitsyn (1918–2008)

This section contains the solutions to the problems from the first three sections of the chapter.

2.6.1 *Single Integrals*

Sir, I have found you an argument. I am not obliged to find you an understanding.

Samuel Johnson (1709–1784)

2.1. The integral equals $1 - \gamma$. We have

$$\int_0^1 \left\{\frac{1}{x}\right\} \mathrm{d}x = \int_1^\infty \frac{\{t\}}{t^2}\mathrm{d}t = \sum_{k=1}^\infty \int_k^{k+1} \frac{t-k}{t^2}\mathrm{d}t = \sum_{k=1}^\infty \left(\ln\frac{k+1}{k} - \frac{1}{k+1}\right) = 1-\gamma.$$

Remark. This integral, which is well known in the mathematical literature ([43, p. 32], [65, pp. 109–111]), is related to a surprising result, due to de la Vallée Poussin [105], which states that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \left\{\frac{n}{k}\right\} = 1-\gamma.$$

In words, if a large integer n is divided by each integer $1 \le k \le n$, then the average fraction by which the quotient n/k falls short of the next integer is not $1/2$, but γ!

2.2. See the solution of Problem **2.6**.

2.3. Using the substitution $1/x = t$ the integral becomes

$$\int_1^\infty \frac{\{t\}^3}{t^2}\mathrm{d}t = \sum_{k=1}^\infty \int_k^{k+1} \frac{(t-k)^3}{t^2}\mathrm{d}t = \sum_{k=1}^\infty \left(3k^2\ln\frac{k+1}{k} + \frac{3}{2} - 3k - \frac{1}{k+1}\right).$$

Let $S_n = \sum_{k=1}^n \left(3k^2\ln(k+1)/k + 3/2 - 3k - 1/(k+1)\right)$ be the nth partial sum of the series. A calculation shows that

$$S_n = 1 - \left(1+\frac{1}{2}+\cdots+\frac{1}{n+1} - \ln n\right) - \frac{3}{2}n^2 - \ln n + 3\sum_{k=1}^n k^2 \ln\frac{k+1}{k}.$$

On the other hand,

$$\sum_{k=1}^n k^2\ln\frac{k+1}{k} = \ln\prod_{k=1}^n \left(\frac{k+1}{k}\right)^{k^2} = \ln\left(\frac{(n+1)^{n^2} n!}{(2^2 3^3\cdots n^n)^2}\right)$$

and

$$-\frac{3}{2}n^2 - \ln n + 3\sum_{k=1}^n k^2\ln\frac{k+1}{k} = \ln\left(\frac{(n+1)^{3n^2}(n!)^3}{(2^2 3^3\cdots n^n)^6 e^{\frac{3n^2}{2}} n}\right).$$

Let

$$x_n = \frac{(n+1)^{3n^2}(n!)^3}{(2^2 3^3\cdots n^n)^6 e^{\frac{3n^2}{2}} n} = \frac{n^{3n^2+3n+\frac{1}{2}} e^{\frac{-3n^2}{2}}}{(2^2 3^3\cdots n^n)^6} \cdot \frac{(n+1)^{3n^2}(n!)^3}{n^{3n^2+3n+\frac{3}{2}}}.$$

The first fraction converges to $1/A^6$, and for calculating the limit of the second fraction, we have, based on Stirling's formula, $n! \sim \sqrt{2\pi n}\,(n/e)^n$, that

$$\frac{(n+1)^{3n^2}(n!)^3}{n^{3n^2+3n+\frac{3}{2}}} \sim (2\pi)^{\frac{3}{2}}\left(\left(\frac{n+1}{n}\right)^n\frac{1}{e}\right)^{3n} \to (2\pi)^{\frac{3}{2}}\,e^{-\frac{3}{2}}.$$

Thus, $x_n \to (2\pi)^{\frac{3}{2}}e^{-\frac{3}{2}}/A^6$, which implies that $\lim\limits_{n\to\infty} S_n = -\frac{1}{2}-\gamma+\frac{3}{2}\ln(2\pi)-6\ln A$.

2.4. See the solution of Problem 2.22.

2.5. (a) Using the substitution $k/x = t$, we get that

$$\int_0^1 \left\{\frac{k}{x}\right\}\mathrm{d}x = k\int_k^\infty \frac{\{t\}}{t^2}\mathrm{d}t = k\sum_{l=k}^\infty \int_l^{l+1}\frac{t-l}{t^2}\mathrm{d}t = k\sum_{l=k}^\infty\left(\ln\frac{l+1}{l}-\frac{1}{l+1}\right).$$

A calculation shows that

$$S_n = \sum_{l=k}^n\left(\ln\frac{l+1}{l}-\frac{1}{l+1}\right) = -\ln k - \left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{1+n}-\ln(n+1)\right),$$

and this implies that $\lim_{n\to\infty} S_n = 1+1/2+\cdots+1/k-\ln k-\gamma$.
(b) Using the substitution $x = q/y$, we obtain that

$$I = \int_0^1\left\{\frac{q}{x}\right\}\mathrm{d}x = q\int_q^\infty \frac{\{y\}}{y^2}\mathrm{d}y.$$

We distinguish here two cases.

Case 1. $q \le 1$. We have

$$\begin{aligned}
I &= q\left(\int_q^1 \frac{1}{y}\mathrm{d}y + \sum_{k=1}^\infty \int_k^{k+1}\frac{y-k}{y^2}\mathrm{d}y\right)\\
&= q\left(\ln\frac{1}{q}+\sum_{k=1}^\infty\left(\ln\frac{k+1}{k}-\frac{1}{k+1}\right)\right)\\
&= q\,(1-\gamma-\ln q)\,.
\end{aligned}$$

Case 2. $q > 1$. A calculation shows that

$$\begin{aligned}
I &= q\left(\int_q^{\lfloor q\rfloor+1}\frac{y-\lfloor q\rfloor}{y^2}\mathrm{d}y + \int_{\lfloor q\rfloor+1}^\infty \frac{\{y\}}{y^2}\mathrm{d}y\right)\\
&= q\left(1+\frac{1}{2}+\cdots+\frac{1}{1+\lfloor q\rfloor}-\gamma-\ln q+\frac{\lfloor q\rfloor(\{q\}-1)}{q(1+\lfloor q\rfloor)}\right).
\end{aligned}$$

Remark. If $k \geq 2$ is an integer, then

$$\int_0^1 \left(\left\lfloor \frac{k}{x} \right\rfloor - k \cdot \left\lfloor \frac{1}{x} \right\rfloor \right) \mathrm{d}x = k \ln k - k \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right),$$

and this generalizes an integral from Pólya and Szegö (see [104, p. 43]).

2.6. Using the substitution $k/x = t$, the integral becomes

$$\int_0^1 \left\{ \frac{k}{x} \right\}^2 \mathrm{d}x = k \int_k^\infty \frac{\{t\}^2}{t^2} \mathrm{d}t = k \sum_{l=k}^{\infty} \left(2 - 2l \ln \frac{l+1}{l} - \frac{1}{l+1} \right).$$

Let S_n be the nth partial sum of the series. We have

$$\sum_{l=k}^{n} \left(2 - 2l \ln \frac{l+1}{l} - \frac{1}{l+1} \right) = 2(n-k+1) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{n+1} \right)$$
$$- 2n \ln(n+1) + 2k \ln k + 2 \ln n! - 2 \ln k!.$$

Since $2 \ln n! \sim \ln(2\pi) + (2n+1) \ln n - 2n$, it follows that S_n is approximated by

$$2(1-k) + \ln(2\pi) + 2k \ln k - 2 \ln k! - 2n \ln \frac{n+1}{n} - \left(\frac{1}{k+1} + \cdots + \frac{1}{n+1} - \ln n \right)$$

and hence, $\lim_{n\to\infty} S_n = \ln(2\pi) - \gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + 2k \ln k - 2k - 2 \ln k!$.

2.7. We have, based on the substitution $x = 1/y^k$, that

$$\int_0^1 \left\{ \frac{1}{\sqrt[k]{x}} \right\} \mathrm{d}x = k \int_1^\infty \frac{\{y\}}{y^{k+1}} \mathrm{d}y = k \sum_{m=1}^{\infty} \int_m^{m+1} \frac{\{y\}}{y^{k+1}} \mathrm{d}y = k \sum_{m=1}^{\infty} \int_m^{m+1} \frac{y-m}{y^{k+1}} \mathrm{d}y$$
$$= k \sum_{m=1}^{\infty} \left(\frac{y^{-k+1}}{-k+1} \Bigg|_m^{m+1} + m \frac{y^{-k}}{k} \Bigg|_m^{m+1} \right)$$
$$= \frac{k}{k-1} \sum_{m=1}^{\infty} \left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}} \right) + \sum_{m=1}^{\infty} m \left(\frac{1}{(m+1)^k} - \frac{1}{m^k} \right)$$
$$= \frac{k}{k-1} + \sum_{m=1}^{\infty} \left(\frac{1}{(m+1)^{k-1}} - \frac{1}{m^{k-1}} \right) - \sum_{m=1}^{\infty} \frac{1}{(m+1)^k}$$
$$= \frac{k}{k-1} - \zeta(k).$$

2.8. Using the substitution $x = k^k/y^k$ we obtain that

$$\begin{aligned}
\int_0^1 \left\{\frac{k}{\sqrt[k]{x}}\right\} \mathrm{d}x &= k^{k+1}\int_k^{\infty} \frac{\{y\}}{y^{k+1}}\mathrm{d}y = k^{k+1}\sum_{m=k}^{\infty}\left(\int_m^{m+1} \frac{y-m}{y^{k+1}}\mathrm{d}y\right)\\
&= k^{k+1}\sum_{m=k}^{\infty}\left(\frac{y^{-k+1}}{-k+1}\bigg|_m^{m+1} + m\frac{y^{-k}}{k}\bigg|_m^{m+1}\right)\\
&= \frac{k^{k+1}}{k-1}\sum_{m=k}^{\infty}\left(\frac{1}{m^{k-1}} - \frac{1}{(m+1)^{k-1}}\right) + k^k\sum_{m=k}^{\infty} m\left(\frac{1}{(m+1)^k} - \frac{1}{m^k}\right)\\
&= \frac{k^2}{k-1} + k^k\sum_{m=k}^{\infty}\left(\frac{1}{(m+1)^{k-1}} - \frac{1}{m^{k-1}} - \frac{1}{(m+1)^k}\right)\\
&= \frac{k^2}{k-1} - k - k^k\left(\zeta(k) - \frac{1}{1^k} - \frac{1}{2^k} - \cdots - \frac{1}{k^k}\right)\\
&= \frac{k}{k-1} - k^k\left(\zeta(k) - \frac{1}{1^k} - \frac{1}{2^k} - \cdots - \frac{1}{k^k}\right).
\end{aligned}$$

2.9. The substitution $x = 1/(k^k y^k)$ implies that

$$\begin{aligned}
\int_0^1 \left\{\frac{1}{k\sqrt[k]{x}}\right\} \mathrm{d}x &= \frac{1}{k^{k-1}}\int_{\frac{1}{k}}^{\infty} \frac{\{y\}}{y^{k+1}}\mathrm{d}y = \frac{1}{k^{k-1}}\left(\int_{\frac{1}{k}}^{1} \frac{y}{y^{k+1}}\mathrm{d}y + \int_1^{\infty}\frac{\{y\}}{y^{k+1}}\mathrm{d}y\right)\\
&= \frac{1}{k^{k-1}}\left(\frac{y^{1-k}}{1-k}\bigg|_{\frac{1}{k}}^{1} + \frac{1}{k-1} - \frac{\zeta(k)}{k}\right)\\
&= \frac{1}{k^{k-1}}\left(\frac{1}{1-k} - \frac{1}{1-k}\left(\frac{1}{k}\right)^{1-k} + \frac{1}{k-1} - \frac{\zeta(k)}{k}\right)\\
&= \frac{1}{k-1} - \frac{\zeta(k)}{k^k}.
\end{aligned}$$

We used that (see the solution of Problem 2.7)

$$\int_1^{\infty} \frac{\{y\}}{y^{k+1}}\mathrm{d}y = \frac{1}{k-1} - \frac{\zeta(k)}{k}.$$

2.10. The substitution $x = 1/t$ implies that

$$\begin{aligned}
I = \int_0^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\mathrm{d}x &= \int_1^{\infty}\frac{\{t\}}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t\\
&= \int_1^2 \frac{\{t\}}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t + \sum_{k=2}^{\infty}\int_k^{k+1}\frac{t-k}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t.
\end{aligned}$$

We calculate the integral and the sum separately. We have that

$$\begin{aligned}\sum_{k=2}^{\infty}\int_k^{k+1}\frac{t-k}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t&=\sum_{k=2}^{\infty}\int_k^{k+1}\frac{t-k}{t^2}\left(\frac{t}{t-1}-1\right)\mathrm{d}t\\&=\sum_{k=2}^{\infty}\int_k^{k+1}\frac{t-k}{t^2}\cdot\frac{\mathrm{d}t}{t-1}\\&=\sum_{k=1}^{\infty}\left(k\ln\frac{k}{k+1}+k\ln\frac{k+2}{k+1}+\frac{1}{k+2}\right).\end{aligned}$$

On the other hand, the substitution $t-1=u$ implies that

$$\begin{aligned}\int_1^2\frac{\{t\}}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t&=\int_0^1\frac{u}{(u+1)^2}\left\{\frac{u+1}{u}\right\}\mathrm{d}u=\int_0^1\frac{u}{(u+1)^2}\left\{\frac{1}{u}\right\}\mathrm{d}u\\&=\int_1^{\infty}\frac{\{t\}}{t\,(1+t)^2}\mathrm{d}t=\sum_{k=1}^{\infty}\int_k^{k+1}\frac{t-k}{t\,(1+t)^2}\mathrm{d}t\\&=\sum_{k=1}^{\infty}\left(k\ln\frac{k}{k+1}+k\ln\frac{k+2}{k+1}+\frac{1}{k+2}\right).\end{aligned}$$

Putting all these together we get that

$$I=2\sum_{k=1}^{\infty}\left(k\ln\frac{k}{k+1}+k\ln\frac{k+2}{k+1}+\frac{1}{k+2}\right).$$

Let S_n be the nth partial sum of the series. A calculation shows that

$$\begin{aligned}S_n&=2\sum_{k=1}^{n}\left(k\ln\frac{k}{k+1}+k\ln\frac{k+2}{k+1}+\frac{1}{k+2}\right)\\&=2\left(n\ln\frac{n+2}{n+1}-\ln(n+1)+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n+2}\right),\end{aligned}$$

and this implies $\lim_{n\to\infty}S_n=2\gamma-1$.

Remark. It is worth mentioning that the following integral formulae hold

$$\int_0^{\frac{1}{2}}\left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\mathrm{d}x=\int_{\frac{1}{2}}^1\left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\mathrm{d}x=\gamma-\frac{1}{2}.$$

2.11. For a solution of this problem, see [45].

2.12. We need the following lemma:

Lemma 2.1. *The following formulae hold:*

1. $\lim_{n\to\infty}n\left(1-(n+2)\ln\frac{n+2}{n+1}\right)=-\frac{1}{2}$.

2. $\int_k^{k+1} \frac{x-k}{(x+1)^2x^2}\mathrm{d}x = -\frac{1}{k+1} - \frac{1}{k+2} + (2k+1)\ln\frac{k+1}{k} - (2k+1)\ln\frac{k+2}{k+1}, \qquad k \geq 1.$
3. $\int_k^{k+1} \frac{(x-k)^2}{x^2(x-1)}\mathrm{d}x = -\frac{k}{k+1} + (k-1)^2\ln\frac{k}{k-1} - (k-2)k\ln\frac{k+1}{k}, \qquad k > 1.$

The lemma can be proved by elementary calculations.

Using the substitution $x = 1/t$, the integral becomes

$$\begin{aligned} I &= \int_1^{\infty} \frac{\{t\}^2}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t \\ &= \int_1^2 \frac{\{t\}^2}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t + \sum_{k=2}^{\infty}\int_k^{k+1}\frac{(t-k)^2}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t. \end{aligned} \tag{2.1}$$

We calculate the integral and the sum separately. We have

$$\begin{aligned} S &= \sum_{k=2}^{\infty}\int_k^{k+1}\frac{(t-k)^2}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t \\ &= \sum_{k=2}^{\infty}\int_k^{k+1}\frac{(t-k)^2}{t^2}\left(\frac{t}{t-1}-1\right)\mathrm{d}t \\ &= \sum_{k=2}^{\infty}\int_k^{k+1}\frac{(t-k)^2}{t^2}\cdot\frac{\mathrm{d}t}{t-1}. \end{aligned}$$

Using part (3) of Lemma 2.1 we get that

$$\begin{aligned} S &= \sum_{k=2}^{\infty}\left((k-1)^2\ln\frac{k}{k-1} + (2k-k^2)\ln\frac{k+1}{k} - \frac{k}{k+1}\right) \\ &= \sum_{k=1}^{\infty}\left(k^2\ln\frac{k+1}{k} + (1-k^2)\ln\frac{k+2}{k+1} - \frac{k+1}{k+2}\right). \end{aligned} \tag{2.2}$$

On the other hand, we have based on the substitution $t-1=u$ that

$$J = \int_1^2 \frac{\{t\}^2}{t^2}\left\{\frac{t}{t-1}\right\}\mathrm{d}t = \int_0^1 \frac{\{u+1\}^2}{(u+1)^2}\left\{\frac{u+1}{u}\right\}\mathrm{d}u = \int_0^1 \frac{u^2}{(u+1)^2}\left\{\frac{1}{u}\right\}\mathrm{d}u.$$

The substitution $1/u = t$, combined with part (2) of Lemma 2.1, shows that

$$\begin{aligned} J &= \int_1^{\infty}\frac{\{t\}}{t^2(1+t)^2}\mathrm{d}t = \sum_{k=1}^{\infty}\int_k^{k+1}\frac{t-k}{t^2(1+t)^2}\mathrm{d}t \\ &= \sum_{k=1}^{\infty}\left(-\frac{1}{k+1} - \frac{1}{k+2} + (2k+1)\ln\frac{k+1}{k} - (2k+1)\ln\frac{k+2}{k+1}\right). \end{aligned} \tag{2.3}$$

Combining (2.1)–(2.3), we get that

$$\int_0^1 \left\{\frac{1}{x}\right\}^2 \left\{\frac{1}{1-x}\right\} \mathrm{d}x = \sum_{k=1}^{\infty} \left((k+1)^2 \ln \frac{k+1}{k} - (k^2+2k) \ln \frac{k+2}{k+1} - 1 - \frac{1}{k+1} \right).$$

Let S_n be the nth partial sum of the series, i.e.,

$$S_n = \sum_{k=1}^{n} \left((k+1)^2 \ln \frac{k+1}{k} - (k^2+2k) \ln \frac{k+2}{k+1} - 1 - \frac{1}{k+1} \right).$$

A calculation shows

$$\begin{aligned}
&\sum_{k=1}^{n} (k+1)^2 \ln \frac{k+1}{k} - \sum_{k=1}^{n} (k^2+2k) \ln \frac{k+2}{k+1} \\
&= \sum_{k=1}^{n} (k+1)^2 \ln \frac{k+1}{k} - \sum_{k=1}^{n} (k^2-1) \ln \frac{k+1}{k} - n(n+2) \ln \frac{n+2}{n+1} \\
&= \sum_{k=1}^{n} 2(k+1) \ln \frac{k+1}{k} - n(n+2) \ln \frac{n+2}{n+1} \\
&= 2(n+1) \ln(n+1) - 2 \ln n! - n(n+2) \ln \frac{n+2}{n+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
S_n &= 2(n+1) \ln(n+1) - 2 \ln n! - (n^2+2n) \ln \frac{n+2}{n+1} - n \\
&\quad - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+2} - \ln(n+1) \right) - \ln(n+1) \\
&= (2n+1) \ln(n+1) - 2 \ln n! - (n^2+2n) \ln \frac{n+2}{n+1} - n \\
&\quad - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+2} - \ln(n+1) \right).
\end{aligned}$$

Since $\ln n! = (\ln 2\pi)/2 + (n+1/2) \ln n - n + O(1/n)$, we get that

$$\begin{aligned}
S_n &= (2n+1) \ln \frac{n+1}{n} - \ln(2\pi) + n \left(1 - (n+2) \ln \frac{n+2}{n+1} \right) \\
&\quad - \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+2} - \ln(n+1) \right) + O\left(\frac{1}{n}\right),
\end{aligned}$$

and this implies that $\lim_{n\to\infty} S_n = 5/2 - \ln(2\pi) - \gamma$.

Remark. We have, due to symmetry reasons, that the following evaluations hold

$$\int_0^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}^2 \mathrm{d}x = \frac{5}{2} - \ln(2\pi) - \gamma.$$

$$\int_0^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\left(\left\{\frac{1}{x}\right\}+\left\{\frac{1}{1-x}\right\}\right) \mathrm{d}x = 5 - 2\ln(2\pi) - 2\gamma.$$

$$\int_0^{\frac{1}{2}} \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\left(\left\{\frac{1}{x}\right\}+\left\{\frac{1}{1-x}\right\}\right) \mathrm{d}x = \frac{5}{2} - \ln(2\pi) - \gamma.$$

$$\int_{\frac{1}{2}}^1 \left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\left(\left\{\frac{1}{x}\right\}+\left\{\frac{1}{1-x}\right\}\right) \mathrm{d}x = \frac{5}{2} - \ln(2\pi) - \gamma.$$

2.13. Using the substitution $1/x = y$, the integral becomes $I = \int_1^\infty \left\{(-1)^{\lfloor y \rfloor} y\right\}/y^2 \mathrm{d}y$. Since for a positive real number y, which is not an integer, one has $\{-y\} = 1 - \{y\}$, we get that

$$\begin{aligned}
I &= \int_1^\infty \frac{\left\{(-1)^{\lfloor y \rfloor} y\right\}}{y^2}\mathrm{d}y = \sum_{k=1}^\infty \int_k^{k+1} \frac{\left\{(-1)^{\lfloor y \rfloor} y\right\}}{y^2}\mathrm{d}y = \sum_{k=1}^\infty \int_k^{k+1} \frac{\left\{(-1)^k y\right\}}{y^2}\mathrm{d}y \\
&= \sum_{p=1}^\infty \int_{2p-1}^{2p} \frac{\{-y\}}{y^2}\mathrm{d}y + \sum_{p=1}^\infty \int_{2p}^{2p+1} \frac{\{y\}}{y^2}\mathrm{d}y \\
&= \sum_{p=1}^\infty \int_{2p-1}^{2p} \frac{1-\{y\}}{y^2}\mathrm{d}y + \sum_{p=1}^\infty \int_{2p}^{2p+1} \frac{y-2p}{y^2}\mathrm{d}y \\
&= \sum_{p=1}^\infty \int_{2p-1}^{2p} \frac{1-(y-(2p-1))}{y^2}\mathrm{d}y + \sum_{p=1}^\infty \left(\ln y + \frac{2p}{y}\right)\Bigg|_{2p}^{2p+1} \\
&= \sum_{p=1}^\infty \int_{2p-1}^{2p} \frac{2p-y}{y^2}\mathrm{d}y + \sum_{p=1}^\infty \left(\ln\frac{2p+1}{2p} - \frac{1}{2p+1}\right) \\
&= \sum_{p=1}^\infty \left(\frac{1}{2p-1} - \ln\frac{2p}{2p-1}\right) + \sum_{p=1}^\infty \left(\ln\frac{2p+1}{2p} - \frac{1}{2p+1}\right) \\
&= \sum_{p=1}^\infty \left(\frac{1}{2p-1} - \frac{1}{2p+1} + \ln\frac{(2p+1)(2p-1)}{(2p)^2}\right) \\
&= 1 + \ln\left(\prod_{p=1}^\infty \frac{(2p+1)(2p-1)}{(2p)^2}\right) \\
&= 1 + \ln\frac{2}{\pi},
\end{aligned}$$

where the last equality follows from the Wallis product formula.

2.14. (a) Using the substitution $1/x = t$ the integral becomes

$$I = \int_0^1 x\left\{\frac{1}{x}\right\}\left\lfloor\frac{1}{x}\right\rfloor \mathrm{d}x = \int_1^\infty \frac{\{t\}\cdot\lfloor t\rfloor}{t^3}\mathrm{d}t = \sum_{k=1}^\infty \int_k^{k+1} \frac{(t-k)\cdot k}{t^3}\mathrm{d}t$$
$$= \sum_{k=1}^\infty \int_k^{k+1}\left(\frac{k}{t^2}-\frac{k^2}{t^3}\right)\mathrm{d}t = \frac{1}{2}\sum_{k=1}^\infty \frac{1}{(k+1)^2} = \frac{1}{2}\left(\frac{\pi^2}{6}-1\right).$$

(b) The integral converges if and only if $\alpha+1>\gamma$. Making the substitution $1/x=t$, the integral equals

$$I(\alpha,\beta,\gamma) = \int_0^1 x^\alpha \left\{\frac{1}{x}\right\}^\beta \left\lfloor\frac{1}{x}\right\rfloor^\gamma \mathrm{d}x = \int_1^\infty \frac{\{t\}^\beta\cdot\lfloor t\rfloor^\gamma}{t^{\alpha+2}}\mathrm{d}t$$
$$= \sum_{k=1}^\infty \int_k^{k+1} \frac{(t-k)^\beta k^\gamma}{t^{\alpha+2}}\mathrm{d}t \overset{t-k=u}{=} \sum_{k=1}^\infty k^\gamma \int_0^1 \frac{u^\beta}{(k+u)^{\alpha+2}}\mathrm{d}u.$$

We have, since $k^{\alpha+2}<(k+u)^{\alpha+2}<(k+1)^{\alpha+2}$, that

$$\sum_{k=1}^\infty \frac{k^\gamma}{(\beta+1)(k+1)^{\alpha+2}} < I(\alpha,\beta,\gamma) < \sum_{k=1}^\infty \frac{k^\gamma}{(\beta+1)k^{\alpha+2}}.$$

Thus, the integral converges if and only if $\alpha+1>\gamma$.

Remark. The convergence of the integral is independent of the parameter β, which can be chosen to be any real number strictly greater than -1.

2.15. We have

$$I = \int_0^1 \left\{\frac{1}{x}\right\}\frac{x}{1-x}\mathrm{d}x = \int_0^1 \left\{\frac{1}{x}\right\}\left(\sum_{k=1}^\infty x^k\right)\mathrm{d}x = \sum_{k=1}^\infty \int_0^1 \left\{\frac{1}{x}\right\} x^k \mathrm{d}x.$$

Let $J_k = \int_0^1 \left\{\frac{1}{x}\right\} x^k \mathrm{d}x$. Using the substitution $1/x=y$ we obtain that

$$J_k = \int_1^\infty \frac{\{y\}}{y^{k+2}}\mathrm{d}y = \sum_{m=1}^\infty \int_m^{m+1} \frac{\{y\}}{y^{k+2}}\mathrm{d}y = \sum_{m=1}^\infty \int_m^{m+1} \frac{y-m}{y^{k+2}}\mathrm{d}y$$
$$= \sum_{m=1}^\infty \int_m^{m+1}\left(\frac{1}{y^{k+1}}-\frac{m}{y^{k+2}}\right)\mathrm{d}y = \sum_{m=1}^\infty \left(\frac{1}{-ky^k}+\frac{m}{(k+1)y^{k+1}}\right)\Bigg|_m^{m+1}$$
$$= \sum_{m=1}^\infty \left(\frac{1}{km^k}-\frac{1}{k(m+1)^k}\right) + \sum_{m=1}^\infty \left(\frac{m}{(k+1)(m+1)^{k+1}}-\frac{1}{(k+1)m^k}\right)$$
$$= \frac{1}{k}+\frac{1}{k+1}\sum_{m=1}^\infty \left[\left(\frac{1}{(m+1)^k}-\frac{1}{m^k}\right)-\frac{1}{(m+1)^{k+1}}\right]$$

$$= \frac{1}{k} + \frac{1}{k+1}(-1-(\zeta(k+1)-1))$$
$$= \frac{1}{k} - \frac{\zeta(k+1)}{k+1}.$$

Thus,

$$I = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{\zeta(k+1)}{k+1}\right) = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \sum_{k=1}^{\infty}\frac{\zeta(k+1)-1}{k+1} = \gamma,$$

since $\sum_{k=1}^{\infty}(\zeta(k+1)-1)/(k+1) = 1-\gamma$ [122, Entry 135, p. 173].

For an alternative solution, see [31].

2.16. The solution is given in [51].

2.17. For a solution see [50].

2.18. The limit equals $1/(k+1)$. Let $a_n = \int_0^1 \{n/x\}^k \, dx$. Using the substitution $n/x = t$, we get that

$$a_n = n\int_n^{\infty}\frac{\{t\}^k}{t^2}dt = \frac{\int_n^{\infty}\frac{\{t\}^k}{t^2}dt}{\frac{1}{n}} = \frac{b_n}{c_n},$$

where $b_n = \int_n^{\infty}\{t\}^k/t^2 dt$ and $c_n = 1/n$, and we note that b_n converges to 0 since $b_n = \int_n^{\infty}\{t\}^k/t^2 dt < \int_n^{\infty} 1/t^2 dt = 1/n$. For calculating $\lim_{n\to\infty} a_n$, we use Stolz–Cesàro lemma (the $0/0$ case) and we get that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{b_{n+1}-b_n}{c_{n+1}-c_n} = \lim_{n\to\infty}\frac{-\int_n^{n+1}\frac{\{t\}^k}{t^2}dt}{-\frac{1}{n(n+1)}} = \lim_{n\to\infty} n(n+1)\int_n^{n+1}\frac{\{t\}^k}{t^2}dt$$
$$= \lim_{n\to\infty} n(n+1)\int_n^{n+1}\frac{(t-n)^k}{t^2}dt = \lim_{n\to\infty} n(n+1)\int_0^1\frac{y^k}{(n+y)^2}dy$$
$$= \lim_{n\to\infty}\int_0^1 y^k\frac{n(n+1)}{(n+y)^2}dy = \int_0^1 y^k dy = \frac{1}{k+1}.$$

For an alternative solution, see Problem **1.70**.

2.19. We prove that $L=0$. Let $x_n = \int_0^1 \{n/x\}^n \, dx$. Using the substitution $n/x = t$, we get that

$$x_n = n\int_n^{\infty}\frac{\{t\}^n}{t^2}dt = n\sum_{k=n}^{\infty}\int_k^{k+1}\frac{\{t\}^n}{t^2}dt = n\sum_{k=n}^{\infty}\int_k^{k+1}\frac{(t-k)^n}{t^2}dt$$
$$= n\sum_{k=n}^{\infty}\int_0^1\frac{y^n}{(k+y)^2}dy = n\int_0^1 y^n\left(\sum_{k=n}^{\infty}\frac{1}{(k+y)^2}\right)dy.$$

Since $1/(k+1)^2 < 1/(k+y)^2 < 1/k^2$, for positive integers k, and $y \in (0,1)$, it follows that

$$n\int_0^1 y^n \left(\sum_{k=n}^{\infty} \frac{1}{(k+1)^2}\right) \mathrm{d}y < x_n < n\int_0^1 y^n \left(\sum_{k=n}^{\infty} \frac{1}{k^2}\right) \mathrm{d}y.$$

Thus,

$$\frac{n}{n+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}\right) < x_n < \frac{n}{n+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}\right),$$

which implies that $\lim_{n\to\infty} x_n = 0$.

The second limit equals 1. We have, based on the preceding inequalities, that

$$\frac{n^2}{n+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2}\right) < nx_n < \frac{n^2}{n+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}\right),$$

and the result follows since $\lim_{n\to\infty} n\left(\pi^2/6 - \sum_{k=1}^{n} 1/k^2\right) = 1$.

2.20. The integral equals $1/m - \zeta(m+1)/(m+1)$. See the calculation of J_k from the solution of Problem **2.15**.

2.21. and **2.22.** Let

$$\begin{aligned} V_{k,m} &= \int_0^1 x^m \left\{\frac{1}{x}\right\}^k \mathrm{d}x = \int_1^{\infty} \frac{\{t\}^k}{t^{m+2}} \mathrm{d}t = \sum_{p=1}^{\infty} \int_p^{p+1} \frac{\{t\}^k}{t^{m+2}} \mathrm{d}t = \sum_{p=1}^{\infty} \int_p^{p+1} \frac{(t-p)^k}{t^{m+2}} \mathrm{d}t \\ &\overset{t-p=y}{=} \sum_{p=1}^{\infty} \int_0^1 \frac{y^k}{(p+y)^{m+2}} \mathrm{d}y = \int_0^1 y^k \left(\sum_{p=1}^{\infty} \frac{1}{(p+y)^{m+2}}\right) \mathrm{d}y. \end{aligned} \tag{2.4}$$

Since

$$\frac{1}{(p+y)^{2+m}} = \frac{1}{(m+1)!}\int_0^{\infty} e^{-(p+y)u} u^{m+1} \,\mathrm{d}u,$$

we have that

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{(p+y)^{m+2}} &= \frac{1}{(m+1)!} \sum_{p=1}^{\infty} \int_0^{\infty} e^{-(p+y)u} u^{m+1} \mathrm{d}u \\ &= \frac{1}{(m+1)!} \int_0^{\infty} u^{m+1} e^{-yu} \left(\sum_{p=1}^{\infty} e^{-pu}\right) \mathrm{d}u \\ &= \frac{1}{(m+1)!} \int_0^{\infty} \frac{u^{m+1} e^{-yu}}{e^u - 1} \mathrm{d}u. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), one has that

$$V_{k,m} = \frac{1}{(m+1)!}\int_0^1 y^k\left(\int_0^\infty \frac{u^{m+1}e^{-yu}}{e^u-1}\mathrm{d}u\right)\mathrm{d}y$$
$$= \frac{1}{(m+1)!}\int_0^\infty \frac{u^{m+1}}{e^u-1}\left(\int_0^1 y^k e^{-yu}\mathrm{d}y\right)\mathrm{d}u.$$

Let $J_k = \int_0^1 y^k e^{-yu}\mathrm{d}y$. Integrating by parts we get the recurrence formula $J_k = -e^{-u}/u + (k/u)J_{k-1}$. Let $a_k = J_k u^k/k!$ and we note that $a_k = -\frac{e^{-u}}{u}\cdot\frac{u^k}{k!} + a_{k-1}$. This implies that

$$a_k = -\frac{e^{-u}}{u}\left(\frac{u^k}{k!} + \frac{u^{k-1}}{(k-1)!} + \cdots + \frac{u}{1!}\right) + \frac{1-e^{-u}}{u}$$
$$= \frac{e^{-u}}{u}\left(e^u - \left(1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots + \frac{u^k}{k!}\right)\right)$$
$$= \frac{e^{-u}}{u}\sum_{j=1}^\infty \frac{u^{k+j}}{(k+j)!}.$$

Thus,

$$J_k = k!e^{-u}\sum_{j=1}^\infty \frac{u^{j-1}}{(k+j)!},$$

and it follows that

$$V_{k,m} = \frac{1}{(m+1)!}\sum_{j=1}^\infty \frac{k!}{(k+j)!}\int_0^\infty \frac{u^{m+j}e^{-u}}{e^u-1}\mathrm{d}u.$$

On the other hand,

$$\int_0^\infty \frac{u^{m+j}e^{-u}}{e^u-1}\mathrm{d}u = \int_0^\infty u^{m+j}e^{-2u}\sum_{p=0}^\infty e^{-pu}\mathrm{d}u$$
$$= \sum_{p=0}^\infty \int_0^\infty u^{m+j}e^{-(2+p)u}\mathrm{d}u$$
$$= \sum_{p=0}^\infty \frac{\Gamma(m+j+1)}{(2+p)^{m+j+1}}$$
$$= (m+j)!(\zeta(m+j+1)-1).$$

Hence,

$$V_{k,m}=\frac{k!}{(m+1)!}\sum_{j=1}^{\infty}\frac{(m+j)!}{(k+j)!}(\zeta(m+j+1)-1).$$

When $k=m$, one has that

$$V_{m,m}=\frac{1}{m+1}\sum_{j=1}^{\infty}(\zeta(m+j+1)-1)=1-\frac{\zeta(2)+\zeta(3)+\cdots+\zeta(m+1)}{m+1},$$

since $\sum_{j=1}^{\infty}(\zeta(j+1)-1)=1$.

Part (b) of Problem 2.22 follows from the calculation of $V_{k,m}$ combined with the formula $\sum_{j=1}^{\infty}(\zeta(j+1)-1)/(j+1)=1-\gamma$.

2.6.2 Double Integrals

But just as much as it is easy to find the differential of a given quantity, so it is difficult to find the integral of a given differential. Moreover, sometimes we cannot say with certainty whether the integral of a given quantity can be found or not.

Johann Bernoulli (1667–1748)

2.23. (a) The integral equals $1-\frac{\pi^2}{12}$. We have

$$I=\int_0^1\int_0^1 x\left\{\frac{1}{1-xy}\right\}\mathrm{d}x\mathrm{d}y=\int_0^1 x\left(\int_0^1\left\{\frac{1}{1-xy}\right\}\mathrm{d}y\right)\mathrm{d}x.$$

Using the substitution $xy=t$, in the inner integral, we obtain that

$$I=\int_0^1\left(\int_0^x\left\{\frac{1}{1-t}\right\}\mathrm{d}t\right)\mathrm{d}x.$$

Integrating by parts, with

$$f(x)=\int_0^x\left\{\frac{1}{1-t}\right\}\mathrm{d}t,\quad f'(x)=\left\{\frac{1}{1-x}\right\},\quad g'(x)=1,\quad g(x)=x,$$

we obtain that

$$\begin{aligned}
I &= \left(x \int_0^x \left\{ \frac{1}{1-t} \right\} \mathrm{d}t \right) \Bigg|_0^1 - \int_0^1 x \left\{ \frac{1}{1-x} \right\} \mathrm{d}x \\
&= \int_0^1 \left\{ \frac{1}{1-t} \right\} \mathrm{d}t - \int_0^1 x \left\{ \frac{1}{1-x} \right\} \mathrm{d}x \\
&= \int_0^1 (1-x) \left\{ \frac{1}{1-x} \right\} \mathrm{d}x \\
&= \int_0^1 x \left\{ \frac{1}{x} \right\} \mathrm{d}x \\
&= 1 - \frac{\zeta(2)}{2},
\end{aligned}$$

where the last equality follows from Problem **2.20** when $m = 1$.

(b) The integral equals

$$\frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} (-1)^{j+1} \left(\frac{1}{j} - \frac{\zeta(j+1)}{j+1} \right).$$

Use the same technique as in the solution of part (a) of the problem.

2.24. Let I_m denote the value of the double integral. We have

$$I_m = \int_0^1 \left(\int_0^1 \left\{ \frac{1}{x+y} \right\}^m \mathrm{d}y \right) \mathrm{d}x \overset{x+y=t}{=} \int_0^1 \left(\int_x^{x+1} \left\{ \frac{1}{t} \right\}^m \mathrm{d}t \right) \mathrm{d}x.$$

We calculate the integral by parts, with

$$f(x) = \int_x^{x+1} \left\{ \frac{1}{t} \right\}^m \mathrm{d}t, \quad f'(x) = \left\{ \frac{1}{x+1} \right\}^m - \left\{ \frac{1}{x} \right\}^m,$$

$g'(x) = 1$ and $g(x) = x$, and we get that

$$\begin{aligned}
I_m &= \left(x \int_x^{x+1} \left\{ \frac{1}{t} \right\}^m \mathrm{d}t \right) \Bigg|_{x=0}^{x=1} - \int_0^1 x \left(\left\{ \frac{1}{x+1} \right\}^m - \left\{ \frac{1}{x} \right\}^m \right) \mathrm{d}x \\
&= 2 \int_1^2 \left\{ \frac{1}{t} \right\}^m \mathrm{d}t - \int_1^2 t \left\{ \frac{1}{t} \right\}^m \mathrm{d}t + \int_0^1 x \left\{ \frac{1}{x} \right\}^m \mathrm{d}x \\
&= 2 \int_1^2 t^{-m} \mathrm{d}t - \int_1^2 t^{1-m} \mathrm{d}t + \int_0^1 x \left\{ \frac{1}{x} \right\}^m \mathrm{d}x.
\end{aligned}$$

When $m=1$, one has that

$$I_1 = 2\ln 2 - 1 + \int_0^1 x\left\{\frac{1}{x}\right\}\mathrm{d}x = 2\ln 2 - \frac{\pi^2}{12},$$

where the equality follows from Problem 2.20.
When $m=2$, one has that

$$I_2 = 1 - \ln 2 + \int_0^1 x\left\{\frac{1}{x}\right\}^2\mathrm{d}x = \frac{5}{2} - \ln 2 - \gamma - \frac{\pi^2}{12},$$

where the equality follows from part (b) of Problem 2.22.
When $m \geq 3$, one has that

$$\begin{aligned} I_m &= \frac{m-3}{(m-1)(m-2)} + \frac{2^{2-m}}{(m-1)(m-2)} + \int_0^1 x\left\{\frac{1}{x}\right\}^m\mathrm{d}x \\ &= \frac{m-3}{(m-1)(m-2)} + \frac{2^{2-m}}{(m-1)(m-2)} + \frac{m!}{2}\sum_{j=1}^{\infty}\frac{(j+1)!}{(m+j)!}(\zeta(j+2)-1), \end{aligned}$$

where the equality follows from part (a) of Problem 2.22.

It is worth mentioning that the case when $m=1$ was solved by an alternative method in [106].

2.25. We have, based on the substitution $y = xt$, that

$$I = \int_0^1\int_0^{1-x}\frac{\mathrm{d}x\mathrm{d}y}{\left(\left\lfloor\frac{x}{y}\right\rfloor+1\right)^k} = \int_0^1 x\left(\int_0^{(1-x)/x}\frac{\mathrm{d}t}{\left(\left\lfloor\frac{1}{t}\right\rfloor+1\right)^k}\right)\mathrm{d}x.$$

We integrate by parts, with

$$f(x) = \int_0^{(1-x)/x}\frac{\mathrm{d}t}{\left(\left\lfloor\frac{1}{t}\right\rfloor+1\right)^k}, \qquad f'(x) = \frac{-1}{x^2\left(\left\lfloor\frac{x}{1-x}\right\rfloor+1\right)^k},$$

$g'(x) = x$ and $g(x) = x^2/2$, and we get that

$$\begin{aligned} I &= \left(\frac{x^2}{2}\int_0^{(1-x)/x}\frac{\mathrm{d}t}{\left(\left\lfloor\frac{1}{t}\right\rfloor+1\right)^k}\right)\Bigg|_{x=0}^{x=1} + \frac{1}{2}\int_0^1\frac{\mathrm{d}x}{\left(\left\lfloor\frac{x}{1-x}\right\rfloor+1\right)^k} \\ &= \frac{1}{2}\int_0^1\frac{\mathrm{d}x}{\left(\left\lfloor\frac{x}{1-x}\right\rfloor+1\right)^k} = \frac{1}{2}\int_0^1\frac{\mathrm{d}x}{\left(\left\lfloor\frac{1-x}{x}\right\rfloor+1\right)^k} \\ &= \frac{1}{2}\int_0^1\frac{\mathrm{d}x}{\left\lfloor\frac{1}{x}\right\rfloor^k} = \frac{1}{2}\int_1^{\infty}\frac{\mathrm{d}t}{t^2\lfloor t\rfloor^k} = \frac{1}{2}\sum_{m=1}^{\infty}\int_m^{m+1}\frac{\mathrm{d}t}{t^2 m^k} \end{aligned}$$

$$= \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m^k} \left(\frac{1}{m} - \frac{1}{m+1} \right)$$
$$= \frac{1}{2} \zeta(k+1) - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m^k (m+1)}.$$

Let $S_k = \sum_{m=1}^{\infty} 1/(m^k(m+1))$. Since $\frac{1}{m^k(m+1)} = \frac{1}{m^k} - \frac{1}{m^{k-1}(m+1)}$, one has that $S_k = \zeta(k) - S_{k-1}$. This implies that $(-1)^k S_k = (-1)^k \zeta(k) + (-1)^{k-1} S_{k-1}$, and it follows that $S_k = (-1)^{k+1} + \sum_{j=2}^{k} (-1)^{k+j} \zeta(j)$. Thus,

$$I = \frac{1}{2} \left((-1)^k - \sum_{j=2}^{k+1} (-1)^{k+j} \zeta(j) \right).$$

Remark. We mention that the case when $k = 2$ is due to Paolo Perfetti [99].

2.26. The integral equals $1 + \frac{1}{2}(k - \zeta(2) - \zeta(3) - \cdots - \zeta(k+1))$. We have, based on the substitution $y = xt$, in the inner integral, that

$$I = \int_0^1 \left(\int_0^1 \frac{\mathrm{d}y}{(\lfloor x/y \rfloor + 1)^k} \right) \mathrm{d}x = \int_0^1 x \left(\int_0^{1/x} \frac{\mathrm{d}t}{(\lfloor 1/t \rfloor + 1)^k} \right) \mathrm{d}x.$$

We integrate by parts, with

$$f(x) = \int_0^{1/x} \frac{\mathrm{d}t}{(\lfloor 1/t \rfloor + 1)^k}, \quad f'(x) = -\frac{1}{x^2 (\lfloor x \rfloor + 1)^k},$$

$g'(x) = x$ and $g(x) = x^2/2$, and we get that

$$I = \left(\frac{x^2}{2} \int_0^{1/x} \frac{\mathrm{d}t}{(\lfloor 1/t \rfloor + 1)^k} \right) \Bigg|_{x=0}^{x=1} + \frac{1}{2} \int_0^1 \frac{\mathrm{d}x}{(\lfloor x \rfloor + 1)^k}$$
$$= \frac{1}{2} \int_0^1 \frac{\mathrm{d}t}{(\lfloor 1/t \rfloor + 1)^k} + \frac{1}{2} \int_0^1 \mathrm{d}x$$
$$= \frac{1}{2} \int_1^{\infty} \frac{\mathrm{d}u}{u^2 (\lfloor u \rfloor + 1)^k} + \frac{1}{2}$$
$$= \frac{1}{2} \sum_{m=1}^{\infty} \int_m^{m+1} \frac{\mathrm{d}u}{u^2 (m+1)^k} + \frac{1}{2}$$
$$= \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{(m+1)^{k+1} m} + \frac{1}{2},$$

and the result follows from Problem **3.8**.

2.27. See [88] or the solutions of Problems **2.28** or **2.29**.

2.28. We have, based on the substitution $kx/y = t$, that

$$I = \int_0^1 \left(\int_0^1 \left\{ k\frac{x}{y} \right\} \mathrm{d}y \right) \mathrm{d}x = k \int_0^1 x \left(\int_{kx}^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t \right) \mathrm{d}x.$$

Integrating by parts, with

$$f(x) = \int_{kx}^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t, \quad f'(x) = -\frac{\{kx\}}{kx^2},$$

$g'(x) = x$ and $g(x) = x^2/2$, we get that

$$\begin{aligned} I &= k \left(\left(\frac{x^2}{2} \int_{kx}^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t \right) \Bigg|_0^1 + \frac{1}{2k} \int_0^1 \{kx\}\,\mathrm{d}x \right) \\ &= k \left(\frac{1}{2} \int_k^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t + \frac{1}{2k} \int_0^1 \{kx\}\,\mathrm{d}x \right). \end{aligned} \tag{2.6}$$

A calculation shows that

$$\int_0^1 \{kx\}\,\mathrm{d}x = \frac{1}{k} \int_0^k \{y\}\,\mathrm{d}y = \frac{1}{k} \sum_{j=0}^{k-1} \int_j^{j+1} \{y\}\,\mathrm{d}y = \frac{1}{k} \sum_{j=0}^{k-1} \int_j^{j+1} (y-j)\mathrm{d}y = \frac{1}{2}. \tag{2.7}$$

On the other hand,

$$\int_k^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t = \sum_{j=k}^{\infty} \left(\ln \frac{j+1}{j} - \frac{1}{j+1} \right) = 1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln k - \gamma. \tag{2.8}$$

Combining (2.6)–(2.8), we get that the integral is calculated and the problem is solved.

Remark. We also have, based on symmetry reasons, that

$$\int_0^1 \int_0^1 \left\{ k\frac{y}{x} \right\} \mathrm{d}x\mathrm{d}y = \frac{k}{2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \ln k - \gamma \right) + \frac{1}{4}.$$

2.29. We have

$$I = \int_0^1 \int_0^1 \left\{ \frac{mx}{ny} \right\} \mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_0^1 \left\{ \frac{mx}{ny} \right\} \mathrm{d}y \right) \mathrm{d}x \overset{t=\frac{mx}{ny}}{=} \frac{m}{n} \int_0^1 x \left(\int_{\frac{mx}{n}}^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t \right) \mathrm{d}x.$$

Integrating by parts, with

$$f(x) = \int_{\frac{mx}{n}}^{\infty} \frac{\{t\}}{t^2} \mathrm{d}t, \quad f'(x) = -\frac{n}{m} \cdot \frac{\left\{ \frac{mx}{n} \right\}}{x^2}, \quad g'(x) = x, \quad g(x) = \frac{x^2}{2},$$

we obtain, since $\left\{\frac{mx}{n}\right\}=\frac{mx}{n}$ for $x\in[0,1)$, that

$$\begin{aligned}
I &= \frac{m}{n}\left(\frac{x^2}{2}\int_{\frac{mx}{n}}^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t\,\bigg|_0^1+\frac{n}{2m}\int_0^1\left\{\frac{mx}{n}\right\}\mathrm{d}x\right)\\
&= \frac{m}{n}\left(\frac{1}{2}\int_{\frac{m}{n}}^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t+\frac{n}{2m}\int_0^1\frac{mx}{n}\mathrm{d}x\right)\\
&= \frac{m}{2n}\left(\int_{\frac{m}{n}}^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t+\frac{1}{2}\right)\\
&= \frac{m}{2n}\left(\int_{\frac{m}{n}}^{1}\frac{\{t\}}{t^2}\mathrm{d}t+\int_1^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t+\frac{1}{2}\right)\\
&= \frac{m}{2n}\left(\int_{\frac{m}{n}}^{1}\frac{1}{t}\mathrm{d}t+1-\gamma+\frac{1}{2}\right)\\
&= \frac{m}{2n}\left(\ln\frac{n}{m}+\frac{3}{2}-\gamma\right).
\end{aligned}$$

Remark. It is worth mentioning that, when $m>n$, the integral equals

$$\frac{m}{2n}\left(\ln\frac{n}{m}+1+\frac{1}{2}+\cdots+\frac{1}{q}-\gamma+\frac{qn^2+r^2+2mr}{2m^2}\right),$$

where q and r are the integers defined by $m=nq+r$ with $r<n$.

2.30. Using the substitution $t=x^k/y$, in the inner integral, we obtain that

$$I=\int_0^1\int_0^1\left\{\frac{x^k}{y}\right\}\mathrm{d}x\mathrm{d}y=\int_0^1\left(\int_0^1\left\{\frac{x^k}{y}\right\}\mathrm{d}y\right)\mathrm{d}x\overset{t=\frac{x^k}{y}}{=}\int_0^1 x^k\left(\int_{x^k}^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t\right)\mathrm{d}x.$$

Integrating by parts, with

$$f(x)=\int_{x^k}^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t,\quad f'(x)=-\frac{k\left\{x^k\right\}}{x^{k+1}},\quad g'(x)=x^k,\quad g(x)=\frac{x^{k+1}}{k+1},$$

we obtain that

$$\begin{aligned}
I &= \left(\frac{x^{k+1}}{k+1}\int_{x^k}^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t\right)\bigg|_0^1+\frac{k}{k+1}\int_0^1\left\{x^k\right\}\mathrm{d}x\\
&= \frac{1}{k+1}\int_1^{\infty}\frac{\{t\}}{t^2}\mathrm{d}t+\frac{k}{k+1}\int_0^1 x^k\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-\gamma}{k+1} + \frac{k}{(k+1)^2} \\
&= \frac{2k+1}{(k+1)^2} - \frac{\gamma}{k+1}.
\end{aligned}
$$

2.31. and **2.32.** We have, based on the substitution $x/y = t$, that

$$
I_k = \int_0^1 \left(\int_0^1 \left\{ \frac{x}{y} \right\}^k \mathrm{d}y \right) \mathrm{d}x = \int_0^1 x \left(\int_x^\infty \frac{\{t\}^k}{t^2} \mathrm{d}t \right) \mathrm{d}x.
$$

We integrate by parts, with

$$
f(x) = \int_x^\infty \frac{\{t\}^k}{t^2} \mathrm{d}t, \quad f'(x) = -\frac{\{x\}^k}{x^2}, \quad g'(x) = x, \quad g(x) = \frac{x^2}{2},
$$

and we get that

$$
\begin{aligned}
I_k &= \left(\frac{x^2}{2} \int_x^\infty \frac{\{t\}^k}{t^2} \mathrm{d}t \right) \Bigg|_{x=0}^{x=1} + \frac{1}{2} \int_0^1 \{x\}^k \mathrm{d}x = \frac{1}{2} \int_1^\infty \frac{\{t\}^k}{t^2} \mathrm{d}t + \frac{1}{2(k+1)} \\
&= \frac{1}{2} V_{k,0} + \frac{1}{2(k+1)} = \frac{1}{2} \sum_{j=1}^\infty \frac{\zeta(j+1)-1}{\binom{k+j}{j}} + \frac{1}{2(k+1)},
\end{aligned}
\tag{2.9}
$$

where the last equality follows from the calculation of $V_{k,m}$ (see the solutions of Problems **2.21** and **2.22**).

On the other hand, we have, based on the substitution $1/x = t$, that $\int_0^1 \left\{\frac{1}{x}\right\}^k \mathrm{d}x = \int_1^\infty \frac{\{t\}^k}{t^2} \mathrm{d}t$, and the equality

$$
\int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\}^k \mathrm{d}x\mathrm{d}y = \frac{1}{2} \int_0^1 \left\{ \frac{1}{x} \right\}^k \mathrm{d}x + \frac{1}{2(k+1)},
$$

follows from the first line of (2.9).

The case $k = 2$. (this is Problem **2.31**)

$$
\begin{aligned}
\int_0^1 \int_0^1 \left\{ \frac{x}{y} \right\}^2 \mathrm{d}x\mathrm{d}y &\overset{(2.9)}{=} \sum_{p=1}^\infty \frac{\zeta(p+1)-1}{(p+1)(p+2)} + \frac{1}{6} \\
&= \sum_{p=1}^\infty \frac{\zeta(p+1)-1}{p+1} - \sum_{p=1}^\infty \frac{\zeta(p+1)-1}{p+2} + \frac{1}{6} \\
&= -\frac{1}{3} - \frac{\gamma}{2} + \frac{\ln(2\pi)}{2},
\end{aligned}
$$

since $\sum_{p=1}^\infty (\zeta(p+1)-1)/(p+2) = 3/2 - \gamma/2 - \ln(2\pi)/2$ [122, p. 213] and $\sum_{p=2}^\infty (\zeta(p)-1)/p = 1-\gamma$ [122, p. 173].

2.33.–2.35. We have, based on the substitution $x/y = t$, that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \frac{y^m}{x^p} \mathrm{d}x\mathrm{d}y = \int_0^1 \frac{1}{x^p} \left(\int_0^1 \left\{\frac{x}{y}\right\}^k y^m \mathrm{d}y \right) \mathrm{d}x$$
$$= \int_0^1 x^{m+1-p} \left(\int_x^\infty \frac{\{t\}^k}{t^{m+2}} \mathrm{d}t \right) \mathrm{d}x.$$

We integrate by parts, with

$$f(x) = \int_x^\infty \frac{\{t\}^k}{t^{m+2}} \mathrm{d}t, \quad f'(x) = -\frac{\{x\}^k}{x^{m+2}}, \quad g'(x) = x^{m+1-p}, \quad g(x) = \frac{x^{m+2-p}}{m+2-p},$$

and we get that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \frac{y^m}{x^p} \mathrm{d}x\mathrm{d}y = \left(\frac{x^{m+2-p}}{m+2-p} \int_x^\infty \frac{\{t\}^k}{t^{m+2}} \mathrm{d}t \right) \Bigg|_{x=0}^{x=1} + \frac{1}{m+2-p} \int_0^1 x^{k-p} \mathrm{d}x$$
$$= \frac{1}{m+2-p} \int_1^\infty \frac{\{t\}^k}{t^{m+2}} \mathrm{d}t + \frac{1}{(m+2-p)(k-p+1)}$$
$$= \frac{V_{k,m}}{m+2-p} + \frac{1}{(m+2-p)(k-p+1)}$$
$$= \frac{k!}{(m+2-p)(m+1)!} \sum_{j=1}^\infty \frac{(m+j)!}{(k+j)!} (\zeta(m+j+1) - 1)$$
$$+ \frac{1}{(m-p+2)(k-p+1)}.$$

When $m = k$ (this is Problem **2.34**), we have that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \frac{y^k}{x^p} \mathrm{d}x\mathrm{d}y = \frac{1}{k-p+1} - \frac{\zeta(2) + \zeta(3) + \cdots + \zeta(k+1)}{(k+2-p)(k+1)}.$$

When $m = p = k$ (this is Problem **2.33**), we have that

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^k \left(\frac{y}{x}\right)^k \mathrm{d}x\mathrm{d}y = 1 - \frac{\zeta(2) + \zeta(3) + \cdots + \zeta(k+1)}{2(k+1)}.$$

2.36. See [3].

2.37. See the solution of Problem **2.38**.

2.38. Recall that (see the calculation of J_k in the solution of Problem **2.15**) if $\alpha > 0$ is a real number, then

$$\int_0^1 t^{\alpha} \left\{\frac{1}{t}\right\} \mathrm{d}t = \frac{1}{\alpha} - \frac{\zeta(\alpha+1)}{\alpha+1}. \tag{2.10}$$

We calculate the double integral by making the substitution $y = xt$, in the inner integral, and we get that

$$I = \int_0^1 x^n \left(\int_0^1 y^n \left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\} \mathrm{d}y\right) \mathrm{d}x = \int_0^1 x^{2n+1} \left(\int_0^{1/x} t^n \{t\} \left\{\frac{1}{t}\right\} \mathrm{d}t\right) \mathrm{d}x.$$

We integrate by parts, with

$$f(x) = \int_0^{1/x} t^n \{t\} \left\{\frac{1}{t}\right\} \mathrm{d}t, \quad f'(x) = -\frac{1}{x^{n+2}} \{x\} \left\{\frac{1}{x}\right\},$$

$g'(x) = x^{2n+1}$ and $g(x) = x^{2n+2}/(2n+2)$, and we get, based on (2.10) with $\alpha = n+1$, that

$$\begin{aligned}
I &= \left(\frac{x^{2n+2}}{2n+2}\int_0^{1/x} t^n \{t\}\left\{\frac{1}{t}\right\} \mathrm{d}t\right)\Bigg|_0^1 + \frac{1}{2n+2}\int_0^1 x^n \{x\}\left\{\frac{1}{x}\right\} \mathrm{d}x \\
&= \frac{1}{2n+2}\int_0^1 t^n \{t\}\left\{\frac{1}{t}\right\} \mathrm{d}t + \frac{1}{2n+2}\int_0^1 x^n \{x\}\left\{\frac{1}{x}\right\} \mathrm{d}x \\
&= \frac{1}{n+1}\int_0^1 x^{n+1}\left\{\frac{1}{x}\right\} \mathrm{d}x \\
&= \frac{1}{n+1}\left(\frac{1}{n+1} - \frac{\zeta(n+2)}{n+2}\right).
\end{aligned}$$

2.39. See [132].

2.40. and **2.41.** We have, based on the substitution $x/y = t$, that

$$I_{m,k} = \int_0^1 \left(\int_0^1 \left\{\frac{x}{y}\right\}^m \left\{\frac{y}{x}\right\}^k \mathrm{d}y\right) \mathrm{d}x = \int_0^1 x \left(\int_x^{\infty} \{t\}^m \left\{\frac{1}{t}\right\}^k \frac{\mathrm{d}t}{t^2}\right) \mathrm{d}x.$$

We integrate by parts, with

$$f(x) = \int_x^{\infty} \{t\}^m \left\{\frac{1}{t}\right\}^k \frac{\mathrm{d}t}{t^2}, \quad f'(x) = -\{x\}^m \left\{\frac{1}{x}\right\}^k \frac{1}{x^2},$$

$g'(x) = x$ and $g(x) = x^2/2$, and we get that

$$\begin{aligned}
I_{m,k} &= \left(\frac{x^2}{2} \int_x^{\infty} \{t\}^m \left\{\frac{1}{t}\right\}^k \frac{\mathrm{d}t}{t^2} \right) \Bigg|_{x=0}^{x=1} + \frac{1}{2} \int_0^1 x^m \left\{\frac{1}{x}\right\}^k \mathrm{d}x \\
&= \frac{1}{2} \int_0^1 x^k \left\{\frac{1}{x}\right\}^m \mathrm{d}x + \frac{1}{2} \int_0^1 x^m \left\{\frac{1}{x}\right\}^k \mathrm{d}x \\
&= \frac{1}{2} V_{m,k} + \frac{1}{2} V_{k,m}. \qquad (2.11)
\end{aligned}$$

Recall that (see the solutions of Problems 2.21 and 2.22)

$$V_{k,m} = \int_0^1 x^m \left\{\frac{1}{x}\right\}^k \mathrm{d}x = \frac{k!}{(m+1)!} \sum_{j=1}^{\infty} \frac{(m+j)!}{(k+j)!} (\zeta(m+j+1)-1).$$

Thus,

$$\begin{aligned}
I_{k,m} = \frac{1}{2} V_{m,k} + \frac{1}{2} V_{k,m} &= \frac{m!}{2(k+1)!} \sum_{j=1}^{\infty} \frac{(k+j)!}{(m+j)!} (\zeta(k+j+1)-1) \\
&+ \frac{k!}{2(m+1)!} \sum_{j=1}^{\infty} \frac{(m+j)!}{(k+j)!} (\zeta(m+j+1)-1).
\end{aligned}$$

When $k=m$ (this is Problem 2.40), one has

$$\begin{aligned}
\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^m \left\{\frac{y}{x}\right\}^m \mathrm{d}x\mathrm{d}y &= \frac{1}{m+1} \sum_{j=1}^{\infty} (\zeta(m+j+1)-1) \\
&= 1 - \frac{\zeta(2)+\zeta(3)+\cdots+\zeta(m+1)}{m+1},
\end{aligned}$$

since $\sum_{p=1}^{\infty}(\zeta(p+1)-1)=1$ (see [122, p. 178]).

2.42. 1. Let $m=\lfloor a \rfloor$ be the floor of a. We have

$$\begin{aligned}
\int_a^{a+k} \{x\}\,\mathrm{d}x &= \int_a^{m+1} \{x\}\,\mathrm{d}x + \sum_{j=m+1}^{k+m-1} \left(\int_j^{j+1} \{x\}\,\mathrm{d}x \right) + \int_{k+m}^{k+a} \{x\}\,\mathrm{d}x \\
&= \int_a^{m+1} (x-m)\mathrm{d}x + \sum_{j=m+1}^{k+m-1} \left(\int_j^{j+1} (x-j)\mathrm{d}x \right) + \int_{k+m}^{k+a} (x-k-m)\mathrm{d}x \\
&= \frac{k}{2}.
\end{aligned}$$

2. The integral equals $(1/2)a_1\cdots a_n$. Let I_n be the value of the integral. We have

$$I_n=\int_0^{a_1}\cdots\int_0^{a_{n-1}}\left(\int_0^{a_n}\{k(x_1+x_2+\cdots+x_n)\}\,\mathrm{d}x_n\right)\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}.$$

Using the substitution $k(x_1+\cdots+x_n)=y$, in the inner integral, we get that

$$\begin{aligned}\int_0^{a_n}\{k(x_1+x_2+\cdots+x_n)\}\,\mathrm{d}x_n&=\frac{1}{k}\int_{k(x_1+\cdots+x_{n-1})}^{k(x_1+\cdots+x_{n-1})+ka_n}\{y\}\,\mathrm{d}y\\&=\frac{a_n}{2},\end{aligned}$$

where the last equality follows based on the first part of the problem. Thus,

$$I_n=\int_0^{a_1}\cdots\int_0^{a_{n-1}}\frac{a_n}{2}\mathrm{d}x_1\cdots\mathrm{d}x_{n-1}=\frac{1}{2}a_1\cdots a_n.$$

2.43. See [135].

2.6.3 Quickies

A mathematician's reputation rests on the number of bad proofs he has given.

A.S. Besicovitch

2.44. We have

$$\int_0^1\{nx\}^k\,\mathrm{d}x\overset{nx=y}{=}\frac{1}{n}\int_0^n\{y\}^k\,\mathrm{d}y=\frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1}(y-j)^k\mathrm{d}y=\frac{1}{n}\sum_{j=0}^{n-1}\int_0^1u^k\mathrm{d}u=\frac{1}{k+1}.$$

2.45. The integral equals $1/2$. Let I denote the value of the integral. We note that if x is a real number and x is not an integer, then

$$\{x\}+\{-x\}=1. \tag{2.12}$$

We have, based on the substitution $x=1-y$, that

$$I=\int_0^1\left\{\frac{1}{1-y}-\frac{1}{y}\right\}\mathrm{d}y=\int_0^1\left(1-\left\{\frac{1}{y}-\frac{1}{1-y}\right\}\right)\mathrm{d}y=1-I.$$

2.46. The integral equals $\gamma - 1/2$. We have

$$I = \int_0^1 \left\{\frac{1}{x} - \frac{1}{1-x}\right\}\left\{\frac{1}{x}\right\}\left\{\frac{1}{1-x}\right\}\mathrm{d}x \overset{x=1-y}{=} \int_0^1 \left\{\frac{1}{1-y} - \frac{1}{y}\right\}\left\{\frac{1}{1-y}\right\}\left\{\frac{1}{y}\right\}\mathrm{d}y$$
$$\overset{(2.12)}{=} \int_0^1 \left(1 - \left\{\frac{1}{y} - \frac{1}{1-y}\right\}\right)\left\{\frac{1}{1-y}\right\}\left\{\frac{1}{y}\right\}\mathrm{d}y = \int_0^1 \left\{\frac{1}{1-y}\right\}\left\{\frac{1}{y}\right\}\mathrm{d}y - I,$$

and the result follows from Problem **2.10**.

2.47. The integral equals $(m!)^2/(2(2m+1)!)$. We have

$$\begin{aligned}
I &= \int_0^1 \left\{\left(\frac{1}{x}\right)^k - \left(\frac{1}{1-x}\right)^k\right\} x^m(1-x)^m \mathrm{d}x \\
&\overset{x=1-y}{=} \int_0^1 \left\{\left(\frac{1}{1-y}\right)^k - \left(\frac{1}{y}\right)^k\right\} y^m(1-y)^m \mathrm{d}y \\
&\overset{(2.12)}{=} \int_0^1 \left(1 - \left\{\left(\frac{1}{y}\right)^k - \left(\frac{1}{1-y}\right)^k\right\}\right) y^m(1-y)^m \mathrm{d}y \\
&= \int_0^1 y^m(1-y)^m \mathrm{d}y - I = \mathrm{B}(m+1, m+1) - I,
\end{aligned}$$

where B denotes the Beta function.

2.48. First, we note that if n is a positive integer, then $\{n(1-y)\} = 1 - \{ny\}$, for all $y \in [0,1]$ except for $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$. Since the Riemann integral does not depend on sets of Lebesgue measure zero (or sets with a finite number of elements), we get, based on the substitution $x = 1-y$, that

$$\begin{aligned}
I &= \int_0^1 \left(x - x^2\right)^k \{nx\}\, \mathrm{d}x = \int_0^1 \left(y - y^2\right)^k \{n(1-y)\}\, \mathrm{d}y \\
&= \int_0^1 \left(y - y^2\right)^k (1 - \{ny\})\, \mathrm{d}y = \int_0^1 \left(y - y^2\right)^k \mathrm{d}y - I \\
&= \mathrm{B}(k+1, k+1) - I \\
&= \frac{(k!)^2}{(2k+1)!} - I.
\end{aligned}$$

Remark. It is worth mentioning that if $f : [0,1] \to \mathbb{R}$ is an integrable function, then

$$\int_0^1 f(x - x^2)\, \{nx\}\, \mathrm{d}x = \frac{1}{2}\int_0^1 f(x - x^2) \mathrm{d}x.$$

2.49. Use the substitution $x = 1 - y$ and identity (2.12).

2.50. The first solution. The integral equals $1/2$. We have

$$I = \int_0^1 \left(\int_0^x \{x-y\}\,\mathrm{d}y + \int_x^1 \{x-y\}\,\mathrm{d}y \right) \mathrm{d}x.$$

We note that when $0 \le y \le x$, then $0 \le x-y \le 1$, and hence $\{x-y\} = x-y$, and when $x \le y \le 1$, then $-1 \le x-y \le 0$, and hence, $\{x-y\} = x-y-\lfloor x-y \rfloor = x-y+1$. It follows that

$$I = \int_0^1 \left(\int_0^x (x-y)\mathrm{d}y + \int_x^1 (x-y+1)\mathrm{d}y \right) \mathrm{d}x = \frac{1}{2}.$$

The second solution. By symmetry,

$$I = \int_0^1 \int_0^1 \{x-y\}\,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \{y-x\}\,\mathrm{d}x\mathrm{d}y.$$

Hence,

$$I = \tfrac{1}{2}(I+I) = \tfrac{1}{2}\int_0^1\int_0^1 \left(\{x-y\}+\{y-x\}\right)\mathrm{d}x\mathrm{d}y \overset{(2.12)}{=} \tfrac{1}{2}\int_0^1\int_0^1 \mathrm{d}x\mathrm{d}y = \tfrac{1}{2},$$

because the set on which the integrand is 0 is a set of measure 0.

2.51. We have, based on symmetry, that

$$I = \int_0^1 \int_0^1 \left\{\frac{x-y}{x+y}\right\} \mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \left\{\frac{y-x}{x+y}\right\} \mathrm{d}x\mathrm{d}y.$$

Thus,

$$I = \frac{1}{2}(I+I) = \frac{1}{2}\int_0^1\int_0^1 \left(\left\{\frac{x-y}{x+y}\right\} + \left\{\frac{y-x}{x+y}\right\} \right) \mathrm{d}x\mathrm{d}y \overset{(2.12)}{=} \frac{1}{2}\int_0^1\int_0^1 \mathrm{d}x\mathrm{d}y = \frac{1}{2},$$

because the set on which the integrand is 0 is a set of measure 0.

2.52. Use symmetry and identity (2.12) and Problem **2.1**.

2.53. The integral equals $1/2$. Use symmetry and identity (2.12).

2.54. The integral equals $1/(2(m+1)^2)$. Use symmetry and identity (2.12).

2.55. The integral equals $1/(2(m+1)^2)$. First, we note that if n is a fixed positive integer, then $\{n(1-a)\} = 1 - \{na\}$, for all $a \in [0,1]$ except for $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$. We have, by symmetry reasons, that

$$I = \int_0^1 \int_0^1 \left\{\frac{nx}{x+y}\right\} x^m y^m \mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^1 \left\{\frac{ny}{x+y}\right\} x^m y^m \mathrm{d}x\mathrm{d}y,$$

and hence,

$$I = \frac{1}{2}(I+I) = \frac{1}{2}\int_0^1\int_0^1 \left(\left\{\frac{nx}{x+y}\right\} + \left\{\frac{ny}{x+y}\right\}\right) x^m y^m \mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1\int_0^1 x^m y^m \mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_0^1 x^m \mathrm{d}x \int_0^1 y^m \mathrm{d}y = \frac{1}{2(m+1)^2}.$$

We used that

$$\left\{\frac{nx}{x+y}\right\} + \left\{\frac{ny}{x+y}\right\} = 1,$$

for all $(x,y) \in [0,1]^2$, except for the points of a set A of Lebesgue area measure zero. The set A is the set of points for which $\frac{y}{x+y} = \frac{k}{n}$, $k = 0, 1, \ldots, n$, and hence, A turns out to be the union of $n+1$ lines through the origin.

2.56. The integral equals $1/(2(m+1)^2)$. See the solution of Problem **2.55**.

2.57. Use symmetry and identity (2.12).

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