Preface

There is no subject so old that something new cannot be said about it.

Fyodor Dostoevsky (1821–1881)

This book is the fruit of my work in the last decade teaching, researching, and solving problems. This volume offers an unusual collection of problems specializing in three topics of mathematical analysis: limits, series, and fractional part integrals. The book is divided into three chapters, each dealing with a specific topic, and two appendices. The first chapter of the book collects non-standard problems on limits of special sequences and integrals. Why limits? First, because in analysis, most things reduce to the calculation of a limit; and second, because limits are the most fundamental basic problems of analysis. Why non-standard limits? Because the standard problems on limits are known, if not very well known, and have been recorded in other books, they might not be so attractive and interesting anymore. The problems vary in difficulty and specialize in different aspects of calculus: from the study of the asymptotic behavior of a sequence to the evaluation of a limit involving a special function, an integral or a finite sum.

The second chapter of the book introduces the reader to a collection of problems that are rarely seen: the evaluation of exotic integrals involving a fractional part term, called fractional part integrals. The problems of this chapter were motivated by the interesting formula $\int_0^1 \{1/x\}\,\mathrm{d}x = 1-\gamma$, which connects an exotic integral to the Euler–Mascheroni constant. One may wonder: are there any other similar formulae? What happens when the integrand function is changed from $\{1/x\}$ to $\{1/x\}^2$? Is the integral $\int_0^1 \{1/x\}^2\,\mathrm{d}x$ calculable in terms of exotic constants or this integral formula is a singular case? Is it possible to extend this equality from the one-dimensional case to the multiple case? The reader will find the answer to these questions by going through the problems of this chapter. The novelty of the problems stands in the fact that, comparatively to the classical integrals that can be calculated by the well-known techniques, integration by parts or substitution, many of these integrals invite the reader to use a host of mathematical techniques that involve elegant connections between integrals, infinite series, exotic constants, and special functions. This chapter has a special section called "Quickies" which contains problems that have an unexpected succint solution. The quickies are solved by using symmetry combined with tricks involving properties of the fractional part function.

The last chapter of the book offers the reader a bouquet of problems with a flavor towards the computational aspects of infinite series and special products, many of these problems being new in the literature. These series, linear or quadratic, single or multiple, involve combinations of exotic terms, special functions, and harmonic numbers and challenge the reader to explore the ability to evaluate an infinite sum, to discover new connections between a series and an integral, to evaluate a sum by using the modern tools of analysis, and to investigate further. In general, the classes of series that can be calculated exactly are widely known and such problems appear in many standard books that have topics involving infinite sums, so by this chapter we offer the reader a collection of interesting and unconventional problems for solutions.

Each chapter contains a section of difficult problems, motivated by other problems in the book, which are collected in a special section entitled "Open problems" and few of them are listed in the order in which the problems appear in the book. These problems may be considered as research problems or projects for students with a strong background in calculus and for the readers who enjoy mathematical research and discovery in mathematics. The intention of having the open problems recorded in the book is to stimulate creativity and the discovery of original methods for proving known results and establishing new ones.

There are two appendices which contain topics of analysis and special function theory that appear throughout the book. In the first part of Appendix A we review the special constants involved in the computations of series and products, the second part of Appendix A contains a bouquet of special functions, from Euler's Gamma function to the celebrated Riemann zeta function, and in the last part of Appendix A we collect some lemmas and theorems from integration theory concerning the calculation of limits of integrals. Appendix B is entirely devoted to the Stolz–Cesàro lemma, a classical tool in analysis, which has applications to the calculations of limits of sequences involving sums.

This volume contains a collection of challenging problems; many of them are new and original. I do not claim originality of all the problems included in the book and I am aware that some may be either known or very old. Other problems, by this volume, are revived and brought into light. Most of the problems are statements to be proved and others are challenges: *calculate, find*. Each chapter contains a very short section, consisting of hints. The hints help the reader to point to the heart of the problem. Detailed solutions are given for nearly all of the problems and for the remaining problems references are provided. I would like to hear about other solutions as well as comments, remarks, and generalizations on the existing ones.

I have not attempted to document the source of every problem. This would be a difficult task: on the one hand, many of the problems of this volume have been discovered by the author over the last decade, some of them have been published in various journals with a problem column, and others will see the light of publication for the first time. Also, there are problems whose history is either lost, with the passing of time, or the author was not aware of it. I have tried to avoid collecting too many problems that are well known or published elsewhere, in order to keep a high level of originality. On the other hand, other problems of this book arose in a natural way: either as generalizations or motivated by known results that have long been forgotten; see the nice alternating series, due to Hardy, recorded in the first part of Problem **3.35**. For such problems, when known, the source of the problem is mentioned, either as a remark, included in the solution, or as a small footnote which contains a brief comment on it.

As I mentioned previously, this book specializes on three selected topics of mathematical analysis: limits, series, and special integrals. I have not collected problems on all topics of analysis because of many problem books, both at the elementary and at the advanced level, that cover such topics. Instead, I tried to offer the reader problems that don't overlap with the existing ones in the literature and others that have received little or no coverage in other texts. Whether I succeeded or not in accomplishing this task is left to the reader to decide, I accept the criticism.

The level of the problems is appropriate for Putnam exams and for problem sections of journals like The American Mathematical Monthly and other journals that have problem sections addressed to undergraduate students. The problems require thorough familiarity with sequences, limits, Riemann integrals, and infinite series and no advanced topics of analysis are required. Anyone with strong knowledge in calculus should be ready for almost everything to be found here.

This book is mainly addressed to undergraduate students with a strong background in analysis, acquired through an honors calculus class, who prepare for prize exams like the Putnam exam and other high-level mathematical contests. Mathematicians and students interested in problem solving will find this collection of topics appealing. This volume is a must-have for instructors who are involved in math contests as well as for individuals who wish to enrich and test their knowledge by solving problems in analysis. It could also be used by anyone for independent study courses. This book can be used by students in mathematics, physics, and engineering and by anyone who wants to explore selected topics of mathematical analysis.

I also address this work to the first and the second year graduate students who want to learn more about the application of certain techniques, to do calculations which happen to have interesting results. Pure and applied mathematicians, who confront certain difficult computations in their research, might find this book attractive.

This volume is accessible to anyone who knows calculus well and to the reader interested in solving challenging problems at the monthly level. However, it is not expected that the book will be an easy reading for students who don't have at their fingertips some classical results of analysis.

I would like to express my great appreciation to Alina Sîntămărian, who read substantial portions of the manuscript and provided many helpful comments and spotted numerous misprints.

I also thank my parents for all of their support during the preparation of this manuscript. Without their effort this book would not have been written.

I am grateful to anonymous referees for their comments and suggestions that led to the improvement of the presentation of the final version of this volume.

Thank you all!

In conclusion, I say to the readers who may use this book: good luck in problem solving since there is no better way to approach mathematics.

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December 2012

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Springer

http://www.springer.com/978-1-4614-6761-8

Limits, Series, and Fractional Part Integrals
Problems in Mathematical Analysis
Furdui, O.
2013, XVIII, 274 p., Hardcover
ISBN: 978-1-4614-6761-8