2
From Points to Sets

> He made loops of blue on the edge of the outermost curtain of the first set; likewise he made them on the edge of the outermost curtain of the second set; he made fifty loops on the one curtain, and he made fifty loops on the edge of the curtain that was in the second set; the loops were opposite one another.
>
> — *Exodus* 36:11–12

Congregatio: Set-Valued Analysis

2.1 Set-Valued Mapping From the forms of the 'point-to-set mappings' $F : \bullet \mapsto \{\cdots\}$ in Section 1.26 (*cf.* (21), (23), (25), and (27) therein), one may naturally proceed to define a set-valued mapping thus:

Definition A A *set-valued mapping* from set X to set Y is a relation $F \subset X \times Y$ (Definition 1.3). It may be denoted

(1) $$F : X \multimap Y\,,$$

such that for each $x \in X$,

(2) $$F(x) = \{\, y \in Y : (x, y) \in F \,\} \subset Y\,.$$

Note the *point-to-set* nature of a set-valued mapping (as opposed to 'point-to-point' for a standard mapping; *cf.* Section 1.6). This relaxation of characteristic 1.24.ii thus includes, when $F(x)$ contains more than one element, Hardy's allowance of mappings in which to a point may plurally "correspond *values* of y". Note, also, the possibility that for some $x \in X$, it may happen that $F(x) = \varnothing$. This relaxation of characteristic 1.24.i thus includes Hardy's allowance of mappings in which values may correspond to only "*some* values of x".

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Note the special 'forked arrow' $-\!\!\sqsubset$ that I have chosen to denote set-valued mappings, in distinction from $\rightarrow$ for a standard (single-valued) mapping. In this chapter when I introduce the concept of set-valued mapping and its properties, I shall also use capital letters to denote set-valued mappings, e.g., $F : X -\!\!\sqsubset Y$, while use lowercase letters to denote standard mappings, e.g., $f : X \rightarrow Y$. This F–versus–f distinction may not, however, necessarily continue in later chapters, but the two different arrows will remain as the characterizing form.

In a set-valued mapping's element-chasing form, one may write

$$F : x \mapsto F(x). \tag{3}$$

The 'source' of F is still a *point* $x \in X$, but now the *value* of the mapping F at the element x is a *set* $F(x) \subset Y$. The source (material cause) and the value (final cause) of a set-valued mapping are thus different in kind from each other, they belonging to different hierarchical levels ('point' versus 'set'). (For a review of the identification of Aristotle's four causes with components of a mapping, see *ML*: Chapter 5.)

A standard (single-valued) mapping (as defined in 1.4) $f : X \rightarrow Y$ may be considered a very specialized set-valued mapping $F : X -\!\!\sqsubset Y$ such that, for each $x \in X$, the value

$$F(x) = \{f(x)\} \tag{4}$$

is a singleton set. Indeed, one can make the formal definition: a set-valued mapping $F : X -\!\!\sqsubset Y$ is called *single-valued* if for each $x \in X$, $F(x)$ is a singleton set. A 'single-valued set-valued mapping' $F : X -\!\!\sqsubset Y$ therefore defines a 'standard' mapping $f : X \rightarrow Y$ by $f : x \mapsto$ the single element in $F(x)$. Thus 'single-valued set-valued mapping' and 'mapping' are equivalent terms.

Since a set-valued mapping $F : X -\!\!\sqsubset Y$ takes its values in the family of subsets of Y (i.e., the power set $\wp Y$ of Y), one may alternatively consider

Definition B A *set-valued mapping* from set X to set Y is a (single-valued) mapping $F : X \rightarrow \wp Y$.

In algebraic terms, the two definitions are equivalent. In topological terms (*cf.* Hadamard's property iii in 1.25), however, because of the complicated power-set topology of $\wp Y$ induced by the topology of Y, it is often advantageous to use Definition A.

2.2 Definition Let $F : X -\!\!\sqsubset Y$ be a set-valued mapping. The *graph* of F is defined as F in its relational form; i.e.,

(5) $$F=\{(x,y)\in X\times Y: y\in F(x)\}=\{(x,y)\in X\times Y:(x,y)\in F\}\subset X\times Y.$$

(Compare this with the 'graph of f' in Section 1.6.)

2.3 Domain The *domain* of the set-valued mapping $F: X \multimap Y$ is the set X, denoted by $\operatorname{dom}(F)$.

The word 'domain' is from the Latin *domus*, 'house, home'. Thus the domain of a mapping is the set of values for which the mapping 'feels at home' (in the idyllic and idealistic sense of the set of values that 'do not cause the mapping any trouble'). In addition, the related Latin word *dominus* means 'lord, master' literally 'one who rules the home', or 'one who owns the domain'. Thus the domain of a mapping is the set of values that the mapping 'owns' or 'has control of'.

There is a subtle difference in the definitions of 'domain' of a set-valued mapping and a (single-valued) mapping, as respectively given in 2.3 and 1.5. When a mapping is considered as a relation $f\subset X\times Y$, one has $\operatorname{dom}(f)\subset X$. But, as I mentioned in Section 1.24, in the notation $f: X\to Y$ for a standard mapping, the convention is that one implicitly takes $\operatorname{dom}(f)=X$ (whence for every $x\in X$, $f(x)$ is defined and it is a single element in Y). Contrariwise, for a set-valued mapping $F: X \multimap Y$, F is still defined at those $x\in X$ for which $F(x)=\varnothing$. One has $\operatorname{dom}(F)=X$ in both interpretations of F, as the relation $F\subset X\times Y$ and as the point-to-set mapping $F: x\mapsto F(x)$ from X to $\mathcal{P}Y$.

2.4 Definition The projections of the graph of F onto its first and second components are, respectively, the *corange* and the *range* of F,

(6) $$\operatorname{cor}(F)=\{x\in X: F(x)\neq\varnothing\},$$

(7) $$\operatorname{ran}(F)=\{y\in Y: y\in F(x) \text{ for some } x\in X\}.$$

Thus $\operatorname{cor}(F)\subset X$ and $\operatorname{ran}(F)\subset Y$, and both inclusions may be proper.

$X\sim\operatorname{cor}(F)=\operatorname{dom}(F)\sim\operatorname{cor}(F)$ is the subset of X that contains all those $x\in X$ at which $F(x)=\varnothing$. Note that some authors, however, define the domain of F as $\{x\in X: F(x)\neq\varnothing\}$ instead of X itself. But there are category-theoretic advantages in allowing $F(x)=\varnothing$ for $x\in\operatorname{dom}(F)$. (I shall return to this point when I presently introduce the category **Rel** of sets and relations.) The range of F may also be expressed as

(8) $$\operatorname{ran}(F) = \bigcup_{x \in X} F(x) \subset Y .$$

F (as a relation in $X \times Y$) is thus a subset of the product $\operatorname{cor}(F) \times \operatorname{ran}(F)$. $x \in \operatorname{cor}(F)$ means there exists $y \in \operatorname{ran}(F)$ such that $(x, y) \in F$; dually, $y \in \operatorname{ran}(F)$ means there exists $x \in \operatorname{cor}(F)$ such that $(x, y) \in F$.

If there exists a subset C of Y such that $F(x) = C$ for all $x \in X$, then F is called a *constant set-valued mapping*. As a relation in $X \times Y$, F is the subset $X \times C$. The constant mapping $f : x \mapsto c$ (where $c \in Y$) thus defines the constant set-valued mapping $F : x \mapsto \{c\}$. The universal relation $U = X \times Y$ from X to Y (*cf.* Section 1.3) is the constant set-valued mapping $U : X \multimap\!\sqsubset Y$ that sends everything to the set Y, i.e., such that $F(x) = Y$ for all $x \in X$.

2.5 The Constant Empty-Set-Valued Mapping The constant set-valued mapping $F : X \multimap\!\sqsubset Y$ that sends everything to the empty set, i.e., such that

(9) $$F(x) = \varnothing \quad \text{for all } x \in X ,$$

has

(10) $$\operatorname{cor}(F) = \{x \in X : F(x) \neq \varnothing\} = \varnothing ,$$

(11) $$\operatorname{ran}(F) = \varnothing ,$$

and

(12) $$\{x \in X : F(x) = \varnothing\} = X \sim \operatorname{cor}(F) = X .$$

As a relation in $X \times Y$, F is thus the 'empty relation' $\varnothing$ (*cf.* Section 1.3).

Note that the 'empty relation' $\varnothing$ is a legitimate set-valued mapping from set X to set Y, for all sets X and Y. This is in contrast to standard mappings, when the 'empty mapping' $\varnothing : X \to Y$ is only a mapping when $X = \varnothing$. Recall (*ML*: A.4) that by convention $Y^{\varnothing} = \{\varnothing\}$; thus the 'empty mapping' $\varnothing$ is the only mapping from the empty set to any set Y. If $X \neq \varnothing$, however, then $f(X) \neq \varnothing$ for any mapping f with $\operatorname{dom}(f) = X$, whence $\operatorname{ran}(f) \neq \varnothing$; so one has $\varnothing^X = \varnothing$ whence $\varnothing \notin \varnothing^X$.

It is interesting to note that for any two sets X and Y, whatever their nature, the constant empty-set-valued mapping $\varnothing : X \dashv\sqsubset Y$ is the same one. There is only one constant empty-set-valued mapping because there is only one empty set. Suppose $\varnothing_1$ and $\varnothing_2$ are two empty sets. Then $x \in \varnothing_1 \Rightarrow x \in \varnothing_2$, since there is no $x \in \varnothing_1$ to contradict this statement; thus $\varnothing_1 \subset \varnothing_2$. Likewise, $\varnothing_2 \subset \varnothing_1$. Therefore, $\varnothing_1 = \varnothing_2$.

The map that is a 'perfect and absolute blank' of Lewis Carroll's Bellman is an example of a constant empty-set-valued mapping (indeed, a manifestation of *the* empty set) $\varnothing$. As a material system, a blank sheet of paper is, of course, *structurally* nonempty, but, as a map, it *functions* as the empty set.

2.6 Definition For a set-valued mapping $F : X \dashv\sqsubset Y$, the set Y is called the *codomain* of F, denoted by $\operatorname{cod}(F)$.

Thus one has the dual relations

(13) $$\operatorname{ran}(F) \subset \operatorname{cod}(F) = Y, \qquad \operatorname{cor}(F) \subset \operatorname{dom}(F) = X.$$

2.7 Definition A set-valued mapping $F : X \dashv\sqsubset Y$ is:

i. *Surjective* if

(14) $$\operatorname{ran}(F) = \operatorname{cod}(F) = Y$$

ii. *Semi-single-valued* if

(15) $$F(x_1) \cap F(x_2) \neq \varnothing \quad \Rightarrow \quad F(x_1) = F(x_2)$$

iii. *Injective* if

(16) $$x_1 \neq x_2 \quad \Rightarrow \quad F(x_1) \cap F(x_2) = \varnothing$$

(which is contrapositively equivalent to

(17) $$F(x_1) \cap F(x_2) \neq \varnothing \quad \Rightarrow \quad x_1 = x_2)$$

A semi-single-valued mapping $F : X \dashv\sqsubset Y$ defines a *partition* of its range $\operatorname{ran}(F)$; its distinct values are pairwise disjoint subsets of Y, forming the *blocks* of the partition. It also defines a partition of its domain X: one block is $X \sim \operatorname{cor}(F)$ (which contains all those $x \in X$ for which $F(x) = \varnothing$), and then

$\operatorname{cor}(F)$ is partitioned into blocks that are in one-to-one correspondence with the blocks of $\operatorname{ran}(F)$.

A single-valued mapping $f: X \to Y$ is clearly also a semi-single-valued mapping, and the blocks of the partition of its range $f(X)$ are the singleton sets $\{f(x)\}$. The mapping f induces an equivalence relation R_f on X ($x_1 R_f x_2$ iff $f(x_1) = f(x_2)$), whence defines the single-valued natural mapping, the projection $\pi_f : X \to X/R_f$, which sends $x \in X$ to its R_f-equivalence class $[x]_{R_f} \in X/R_f$ (*cf.* *ML*: 2.19–2.21). The single-valued natural mapping $\pi_f : x \mapsto [x]_{R_f}$ may alternatively be formulated as the set-valued mapping $\Pi_f : X -\!\!\sqsubset X$ defined by $\Pi_f : x \mapsto [x]_{R_f}$, which sends $x \in X$ to $[x]_{R_f} \subset X$. Π_f is semi-single-valued, since equivalence classes are mutually exclusive.

It is also evident that an injective set-valued mapping is semi-single-valued. Each block of the partition of the *corange* of an injective set-valued mapping is a singleton set. An injective single-valued mapping is an injective set-valued mapping.

2.8 Embedding For $A \subset X$, the injective set-valued mapping $i: A -\!\!\sqsubset X$ defined by $i(x) = \{x\}$ for all $x \in A$ is called the *inclusion map* (or the *embedding*) of A in X. The inclusion map of X in X is called the *identity map* on X, denoted 1_X (whence $1_X : x \mapsto \{x\}$). These match their definitions as (single-valued) mappings (*cf.* Section 1.13).

As relations, the inclusion map $i: A -\!\!\sqsubset X$ is the set $i = \{(x,x) : x \in A\} \subset A \times X (\subset X \times X)$, and the identity map $1_X : X -\!\!\sqsubset X$ is the set $1_X = \{(x,x) : x \in X\} \subset X \times X$. Thus each is a member of $\mathcal{P}(X \times X)$ that consists of all the 'diagonal elements' corresponding to the embedded set.

From Sets to Sets

2.9 Definition Let F be a set-valued mapping from X to Y. If $E \subset X$, the *image* of E under F is defined as the set

$$F(E) = \bigcup_{x \in E} F(x) \subset Y. \tag{18}$$

This is the natural extension of Definition 1.8 of image of a (single-valued) mapping, whence the mapping in (18) is in fact the 'power-set map' $\mathcal{P}F : \mathcal{P}X \to \mathcal{P}Y$. It is also evident that

(19) $$F(X) = \operatorname{ran}(F),$$

whence, in particular, that surjective means $Y = F(X)$.

While the definition of a 'set-to-set' mapping $\mathrm{P}f : \mathrm{P}X \to \mathrm{P}Y$ from a 'point-to-point' mapping $f : X \to Y$ only goes one way (as explained in Section 1.18), the definition of a 'set-to-set' mapping $\mathrm{P}F : \mathrm{P}X \to \mathrm{P}Y$ from a 'point-to-set' mapping $F : X \multimap Y$ is reversible. Given a mapping $g : \mathrm{P}X \to \mathrm{P}Y$ of power sets, the assignment, for $x \in X$, $F(x) = g(\{x\})$ naturally defines a set-valued mapping $F : X \multimap Y$ for which $g = \mathrm{P}f$.

The 'set-to-set' mapping $\mathrm{P}F : \mathrm{P}X \to \mathrm{P}Y$ has the following properties (*cf.* Theorem 1.19):

2.10 Theorem *Let $F : X \multimap Y$ be a set-valued mapping and $A, B \subset X$. Then:*

i. $A = \varnothing \;\Rightarrow\; F(A) = \varnothing$.
ii. $A \subset B \;\Rightarrow\; F(A) \subset F(B)$.
iii. $F(A \cup B) = F(A) \cup F(B)$.
iv. $F(A \cap B) \subset F(A) \cap F(B)$.
v. $F(B \sim A) \supset F(B) \sim F(A)$.

Since a (single-valued) mapping is a specialized set-valued mapping through the correspondence (4), whatever properties that set-valued mappings have cannot be contradictory to (but may be weaker than) their analogs for mappings. Properties 2.10.ii–v are identical to their counterparts in Theorem 1.19. But since $F(x) = \varnothing$ is allowed, the implication in property 2.10.i now only goes one way.

2.11 Theorem *Let $F : X \multimap Y$ be a set-valued mapping. The following are equivalent:*

i. *F is injective.*
ii. *For all $A, B \subset X$, $A \cap B = \varnothing \;\Rightarrow\; F(A) \cap F(B) = \varnothing$.*
iii. *For all $A, B \subset X$, $F(A \cap B) = F(A) \cap F(B)$.*
iv. *For all $A, B \subset X$, $F(B \sim A) = F(B) \sim F(A)$.*
v. *For all $A \subset X$, $F(X \sim A) \subset Y \sim F(A)$.*

2.12 Theorem *Let $F : X \multimap Y$ be a set-valued mapping. The following are equivalent:*

i. *F is surjective.*
ii. *For all $A \subset X$, $Y \sim F(A) \subset F(X \sim A)$.*

Inverse Mapping

2.13 Definition Given a set-valued mapping $F : X \multimap Y$, its *inverse* is the set-valued mapping $F^{-1} : Y \multimap X$ (equivalently, the relation $F^{-1} \subset Y \times X$) defined by interchanging the ordered components in the graph (5) of F:

$$F^{-1} = \{(y,x) \in Y \times X : y \in F(x)\} = \{(y,x) \in Y \times X : (x,y) \in F\} \subset Y \times X \,. \tag{20}$$

A (single-valued) mapping is not necessarily injective, and so its inverse is not necessarily single-valued and hence not (well defined as) a mapping. But the inverse of a set-valued mapping is always a set-valued mapping. Note, however, that F^{-1} is itself a point-to-set mapping (not a 'set-to-point mapping', as a direct reversal-of-roles 'inverse' of a point-to-set mapping would have been), with its value at the point $y \in Y$ defined as the set

$$F^{-1}(y) = \{x \in X : (x,y) \in F\} \subset X \,. \tag{21}$$

Indeed, since both $F(x)$ and $F^{-1}(y)$ are defined by the membership $(x,y) \in F$ (*cf.* (2) and (21)), one trivially has

2.14 Lemma *Let $F : X \multimap Y$, $x \in X$, and $y \in Y$. Then*

$$y \in F(x) \;\; \textit{iff} \;\; x \in F^{-1}(y) \,. \tag{22}$$

While F maps points in X to subsets of Y, the inverse F^{-1} maps points in Y to subsets of X; so the involvements of the sets X and Y in F and F^{-1} are asymmetric. The situation is more evident if one considers the maps in terms of Definition 2.1B:

$$F : X \to \wp Y \,, \qquad F^{-1} : Y \to \wp X \,. \tag{23}$$

There is, however, symmetry in corange and range:

$$\operatorname{cor}(F) = \operatorname{ran}(F^{-1}) = F^{-1}(Y) \,, \qquad F(X) = \operatorname{ran}(F) = \operatorname{cor}(F^{-1}) \,. \tag{24}$$

Note also that

(25) $$\operatorname{dom}(F) = \operatorname{cod}(F^{-1}) = X, \qquad Y = \operatorname{cod}(F) = \operatorname{dom}(F^{-1}).$$

And that

(26) $$(F^{-1})^{-1} = F.$$

For $F : X \multimap Y$, all the $x \in X$ for which $F(x) = \varnothing$ are not members of $\operatorname{cor}(F)$ and, therefore, not members of $\operatorname{ran}(F^{-1})$. In other words, when $X \sim \operatorname{cor}(F) \neq \varnothing$, F^{-1} is not surjective. If $y \in Y \sim \operatorname{ran}(F)$, then $F^{-1}(y) = \varnothing$. Consider the simple example of $F : \{1,2\} \multimap \{p,q\}$ with $F(1) = \{p,q\}$ and $F(2) = \{q\}$; then $F^{-1}(p) = \{1\}$ and $F^{-1}(q) = \{1,2\}$. This F^{-1} is not semi-single-valued and (hence) not injective. Thus, in contrast to an inverse mapping f^{-1} (which is only defined from $\operatorname{ran}(f)$ to X but is both injective and surjective thence, *cf.* Section 1.15), an inverse set-valued mapping F^{-1} is defined from Y to X, but is not necessarily either injective or surjective.

2.15 Theorem *Let $F : X \multimap Y$ and $F^{-1} : Y \multimap X$ be its inverse. Then:*

i. *If F is single-valued, F^{-1} is injective.*
ii. *If F is injective, F^{-1} is single-valued.*
iii. *If F is semi-single-valued, F^{-1} is semi-single-valued.*

Inverse Images

If $f : X \to Y$ is a mapping and $E \subset Y$, the inverse image of E under f, the set $f^{-1}(E) = \{x \in X : f(x) \in E\}$, may be considered in two equivalent ways:

i. As the set $\{x \in X : \{f(x)\} \cap E \neq \varnothing\}$
ii. As the set $\{x \in X : \{f(x)\} \subset E\}$

When these two sets are interpreted in set-valued mapping terms (recalling that f defines the special singleton-set-valued mapping $x \mapsto \{f(x)\}$), they give two different notions of the inverse image of a set $E \subset Y$:

2.16 Definition For a set-valued mapping $F : X \multimap Y$ and $E \subset Y$,

i. The *inverse image* of E by F is the set

$$F^{-1}(E) = \begin{cases} \{x \in X : F(x) \cap E \neq \varnothing\} & \text{if } E \neq \varnothing \\ \varnothing & \text{if } E = \varnothing \end{cases}. \tag{27}$$

ii. The *core* of E by F is the set

$$F^{+1}(E) = \{x \in X : F(x) \subset E\}. \tag{28}$$

The two notions i and ii coincide (and are identical to the inverse image in Definition 1.12) when the mapping is single-valued, since $F(x) \cap E \neq \varnothing$ iff $F(x) \subset E$ when $F(x)$ is a singleton set.

Note that when $F^{-1} : Y \multimap X$ is considered a set-valued mapping itself (as opposed to its role as the inverse of another set-valued mapping), for $E \subset Y$ the set $F^{-1}(E)$, the image of E under F^{-1}, has already been defined in 2.9. It is the set

$$F^{-1}(E) = \bigcup_{y \in E} F^{-1}(y) \subset X. \tag{29}$$

One may verify that this defines the same set as in (27), so the notation is consistent. In particular, for $y \in Y$, $F(x) \cap \{y\} \neq \varnothing$ iff $y \in F(x)$ iff $(x, y) \in F$, thus $F^{-1}(\{y\})$ as defined by (27) when $E = \{y\}$ is identical to $F^{-1}(y)$ as defined in (21).

The similarity of the word 'core' to the symbol 'cor' for corange may lead to confusion, so it is perhaps opportune to clarify here at the outset. For a set-valued mapping $F : X \multimap Y$ and $E \subset Y$, both the corange of F and the core of E by F are subsets of the domain X of F:

$$\operatorname{cor}(F) \subset X \quad \text{and} \quad F^{+1}(E) \subset X. \tag{30}$$

But there are no general inclusion relations between $\operatorname{cor}(F)$ and $F^{+1}(E)$. Other than having the first three letters of their names in common, corange and core are very different entities: $\operatorname{cor}(\bullet)$, the corange of $\bullet$, accepts one argument F that is a set-valued mapping, whereas $\bullet^{+1}(\bullet)$, the core of $\bullet$ by $\bullet$, accepts two arguments, the first being a set-valued mapping F and the second being a subset E of the mapping's codomain.

The definition of $F^{+1}(E)$ implies that

(31) $$F^{+1}(\varnothing) = \{x \in X : F(x) = \varnothing\} = X \sim \mathrm{cor}(F) = \mathrm{dom}(F) \sim \mathrm{cor}(F);$$

i.e., $F^{+1}(\varnothing)$ is the subset of X that contains all those $x \in X$ at which $F(x) = \varnothing$, and it is not necessarily the empty set. Equivalently, (31) says

(32) $$\mathrm{cor}(F) = \{x \in X : F(x) \neq \varnothing\} = X \sim F^{+1}(\varnothing) = \mathrm{dom}(F) \sim F^{+1}(\varnothing).$$

Note that for every $E \subset Y$,

(33) $$F^{+1}(\varnothing) \subset F^{+1}(E),$$

and

(34) $$F^{-1}(E) \subset X \sim F^{+1}(\varnothing).$$

This last inclusion says that $F^{-1}(E) \cap F^{+1}(\varnothing) = \varnothing$, which means if $x \in F^{-1}(E)$, then $F(x) \neq \varnothing$.

Consider the simple example of $F : \{1,2\} \multimap \{p,q\}$ with $F(1) = \{p,q\}$ and $F(2) = \varnothing$; then $\mathrm{cor}(F) = \{1\}$, $F^{-1}(\{p\}) = \{1\}$, and $F^{+1}(\{p\}) = \{2\}$. This shows that in general there are no inclusion relations between $\mathrm{cor}(F)$ and $F^{+1}(E)$ and between $F^{-1}(E)$ and $F^{+1}(E)$.

The same authors who define the domain of F as $\{x \in X : F(x) \neq \varnothing\}$ (i.e., my $\mathrm{cor}(F)$) also define their alternate inverse (sometimes called *upper inverse*) accordingly, for $E \subset Y$, as

(35) $$F^{\wedge 1}(E) = \begin{cases} \{x \in X : F(x) \neq \varnothing \text{ and } F(x) \subset E\} & \text{if } E \neq \varnothing \\ \varnothing & \text{if } E = \varnothing \end{cases}.$$

This puts, for all $E \subset Y$,

(36) $$F^{\wedge 1}(E) \subset X \sim F^{+1}(\varnothing) = \mathrm{cor}(F).$$

One sees that

(37) $$F^{+1}(E) = F^{\wedge 1}(E) \cup F^{+1}(\varnothing) \text{ and } F^{\wedge 1}(E) \cap F^{+1}(\varnothing) = \varnothing$$

(i.e., $F^{+1}(E)$ is the union of the disjoint sets $F^{\wedge 1}(E)$ and $F^{+1}(\varnothing)$), and

(38) $$F^{\wedge 1}(E) \subset F^{-1}(E).$$

Also

(39) $$F^{-1}(E) \cap F^{+1}(\varnothing) = \varnothing .$$

In particular,

(40) $$F^{-1}(Y) = F^{\wedge 1}(Y) = X \sim F^{+1}(\varnothing) = \operatorname{cor}(F)$$

and

(41) $$F^{+1}(Y) = X .$$

2.17 Lemma *For a set-valued mapping $F : X \multimap Y$ and $E \subset Y$,*

(42) $$F^{-1}(Y \sim E) = X \sim F^{+1}(E);$$

(43) $$F^{+1}(Y \sim E) = X \sim F^{-1}(E).$$

With the identities (32) and (37), one has

2.18 Corollary *For a set-valued mapping $F : X \multimap Y$ and $E \subset Y$,*

(44) $$F^{-1}(Y \sim E) = \operatorname{cor}(F) \sim F^{\wedge 1}(E);$$

(45) $$F^{\wedge 1}(Y \sim E) = \operatorname{cor}(F) \sim F^{-1}(E).$$

Note that among the three varieties of 'inverse images' that I have defined, inverse image $F^{-1}(E)$, core $F^{+1}(E)$, and upper inverse $F^{\wedge 1}(E)$, only the first is associated with an 'inverse mapping', viz., $F^{-1} : Y \multimap X$, with

(46) $$F^{-1}(y) = F^{-1}(\{y\}) = \{x \in X : F(x) \cap \{y\} \neq \varnothing\} = \{x \in X : y \in F(x)\} .$$

While one may similarly define $F^{+1}(y)$ and $F^{\wedge 1}(y)$,

(47) $$F^{+1}(y) = F^{+1}(\{y\}) = \{x \in X : F(x) \subset \{y\}\},$$

(48) $$F^{\wedge 1}(y) = F^{\wedge 1}(\{y\}) = \{x \in X : F(x) \neq \varnothing \text{ and } F(x) \subset \{y\}\},$$

the resulting mappings are not very useful. The restriction (48) means $F^{\wedge 1}(y)$ would contain only those $x \in X$ for which $F(x)$ is the singleton set $\{y\}$, and (47) just means $F^{+1}(y)$ would contain all those $x \in X$ for which $F(x)$ is either the empty set $\varnothing$ or the singleton set $\{y\}$, i.e., $F^{+1}(y) = F^{+1}(\varnothing) \cup F^{\wedge 1}(y)$.

2.19 Theorem *Let $F : X \multimap Y$ and $A, B \subset Y$. Then:*

i. $A = \varnothing \;\Rightarrow\; F^{-1}(A) = \varnothing$.
ii. $A \subset B \;\Rightarrow\; F^{-1}(A) \subset F^{-1}(B)$.
iii. $F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B)$.
iv. $F^{-1}(A \cap B) \subset F^{-1}(A) \cap F^{-1}(B)$.
v. $F^{-1}(B \sim A) \supset F^{-1}(B) \sim F^{-1}(A)$.

Compare this with Theorem 2.10 for the corresponding properties of F. Note that the two theorems are in fact the same (with the obvious corresponding changes due to the replacement of $F : X \multimap Y$ in 2.10 by $F^{-1} : Y \multimap X$ in 2.19). I put Theorem 2.19 here to emphasize the point that $F^{-1} : Y \multimap X$ is a 'general' set-valued mapping like any other, without any inherent special properties. This fact is different from the case of (single-valued) mappings, for which f^{-1} only exists when f is injective, and this specialization of f gives f^{-1} stronger properties that f^{-1} is necessarily bijective. Compare Theorems 2.10 and 2.19 with their counterparts for f and f^{-1}, Theorems 1.19 and 1.20.

Iterated Mappings of Sets

Theorem 1.21 lists three properties of the combination of a mapping $f : X \to Y$ and its (not-necessarily-a-mapping) inverse f^{-1} when they map sets. I shall examine their counterparts for set-valued mappings.

2.20 First Combination Theorem 1.21.i says that for $A \subset X$, one has $A \subset f^{-1}(f(A))$. But there is no corresponding property $A \subset F^{-1}(F(A))$ for a set-valued mapping $F : X \multimap Y$. Consider the simple example of $F : \{1,2\} \multimap$

$\{p,q\}$ with $F(1)=\{p\}$ and $F(2)=\varnothing$; then $F^{-1}(F(\{1,2\}))=F^{-1}(\{p\})$ $=\{1\}$, so $\{1,2\}\not\subset F^{-1}(F(\{1,2\}))$.

Suppose $F : X \multimap Y$ and $A \subset X$. Recall ((31) above) $F^{+1}(\varnothing)$ is the subset of X that contains all those $x \in X$ at which $F(x)=\varnothing$. It follows from (39) above that $F^{-1}(F(A)) \cap F^{+1}(\varnothing) = \varnothing$; i.e., $F^{-1}(F(A)) \subset \operatorname{cor}(F)$. Now $F^{-1}(F(A))$ is the set $\{x \in X : F(x) \cap F(A) \neq \varnothing\}$, so if $x \in A$ and $F(x) \neq \varnothing$, then $x \in F^{-1}(F(A))$. Thus $A \sim F^{+1}(\varnothing) \subset F^{-1}(F(A))$. Stated otherwise, if one restricts to subsets $A \subset \operatorname{cor}(F) = X \sim F^{+1}(\varnothing)$, then one does have $A \subset F^{-1}(F(A))$. Conversely, since $F^{-1}(F(A)) \subset \operatorname{cor}(F)$, if $A \subset F^{-1}(F(A))$, then *a fortiori* $A \subset \operatorname{cor}(F)$. Thus

2.21 Lemma *Let* $F : X \multimap Y$. *Then* $A \subset F^{-1}(F(A))$ *iff* $A \subset \operatorname{cor}(F)$.

For $F : X \multimap Y$, $G(x) = F^{-1}(F(x))$ is a set-valued mapping from X to X. Lemma 2.21 says that if $x \in \operatorname{cor}(F)$ (i.e., if $F(x) \neq \varnothing$), then $x \in F^{-1}(F(x))$, which means the 'diagonal element' $(x,x) \in G$. Let $i : \operatorname{cor}(F) \multimap X$ be the inclusion map of $\operatorname{cor}(F)$ in X (Section 2.8). Then as a relation $i = \{(x,x) : x \in \operatorname{cor}(F)\} \subset \operatorname{cor}(F) \times X \,(\subset X \times X)$, whence $i \subset G$.

Subsets $A \subset X$ for which $A = F^{-1}(F(A))$ are special:

2.22 Definition Let $F : X \multimap Y$. A subset $A \subset X$ for which $A = F^{-1}(F(A))$ is called a *stable subset* (of X under F).

It follows from Lemma 2.21 that a stable subset must be a subset of $\operatorname{cor}(F)$.

2.23 Theorem *Let* $F : X \multimap Y$. *The stable subsets form a complemented lattice* $\mathfrak{S}$ (*a complemented sublattice of the power-set lattice* $\mathrm{P}(\operatorname{cor}(F))$).

Proof Note that $F^{-1}(F(\varnothing)) = \varnothing$ and $F^{-1}(F(\operatorname{cor}(F))) = F^{-1}(\operatorname{ran}(F))$ $= \operatorname{cor}(F)$, so $\varnothing \in \mathfrak{S}$ and $\operatorname{cor}(F) \in \mathfrak{S}$ (respectively the least element and the greatest element of $\mathfrak{S}$).

Let $A \subset \operatorname{cor}(F)$, whence $A \subset F^{-1}(F(A))$ by Lemma 2.21. A nonempty $F^{-1}(F(A)) \sim A$ means the existence of an element $x \in F^{-1}(F(A)) \sim A$. This element x must be in $\operatorname{cor}(F) \sim A$, which means $F(x) \cap F(\operatorname{cor}(F) \sim A) \neq \varnothing$. At the same time, $x \in F^{-1}(F(A)) \sim A$, so *a fortiori* $x \in F^{-1}(F(A))$, which means $F(x) \cap F(A) \neq \varnothing$. In other words, $A = F^{-1}(F(A))$ iff there is no $x \in \operatorname{cor}(F)$ such that $F(x) \cap F(A) \neq \varnothing$ and $F(x) \cap F(\operatorname{cor}(F) \sim A) = \varnothing$. But this equivalent condition is the same when A is replaced by $\operatorname{cor}(F) \sim A$, since $\operatorname{cor}(F) \sim (\operatorname{cor}(F) \sim A) = A$, whence it also defines the conditions under which $\operatorname{cor}(F) \sim A = F^{-1}(F(\operatorname{cor}(F) \sim A))$. Thus $A = F^{-1}(F(A))$ iff $\operatorname{cor}(F) \sim A = F^{-1}(F(\operatorname{cor}(F) \sim A))$, and this says $A \in \mathfrak{S}$ iff $\operatorname{cor}(F) \sim A \in \mathfrak{S}$. $\mathfrak{S}$ is therefore complemented.

Let $A, B \in \mathfrak{S}$. Then, using Theorem 2.10.iii and Theorem 2.19.iii,

$$F^{-1}(F(A \cup B)) = F^{-1}(F(A)) \cup F^{-1}(F(B)) = A \cup B, \tag{49}$$

so $A \cup B \in \mathfrak{S}$. Since $\operatorname{cor}(F) \sim (A \cap B) = (\operatorname{cor}(F) \sim A) \cup (\operatorname{cor}(F) \sim B) \in \mathfrak{S}$, one also has $A \cap B \in \mathfrak{S}$. □

Theorem 1.22 says that a mapping $f : X \to Y$ is injective iff $A = f^{-1}(f(A))$ for all $A \subset X$. Correspondingly, one has

2.24 Lemma *A set-valued mapping $F : X \multimap Y$ is injective iff $A = F^{-1}(F(A))$ for all $A \subset \operatorname{cor}(F)$.*

When a set-valued mapping $F : X \multimap Y$ is injective, every subset of $\operatorname{cor}(F)$ is stable; the complemented lattice $\mathfrak{S}$ of stable subsets is thus all of $\mathfrak{P}(\operatorname{cor}(F))$.

An injective mapping $f : X \to Y$ means $f^{-1} \circ f = 1_X$, the identity mapping $x \mapsto x$ on the domain X. But an injective set-valued mapping $F : X \multimap Y$ means

$$F^{-1}(F(x)) = \begin{cases} \{x\} & \text{if } x \in \operatorname{cor}(F) \\ \varnothing & \text{if } F(x) = \varnothing \end{cases}, \tag{50}$$

i.e., $x \mapsto F^{-1}(F(x))$ is a disjoint union, the concatenation of the inclusion map i of $\operatorname{cor}(F)$ in X and the constant empty-set-valued mapping $\varnothing$ on $F^{+1}(\varnothing) = X \sim \operatorname{cor}(F)$. So even for an injective F, the combination $G(x) = F^{-1}(F(x))$ is still not quite the identity mapping on X (unless $F^{+1}(\varnothing) = \varnothing$). When i and G are considered as subsets of $X \times X$, $G = i = \{(x,x) : x \in A\}$ (but not necessarily $G = 1_X = \{(x,x) : x \in X\}$).

2.25 Second Combination Given a mapping $f : X \to Y$ and $B \subset Y$, one has $B \supset f(f^{-1}(B))$ (Theorem 1.21.ii). But there is no containment relation between B and $F(F^{-1}(B))$ for a set-valued mapping $F : X \multimap Y$. Consider the example $F : \{1,2\} \multimap \{p,q,r\}$ with $F(1) = \{p,q\}$ and $F(2) = \varnothing$; then $F(F^{-1}(\{q,r\}))$ $= F(\{1\}) = \{p,q\}$. This time, even a restriction to $B \subset \operatorname{ran}(F)$ (dual to $A \subset$ $\operatorname{cor}(F)$ in Lemma 2.21) does not help: in the example, $\{p\} \not\supset F(F^{-1}(\{p\}))$ = $F(\{1\})$ = $\{p,q\}$. Neither does the specialization to surjections: Theorem 1.23 says that a mapping $f : X \to Y$ is surjective iff $B = f(f^{-1}(B))$ for all $B \subset Y$. But the same F in my example is a surjective set-valued mapping from $\{1,2\}$ onto $\{p,q\}$, and still $\{p\} \neq F(F^{-1}(\{p\}))$ = $F(\{1\})$ = $\{p,q\}$.

Since neither $x \mapsto F^{-1}(F(x))$ nor $y \mapsto F(F^{-1}(y))$ is necessarily the identity mapping on its respective domain, one must understand the usage of the term '*inverse* set-valued mapping' with this in mind: it is not the usual algebraic definition in connection with a 'reversal entity for the recovery of the identity'. For this reason, some authors call $F^{-1} : Y \multimap X$ the 'converse' of $F : X \multimap Y$ instead of the 'inverse'.

2.26 Third Combination Given a mapping $f : X \to Y$, $A \subset X$, and $B \subset Y$, one has $f(A \cap f^{-1}(B)) = f(A) \cap B$ (Theorem 1.21.iii).

Consider the example $F : \{1,2\} \multimap \{p,q\}$ with $F(1) = \{p,q\}$ and $F(2)$ $= \{q\}$; then $F(\{1\} \cap F^{-1}(\{p\}))$ = $F(\{1\} \cap \{1\})$ = $\{p,q\}$, but $F(\{1\}) \cap \{p\}$ = $\{p,q\} \cap \{p\}$ = $\{p\}$. So they are not equal. But one does have inclusion:

2.27 Theorem *Let* $F : X \multimap Y$, $A \subset X$, *and* $B \subset Y$. *Then* $F(A \cap F^{-1}(B))$ $\supset F(A) \cap B$.

Operations on Set-Valued Mappings

2.28 Definition If $F : X -\!\!\sqsubset Y$ and $G : X -\!\!\sqsubset Y$ are two set-valued mappings, then:

i. Their *union* is the mapping $F \bigcup G : X -\!\!\sqsubset Y$ defined by $(F \bigcup G)(x) = F(x) \bigcup G(x)$.

ii. Their *intersection* is the mapping $F \bigcap G : X -\!\!\sqsubset Y$ defined by $(F \bigcap G)(x) = F(x) \bigcap G(x)$.

iii. Their *Cartesian product* is the mapping $F \times G : X -\!\!\sqsubset Y \times Y$ defined by $(F \times G)(x) = F(x) \times G(x)$.

2.29 Theorem *Let $F : X -\!\!\sqsubset Y$ and $G : X -\!\!\sqsubset Y$. Then, for $A \subset X$:*

i. $(F \bigcup G)(A) = F(A) \bigcup G(A)$.

ii. $(F \bigcap G)(A) \subset F(A) \bigcap G(A)$.

iii. $(F \times G)(A) \subset F(A) \times G(A)$.

Recall (Section 2.4) that $F : X -\!\!\sqsubset Y$ is a constant (set-valued) mapping if there exists a subset C of Y such that $F(x) = C$ for all $x \in X$. This implies $F(A) = C$ for all $A \subset X$.

2.30 Corollary *Let $F : X -\!\!\sqsubset Y$ be a constant mapping. Let $G : X -\!\!\sqsubset Y$ and $A \subset X$. Then $(F \bigcap G)(A) = F(A) \bigcap G(A)$.*

2.31 Theorem *If both $F : X -\!\!\sqsubset Y$ and $G : X -\!\!\sqsubset Y$ are semi-single-valued, then the set-valued mappings $F \bigcap G : X -\!\!\sqsubset Y$ and $F \times G : X -\!\!\sqsubset Y \times Y$ are semi-single-valued.*

2.32 Theorem *If one of $F : X -\!\!\sqsubset Y$ and $G : X -\!\!\sqsubset Y$ is injective, then the set-valued mappings $F \bigcap G : X -\!\!\sqsubset Y$ and $F \times G : X -\!\!\sqsubset Y \times Y$ are injective.*

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