Chapter 2
Directed Transport in a Stochastic Layer

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Abstract We consider a problem of transport in a spatially periodic potential under the influence of a slowly time-dependent unbiased periodic external force. Using methods of the adiabatic perturbation theory we show that for a periodic external force of general kind the system demonstrates directed (ratchet) transport in the chaotic domain on very long time intervals and obtain a formula for the average velocity of this transport. Two cases are studied: the case of the external force of small amplitude and the case of the external force with amplitude of order one.

2.1 Introduction

In recent years, studies of transport phenomena in nonlinear systems have been attracting a growing interest. In particular, a large and constantly growing number of papers are devoted to dynamics in systems which allow for directed (on average) motion under unbiased external forces and are referred to as ratchet systems. (The name comes from the famous Feynmann's lecture [1] on impossibility to obtain a directed motion and usable work with a system in the state of thermodynamic equilibrium.) Intensive study of ratchet systems was motivated by problems of motion of Brownian particles in spatially periodic potentials, unidirectional transport of molecular motors in biological systems, and recognition of "ratchet effects" in quantum physics (see review [2] and references therein). Generally speaking, ratchet phenomena occur due to lack of symmetry in the spatially periodic potential and/or the external forcing. It is interesting, however, to understand microscopic mechanisms leading to these phenomena. A possible approach is to neglect dissipation and noise terms arriving at a Hamiltonian system with deterministic forcing.

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Thus, one can make use of results obtained and methods developed in the theory of Hamiltonian chaos. Many papers studying chaotic transport in such Hamiltonian ratchets appeared in the last years (see, e.g., [3–9]).

Roughly speaking, Hamiltonian ratchets are related to an equation of the kind,

$$\ddot{q} + \frac{\partial U}{\partial q} = f(t),$$

with $2\pi-$periodic potential $U(q+2\pi) = U(q)$ and time-periodic external force $f(t+\tilde{T}) = f(t)$ with zero time average:

$$\int_0^{\tilde{T}} f \mathrm{d}t = 0.$$

Typically, a phase space of such a system contains invariant tori carrying regular motions and domains where motion is chaotic (stochastic layers). A most interesting fact found numerically (see the references above) is that in general for a phase trajectory in a stochastic layer there exists a nonzero limit:

$$V_q = \lim_{t\to\infty} q(t)/t \neq 0,$$

which means that there is directed transport (sometimes referred to as ratchet current) in stochastic layer in such systems. This phenomenon has been widely investigated, yet only few explicit analytical results were obtained. In particular, in [7] the ratchet current is estimated in the case when there are stability islands in the chaotic domain in the phase space of the system. The borders of such islands are "sticky" [10] and this stickiness together with desymmetrization of the islands is responsible for the occurrence of the ratchet transport.

We consider the problem of motion of a particle in a periodic potential under the influence of unbiased time-periodic external forcing. In numerics, we take $U(q) = \omega_0^2 \cos q$, where q is the coordinate and $\omega_0 = \text{const}$. Thus the equations are the same as in the paradigmatic model of a nonlinear pendulum under the action of external torque with zero time average. We study the case when the external forcing is time periodic with a large period of order ε^{-1}, $0 < \varepsilon \ll 1$, and use results and methods of the adiabatic perturbation theory. If ε is small enough, there are no stability islands in the domain of chaotic dynamics (see [11]). Thus, the mechanism of ratchet transport in this system differs from one suggested in [7].

The main objective of this chapter is to find a formula for the average velocity $V_q = \langle \dot{q} \rangle$ of a particle in the chaotic domain on very large time intervals. We study two cases: of the external force with the amplitude of order 1 and of the external force of small amplitude of order ε. We show that in the first case chaos develops as a result of multiple passages through a resonance. Each passage produces a small variation (a jump) of the value of the adiabatic invariant of the system. These jumps result in effective mixing and uniform distribution of the adiabatic invariant along a trajectory in the chaotic domain. On the other hand, direction and value of velocity depend on the immediate value of the action. Thus, to find the average velocity of transport on time intervals of order or larger than the mixing time, we find formulas for displacement in q at a given value of the action and then integrate them over the interval of values of the action corresponding to the chaotic domain. The situation is similar in the case of small external force. In this case, a typical phase trajectory repeatedly crosses a separatrix on the phase portrait. At each crossing the adiabatic invariant undergoes a quasi-random jump (see [12, 13]). Like in the first case, these jumps produce chaotic dynamics in the domain of separatrix crossings. In both cases, we demonstrate that for an external force of general kind (i.e. with zero time average but lowered time symmetry, cf. [6]), there is directed transport in the chaotic domain and obtain an analytic formula for the average velocity V_q of this transport. In both cases the width of the chaotic domain in the phase space is large: in the case of forcing of order one the width is $\sim \varepsilon^{-1}$, and in the case of small forcing the width is ~ 1. This is a common situation in systems with adiabatic chaos. Indeed, chaos in such systems develops in the domain filled with phase trajectories that repeatedly cross the resonance or the separatrix (see for particular examples, e.g., [13–18]). Thus the total phase flux due to the directed transport in the considered system is also large.

The chapter is based on results obtained in [19] by Leoncini, Neishtadt, and the author.

2.2 External Forcing of Order One

In this section we study the case when the external forcing is not small. The equation of motion has the form

$$\ddot{q} + \frac{\partial U}{\partial q} = f(\tau),$$

where a dot denotes t-derivative, $0 < \varepsilon \ll 1$ is a small parameter, $\tau = \varepsilon t$ is called the "slow time", function $f(\tau)$ is periodic with period T, i.e. $f(\tau + T) = f(\tau)$, and has zero time average. The system can be rewritten in the form

$$\dot{q} = p,\ \dot{p} = -\frac{\partial U}{\partial q} + f(\tau),\ \dot{\tau} = \varepsilon. \tag{2.1}$$

This is a Hamiltonian system with time-dependent Hamiltonian:

$$H = \frac{p^2}{2} + U(q) - f(\tau)q.$$

One can see from the second equation in (2.1) that magnitude of momentum p can reach values of order ε^{-1}. Make the canonical transformation of variables $(p,q) \mapsto (\bar{p},\bar{q})$ with generating function $W = (\bar{p} - \varepsilon^{-1}F(\varepsilon t))q$, where $F(\tau)$ is defined as $F(\tau) = -\int_0^\tau f(x)\mathrm{d}x + C$. Here C is a constant which we are free to choose.

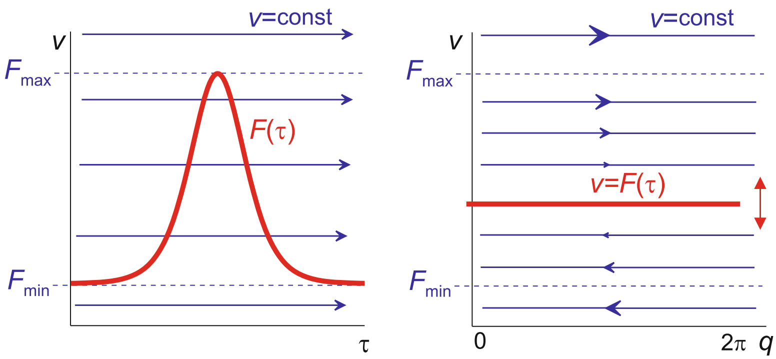


Fig. 2.1 *Left panel*: phase portrait of system (2.4). The *bold line* represents one period of function $F(\tau)$. *Right panel*: dynamics on (q,v)-plane

To simplify the following discussion, we take it large enough to make F positive at all values of τ and assume that function F has only one minimum and one maximum on its period. Note that $\bar{q} \equiv q$. After this transformation, Hamiltonian of the system acquires the form (bars over q are omitted):

$$H = \frac{(\bar{p} - \varepsilon^{-1} F(\tau))^2}{2} + U(q). \tag{2.2}$$

Introduce rescaled momentum $v = \varepsilon \bar{p}$ and rescaled time $\theta = \varepsilon^{-1} t$. We denote the derivative with respect to θ with prime and thus obtain:

$$q' = v - F(\tau),\ v' = -\varepsilon^2 \frac{\partial U}{\partial q},\ \tau' = \varepsilon^2. \tag{2.3}$$

This is a system in a typical form for application of the averaging method. Variable q is fast, and variables v and τ are slow. Take into account that U is a 2π-periodic function of q and thus its q-derivative has zero q-average. We average over fast variable q and obtain the averaged system:

$$v' = 0,\ \tau' = \varepsilon^2. \tag{2.4}$$

Thus, v is constant along a phase trajectory of the averaged system and is an adiabatic invariant of the exact system. The approximation $v = \text{const}$ is called adiabatic. The averaged system describes the dynamics adequately everywhere in the phase space except for a small neighborhood of the resonance at $v - F(\tau) = 0$, where the "fast" variable q is not fast.

In Fig. 2.1, left, a phase portrait of system (2.4) is shown. The horizontal lines are phase trajectories of the averaged system. Along every trajectory $v = \text{const}$. The bold line represents one period of function $F(\tau)$. When a phase trajectory of the averaged system crosses the resonance $v = F(\tau)$, value of the adiabatic invariant undergoes a quasi-random jump of typical order $\sqrt{\varepsilon^2} = \varepsilon$ (see, e.g., [20, 21] and references therein). A jump of v at the resonance crossing can be expressed as $\Delta v = \varepsilon G(q_*)$, where $q_* \in (0, 2\pi)$ is the value of $q \bmod 2\pi$ at the resonance crossing in the adiabatic approximation, and $G(q_*) \sim 1$ is a smooth function on the interval $(0, 2\pi)$. Magnitude of the jump should be considered as a quasi-random value, because a small variation of initial conditions results generally in large, of order one, variation of q_*. Consider two successive resonance crossings, corresponding to $q_* = q_1$ and $q_* = q_2$. It follows from the second equation in (2.4) that time interval (in terms of θ) between these crossings is a value of order ε^{-2}. A small variation δq_1 in q_1 produces variation of order ε in Δv. This latter variation after a long time interval $\sim \varepsilon^{-2}$ results in large variation of q_2: $\delta q_2 \sim \delta q_1/\varepsilon$. Therefore, jumps of the adiabatic invariant at successive resonant crossings can be considered as statistically independent random values.

Another representation of dynamics is shown in Fig. 2.1, right, on the plane (q, v). The bold line correspond to the position of the resonance $v = F(\tau)$. It slowly moves upwards and downwards in the picture, oscillating between $v = F_{\max}$ and $v = F_{\min}$, i.e. the maximal and the minimal values of $F(\tau)$. Note that $\dot{q} = 0$ on the line $v = F(\tau)$, $\dot{q} > 0$ above this line, and $\dot{q} < 0$ under this line. The region swept by this line in its slow motion is the region where resonance crossings occur. Uncorrelated jumps of adiabatic invariant v result in stochastization of dynamics in this region. The dynamics can be considered as a random walk between level lines of v. On a period of function $F(\tau)$ (after two resonance crossings) v changes by a value of order ε. Hence, after $N \sim \varepsilon^{-2}$ separatrix crossings, v varies by a value of order one. As a result, in time of order $t_{\text{diff}} \sim \varepsilon^{-3}$, the value of adiabatic invariant is distributed in all the range of values corresponding to the domain of resonance crossings. Phase trajectories of the averaged system that cross the resonance correspond to values of $\tilde{p}$ belonging to the interval $(F_{\min}, F_{\max})$. Therefore the chaotic domain of the exact system is, in the main approximation, a strip $F_{\min} \le v \le F_{\max}$. It is reasonable to assume that distribution of values of v in the stochastic layer is uniform. Captures into the resonance followed by escapes from the resonance (see [20, 21]) are also possible in this system. However, probability of a capture is small, of order ε, and hence impact of these phenomena on the transport is small.

To check these conclusions, we take $U(q) = -\omega_0^2 \cos q$ and $F(\tau) = A\left(1 + 2\exp\left[-\alpha(\sin\tau)^2\right]\right)$. A plot of $F(\tau)$ is shown in Fig. 2.2, left; the plot of the corresponding function $f(\tau)$ is shown in Fig. 2.2, right (recall that $f = \mathrm{d}F/\mathrm{d}\tau$). In Fig. 2.3 we represent a sample of Poincaré section of a long phase trajectory of (2.2) and the corresponding histogram of $\bar{p}$ for this trajectory. The plots show that the distribution of $\bar{p}$ is close to the uniform one.

To find the mean velocity V_q in q-coordinate, we take into account that v is uniformly distributed in the chaotic domain. Thus V_q is velocity at a fixed value of v averaged over all v-s within the stochastic layer, i.e. over interval $(F_{\min}, F_{\max})$. In other words, one can find the value of displacement Δq on one period of

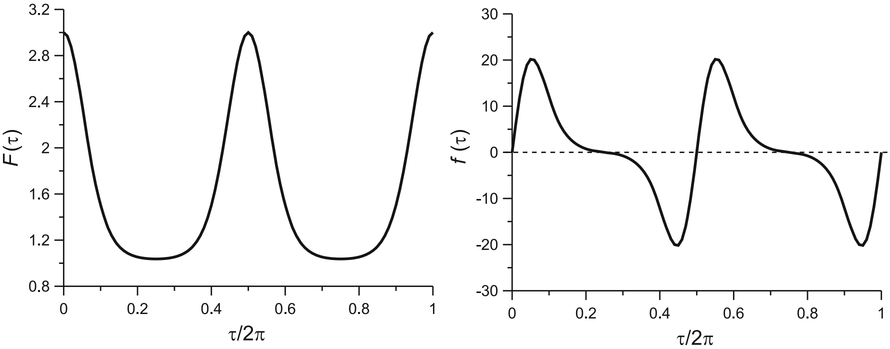


Fig. 2.2 *Left panel*: plot of the function $F(\tau)$ used in numerics. *Right panel*: plot of the external forcing $f(\tau)$

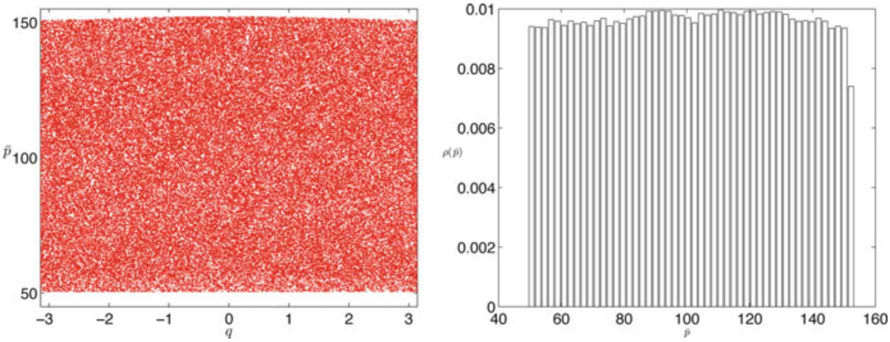


Fig. 2.3 *Left panel*: Poincaré section at $\tau = 0 \bmod 2\pi$ of a long phase trajectory ($5 \cdot 10^4$ dots) of system (2.2). All the points are mapped onto the interval $q \in (-\pi, \pi)$. $U(q) = -\omega_0^2 \cos q$, $F(\tau) = A\left(1 + 2\exp\left[-\alpha(\sin\tau)^2\right]\right)$ with $A = 0.5, \alpha = 4, \varepsilon = 0.01, \omega_0 = 1$. *Right panel*: histogram of $\bar{p}$ along the same phase trajectory

perturbation, then average it over the range of adiabatic invariant v corresponding to the chaotic domain, and find the average velocity of transport. Thus we find

$$\Delta q = \int_0^{T/\varepsilon} p\,\mathrm{d}t = \int_0^{T/\varepsilon} \left(\bar{p} - \varepsilon^{-1} F(\tau)\right) \mathrm{d}t = \frac{1}{\varepsilon^2} \int_0^T (v - F(\tau))\,\mathrm{d}\tau. \tag{2.5}$$

To find V_q, we have to integrate this expression over v from $F_{\min}$ to $F_{\max}$ and divide the result by $(F_{\max} - F_{\min})$ and by the length of the period of the external forcing T/ε. Thus we obtain

$$V_q = \frac{\varepsilon}{T(F_{\max} - F_{\min})} \int_{F_{\min}}^{F_{\max}} \Delta q\,\mathrm{d}v. \tag{2.6}$$

Table 2.1 Numerically found values of εV_q corresponding to various values of parameters ε, α in system (2.2) for $F(\tau) = A\left(1+2\exp\left[-\alpha(\sin\tau)^2\right]\right)$ (four upper rows, $A = 0.5$, $\omega_0 = 1$)

	$\alpha = 1$	$\alpha = 2$	$\alpha = 4$
$\varepsilon = 0.1$	0.046	0.128	0.253
$\varepsilon = 0.05$	0.046	0.112	0.225
$\varepsilon = 0.01$	0.0353	0.1050	0.2044
$\varepsilon = 0.005$	0.0369	0.1081	0.1916
$\varepsilon V_q^{\text{theor}}$	0.0389	0.1018	0.2006

In the bottom row theoretical values $\varepsilon V_q^{\text{theor}}$ obtained according to (2.8) are shown

Substituting Δq from (2.5) and integrating, one straightforwardly obtains

$$V_q = \frac{1}{2T\varepsilon}\int_0^{2\pi}\left(F_{\max} + F_{\min} - 2F(\tau)\right)\mathrm{d}\tau. \tag{2.7}$$

Note that formula (2.7) can be rewritten in a more elegant form as

$$V_q = \frac{1}{\varepsilon}\left(\frac{F_{\max} + F_{\min}}{2} - \langle F(\tau)\rangle\right), \tag{2.8}$$

where the angle brackets denote time average. The results of numerical checks of the formula are represented in Table 2.1. To obtain values presented in the table we integrated the system with Hamiltonian (2.2) on a long time interval $\Delta t = 2\pi \cdot 10^6/\varepsilon$.

Remarkably, formula (2.8) is the same for any smooth 2π-periodic potential (not necessarily harmonic). The potential may also depend periodically on time with the same period as that of the external force.

2.3 Small External Forcing

In the case considered in the previous section, chaotization of motion in the stochastic layer was a result of multiple resonance crossings. In the case to be studied in this section, the chaos is due to *separatrix* crossings. This produces somehow different estimates of the diffusion time. Besides, it results in a more complicated formula for the mean transport velocity.

2.3.1 Main Equations: Diffusion of the Adiabatic Invariant

Consider now the case when the external forcing is small, of order ε. The Hamiltonian equations of motion are

$$\dot{q} = p, \ \dot{p} = -\frac{\partial U}{\partial q} + \varepsilon f(\tau), \ \dot{\tau} = \varepsilon. \tag{2.9}$$

The time-dependent Hamiltonian function is

$$H = \frac{p^2}{2} + U(q) - \varepsilon f(\tau) q. \tag{2.10}$$

Consider for definiteness $U(q) = -\omega_0^2 \cos q$ (qualitative results do not depend on this choice). Similar to the previous section, we make a canonical transformation of variables $(p,q) \mapsto (\bar{p},\bar{q})$ using generating function $W_1 = (\bar{p} - F(\varepsilon t))q$, where $F(\tau)$ was defined in Sect. 2.2. Thus, $F(\tau)$ is again a periodic function defined up to an additive constant, which we are free to choose. To make the following presentation more clear, we choose this constant in such a way that the minimal value of F is $F_{\min} > 4\omega_0/\pi$. Note that $\bar{q} \equiv q$. After this transformation of variables, Hamiltonian of the system acquires the form (bars over q are omitted):

$$H = \frac{(\bar{p} - F(\tau))^2}{2} - \omega_0^2 \cos q. \tag{2.11}$$

This is a system explicitly depending on the slow time τ. A standard approach to study such a system is to consider it first at frozen τ, i.e. at $\tau = \text{const}$. Phase portrait of the system at a frozen value of τ (we call it the unperturbed system) is shown in Fig. 2.4. There is a separatrix on the portrait. It divides the phase space into the domains of direct rotations (above the upper branch of the separatrix), oscillations (between the separatrix branches), and reverse rotations (below the lower branch of the separatrix). Introduce the "action" I associated with a phase trajectory of the unperturbed system on this portrait. In the domains of rotation, I equals an area between the trajectory, the lines $q = -\pi$, $q = \pi$, and the axis $\bar{p} = 0$, divided by 2π; in the domain of oscillations, this is an area surrounded by the trajectory divided by 2π. It is known that I is an adiabatic invariant of (2.11): far from the separatrix its value is preserved along a phase trajectory with the accuracy of order ε on long time intervals (see, e.g., [21]).

Location of the separatrix on the $(q,\bar{p})$-plane depends on the value of $F(\tau)$. As τ slowly varies, the separatrix slowly moves up and down, and phase points cross the separatrix and switch its regime of motion from direct rotations to reverse rotations and vice versa. Recall known results on variation of the adiabatic invariant when a phase point crosses the separatrix. The area surrounded by the separatrix is constant, and hence, capture into the domain of oscillations is impossible in the first approximation (in the exact system, only a small measure of initial conditions correspond to phase trajectories that spend significant time in this domain; thus their influence on the transport is small). To be definite, consider the situation when the separatrix on the phase portrait slowly moves down. Thus, phase points cross the separatrix and change their mode of motion from reverse rotation to direct rotation. Let the action before the separatrix crossing at a distance of order 1 from the

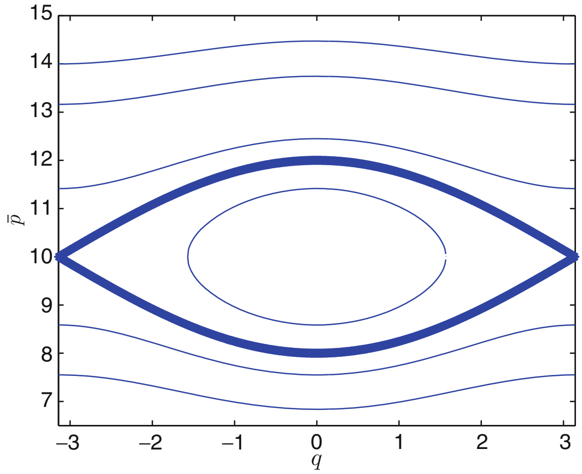


Fig. 2.4 Phase portrait of system (2.11) at frozen τ. The *bold line* is the separatrix

separatrix be $I = I_-$ and let the action after the crossing (also at a distance of order 1 from the separatrix) be $I = I_+$. In the first approximation, we have $I_+ = I_- + 8\omega_0/\pi$, i.e. the action increases by the value of the area inside the separatrix divided by 2π (see, e.g., [22, 23]). We shall call this change in the action a "geometric jump". If the separatrix contour slowly moves up and a phase point goes from the mode of direct rotation to the mode of reverse rotation, the corresponding value of the action decreases by the same value $8\omega_0/\pi$. Thus, in this approximation, the picture of motion looks as follows. While a phase point is in the domain of reverse rotation, the value of I along its trajectory stays constant: $I = I_-$. After transition to the domain of direct rotation, this value changes by the value of the geometric jump. The transition itself in this approximation occurs instantaneously. After the next separatrix crossing, the adiabatic invariant changes again by the value of the geometric jump, with the opposite sign, and returns to its initial value I_-. We call this approximation adiabatic.

In the next approximation, the value of action at the separatrix crossing undergoes a small additional jump. Consider for definiteness the case when the separatrix contour on the phase portrait moves down, and I_- and I_+ are measured when it is in its uppermost and lowermost positions, accordingly. Results of [12, 13] imply the following formula for the jump in the adiabatic invariant:

$$\begin{aligned}2\pi(I_+ - I_-) = 16\omega_0 + 2a(1-\xi)\varepsilon\Theta\ln(\varepsilon\Theta)\\ + a\varepsilon\Theta\ln\frac{2\pi(1-\xi)}{\Gamma^2(\xi)} - 2b\varepsilon\Theta(1-\xi),\end{aligned}$$

where $a = \omega_0^{-1}$, $b = \omega_0^{-1}\ln(32\omega_0^2)$, and $\Theta = 2\pi F'(\tau_*)$. Here F' is the τ-derivative of F, τ_* is the value of τ at the separatrix crossing found in the adiabatic approximation, and $\Gamma(\cdot)$ is the gamma-function. Value ξ is a so-called pseudo-phase

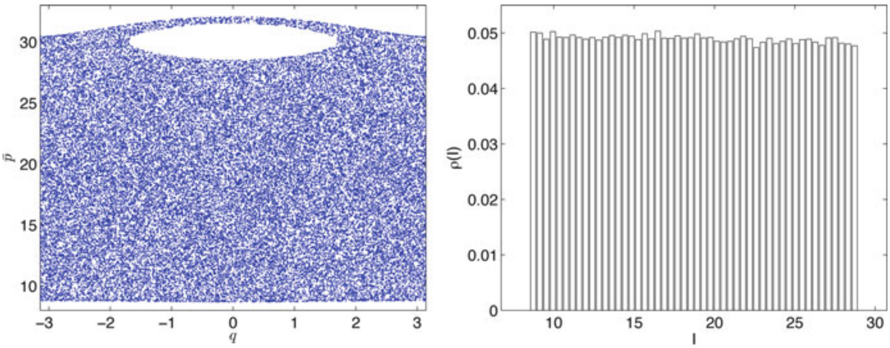


Fig. 2.5 *Left panel*: Poincaré section at $\tau = 0 \bmod 2\pi$ of a long phase trajectory ($5 \cdot 10^4$ dots). All the points are mapped onto the interval $q \in (-\pi, \pi)$. $F(\tau) = A\left(1 + 2\exp\left[-\alpha(\sin\tau)^2\right]\right)$ with $A = 10, \alpha = 16, \varepsilon = 0.005, \omega_0 = 1$. The empty region in the chaotic sea corresponds to phase points eternally locked in the domain of oscillations; they never enter the chaotic domain and do not participate in the transport. *Right panel*: histogram of I on the segment $(I_{\min}, I_{\max} - 8\omega_0/\pi)$ along the same phase trajectory

of the separatrix crossing; it strongly depends on the initial conditions and can be considered as a random variable uniformly distributed on interval $(0,1)$ (see, e.g., [13]). Thus, value of the jump in the adiabatic invariant at the separatrix crossings has a quasi-random component of order $\varepsilon \ln \varepsilon$.

Similar to the case considered in Sect. 2.2, accumulation of small quasi-random jumps due to multiple separatrix crossings produces diffusion of adiabatic invariant (see, e.g., [13]). On a period of $F(\tau)$ (after two separatrix crossings), the action changes by a value of order $\varepsilon \ln \varepsilon$. Hence, after $N \sim \varepsilon^{-2}(\ln\varepsilon)^{-2}$ separatrix crossings, the adiabatic invariant varies by a value of order one. As a result, in time of order $t_{\text{diff}} \sim \varepsilon^{-3}(\ln\varepsilon)^{-2}$, the value of adiabatic invariant is distributed in all the range of values corresponding to the domain where phase points cross the separatrix on the phase plane; its distribution is close to the uniform one. We have checked this fact numerically for the same sample function $F(\tau)$ as in Sect. 2.2 at various parameter values. Poincaré sections and distribution histograms of I in all the cases look similar; see an example in Fig. 2.5.

2.3.2 Average Velocity of the Transport

Our aim is to find a formula for average velocity V_q along a phase trajectory on time intervals of order t_{diff} or larger. We first only take into consideration the geometric jumps, and afterwards, to obtain the final result, we take into account the mixing due to small quasi-random jumps. To simplify the consideration, assume again that function $F(\tau)$ has one local minimum $F_{\min}$ and one local maximum $F_{\max}$ on the interval $(0,T)$. The main results are valid without this assumption.

Introduce $\tilde{I}$, defined in the domains of rotation, as follows: it equals the area bordered by the trajectory, the line $\bar{p} = F(\tau)$, and the lines $q = -\pi$, $q = \pi$, divided by 2π. Thus, $\tilde{I} = |F(\tau) - I|$. Frequency of motion in the domains of rotation is $\omega(\tilde{I})$, where $\omega(\tilde{I})$ at $\tilde{I} > 4\omega_0/\pi$ is the frequency of rotation of a standard nonlinear pendulum with Hamiltonian:

$$H_0 = p^2/2 - \omega_0^2 \cos q,$$

expressed in terms of its action variable $\tilde{I}$. We do not need an explicit expression for function $\omega(\tilde{I})$. From Hamiltonian (2.11) we find $\dot{q} = \bar{p} - F(\tau)$. Consider a phase trajectory of the system frozen at $\tau = \bar{\tau}$ in a domain of rotation. Let the value of action on this trajectory be $I = I_0$ and the period of rotation be T_0 (note that $T_0 = 2\pi/\omega$ by definition). Then the value of $\dot{q}$ averaged over a period of rotation equals

$$\int_0^{T_0} \frac{|\dot{q}|}{T_0}\,\mathrm{d}t = 2\pi/T_0 = \omega(|F(\bar{\tau}) - I_0|).$$

Now consider a long phase trajectory in the case of slowly varying τ. Let on the interval (τ_1, τ_2) a phase point of (2.11) be below the separatrix contour. In the adiabatic approximation, the value I_0 of the adiabatic invariant along its trajectory is preserved on this interval. Hence, at $\tau \in (\tau_1, \tau_2)$ we have

$$2\pi F(\tau) - 2\pi I_0 \geq 8\omega_0, \tag{2.12}$$

and the equality here takes place at $\tau = \tau_1$ and $\tau = \tau_2$. In the process of motion on this time interval, q changes (in the main approximation) by a value:

$$\Delta q_-(I_0) = -\frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2} \omega(F(\tau) - I_0)\mathrm{d}\tau. \tag{2.13}$$

On the interval $(\tau_2, \tau_1 + T)$ the phase trajectory is above the separatrix contour, and the value of the adiabatic invariant equals $\hat{I}_0 = I_0 + 8\omega_0/\pi$ due to the geometric jump. On this interval we have

$$2\pi F(\tau) - 2\pi I_0 \leq 8\omega_0. \tag{2.14}$$

In the process of motion on this time interval, q changes by a value:

$$\Delta q_+(I_0) = \frac{1}{\varepsilon}\int_{\tau_2}^{\tau_1 + T} \omega(|F(\tau) - \hat{I}_0|)\mathrm{d}\tau. \tag{2.15}$$

Total displacement in q on the interval $(\tau_1, \tau_1 + T)$ equals $\Delta q(I_0) = \Delta q_-(I_0) + \Delta q_+(I_0)$, and the average velocity on this interval is $\varepsilon \Delta q(I_0)/T$.

Consider now the motion on a long enough time interval $\Delta t \sim t_{\mathrm{diff}}$. Due to the diffusion in the adiabatic invariant described above, on this time interval, values of I_0, defined as a value of I when the phase point is *below* the separatrix contour, cover the interval $(I_{\min}, I_{\max} - 8\omega_0/\pi)$. Here $I_{\min} = F_{\min} - 4\omega_0/\pi$ and $I_{\max} = F_{\max} + 4\omega_0/\pi$. Assume that the distribution of I on this interval is uniform. To find the average velocity, we integrate $\varepsilon \Delta q(I_0)/T$ over this interval. Integrating (2.13) over I_0 and changing the order of integration we find

$$\int_{I_{\min}}^{I_{\max}-8\omega_0/\pi} \Delta q_- \mathrm{d}I_0 = -\frac{1}{\varepsilon}\int_0^T \mathrm{d}\tau \int_{I_{\min}}^{F(\tau)-4\omega_0/\pi} \omega(F(\tau)-I_0)\mathrm{d}I_0$$
$$= -\frac{1}{\varepsilon}\int_0^T \mathrm{d}\tau \int_{4\omega_0/\pi}^{F(\tau)-I_{\min}} \omega(\eta)\mathrm{d}\eta.$$

Now we take into account the equality

$$\omega(\tilde{I}) = \frac{\partial H_0(\tilde{I})}{\partial \tilde{I}}$$

(recall that $H_0(\tilde{I})$ is the Hamiltonian of a nonlinear pendulum as a function of its action variable) and obtain

$$-\int_0^T \mathrm{d}\tau \int_{4\omega_0/\pi}^{F(\tau)-I_{\min}} \omega(\eta)\mathrm{d}\eta = -\int_0^T (H_0(F(\tau)-I_{\min}) - H_0^s)\,\mathrm{d}\tau, \tag{2.16}$$

where H_0^s is the value of H_0 on the separatrix. Similarly, integrating (2.15) we obtain

$$\int_{I_{\min}}^{I_{\max}-8\omega_0/\pi} \Delta q_+ \mathrm{d}I_0 = \frac{1}{\varepsilon}\int_0^T (H_0(I_{\max}-F(\tau)) - H_0^s)\,\mathrm{d}\tau. \tag{2.17}$$

Adding (2.16) to (2.17) and dividing by $T(F_{\max} - F_{\min})/\varepsilon$ we find the expression for the average velocity V_q of transport on long time intervals:

$$V_q = \frac{1}{T(F_{\max} - F_{\min})} \times \int_0^{2\pi} (H_0(I_{\max} - F(\tau)) - H_0(F(\tau) - I_{\min}))\,\mathrm{d}\tau. \tag{2.18}$$

In (2.18), $H_0(I)$ can be found as the inverse function to $\tilde{I}(h)$, which defines action as a function of energy in domains of rotation of a nonlinear pendulum. For the latter function, the following formula holds (see, e.g., [24]):

$$\tilde{I}(h) = \frac{4}{\pi}\omega_0 \kappa \mathscr{E}(1/\kappa),\ \kappa \geq 1, \tag{2.19}$$

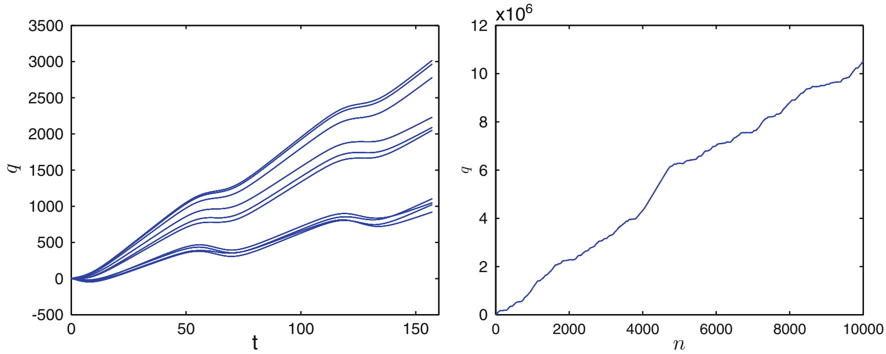
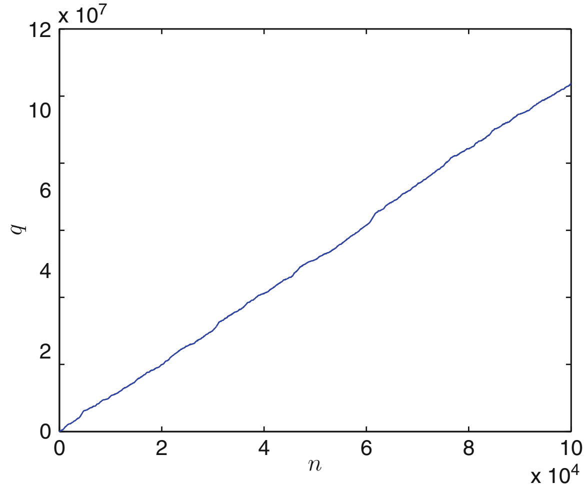


Fig. 2.6 *Left panel*: q against t for ten different initial conditions (comparatively short time interval), $\alpha = 4, \varepsilon = 0.05$. *Right panel*: q against the number of periods of the external force for a sample trajectory (10^4 periods), $\alpha = 16, \varepsilon = 0.05$. Parameter $A = 10$ in both cases

Fig. 2.7 Coordinate q against the number of periods of the external force for the same trajectory as in Fig. 2.6 (10^5 periods), $\alpha = 16, \varepsilon = 0.05, A = 10$



where $\kappa^2 = (1 + h/\omega_0^2)/2$, $\mathscr{E}(\cdot)$ is the complete elliptic integral of the second kind. If function $F(\tau)$ has several local extremes on the interval $(0, 2\pi)$, $F_{\min}$ and $F_{\max}$ in (2.18) are the smallest and largest values of F, respectively.

It can be seen from (2.18) that for function $F(\tau)$ of general type V_q is not zero, and hence there is the directed transport in the system. We checked this formula numerically for the sample function $F(\tau) = A\left(1 + 2\exp\left[-\alpha(\sin\tau)^2\right]\right)$, $\alpha > 0$ at various values of parameters ε and α. Typical plots of q against time t are shown in Figs. 2.6 and 2.7.

The results of numerical checks of formula (2.18) are represented in Table 2.2. To find numerical values of V_q presented in the table, we integrated the system with Hamiltonian (2.11) on a long time interval $\Delta t = 2\pi \cdot 10^6/\varepsilon$ with a constant time

Table 2.2 Numerically found values of V_q corresponding to various values of parameters ε, α (four upper rows, $A = 10, \omega_0 = 1$) and theoretical values V_q^{theor} obtained according to (2.18) (the bottom row)

	$\alpha = 4$	$\alpha = 8$	$\alpha = 16$
$\varepsilon = 0.1$	4.721	6.756	8.363
$\varepsilon = 0.05$	4.446	6.681	8.076
$\varepsilon = 0.01$	4.298	6.211	7.442
$\varepsilon = 0.005$	4.598	6.702	8.202
V_q^{theor}	4.393	6.679	8.110

step of $\pi/100$ (fifth order symplectic scheme [25]). Use of a symplectic scheme for long time simulations of Hamiltonian systems is necessary in order to ensure that creeping numerical error do not end up washing off the invariant tori bounding the chaotic domain. The table demonstrates satisfactory agreement between the formula and the numerics.

Finally, we note that formula (2.18) can be used also in the case of arbitrary (nonharmonic) spatially periodic time-independent potential in place of the term $-\omega_0^2 \cos q$ in (2.10) and (2.11). Of course, in this case function H_0 is different from the Hamiltonian of the nonlinear pendulum, but it always can be found, at least numerically.

2.4 Summary

To summarize, we have considered the phenomenon of the directed transport in a spatially periodic potential adiabatically influenced by a slow periodic in time-unbiased external force. We have shown that for the external force of a general kind the system exhibits directed transport on long time intervals. Direction and average velocity of the transport in the chaotic domain are independent of initial conditions and determined by properties of the external force. We studied two different cases: the case of small amplitude of the external force and the case, when this amplitude is a value of order one. We have obtained an approximate formula for average velocity of the transport and checked it numerically. The final formulas (2.8) and (2.18) are valid for any smooth periodic potential (not necessarily harmonic one).

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