Chapter 2
Generalised Dualities

This chapter introduces the framework upon which we will build the rest of this monograph. We begin by considering geometric duals and Petrials, observing that both Petriality and geometric duality result from local operations on each edge of an embedded graph. These local operations applied to subsets of the edge set result in partial Petriality and partial duality. We provide constructions for partial duals and partial Petrials in various realisations of embedded graphs. The two operations of partial Petriality and partial duality give rise to an action of the symmetric group on embedded graphs with a distinguished set of edges. This group action leads to twisted duality, which assimilates several types of duality from the literature, including geometric duality, direct derivatives, Petrie duals, and partial duality. We concluded by defining the ribbon group and describing how twisted duality can be obtained as orbits under the ribbon group action on the set of edge-ordered embedded graphs.

2.1 Partial Petrials

Recall from Sect. 1.3 that if G is a ribbon graph, then its Petrial, $G^{\times}$, is obtained by adding a half-twist to each edge of G, as in Fig. 1.7. Notice that rather than adding a half-twist to every edge of G, we may add half-twists to only some of the edges of G. The result is a partial Petrial:

Definition 2.1. Let G be a ribbon graph and $A \subseteq E(G)$. Then the *partial Petrial*, $G^{\tau(A)}$, of G with respect to A is the ribbon graph obtained from G by adding a half-twist to each of the edges in A, as in Fig. 1.7.

An example of a partial Petrial is shown in Fig. 2.1.

If $A = \{e\}$, we write $G^{\tau(e)}$ for $G^{\tau(\{e\})}$. Also, in parallel with the terminology for duals and duality, we can regard the act of taking a partial Petrial as an operation on embedded graphs which we call partial Petriality.

J.A. Ellis-Monaghan and I. Moffatt, *Graphs on Surfaces: Dualities, Polynomials, and Knots*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4614-6971-1_2,

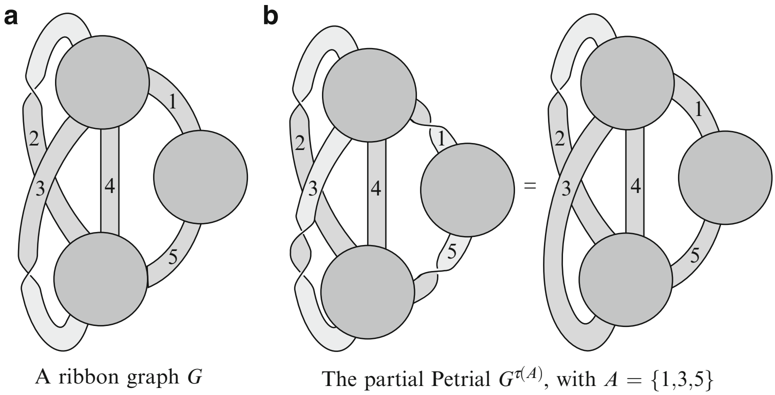


Fig. 2.1 An example of a partial Petrial of a ribbon graph

Partial Petrials have a very simple construction in the setting of arrow presentations. If G is an arrow presentation and $A \subseteq E(G)$, then $G^{\tau(A)}$ is the arrow presentation obtained by, for each $e \in A$, reversing the direction of exactly one of the e-labelled arrows. In terms of signed rotation systems, $G^{\tau(A)}$ is formed by toggling the sign of every edge in A. Finally, partial Petrials for ram graphs and cellularly embedded graphs can be constructed by translating to, say, the language of ribbon graphs, taking the partial Petrial, and then translating back.

We observe a few basic properties of partial Petrials.

Proposition 2.2. *Let G be an embedded graph and $A, B \subseteq E(G)$.*

1. $G^{\tau(\emptyset)} = G$.
2. $G^{\tau(E(G))}$ *is the Petrial* $G^{\times}$ *of* G.
3. $(G^{\tau(A)})^{\tau(B)} = G^{\tau(A \Delta B)}$, *where* $A \Delta B := (A \cup B) \backslash (A \cap B)$ *is the symmetric difference of* A *and* B.
4. *Partial Petriality acts disjointly on components, i.e.,* $(P \sqcup Q)^{\tau(A)} = (P^{\tau(A \cap E(P))}) \sqcup (Q^{\tau(A \cap E(Q))})$.
5. $G^{\tau(\{e,f\})} = (G^{\tau(e)})^{\tau(f)} = (G^{\tau(f)})^{\tau(e)}$. *In particular, partial Petrials can be formed one edge at a time.*
6. *There is a natural 1–1 correspondence between the edges of* G *and the edges of* $G^{\tau(A)}$.

2.2 Partial Duals

We have just seen that partial Petrials arise from the same local operation on the edges as generate Petrials, but by applying it to only a subset of edges of an embedded graph. In this section we show that the same principle applies to duality.

There is a local operation on individual edges that gives rise to geometric duals, and this operation can be applied to a subset of edges of an embedded graph, with a partial dual as the result.

Partial duality was introduced by Chmutov in [16] to relate various recent realisations of the Jones polynomial as evaluations of Bollobás and Riordan's ribbon graph polynomial (this will be discussed in detail in Chap. 5). Since its introduction, partial duality has found a variety of applications to knot theory [3, 13, 14, 16, 45, 77, 79], graph theory [16, 28–30, 44, 76, 78, 80, 96], and physics [42, 63, 89–91]. Partial duality appears to be a fundamental and far-reaching construction for embedded graphs.

As with partial Petrials, partial duals may be described in the various presentations for embedded graphs. Each description has its advantages and disadvantages: some properties can be immediately clear in one construction, but very hard to see in another. We will give descriptions for arrow presentations, ribbon graphs, and ram graphs. As with partial Petrials, partial duals for other expressions may be found by first translating to one of these three descriptions, finding the partial dual, and translating back.

2.2.1 Partial Duality with Respect to an Edge

To see that geometric duality is the result of a local operation applied to each edge of an embedded graph, we begin by drawing an embedded graph G as a band decomposition, giving equal emphasis to the vertices and faces, as in Fig. 2.2b. In Fig. 2.2b, the band decomposition for G has 0-bands A, B, and C; 1-bands 1, 2, 3, and 4; and 2-bands a, b, and c. Since a band decomposition for G^* can be obtained by switching the 0-bands and 2-bands, G^* has 0-bands a, b, and c; 1-bands 1, 2, 3, and 4; and 2-bands A, B, and C. We superimpose the circles of an arrow presentation of G on this drawing (as in Fig. 2.2c), and we superimpose the circles of an arrow presentation of G^* on another copy (as in Fig. 2.2d).

Consider a single edge e in the arrow presentation of both G and G^*. We see that the process of taking the dual is the result of a local operation on e that simply shifts the arrows associated with e from vertex boundaries to facial boundaries, as in Fig. 2.3.

Figure 2.3 therefore suggests an operation on arrow presentations that allows for the formation of the dual of an embedded graph with respect to a *single* edge: simply change the arrow presentation as indicated in the figure. This results in an arrow presentation for a ribbon graph $G^{\delta(e)}$ which is called the partial dual with respect to e. More generally, one may form the dual with respect to a subset of edges of G:

Definition 2.3. Let G be an arrow presentation and $A \subseteq E(G)$. Then the *partial dual*, $G^{\delta(A)}$, of G with respect to A is the arrow presentation obtained as follows. For each $e \in A$, suppose α and β are the two arrows labelled e in the arrow presentation of G. Draw a line segment with an arrow on it directed from the head

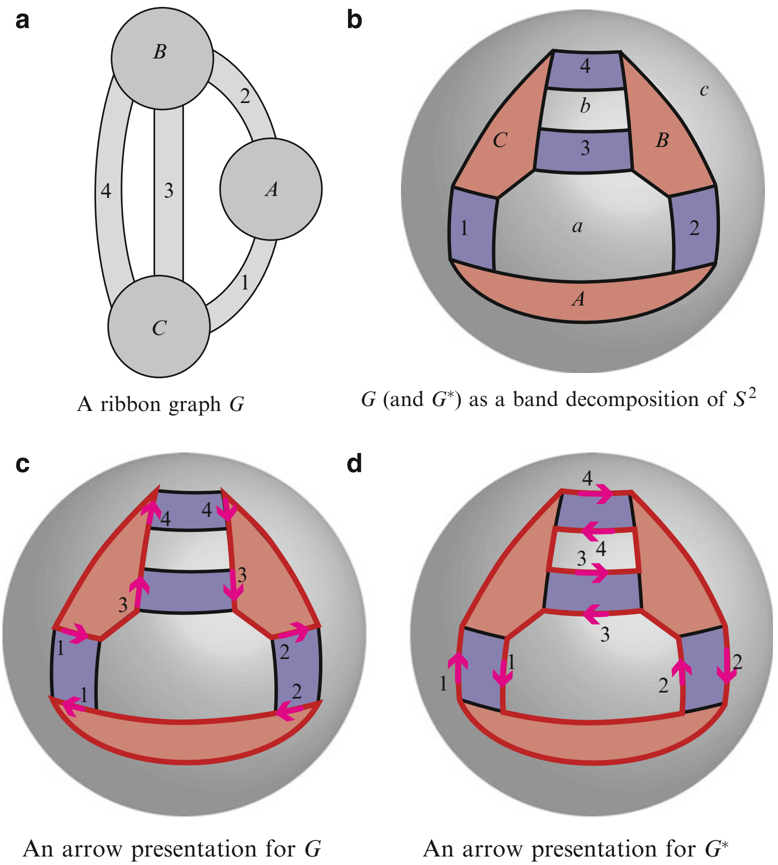
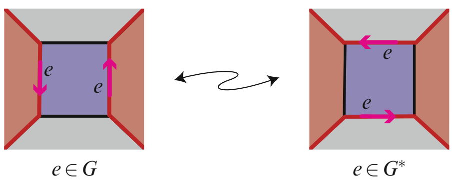


Fig. 2.2 An example of a partial dual

Fig. 2.3 Local dual operation at an edge e of G and G^*



of α to the tail of β, and a line segment with an arrow on it directed from the head of β to the tail of α. Label both of these arrows e and delete α and β and the arcs containing them. This process is illustrated locally at a pair of arrows in Fig. 2.4.

Partial duals for cellularly embedded graphs, ribbon graphs, band decompositions, ram graphs, and signed rotation systems may be defined by translating to the

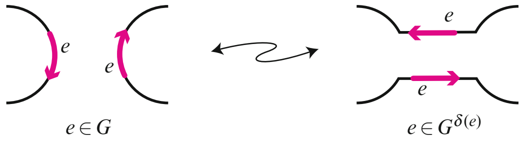


Fig. 2.4 Taking the partial dual of an edge in an arrow presentations

language of arrow presentations, forming the partial dual, and then translating back again. However, in the next section, we will describe some other ways to construct partial duals in these realisations of embedded graphs.

If $A = \{e\}$, we will often write $G^{\delta(e)}$ for $G^{\delta(\{e\})}$.

Figure 2.5 shows the partial dual $G^{\delta(3)}$ (Fig. 2.5b) of an arrow presentation G (Fig. 2.5a). The figure also shows the arrow presentations superimposed on the band decomposition of G and G^* (Fig. 2.5c, d) and also G and $G^{\delta(3)}$ as ribbon graphs (Fig. 2.5e, f). Observe that although both G and $G^{\delta(3)}$ are non-orientable, G is of genus 0, while $G^{\delta(3)}$ is of genus 1. Thus, unlike geometric duals, a partial dual may have a different genus than that of the original embedded graph.

Figure 2.6 gives a second example of a partial dual formed using Definition 2.3. In this figure the partial dual is formed with respect to $A = \{1,3\}$.

2.2.2 *Other Constructions of Partial Duals*

A number of methods, using different representations of embedded graphs, for constructing partial duals have appeared in the literature. Since each of these methods has its advantages, we review them all here and prove their equivalence. These methods typically use a more "global" approach, considering the entire subset of edges in the construction, as opposed to the "local", one edge at a time, approach we have just seen.

2.2.2.1 Partial Duals via Spanning Ribbon Subgraphs

We begin with Chmutov's original definition of partial duality from [16]. Let G be a ribbon graph and $A \subseteq E(G)$. Observe in Fig. 2.3 that in forming the partial dual using an arrow presentation, if $e \in A$, then we can position the arrow presentation so that it follows the boundaries of the edge e of G that coincide with a vertex as in the left-hand side of Fig. 2.3; and if $e \notin A$, then we can position the arrow presentation so that it follows the vertices incident to e of G that coincide with a face as in the right-hand side of Fig. 2.3. Thus we can position the arrow presentation for $G^{\delta(A)}$ so that it lies on the spanning ribbon subgraph $(V(G), A)$ of G. Moreover, the two e-labelled arrows in the arrow presentation all lie on the two arcs where the boundary of the edge e of G intersects the spanning ribbon subgraph. Thus, we may rewrite Definition 2.3 in the following way.

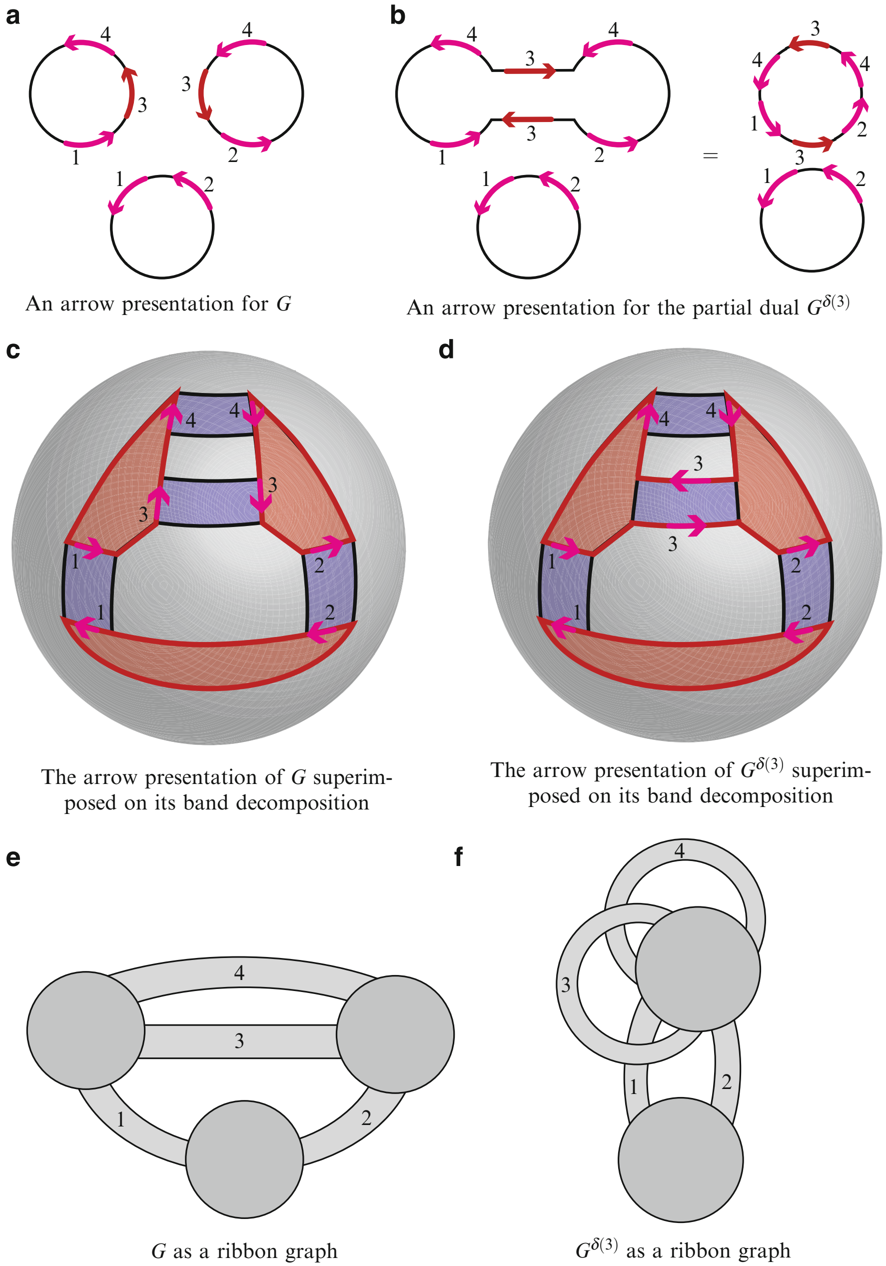


Fig. 2.5 Taking the partial dual with respect to the edge labelled 3

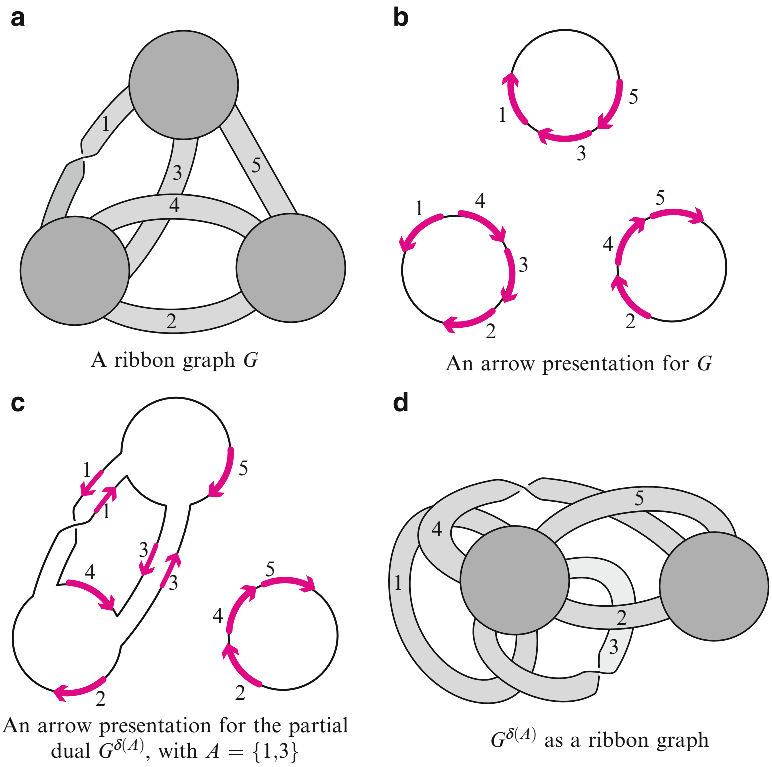


Fig. 2.6 An example of a partial dual formed using arrow presentations

Definition 2.4 (Chmutov [16]). Let G be a ribbon graph and $A \subseteq E(G)$. Arbitrarily orient and label each of the edges of G (the orientation need not extend to an orientation of the ribbon graph). The boundary components of the spanning ribbon subgraph $(V(G),A)$ of G meet the edges of G in disjoint arcs (where the spanning ribbon subgraph is naturally embedded in G). On each of these arcs, place an arrow which points in the direction of the orientation of the edge boundary and is labelled by the edge it meets. The resulting marked boundary components of the spanning ribbon subgraph $(V(G),A)$ define an arrow presentation. The ribbon graph corresponding to this arrow presentation is the partial dual $G^{\delta(A)}$ of G.

Example 2.5. Figure 2.7 provides an example of the formation of a partial dual $G^{\delta(A)}$ with $A = \{2,5\}$ using Definition 2.4. The ribbon graph G is shown in Fig. 2.7a. Figure 2.7b shows the spanning subgraph $(V(G),\{2,5\})$ with marked boundary components. The arrow presentation for $G^{\delta(A)}$ is shown in Fig. 2.7c. The partial dual $G^{\delta(A)}$ is shown in Fig. 2.7d.

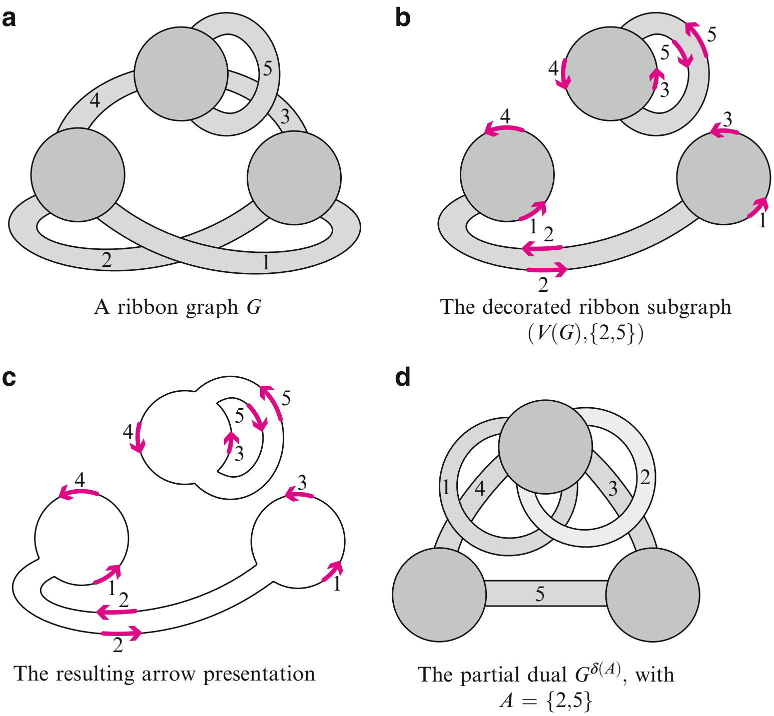


Fig. 2.7 An example of a partial dual formed through spanning ribbon subgraphs

2.2.2.2 Partial Duals via Ram Graphs

Ram graphs lend themselves to a particularly intuitive construction of partial duals. We would like to think of forming $G^{\delta(A)}$ by taking the usual geometric dual of G, but only at the edges in A. We can realise this construction geometrically by "hiding" the edges that are not in A, that is, by starting with the ram graph which has ribbons for the edges in A and arrows representing the edges not in A, and then taking the geometric dual of the underlying ribbon graph as in the following definition.

Definition 2.6. Given a ribbon graph G and $A \subseteq E(G)$, form the ram graph $G\{A^c\}$. Then form the geometric dual of the ribbon graph $G|_A$ contained in $G\{A^c\}$, but *carrying the arrow markings on the boundaries along in the process*. This results in a new ram graph, which we denote $(G\{A^c\})^\dagger$.

Proposition 2.7. *Let G be a ribbon graph and $A \subseteq E(G)$. Then $G^{\delta(A)} = (G\{A^c\})^\dagger$.*

Proof. The equivalence between $(G\{A^c\})^\dagger$ and $G^{\delta(A)}$ as given in Definition 2.4 (and therefore Definition 2.3) can be seen by observing that $(G\{A^c\})^\dagger$ can be obtained from the marked boundary components of the spanning ribbon subgraph $(V(G),A)$ in Definition 2.4 by adding all of the edges corresponding to elements of A. $\square$

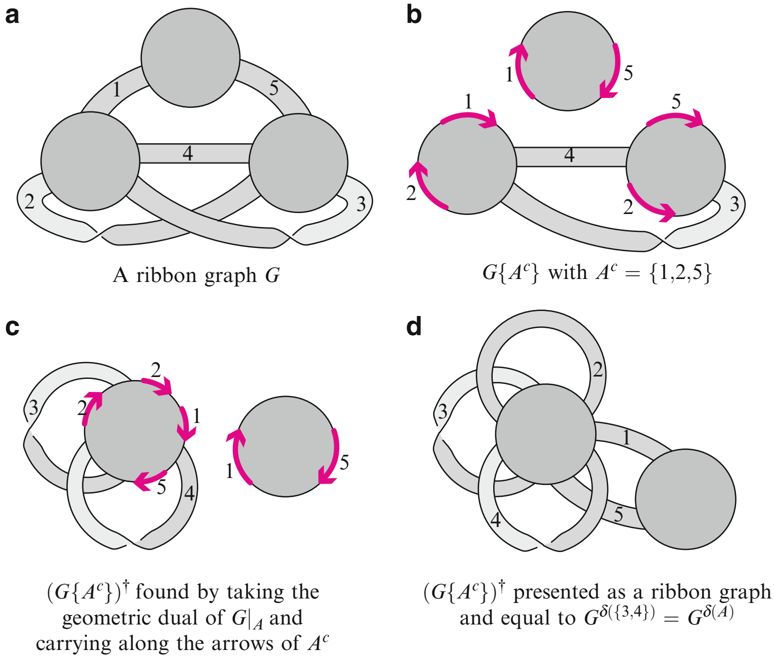


Fig. 2.8 Forming a partial dual using ram graphs

Example 2.8. An example of the formation of a partial dual using Definition 2.6 is given in Figure 2.8.

2.2.2.3 Partial Duals via Ribbon Graphs

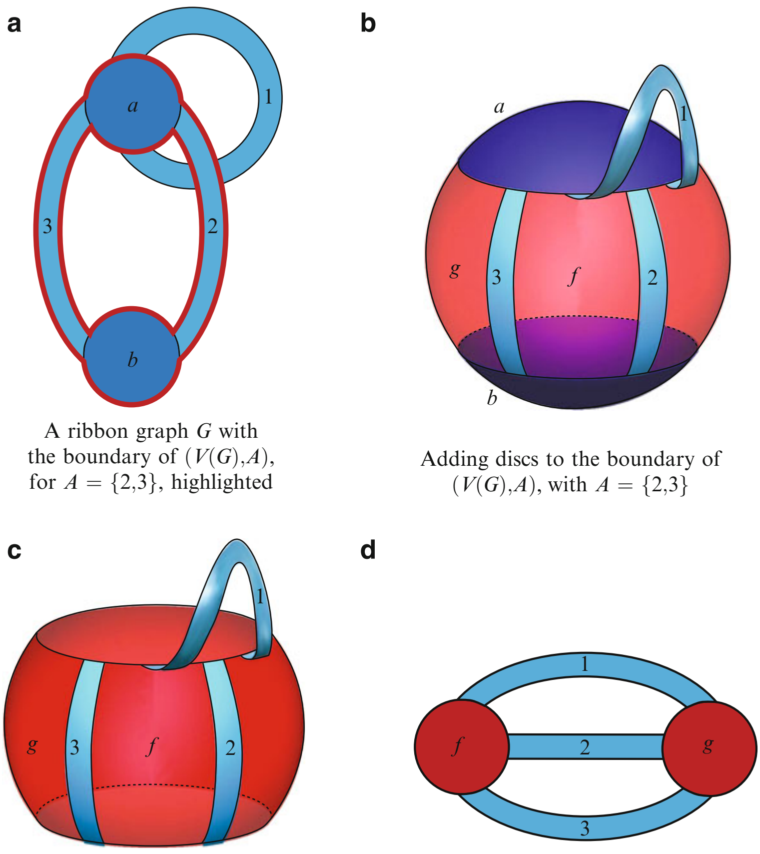
The following construction of a partial dual is purely geometric in that it avoids the use of marking arrows.

Definition 2.9 (Bradford, Butler, and Chmutov [8]). Regard the boundary components of the spanning ribbon subgraph $(V(G),A)$ of G as curves on the surface of G. Glue a disc to G along each connected component of this curve and remove the interior of all vertices of G. We denote the result of this operation by $G(A)^{\ddagger}$.

Proposition 2.10. *Let G be a ribbon graph and $A \subseteq E(G)$. Then $G^{\delta(A)} = G(A)^{\ddagger}$.*

Proof. The construction in Definition 2.9 can be recovered from that of Definition 2.6 by noting that the edges in A^c just encode the marking arrows. $\square$

Definition 2.9 is particularly pleasing from a conceptual viewpoint as we again see directly that the partial dual $G^{\delta(A)}$ is formed by taking the geometric dual of the ribbon graph G as discussed in Sect. 1.4, but ignoring the edges not in A.



a: A ribbon graph G with the boundary of $(V(G),A)$, for $A = \{2,3\}$, highlighted

b: Adding discs to the boundary of $(V(G),A)$, with $A = \{2,3\}$

c: Deleting vertices in $V(G)$ gives $G^{\delta(A)}$

d: The partial dual $G^{\delta(A)}$

Fig. 2.9 An example of a partial dual formed through ribbon subgraphs

Example 2.11. An example of forming a partial dual $G^{\delta(A)}$ for $A = \{2,3\}$ using Definition 2.9 is given in Fig. 2.9. Figure 2.9b shows the structure obtained by adding discs to G (shown in Fig. 2.9a) that follow the boundary components of $(V(G),A)$, where $A = \{2,3\}$. Figure 2.9c shows the ribbon graph obtained by deleting the vertices of G. The ribbon graph $G^{\delta(A)}$ has been redrawn in Fig. 2.9d.

2.2.2.4 The Partial Dual of an Edge of a Ribbon Graph

Definition 2.3 provides a local construction of partial duals: change part of an arrow presentation as indicated in Fig. 2.4. In contrast, all of the other constructions of partial duals given above are global in that they involve the whole of the embedded graph. Giving local constructions of partial duals in presentations of embedded

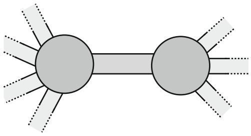
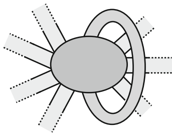
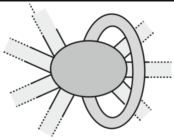
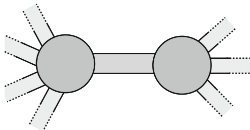
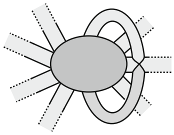
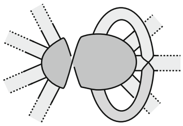
Type of edge	G	$G^{\delta(e)}$
Not a loop		
Orientable loop		
Non-orientable loop		

Fig. 2.10 The partial dual of an edge of a ribbon graph

graphs other than arrow presentations can be a little more involved. For example, Fig. 2.10 describes how partial duality at an edge e changes a ribbon graph G. Note that the cases when e is an orientable loop, non-orientable loop, and is not a loop must be treated separately.

2.2.3 Basic Properties of Partial Duality

The following proposition lists a few basic properties of partial duals. Further properties and applications of partial duals will be discussed later in the text.

Proposition 2.12 (Chmutov [16]). *Let G be a ribbon graph and $A, B \subseteq E(G)$. Then the following properties hold:*

1. $G^{\delta(\emptyset)} = G$.
2. $G^{\delta(E(G))} = G^*$, *where G^* is the geometric dual of G.*
3. $(G^{\delta(A)})^{\delta(B)} = G^{\delta(A \triangle B)}$, *where $A \triangle B := (A \cup B) \backslash (A \cap B)$ is the symmetric difference of A and B.*
4. $G^{\delta(\{e,f\})} = (G^{\delta(e)})^{\delta(f)} = (G^{\delta(f)})^{\delta(e)}$. *In particular, partial duals can be formed one edge at a time.*

5. *G is orientable if and only if $G^{\delta(A)}$ is orientable.*
6. *Partial duality acts disjointly on components, i.e., $(P \sqcup Q)^{\delta(A)} = (P^{\delta(A \cap E(P))}) \sqcup (Q^{\delta(A \cap E(Q))})$.*
7. *There is a natural 1–1 correspondence between the edges of G and the edges of $G^{\delta(A)}$.*

Proof. Items 1 and 2 follow immediately from Definition 2.6. Item 3 is easily seen by considering the arrow presentation construction of partial duals from Definition 2.1 and observing that $(G^{\delta(e)})^{\delta(e)} = G$. Item 4 follows from Item 3.

For Item 5 we use the construction of Definition 2.9. Suppose G is an orientable ribbon graph and $A \subseteq E(G)$. Adding the discs to the boundary components of $(V(G), A)$ as in Definition 2.9 results in a cell complex X that consists of a (not necessarily connected) closed surface Σ with ribbons attached to it. Let R be this set of ribbons. Σ is orientable since the ribbon subgraph $(V(G), A)$ is. Therefore, if there was a closed curve l in X with a regular neighbourhood $N(l)$ homeomorphic to a Möbius band, it would pass through, in order, the sub-complexes $\Sigma_1 R_1 \Sigma_2 R_2 \cdots R_n \Sigma_1 R_1$ of X, where $\Sigma_i \subseteq \Sigma$ and $R_i \in R$. But as each R_i is an edge in G, and there is a path in Σ_i between R_{i-1} and R_i if and only if there is a path between R_{i-1} and R_i in G, it follows that G contains a curve with a neighbourhood homeomorphic to a Möbius band and is therefore non-orientable. It follows that X is orientable and so $G^{\delta(A)} = X - V(G)$ is. For the converse note that if $G^{\delta(A)}$ is orientable, then, by the above, so is $(G^{\delta(A)})^{\delta(A)} = G$.

Items 6 and 7 are trivial. □

2.3 Twisted Duality

In the preceding sections, we described two operations on embedded graphs: partial Petrality and partial duality. Now we will show how these two operations give rise to a generalisation of duality called twisted duality. The material in this section is based on the results of [30], but here our approach is more geometric than that of [30].

2.3.1 Sequences of Partial Duals and Petrials

Let G be an embedded graph and $A \subseteq E(G)$. Also, let $w = w_1 w_2 \cdots w_n$ be a word in the alphabet $\{\delta, \tau\}$. Then we define

$$G^{w(A)} := (\cdots((G^{w_n(A)})^{w_{n-1}(A)})\cdots)^{w_1(A)}. \tag{2.1}$$

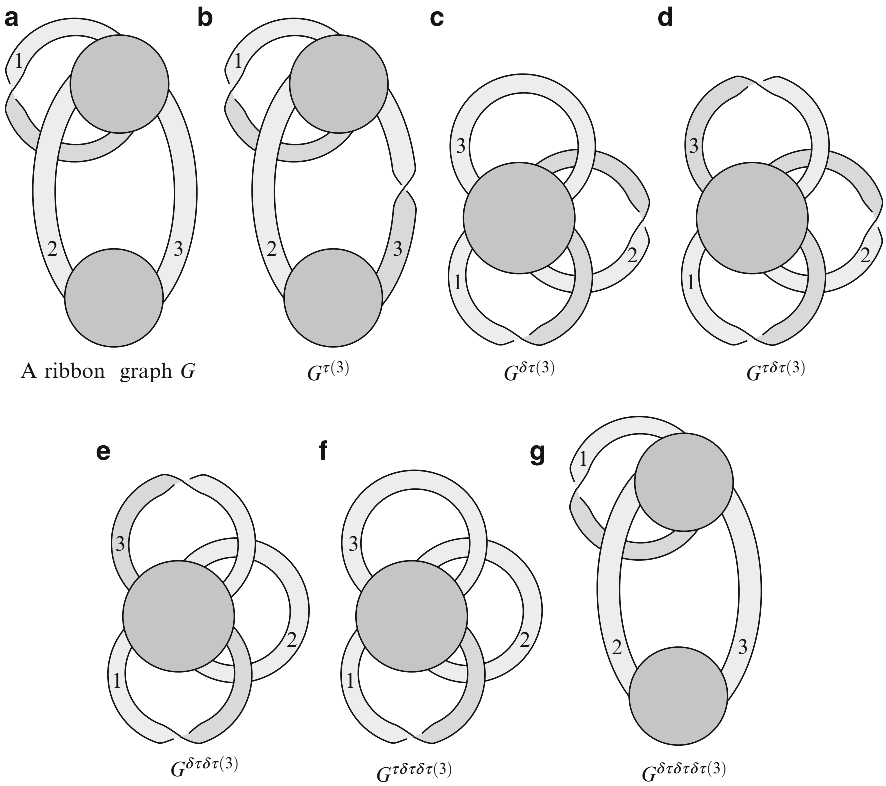


Fig. 2.11 Forming a ribbon graph $G^{\delta\tau\delta\tau\delta\tau(3)}$

In Eq. (2.1), as with function composition, the w_i's in the word w act on A from the right to the left. Also, at each step, we are identifying the edges $A \subseteq E(G)$ with the edges $A \subseteq E((\cdots(G^{w_n(A)})^{w_{n-1}(A)}\cdots)^{w_{n-(i-1)}(A)})$.

Example 2.13. Figure 2.11 illustrates the formation of $G^{w(A)}$ where $w = \delta\tau\delta\tau\delta\tau$ and $A = \{3\}$.

Figure 2.12 shows the actions of δ and τ on an edge of a ribbon graph. Notice that applying some sequences of δ and τ to an edge of G does not change the ribbon graph. This is a consequence of relations among the words in δ and τ:

Proposition 2.14. *Let G be an embedded graph and $A \subseteq E(G)$. Then*

1. $G^{\tau^2(A)} = G$
2. $G^{\delta^2(A)} = G$
3. $G^{(\delta\tau)^3(A)} = G$

Proof. The proposition follows immediately upon considering Fig. 2.12. □

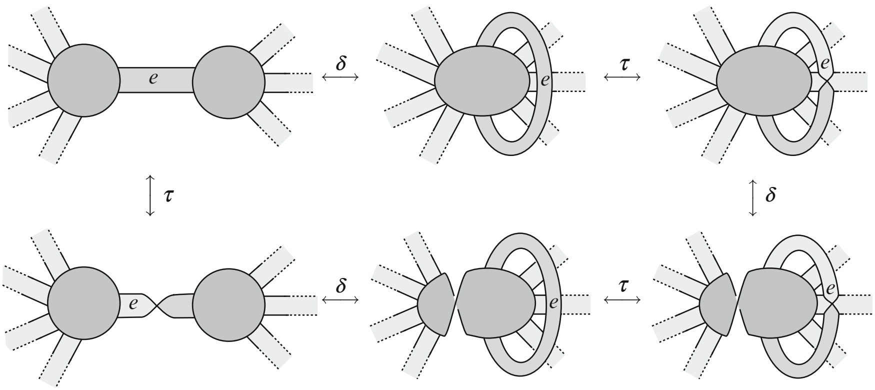


Fig. 2.12 The actions of δ and τ on an edge e of a ribbon graph

The relations on words in δ and τ given in Proposition 2.14 allow us to manipulate a word w in $G^{w(A)}$, but more importantly, these relations allow us to recognise the above operations as an action of the symmetric group S_3 on embedded graphs with specified subsets of edges. For the following theorem, we recall that $\mathscr{G}$ denotes the set of all embedded graphs.

Theorem 2.15. *Let* $\mathscr{X} = \{(G,A) \mid G \in \mathscr{G}, A \subseteq E(G)\}$. *Then there is a group action of*

$$\mathfrak{G} := \langle \delta, \tau \mid \delta^2, \tau^2, (\delta\tau)^3 \rangle \tag{2.2}$$

on $\mathscr{X}$ *given by* $g(G,A) := (G^{g(A)}, A)$ *for* $g \in \mathfrak{G}$. *Moreover, the group action is faithful.*

Proof. It follows immediately from Eq. (2.1) and Proposition 2.14 that setting $g(G,A) = (G^{g(A)}, A)$ defines a group action of $\mathfrak{G}$ on $\mathscr{X}$. It remains to show that δ and τ satisfy no additional relations. Since $\mathfrak{G}$ is a presentation of the symmetric group S_3 and every proper quotient group of $\mathfrak{G}$ is abelian, it is enough to show that $\tau\delta(G,A) \neq \delta\tau(G,A)$ for some (G,A), a fact that is readily verified. □

When the set A consists of a single edge, or even several specified edges, we sometimes simplify the notation slightly, writing $G^{g(e)}$ for $G^{g(\{e\})}$ and $G^{g(e_1,\ldots,e_n)}$ for $G^{g(\{e_1,\ldots,e_n\})}$.

2.3.2 Twisted Duals

We have just seen how a word in δ and τ acts on an embedded graph with a fixed subset of edges. In this section we examine what happens when we apply such words to combinations of subsets of edges. This leads to *twisted duality*, which extends all of the forms of duality we have seen so far (i.e., geometric duality, Petrality, partial

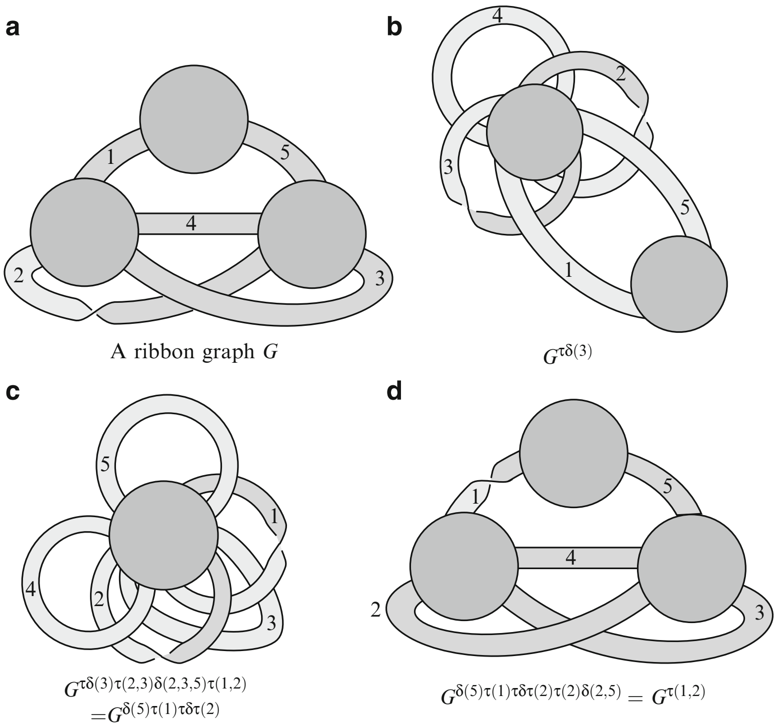


Fig. 2.13 Examples of twisted duals

duality, and partial Petriality). In this section, instead of the formalism of group actions, we focus on operations on embedded graphs, usually presented as ribbon graphs. In Sect. 2.4 we will see how the same twisted duality discussed here can also be obtained through a group action, as it was in [30].

Suppose G is a ribbon graph, $A, B \subseteq E(G)$, and $g, h \in \mathfrak{G} = \langle \delta, \tau \,|\, \delta^2, \tau^2, (\delta\tau)^3 \rangle$. Then we define

$$G^{g(A)h(B)} := \left(G^{g(A)}\right)^{h(B)}.$$

Equipped with this, we are now able to define twisted duality.

Definition 2.16. Embedded graphs G and H are *twisted duals* if there exist $A_1, \ldots, A_n \subseteq E(G)$ and $g_1, \ldots, g_n \in \mathfrak{G}$ such that

$$H = G^{g_1(A_1)g_2(A_2)\cdots g_n(A_n)}.$$

Consistent with our previous terminology we can regard the act of taking a twisted dual as an operation which we call twisted duality.

Figure 2.13 shows the formation of three twisted duals of a ribbon graph G. The set of twisted duals of the plane 2-cycle is shown in Fig. 2.14.

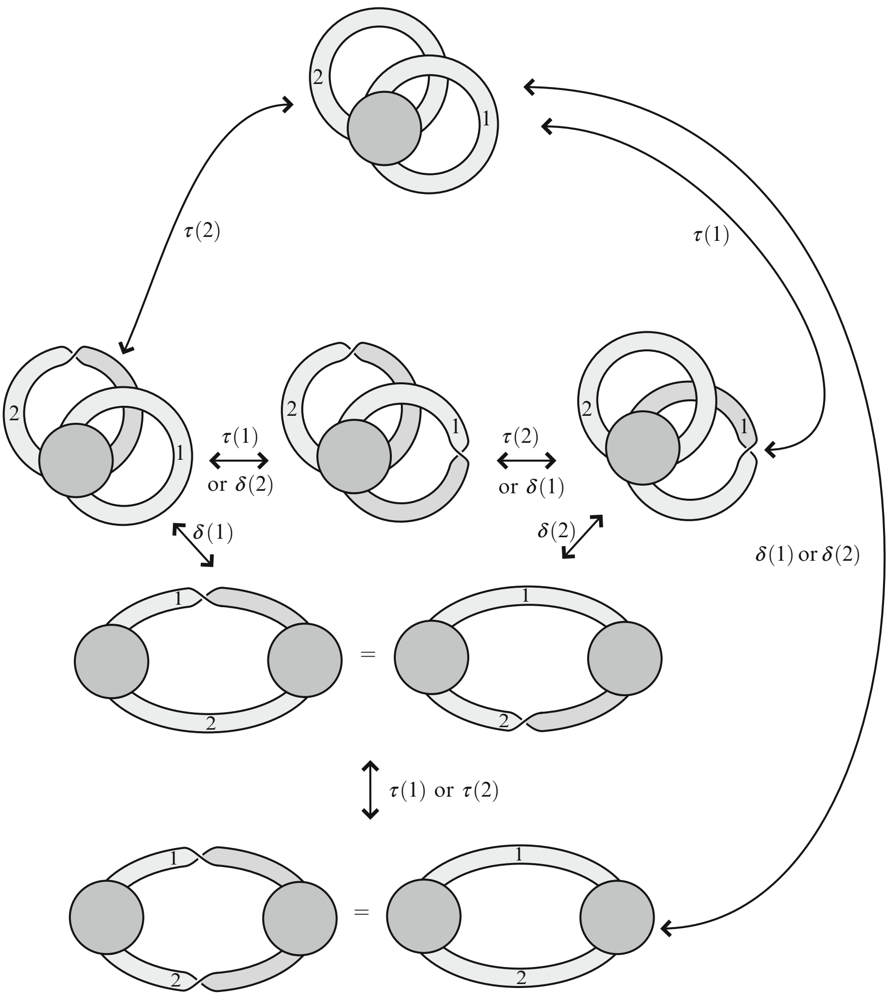


Fig. 2.14 All the twisted duals of the 2-cycle

As can be seen in Figs. 2.13 and 2.14, twisted duals of the same embedded graph can have very different graph theoretical and topological properties from one another. Also, although an embedded graph can have at most $6^{|E(G)|}$ twisted duals many of these twisted duals may be equivalent.

All of the identities in Proposition 2.14 for a single edge also hold for twisted duals, i.e., the group relations hold when applied to a set of edges A. However, when the exponent in a twisted dual involves more than one set, additional identities also hold.

Proposition 2.17. *Let G be an embedded graph, $A, B \subseteq E(G)$ and $g, h \in \mathfrak{G}$. Then the following hold:*

1. *If $A \cap B = \emptyset$, then $G^{g(A)h(B)} = G^{h(B)g(A)}$.*
2. *$G^{g(A)} = (G^{g(e)})^{g(A\setminus\{e\})}$, when $e \in A$, and so twisted duals can be formed one edge at a time.*
3. *Twisted duality acts disjointly on the components of G.*
4. *There is a natural 1–1 correspondence between the edges of G and the edges of each of its twisted duals.*

Proof. These identities hold since they hold for partial Petrials (by Proposition 2.2) and partial duals (by Proposition 2.12). □

Propositions 2.14 and 2.17 allow us to express twisted duals in a generic form, as follows.

Proposition 2.18. *If, for all i belonging to some indexing set, we have $h_i \in \mathfrak{G}$ and $B_i \subseteq E(G)$, then any expression of the form $G^{\prod h_i(B_i)}$, where the product is over the i's, can be written as*

$$G^{1(A_1)\delta(A_2)\tau(A_3)\tau\delta(A_4)\delta\tau(A_5)\tau\delta\tau(A_6)}, \tag{2.3}$$

where the A_i partition $E(G)$. Moreover, this expression is unique.

Proof. The proposition follows by repeated applications of Propositions 2.14 and 2.17. □

Figure 2.13 illustrates how the exponents can be written in the generic form from Proposition 2.18.

Note that, in the notation of Proposition 2.18, the terms in the product $\prod h_i(B_i)$ do not necessarily commute, while the terms in the product $\prod_{i=1}^{6} g_i(A_i)$ do commute with each other since the A_i's are disjoint.

Notation 2.19. We use G^{Γ} to denote a generic expression of the form $G^{g_1(A_1)g_2(A_2)\cdots g_n(A_n)}$, where each $g_i \in \mathfrak{G}$ and $A_i \subseteq E(G)$. We may assume without loss of generality that G^{Γ} is of the form shown in Eq. (2.3), and we can omit factors of the form $1(A_1)$ or $g_i(\emptyset)$.

It is useful to have a taxonomy, given in Table 2.1, for "partial" analogues of other derived graphs such as the Wilson dual (or opposite graph), trialities, and direct derivatives from, for example, [100], especially since these, like partial duals and partial Petrials, arise from applying group elements of $\mathfrak{G}$ to embedded graphs. We also note that when $A = E(G)$, the result is the action of $\mathfrak{G}$ on an embedded graph (usually a regular map) studied by Wilson in [100].

2.4 The Ribbon Group and its Action

In Definition 2.16, twisted duals were formed by applying one of the six elements of $\mathfrak{G}$ to specified subsets of the edges. However, twisted duals can equivalently be defined by a group action of $\mathfrak{G}^n$ on embedded graphs with n edges. The significance

Table 2.1 Taxonomy of classes of twisted duals

Generator(s)	Order of subgroup of $\mathfrak{G}$ generated	Applied to all edges	Applied to a subset of edges
δ	2	Geometric dual	Partial dual
τ	2	Petrie dual or Petrial	Partial Petrial
$\tau\delta\tau$	2	Wilson dual or Wilsonial (or the opposite)	Partial Wilsonial
$\delta\tau$	3	Triality	Partial triality
δ and τ	6	A direct derivative	Twisted dual

of this approach is that the twisted duals of an embedded graph G, with an arbitrary order on its edges, form the orbit of G under this group action, while geometric duality, Petrials, partial duality, and partial Petrials, as well as the trialities, partial trialities, Wilsonials, and partial Wilsonials described above, arise as the orbits of various subgroups of $\mathfrak{G}^n$.

2.4.1 Defining the Group Action

By Theorem 2.15, the group $\mathfrak{G} = \langle \delta, \tau \,|\, \delta^2, \tau^2, (\delta\tau)^3 \rangle$ acts on the set of ribbon graphs with a distinguished edge by $g(G, \{e\}) = (G^{g(e)}, \{e\})$. We now extend this construction to an action of the direct product $\mathfrak{G}^n$ of n copies of $\mathfrak{G}$ on the set of embedded graphs with n linearly ordered edges by making each factor in the direct product act on a specific edge of an embedded graph.

Definition 2.20. We call $\mathfrak{G}^n$ the *ribbon group for embedded graphs with n edges*.

To define the action of the ribbon group, we work with embedded graphs with arbitrarily ordered edges. Recall that $\mathscr{G}_n$ denotes the set of embedded graphs with exactly n edges. Let

$$\mathscr{G}_{\mathrm{or}(n)} = \{(G, \prec) | G \in \mathscr{G}_n \text{ and } \prec \text{ is a linear ordering of } E(G)\}$$

denote the set of embedded graphs with exactly n linearly ordered edges.

Definition 2.21. We define the *ribbon group action* of the ribbon group on $\mathscr{G}_{\mathrm{or}(n)}$ as follows. Given $(g_1, g_2, \dots, g_n) \in \mathfrak{G}_n$ and $(G, e_1 \prec e_2 \prec \cdots \prec e_n) \in \mathscr{G}_n$, then

$$(g_1, g_2, \dots, g_n)(G, e_1 \prec e_2 \prec \cdots \prec e_n) = (G^{g_1(e_1)g_2(e_2)\cdots g_n(e_n)}, e_1 \prec e_2 \prec \cdots \prec e_n). \tag{2.4}$$

The following theorem states that the ribbon group action does indeed define a group action on $\mathscr{G}_{\mathrm{or}(n)}$.

Theorem 2.22. *The action of $\mathfrak{G}^n$ on $\mathcal{G}_{\mathrm{or}(n)}$ defined in Eq. (2.4) is a group action.*

Proof. The result follows easily from Theorem 2.15. □

We record here, without proof, the following result from [30] which provides some properties of the ribbon group action.

Proposition 2.23. *The ribbon group action of $\mathfrak{G}^n$ on $\mathcal{G}_{\mathrm{or}(n)}$ is*

1. *Faithful*
2. *Has no fixed points*
3. *Transitive if and only if $n > 1$*
4. *Not free*

2.4.2 Recovering Dualities from Actions of Subgroups of the Ribbon Group

Here we see how twisted duality and its various specialisations arise as orbits under the action of the ribbon group and its subgroups.

Suppose that a group $\mathfrak{H}$ acts on a set X. Recall that the *orbit* of an element $x \in X$ is the set

$$\mathrm{Orb}_{\mathfrak{H}}(x) := \{hx \,|\, h \in \mathfrak{H}\}.$$

It turns out that twisted duals arise from the orbits of embedded graphs under the ribbon group action.

The ribbon group $\mathfrak{G}^n$ acts on the set $\mathcal{G}_{\mathrm{or}(n)}$ of embedded graphs with n ordered edges. To make contact with twisted duality and its specialisations, which do not require an edge order, we make the following definition.

Definition 2.24. Let G be an embedded graph and $\mathfrak{H}$ be a subgroup of the ribbon group $\mathfrak{G}^n$. Then we define $\mathrm{Orb}_{\mathfrak{H}}(G)$ to be the set of embedded graphs (without edge orders) in $\mathrm{Orb}_{\mathfrak{H}}(G, \prec)$ where $\prec$ is any edge order of G.

In Definition 2.24, the action of $\mathfrak{H}$ on $\mathcal{G}_{\mathrm{or}(n)}$ is the one arising from the ribbon group action.

We can now recover all of the dualities discussed so far via orbits of embedded graphs under the action of the ribbon group and its subgroups.

Theorem 2.25. *Let G and H be embedded graphs.*

1. *G and H are geometric duals if and only if $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$, where $\mathfrak{H}$ is the subgroup of $\mathfrak{G}^n$ generated by $(\delta, \delta, \dots, \delta)$.*
2. *G and H are partial duals if and only if $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$, where $\mathfrak{H}$ is the subgroup of $\mathfrak{G}^n$ generated by $\{(\delta, 1, \dots, 1), (1, \delta, 1, \dots, 1), \dots, (1, 1, \dots, \delta)\}$.*
3. *G and H are Petrials if and only if $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$, where $\mathfrak{H}$ is the subgroup of $\mathfrak{G}^n$ generated by $(\tau, \tau, \dots, \tau)$.*
4. *G and H are partial Petrials if and only if $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$, where $\mathfrak{H}$ is the subgroup of $\mathfrak{G}^n$ generated by $\{(\tau, 1, \dots, 1), (1, \tau, 1, \dots, 1), \dots, (1, 1, \dots, \tau)\}$.*

5. *G and H are Wilsonials if and only if* $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$*, where* $\mathfrak{H}$ *is the subgroup of* $\mathfrak{G}^n$ *generated by* $(\tau\delta\tau, \tau\delta\tau, \dots, \tau\delta\tau)$.
6. *G and H are partial Wilsonials if and only if* $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$*, where* $\mathfrak{H}$ *is the subgroup of* $\mathfrak{G}^n$ *generated by* $\{(\tau\delta\tau, 1, \dots, 1), (1, \tau\delta\tau, 1, \dots, 1), \dots, (1, 1, \dots, \tau\delta\tau)\}$.
7. *G and H are trialities if and only if* $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$*, where* $\mathfrak{H}$ *is the subgroup of* $\mathfrak{G}^n$ *generated by* $(\delta\tau, \delta\tau, \dots, \delta\tau)$.
8. *G and H are partial trialities if and only if* $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$*, where* $\mathfrak{H}$ *is the subgroup of* $\mathfrak{G}^n$ *generated by* $\{(\delta\tau, 1, \dots, 1), (1, \delta\tau, 1, \dots, 1), \dots, (1, 1, \dots, \delta\tau)\}$.
9. *G and H are direct derivatives if and only if* $H \in \mathrm{Orb}_{\mathfrak{H}}(G)$*, where* $\mathfrak{H}$ *is the subgroup of* $\mathfrak{G}^n$ *generated by* $\{(\delta, \delta, \dots, \delta), (\tau, \tau, \dots, \tau)\}$.
10. *G and H are twisted duals if and only if* $H \in \mathrm{Orb}_{\mathfrak{G}^n}(G)$.

Proof. The theorem follows by the construction of the ribbon group action. □

Notation 2.26. Because of the frequency of its use, we will write simply $\mathrm{Orb}(G)$ for $\mathrm{Orb}_{\mathfrak{G}^n}(G)$, the set of all twisted duals of an embedded graph G. In addition, if $g \in \mathfrak{G}$, we define

$$\mathrm{Orb}_{(g)}(G) := \mathrm{Orb}_{\mathfrak{H}}(G),$$

where $\mathfrak{H}$ is the subgroup of $\mathfrak{G}^n$ generated by $\{(g, 1, \dots, 1), (1, g, 1, \dots, 1), \dots, (1, 1, \dots, g)\}$.

Springer
http://www.springer.com/978-1-4614-6970-4

Graphs on Surfaces
Dualities, Polynomials, and Knots
Ellis-Monaghan, J.A.; Moffatt, I.
2013, XI, 139 p. 82 illus., 41 illus. in color., Softcover
ISBN: 978-1-4614-6970-4