Preface

This monograph aims to provide an accessible and fairly comprehensive treatment of recent developments on generalised dualities for graphs on surfaces and their applications. Duality arises in many areas, particularly topological and algebraic graph theory, topology, and physics. The importance of duality derives not only from its inherent properties but also from its interactions with functions on graphs (such as graph polynomials) and knot invariants. Traditionally, most of graph theory concerning polynomials and knot invariants has focused on properties of abstract or plane graphs. However, new research has impelled an analogous theory for graphs on surfaces. Here we examine the full generalisation of duality for embedded graphs and the interactions of this duality with graph polynomials and knot polynomials that resulted from this research. We illustrate some of the advantages of moving from plane and abstract graphs to graphs on surfaces. Although primarily a survey, this text does give new approaches to the material and contains several new results.

We focus on four key interdependent topics:

- Extending geometric duality fully to graphs on surfaces
- Properties of 4-regular graphs and medial graphs
- Relations, identities, and interpretations for polynomials of graphs on surfaces
- Connections between knot theory and graph theory

We begin by briefly cataloguing various descriptions of graphs on surfaces and reviewing the classical constructions of the Petrie dual, $G^{\times}$, geometric dual, G^*, and the medial graph, G_m, of an embedded graph G. This leads to our primary motivation, namely the classical relationships among the medial graphs and the geometric duals of plane graphs. Suppose that G is a plane graph with dual G^* and medial graph G_m. The medial graph of G^* is exactly the medial graph of G, i.e., $(G^*)_m = G_m$, where $=$ denotes equality as plane graphs. In fact, the connection between geometric duals and medial graphs is a little stronger than this. The two graphs G and G^* are the only plane graphs that have G_m as their plane medial graphs, that is,

$$\{G, G^*\} = \{H | H_m = G_m\}. \quad (1)$$

Twisted duality extends the fundamental classical relations among a plane graph, its plane dual, and its medial graph to graphs embedded in arbitrary surfaces. It arises from "localising" the classical constructions of geometric and Petrie duals to individual edges. These local operations lead to a group action, called the ribbon group action, on the set of embedded graphs. The twisted duals of an embedded graph comprise the orbit of it under this group action. Other forms of duality, such as geometric duality, Petriality, and partial duality, appear as actions of different subgroups of the ribbon group. Twisted duality gives a full surface analogue of Eq. (1) in that if G is any embedded graph with medial graph G_m, then its twisted duals are precisely the set of all embedded graphs with medial graphs isomorphic (as abstract graphs) to G_m. Furthermore, in analogy with how a plane graph and its dual may be reconstructed from the medial graph, all the twisted duals of an embedded graph may be constructed from its medial graph. Exploring Eq. (1) further, we find that the type of graph duality on the left-hand side and the type of graph isomorphism on the right-hand side are inextricably connected in that substituting another kind of graph isomorphisms for $=$ in the left-hand side of Eq. (1) corresponds to a particular form of duality on the right-hand side. Isomorphism as embedded graphs corresponds to geometric duality, and isomorphism as abstracts graphs corresponds to twisted duality, but further, we show that twisted duality gives a hierarchy of various forms of graph duality from the literature that correspond, through appropriate analogues of Eq. (1), to a hierarchy of graph isomorphism.

After establishing twisted duality, we turn to its interactions with invariants of graphs on surfaces, particularly graph polynomials. Recently, several graph polynomials that were originally defined for abstract or plane graphs have been extended to graphs on surfaces. These include the transition polynomial, the Penrose polynomial, and several different extensions of the Tutte polynomial. Our main tool here is the topological transition polynomial, which interacts naturally with the ribbon group action and coincides with these other polynomials as well as the Kauffman bracket of knot theory. The ribbon group action leads to new properties of the transition polynomial and from there to a deeper understanding of the properties of, and relationships among, various graph polynomials. The advantages of this approach are particularly well illustrated by the Penrose polynomial. The Penrose polynomial of a plane graph, which encodes colouring information, first appeared implicitly in the work of R. Penrose on diagrammatic tensors. Extending the Penrose polynomial to graphs on surfaces, and using its relation to the transition polynomial, reveals many new properties that simply cannot be realised in the original plane setting. These include deletion–contraction reductions, duality relations, and a restatement of the Four Colour Theorem. Similarly, we show how the ribbon group action and its interaction with the transition polynomial lead to new identities for the ribbon graph and topochromatic polynomials. These identities for graph polynomials subsequently inform applications to knot theory.

Generalising duality to graphs on surfaces has its origins in knot theory. S. Chmutov and I. Pak showed that the Jones polynomial of an alternating checkerboard colourable virtual link diagram is an evaluation of the ribbon graph polynomial of B. Bollobás and O. Riordan (which generalises the Tutte polynomial from abstract to embedded graphs). This result extends a seminal theorem of M. Thistlethwaite which relates the Tutte polynomial of a plane graph and the Jones polynomial of an alternating (classical) link. Chmutov and Pak's paper stimulated considerable research into connections among polynomials of embedded graphs, knot polynomials, and representations of link diagrams as embedded graphs. To connect the various realisations of the Jones polynomial as a graph polynomial, Chmutov introduced an extension of geometric duality called partial duality, one of the inspirations of twisted duality. We use the theory described in this text to unify various connections among dualities, graph polynomials, and knot polynomials. We emphasise the ways in which developments in knot theory lead to developments in graph theory, and vice versa, and take the reader to the forefront of research in this area.

Fundamentally, this text illustrates the interdependency between duality, medial graphs, and knots; how this interdependency is reflected in algebraic invariants of graphs and knots; and how this interdependency can be exploited to solve problems in graph theory and knot theory. Throughout, we take a constructive approach, emphasising how the ideas and constructions described here arise from classical constructions such as geometric duals and Tait graphs, by removing artificial restrictions in these constructions, by localising global operations, or by broadening the setting to embedded graphs. We describe how these adaptations may be accomplished and what the benefits of doing so are.

Our goal is to give a self-contained introduction to graphs on surfaces, twisted duality, and topological graph and knot polynomials that is accessible to both graph theorists and knot theorists. Accordingly, we have assumed familiarity with only basic graph theory and knot theory so that the text should be accessible to graduate students and researchers in either area. Because the area is advancing so rapidly, we have not attempted an exhaustive catalogue but rather tried to give a comprehensive overview, with a robust bibliography, hoping to provide the reader with the necessary knowledge and background to read research papers on these topics as they appear. We hope that the reader will come away from the text convinced of advantages of considering these higher genus analogues of constructions of plane and abstract graphs and with a good understanding of how they arise.

As a final remark, Chaps. 1 and 2 contain the common foundational material for Chaps. 3, 4 and 5. Chapters 4 and 5 do not depend upon Chap. 3 (so a reader interested only in graph polynomials or knots may safely skip Chap. 3). Chapter 5 uses material from Sects. 4.1 to 4.3 and 4.5.

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