Chapter 2

Overconvergence and Convergence in $\mathbb{C}$ of Some Integral Convolutions

This chapter deals with the overconvergence and convergence in $\mathbb{C}$ of some trigonometric convolution operators and with the approximation by some special type of convolutions called complex potentials, generated by the Beta and Gamma functions.

2.1 Complex Convolutions with Trigonometric-Type Kernels

In this section we study the overconvergence and convergence in $\mathbb{C}$ of the convolution operators based on trigonometric-type kernels.

2.1.1 Convolutions with Positive Trigonometric Kernels

This subsection deals with quantitative estimates in the overconvergence phenomenon for the classical convolution operators with positive trigonometric kernels. Also, in the particular cases of the Beatson kernel and their iterates, new shape-preserving properties are presented.

It is well known that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic and if $K_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} cos(kt)$ is a positive, even kernel (with $\rho_{k,n} \in \mathbb{R}$, for all k, n), then one can define the sequence of convolution trigonometric polynomials:

$$P_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) K_n(u-x) du, \; n \in \mathbb{N}. \quad (2.1.1)$$

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Classical choices for $K_n(t)$ are the de la Vallée–Poussin kernel, the Fejér kernel, the Jackson kernel, the generalized Jackson kernel, the Beatson kernel, and the Korovkin kernel, to mention only a few.

The quantitative convergence properties to $f(t)$ (of real variable $t \in \mathbb{R}$) of the above sequence $P_n(f)(t)$, $n = 1, 2, \ldots$, are well studied and can be found in any classical book in approximation theory (see, e.g., Stepanets [135] or DeVore–Lorentz [31]).

Supposing now that f is of complex variable $z \in \mathbb{C}$, the "complexification" of the above type of convolution polynomials can be done in two directions:

1) One replaces $x + t$ in the first expression in (2.1.1) of $P_n(f)$ by ze^{it} (here $i^2 = -1$), obtaining

$$P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) K_n(t) dt,$$

which in the case when f is analytic in a disk D_r centered at origin, $P_n(f)(z)$ becomes a polynomial of degree $\leq m_n$ of complex variable $z \in D_r$. In this direction, the approximation (and geometric) properties of the sequence $(P_n(f)(z))_{n \in \mathbb{N}}$ were intensively studied in Chap. 3, Sect. 3.1, pp. 181–204 of the book of Gal [49], where even exact estimates were obtained.

2) One replaces x from $x+t$ simply by z in the first form in (2.1.1) of $P_n(f)x)$, obtaining

$$P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(z + t) K_n(t) dt,$$

where we have to suppose that f is, for example, at least continuous in a strip of $\mathbb{C}$. In this case, evidently that $P_n(f)(z)$ loses its convolution character, but it would be of interest to study the overconvergence properties of $P_n(f)(z)$ in that strip containing the real axis.

The first goal of the present section is to study the above direction 2). It is easy to observe that if we replace x by z in the second form for $P_n(f)(x)$ in (2.1.1), then we don't obtain an operator with good approximation properties.

Clearly that the approximation properties of $P_n(f)(z)$ depend on the kernel.

For example, we can prove the following local/pointwise estimates.

Theorem 2.1.1. *Let $d > 0$ and suppose that $f : S_d \to \mathbb{C}$ is bounded and uniformly continuous in the strip $S_d = \{z = x + iy \in \mathbb{C}; x \in \mathbb{R},\ |y| \leq d\}$. Also, denote $P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(z + t) K_n(t) dt$:*

(i) If $K_n(t)$ is the de la Vallée–Poussin kernel, that is, $K_n(t) = \frac{1}{2} \cdot \frac{(n!)^2}{(2n)!} (2cos[t/2])^{2n}$, then

$$|P_n(f)(z) - f(z)| \leq 3\omega_1(f; 1/\sqrt{n})_{[z-\pi, z+\pi]},\ \textit{for all } z \in S_d,\ n \in \mathbb{N},$$

where $[z-\pi, z+\pi] = \{(z-\pi)(1-\lambda) + \lambda(z+\pi); \lambda \in [0,1]\}$ *and for* $0 \le \delta \le \pi$

$$\omega_1(f;\delta)_{[z-\pi,z+\pi]} =$$
$$sup\{|f(u+t) - f(u)|; u, u+t \in [z-\pi, z+\pi], t \in \mathbb{R}, |t| \le \delta\}.$$

(ii) If $K_n(t)$ *is the Fejér kernel, that is,* $K_n(t) = \frac{1}{2n} \cdot \left(\frac{sin[nt/2]}{sin[t/2]}\right)^2$, *then*

$$|P_n(f)(z) - f(z)| \le M\omega_1(f; 1/n)_{[z-\pi,z+\pi]}, \text{ for all } z \in S_d,\ n \in \mathbb{N},$$

where $M > 0$ *is a constant independent of* n *and* f.

(iii) If $K_n(t)$ *is the Jackson kernel, that is,* $K_n(t) = \frac{3}{2n(2n^2+1)} \cdot \left(\frac{sin[nt/2]}{sin[t/2]}\right)^4$, *then*

$$|P_n(f)(z) - f(z)| \le M\omega_2(f; 1/n)_{[z-\pi,z+\pi]}, \text{ for all } z \in S_d,\ n \in \mathbb{N},$$

where $M > 0$ *is a constant independent of* n *and* f *and for* $0 \le \delta \le \pi$

$$\omega_2(f;\delta)_{[z-\pi,z+\pi]} =$$
$$\sup\{|f(u+t) - 2f(u) + f(u-t)|; u, u-t, u+t \in [z-\pi, z+\pi], t \in \mathbb{R}, |t| \le \delta\}.$$

Proof. (i) We immediately get

$$|P_n(f)(z) - f(z)| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(z+t) - f(z)| K_n(t)dt$$
$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} \omega_1(f;|t|)_{[z-\pi,z+\pi]} K_n(t)dt$$
$$\le \omega_1(f;1/\sqrt{n})_{[z-\pi,z+\pi]} \int_{-\pi}^{\pi} (1+|t|\sqrt{n})K_n(t)dt \le 3\omega_1(f;1/\sqrt{n})_{[z-\pi,z+\pi]}.$$

(For the last inequality see, e.g., Gal [55], p. 427).

(ii) As above we arrive at the estimate

$$|P_n(f)(z) - f(z)| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(z+t) - f(z)| K_n(t)dt$$
$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} \omega_1(f;|t|)_{[z-\pi,z+\pi]} K_n(t)dt$$
$$\le \omega_1(f;1/n)_{[z-\pi,z+\pi]} \int_{-\pi}^{\pi} (1+|t|n)K_n(t)dt \le M\omega_1(f;1/\sqrt{n})_{[z-\pi,z+\pi]},$$

taking into account that for the Fejér kernel, we have (see, e.g., Gaier [37], Theorem 1) $\int_{-\pi}^{\pi} n|t|K_n(t)dt < M < \infty$, with $M > 0$ independent of n.

(iii) Firstly we easily obtain

$$P_n(f)(z) - f(z) = \frac{1}{\pi}\int_0^{\pi} [f(z+t) - 2f(z) + f(z-t)]K_n(t)dt,$$

which immediately implies

$$|P_n(f)(z) - f(z)| = \frac{1}{\pi}\int_{-\pi}^{\pi} \omega_2(f;t)_{[z-\pi,z+\pi]}K_n(t)dt$$

$$\leq \frac{1}{\pi}\omega_2(f;1/n)_{[z-\pi,z+\pi]}\int_0^{\pi}(1+nt)^2K_n(t)dt \leq M\omega_2(f;1/n)_{[z-\pi,z+\pi]},$$

because by, e.g., Lorentz [97], p. 56, we have $\int_0^{\pi}(1+nt)^2K_n(t)dt \leq C$, with $C > 0$ independent of n. □.

The second goal of the present section is to discuss the shape-preserving properties of the complex convolution operators as defined by the above-mentioned direction 1), in the particular cases of the Beatson kernel and their iterates. At the beginning, we will recall some approximation and shape-preserving properties of these complex convolutions based on the Beatson kernel and their iterates. For that purpose we need some preliminaries, as follows.

Let us consider the open disk $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ and

$$A(\mathbb{D}_1) = \{f : \overline{\mathbb{D}_1} \to \mathbb{C}; f \text{ is analytic on } \mathbb{D}_1, \text{ and } f(0) = 0, f'(0) = 1\}.$$

It is well known that $f \in A(\mathbb{D}_1)$ is called starlike if $f(\mathbb{D}_1)$ is a starlike plane domain with respect to 0 and it is called convex if $f(\mathbb{D}_1)$ is a convex plane domain.

Define now for $n, r \in \mathbb{N}$ the Beatson kernels by (see Beatson [13])

$$B_{n,r}(t) = \frac{n}{2\pi c_{n,r}}\int_{t-\pi/n}^{t+\pi/n} K_{n,r}(s)ds,$$

where $K_{n,r}(t)$ are the Jackson kernels given by

$$K_{n,r}(s) = \left(\frac{sin\frac{ns}{2}}{sin\frac{s}{2}}\right)^{2r},$$

with $c_{n,r}$ chosen such that $\frac{1}{\pi}\int_{-\pi}^{\pi} K_{n,r}(s)ds = c_{n,r}$, and the iterates of the Beatson kernels by (see Gal [40]) $B_{n,r,1}(t) := B_{n,r}(t)$,

$$B_{n,r,2}(t) = \frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n} B_{n,r,1}(s)ds \quad ,\ldots,$$

$$B_{n,r,p}(t) = \frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n} B_{n,r,p-1}(s)ds,$$

$p = 2, 3\ldots,, n, r \in \mathbb{N}$.

Through these trigonometric kernels, one can define the complex convolutions

$$L_{n,r}(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{iu})B_{n,r}(u)du$$

and

$$L_{n,r,p}(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{iu})B_{n,r,p}(u)du.$$

Remark. The approximation properties by these complex convolutions were obtained in terms of the second-order modulus of smoothness in Gal [39], at p. 243 and p. 246.

The following subclasses of functions are important in geometric function theory:

$$S_1 = \{f : \mathbb{D}_1 \to \mathbb{C}; f(z) = z + a_2z^2 + \ldots, \text{ analytic in } \mathbb{D}_1 \text{ and} \sum_{k=2}^{\infty} k|a_k| \leq 1\},$$

$$S_2 =$$

$$= \{f : \overline{\mathbb{D}_1} \to \mathbb{C}; f \text{ is analytic on } \mathbb{D}_1, f(0) = f'(0)-1 = 0, |f''(z)| \leq 1, z \in \mathbb{D}_1\},$$

$$\mathcal{R} =$$

$$= \{f : \overline{\mathbb{D}_1} \to \mathbb{C}; f \text{ is analytic on } \mathbb{D}, f(0) = f'(0)-1 = 0, Ref'(z) > 0, z \in \mathbb{D}_1\},$$

and

$$S_M =$$

$$= \{f : \overline{\mathbb{D}_1} \to \mathbb{C}; f \text{ is analytic on } \mathbb{D}, f(0) = f'(0)-1 = 0, |f'(z)| < M, z \in \mathbb{D}_1\}.$$

According to, e.g., Mocanu, Bulboacă, and Sălăgean [110], p. 97, Exercise 4.9.1, if $f \in S_1$, then $|\frac{zf'(z)}{f(z)} - 1| < 1, z \in \mathbb{D}_1$ and therefore f is starlike (and univalent) on $\mathbb{D}_1$.

By Obradović [116] it follows that $f \in S_2$ implies that f is starlike univalent on $\mathbb{D}_1$.

Also, it is known that $\mathcal{R}$ is called the class of functions with bounded turn (because $f \in \mathcal{R}$ is equivalent to $|argf'(z)| < \frac{\pi}{2}$, $z \in \mathbb{D}_1$) and that $f \in \mathcal{R}$ implies the univalence of f on $\mathbb{D}_1$.

Finally, according to, e.g., Mocanu, Bulboacă, and Sălăgean [110], p. 111, Exercise 5.4.1, $f \in S_M$ implies that f is univalent on $\mathbb{D}_{\frac{1}{M}} = \{z \in \mathbb{C}; |z| < \frac{1}{M}\}$.

We can present the following shape-preserving properties, recalled in Gal and Greiner [65] too:

Theorem 2.1.2. *Let $n, r, p \in \mathbb{N}$.*

(i) (see also Gal [40], Theorem 1, (ii)) If f is convex on $\mathbb{D}_1$, then $L_{n,r,p}(f)$ is close to convex;
(ii) (see also Gal [42], Theorem 3.3, (i)) $L_{n,r,p}(S_1) \subset S_1$;
(iii) (see also Gal [42], Theorem 3.4, (i)) $L_{n,r,p}(S_2) \subset S_2$ and $L_{n,r,p}(\mathcal{R}) \subset \mathcal{R}$;
(iii) (see also [42], Theorem 3.5) $L_{n,r,p}(S_M) \subset S_M$.

Unfortunately, the convolution $L_{n,r}(f)(z)$ does not preserve the convexity of f. More exactly we have:

Theorem 2.1.3 (Gal–Greiner [65]). *Let $f \in A(\mathbb{D})$. The convolution polynomial defined by*

$$L_{n,r}(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{iu}) B_{n,r}(u)du,$$

$z = re^{ix} \in \mathbb{D}$, does not preserve the convexity of f for any $n, r \in \mathbb{N}$.

Proof. Take, for example, the Köebe function $k(z) = \frac{z}{(1-z)^2}$. Then by straightforward calculation, we get that up to a constant we have

$$L_{n+1,1}(k)(z) = \sum_{k=0}^{n} \frac{\sin(k\pi/(n+1))}{\sin(\pi/(n+1))} \cdot \frac{n+1-k}{n+1} \cdot z^k.$$

These polynomials are known to be univalent but not convex in any direction; see Suffridge [137].

Also, plots of $L_{n,r}(k)(z)$, for $r \geq 2$ and $n \in \mathbb{N}$ lead (at least numerically) to polynomials which are not convex in any direction. □

In what follows, connected to a famous problem of Schoenberg, we will give an explanation of the negative result contained by Theorem 2.1.3.

Thus, if $Q(t)$ is a 2π-periodic kernel, $f \in A(\mathbb{D}_1)$ and one defines the convolution

$$L(f)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ze^{it}) Q(t)dt,$$

then in Ruscheweyh and Salinas [127], the complete solution to the problem in Schoenberg [129] is found, by proving the following result.

Theorem 2.1.4 (Ruscheweyh–Salinas [127]). *The convolution defined through the 2π-periodic kernel $Q(t)$ as above (with $Q(t)$ appropriate smooth) is convexity preserving if and only if the following conditions are satisfied:*

(i) $Q(t)$ *is periodically monotone, that is, monotonically increasing on the first subinterval of* $[-\pi, \pi]$ *and decreasing on the second subinterval of* $[-\pi, \pi]$.

(ii) $\log |Q'(t)|$ *is concave in every subinterval of* $\mathbb{R}$ *in which* $Q(t)$ *does not take its minimum or maximum value.*

Remark. It is easy to see that for an appropriate smooth kernel $Q(t)$, the fact that $\log |Q'(t)|$ is concave, one reduces to the inequality

$$[Q''(t)]^2 - Q'(t) \cdot Q'''(t) \geq 0, \text{ for all } t \in \mathbb{R}. \tag{A}$$

The main result connected to the Beatson kernels is the following.

Theorem 2.1.5 (Gal–Greiner [65]). *For any* $n, r \in \mathbb{N}$ *the* n*th Beatson kernel of order* r, $B_{n,r}(t)$, *satisfies*

$$-B'_{n,r}(t) = B'_{n,r}(-t) \geq 0, \textit{for all } t \in [0, \pi], \tag{2.1.2}$$

$$B''_{n,r}(t)^2 - B'_{n,r}(t) B'''_{n,r}(t) \geq 0, \textit{for all } t \in \mathbb{R}. \tag{2.1.3}$$

Proof. The case $n = 1$ is immediate. Fix $n, r \in \mathbb{N}$, $n \geq 2$, and let

$$\begin{aligned}
b(t) &:= \frac{2\pi}{n c_{n,r}} B'_{n,r}(t) = \frac{1}{c_{n,r}} \left(K_{n,r}\left(t + \frac{\pi}{n}\right) - K_{n,r}\left(t - \frac{\pi}{n}\right)\right) \\
&= \left(\frac{\cos \frac{nt}{2}}{\sin \frac{nt+\pi}{2n}} \right)^{2r} - \left(\frac{\cos \frac{nt}{2}}{\sin \frac{nt-\pi}{2n}} \right)^{2r} \\
&= \left(\frac{2\cos \frac{nt}{2}}{\cos \frac{\pi}{n} - \cos t} \right)^{2r} \left(\left(\sin \frac{nt - \pi}{2n} \right)^{2r} - \left(\sin \frac{nt + \pi}{2n} \right)^{2r} \right) \\
&= f(t)^{2r} g(t),
\end{aligned}$$

where

$$\begin{aligned}
f(t) &:= \frac{2\cos \frac{nt}{2}}{\cos \frac{\pi}{n} - \cos t}, \\
g(t) &:= \left(\sin \frac{nt - \pi}{2n} \right)^{2r} - \left(\sin \frac{nt + \pi}{2n} \right)^{2r}.
\end{aligned}$$

(Observe that $f(t)^2$ is up to normalization the nth Fejér–Korovkin kernel.)

Inequality (2.1.2) now immediately follows from $g(t) \leq 0$ for $t \in [0, \pi]$. For the proof of (2.1.3) we are left to show, because of symmetry and periodicity, that

$$\begin{aligned}
0 &\leq b'(t)^2 - b(t) b''(t) \\
&= f(t)^{4r-2} \left[2r \left(f'(t)^2 - f(t) f''(t) \right) g(t)^2 + f(t)^2 \left(g'(t)^2 - g(t) g''(t) \right) \right].
\end{aligned}$$

for all $t \in [0,\pi]$. To this end we shall establish the two inequalities

$$0 \le f'(t)^2 - f(t)f''(t), \tag{2.1.4}$$

$$0 \le g'(t)^2 - g(t)g''(t). \tag{2.1.5}$$

Inequality (2.1.5) is implied by the representation

$$g'(t)^2 - g(t)g''(t) = \frac{r}{2}\left[\left(\sin\frac{nt+\pi}{2n}\right)^{2r-1} + \left(\sin\frac{nt-\pi}{2n}\right)^{2r-1}\right]^2$$
$$+r\left(2r\cos\frac{\pi}{n} + 2r - 1 - \cos t\right)\left(\sin\frac{\pi}{2n}\right)^2\left(\sin\frac{nt+\pi}{2n}\right)^{2r-2}\left(\sin\frac{nt-\pi}{2n}\right)^{2r-2}$$
$$\ge 0.$$

To prove inequality (2.1.4) we first observe

$$\begin{aligned}&\left(\cos t - \cos\frac{\pi}{n}\right)^4 [f'(t)^2 - f(t)f''(t)]\\ &= n^2\left(\cos t - \cos\frac{\pi}{n}\right)^2 - 2\left(1 - \cos t\cos\frac{\pi}{n}\right)(1+\cos nt)\\ &= n^2\left(x - \cos\frac{\pi}{n}\right)^2 - 2\left(1 - x\cos\frac{\pi}{n}\right)(1 + T_n(x)),\end{aligned}$$

where $x = \cos t \in [-1,1]$ and T_n is the nth Chebyshev polynomial of the first kind. Hence we are left to show that

$$p_n(x) := 2\left(1 - x\cos\frac{\pi}{n}\right)(1+T_n(x)) \le n^2\left(\cos\frac{\pi}{n} - x\right)^2, \tag{2.1.6}$$

for $x \in [-1,1]$. If $n = 2,3$ is immediate, for $n \ge 4$, we consider two cases to prove (2.1.6). First, let $x \in [-1, \cos(2\pi/n)]$. Since $1+T_n(x) = 1+T_n(\cos t) = 1+\cos nt \le 2$ and

$$\begin{aligned}\frac{n^2}{2}\frac{(x-\cos\frac{\pi}{n})^2}{1 - x\cos\frac{\pi}{n}} &= \frac{(1+2\cos\frac{\pi}{n})^2}{1+2\cos\frac{\pi}{n}+2(\cos\frac{\pi}{n})^2}\left(n\sin\frac{\pi}{2n}\right)^2\\ &\ge \frac{(1+2\cos\frac{\pi}{4})^2}{1+2\cos\frac{\pi}{4}+2(\cos\frac{\pi}{4})^2}\left(n\frac{2\sqrt{2}}{\pi}\frac{\pi}{2n}\right)^2\\ &= 2+\sqrt{2} > 2\end{aligned}$$

we obtain (2.1.6) for those x.

Now, let $x \in [\cos(2\pi/n), 1]$. Since $p_n(\cos(\pi/n)) = p_n'(\cos(\pi/n)) = 0$, inequality (2.1.6) follows from

$$p_n''(x) \le 2n^2, \text{ for } x \in [\cos(2\pi/n), 1], \tag{2.1.7}$$

by integrating twice. Hence, we are left to establish (2.1.7).

Elementary properties of the Chebyshev polynomials T_n as their differential equation

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0$$

and

$$T_n\left(\cos\frac{k\pi}{n}\right) = (-1)^k \qquad \text{for } k = 0, 1, \dots, n,$$

$$T_n'\left(\cos\frac{k\pi}{n}\right) = 0 \qquad \text{for } k = 1, 2\dots, n-1,$$

imply

$$p_n''\left(\cos\frac{k\pi}{n}\right) = (-1)^{k+1} 2n^2 \frac{1-\cos\frac{k\pi}{n}}{1-(\cos\frac{k\pi}{n})^2} \qquad \text{for } k = 1, 2\dots, n-1$$

and show together with $\lim_{x\to\infty} p_n''(x) = -\infty$ that this polynomial of exact degree $n-1$ has one zero in the interval $(\cos(\pi/n), +\infty)$ and one zero in each of the $n-2$ intervals $(\cos(k\pi/n), \cos((k-1)\pi/n))$, $k = 2, 3, \dots, n-1$. Hence, p_n'' has exactly $n-2$ extremas $x_{n-2} < x_{n-1} < \dots < x_1$, each one of them between two consecutive zeros. In this enumeration, the point x_l is a maximum if l is odd and a minimum if l is even. Particularly we have $x_3 < \cos(2\pi/n)$. Furthermore,

$$p_n^{(3)}\left(\cos\frac{\pi}{n}\right) = 0,$$

$$p_n^{(4)}\left(\cos\frac{\pi}{n}\right) = -\frac{n^2}{12}\frac{1}{(\sin\frac{\pi}{n})^2}\left(n^2 - 1 - \frac{3}{(\sin\frac{\pi}{n})^2}\right) < 0,$$

which imply $x_1 = \cos(\pi/n)$. Therefore, p_n'' attains its maximum in the interval $[\cos(2\pi/n), 1]$ at the point $\cos(\pi/n)$. This proves (2.1.7). □

Remarks. 1) Inequality (2.1.6) also follows by writing $1+T_n(x)$ as a product (knowing its zeros), dividing the left-hand side of (2.1.6) by the right-hand side and an examination of its logarithmic derivative in the interval $(\cos(3\pi/n), 1)$. This may shorten the proof a little bit, depending on how explicit one would like to be.

2) In conclusion, Theorem 2.1.5 combined with Theorem 2.1.3 show that despite of the fact that the kernels $B_{n,r}(t)$ satisfy condition (i) in Theorem 2.1.4 and the above inequality (A), however do not fully satisfy condition (ii) in Theorem 2.1.4, because any $B'_{n,r}(t)$ has roots at which $B_{n,r}(t)$ does not neither take its minimum nor its maximum value.

Notice that it is for the first time when one really has to read the characterization in Theorem 2.2.4 so carefully that the small difference between condition (ii) in Theorem 2.1.4 and the inequality (A) in the Remark after the statement of Theorem 2.1.4 comes up.

Finally, several interesting open problems could be raised, as follows:

Problem 1. Do the inequality (2.1.3) in Theorem 2.1.5 satisfy the iterative Beatson kernels $B_{n,r,p}(t)$ too?

Problem 2. If the answer to the Problem 1 is positive, then for $p \geq 2$, do the iterative Beatson kernels $B_{n,r,p}(t)$ satisfy for the conditions in Theorem 2.1.4?

Problem 3. Do the convolution polynomials $L_{n,r}(f)(z)$ or $L_{n,r,p}(f)(z)$ preserve the subordination and the distortion of $f(z)$?

Problem 4. What other geometric properties could the convolutions have based on the Beatson kernels?

2.1.2 Convolutions with Nonpositive Cosine Kernels

In this subsection we derive approximation properties of the complex convolution operators based on nonpositive kernels of the form

$$K_{p,t}(u) = \int_0^{\infty} e^{-s^p} cos\left\{\frac{us}{t^{1/(p)}}\right\} ds,\ p \in \mathbb{N}, t \in \mathbb{R}_+.$$

More exactly, we deal with the following two types of complex convolutions, which were defined and studied in Gal, Gal, and Goldstein [64]:

$$S_q(t)f(z) =$$

$$= \frac{1}{\pi t^{1/(2q)}} \int_{-\infty}^{+\infty} \left[\int_0^{\infty} e^{-s^{2q}} cos\left\{\frac{us}{t^{1/(2q)}}\right\} ds\right] f(ze^{-iu})du,\ q \geq 2,$$

and

$$T_q(t)f(z) =$$

$$= \frac{1}{\pi t^{1/(2q+1)}} \int_{-\infty}^{+\infty} \left[\int_0^{\infty} e^{-s^{2q+1}} cos\left\{\frac{us}{t^{1/(2q+1)}}\right\} ds\right] f(ze^{-iu})du,\ q \geq 1,$$

where $z \in \overline{\mathbb{D}_1}$, $t \geq 0$ and f is considered analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}_1}$.

In this sense, we present the following.

Theorem 2.1.6 (Gal–Gal–Goldstein [64]). *Let f be analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}_1}$, and $t \in \mathbb{R}$, $t \geq 0$.*

(i) For $q \in \mathbb{N}$, $q \geq 2$, the following estimate holds:

$$|S_q(t)f(z) - f(z)| \leq C_{2q}\omega_1(f; t^{1/(2q)})_{\overline{\mathbb{D}_1}},\ for\ all\ z \in \overline{\mathbb{D}_1},,\ \ t > 0,$$

where $C_{2q} > 0$ *is a constant independent of* t *and* f *and* $\omega_1(f;\delta)_{\overline{\mathbb{D}_1}}$ *denotes the modulus of continuity, defined by*

$$\omega_1(f;\delta)_{\overline{\mathbb{D}_1}} = \sup\{|f(u)-f(v)| : |u-v| \le \delta, u,v \in \overline{\mathbb{D}_1}\}.$$

(ii) For $q \in \mathbb{N}$, $q \ge 1$, *we have*

$$|T_q(t)f(z) - f(z)| \le C_{2q+1}\omega_1(f;t^{1/(2q+1)})_{\overline{\mathbb{D}_1}}, \text{ for all } z \in \overline{\mathbb{D}_1},\ t>0,$$

where $C_{2q+1} > 0$ *is a constant independent of* t *and* f.

Proof. (i) Reasoning exactly as in the proof of Theorem 2.1, (v) in Gal, Gal, and Goldstein [64] (whose details are too long to be reproduced here) and taking into account the maximum modulus theorem, we obtain

$$\begin{aligned}
|S_q(t)f(z) - f(z)| &\le \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\int_0^{\infty} e^{-ts^{2q}} cos(\alpha s)ds\right| |f(ze^{-i\alpha}) - f(z)|d\alpha \\
&\le \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\int_0^{\infty} e^{-ts^{2q}} cos(\alpha s)ds\right| \omega_1(f;|1-e^{-i\alpha}|)_{\overline{\mathbb{D}_1}}d\alpha \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\int_0^{\infty} e^{-ts^{2q}} cos(\alpha s)ds\right| \omega_1\left(f;2\left|\sin\frac{\alpha}{2}\right|\right)_{\overline{\mathbb{D}_1}}d\alpha \\
&\le \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\int_0^{\infty} e^{-ts^{2q}} cos(\alpha s)ds\right| \omega_1(f;|\alpha|)_{\overline{\mathbb{D}_1}}d\alpha \\
&\le C_{2q}\omega_1(f;t^{1/(2q)})_{\overline{\mathbb{D}_1}}.
\end{aligned}$$

(ii) The proof is similar to that from the above point (i). □

Remark. Denoting

$$A(\mathbb{D}_1) = \{f : \mathbb{D}_1 \to \mathbb{C}; f \text{ is analytic in } \mathbb{D}_1 \text{ and continuous on } \overline{\mathbb{D}_1}\},$$

in Gal, Gal, and Goldstein [64], it is proved that $(S_q(t), t \ge 0)$ and $(T_q(t), t \ge 0)$ are (C_0)-semigroups of linear operators on $A(\mathbb{D}_1)$ and $u_q(t,\cdot) = S_q(t)f(\cdot)$ and $v_q(t,\cdot) = T_q(t)f(\cdot)$ are the unique solutions of the following two Cauchy problems for higher-order evolution equations in $\mathbb{C}$:

$$\frac{\partial u_q}{\partial t}(t,z) = (-1)^{q+1}\frac{\partial^{2q}u_q}{\partial\varphi^{2q}}(t,z),\ (t,z) \in (0,+\infty)\times D,\ z = re^{i\varphi},\ z \ne 0,$$

$$u(0,z) = f(z),\ z \in \overline{D},\ f \in A(\mathbb{D}_1)$$

and

$$\frac{\partial^2 v_q}{\partial t^2}(t,z) + \frac{\partial^{2(2q+1)} v_q}{\partial \varphi^{2(2q+1)}}(t,z) = 0,\ (t,z) \in (0,+\infty) \times D,\ z = re^{i\varphi},\ z \neq 0,$$

$$v_q(0,z) = f(z),\ \ z \in \overline{D},\ f \in A(\mathbb{D}_1),$$

respectively.

2.2 Approximation by Complex Potentials of Euler Type

In the real case, the approximation properties of the potentials such as those of Riesz, Bessel, generalized Riesz, generalized Bessel, and Flett have been studied by many authors; see, e.g., Kurokawa [93]; Gadjiev, Aral, and Aliev [36]; Uyhan, Gadjiev, and Aliev [140]; Sezer [131]; Aliev, Gadjiev, and Aral [6]; and their references.

In this section, we obtain some results concerning the approximation by several types of complex potentials generated by the Γ and *Beta* Euler's functions.

Let us recall that in the real case, the classical Bessel-type potential is defined for any $f \in L^p(\mathbb{R}^2)$, $1 \leq p < \infty$, by

$$B^{\alpha}(f)(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} \left[\int_{-\infty}^{\infty} \tau^{(\alpha/2)-1} e^{-\tau} W(y,\tau) f(x-y, t-\tau) dy \right] d\tau,$$

where $\alpha > 0$, $\Gamma(\alpha)$ is the Gamma function and $W(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-y^2/(4\tau)}$ is the Gauss–Weierstrass kernel.

It is known that formally, we can write

$$B^{\alpha}(f)(x,t) = \left(I - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} \right)^{-\alpha/2} f(x,t),$$

and the following convergence properties hold (see Uyhan–Gadjiev–Aliev [140]):

(i) *If* $f \in L^p(\mathbb{R}^2)$, $1 \leq p < \infty$, *is continuous at* $(x,t) \in \mathbb{R}^2$ *then* $\lim_{\alpha \to 0^+} B^{\alpha}(f)(x,t) = f(x,t)$.

(ii) *If* $f \in L^p(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$, *where* $C_0(\mathbb{R}^2)$ *denotes the space of all continuous functions on* $\mathbb{R}^2$ *vanishing at infinity, then* $\lim_{\alpha \to 0^+} B^{\alpha}(f) = f$ *uniformly on* $\mathbb{R}^2$.

(iii) *If* $f \in L^p(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, *where* $C(\mathbb{R}^2)$ *denotes the space of all continuous functions on* $\mathbb{R}^2$, *then* $\lim_{\alpha \to 0^+} B^{\alpha}(f) = f$ *uniformly on every compact* $K \subset \mathbb{R}^2$.

(iv) *In addition, for* f *in some suitable Lipschitz-type classes, quantitative upper estimates of order* $O(\alpha)$ *are obtained.*

Also, let us recall that the classical Flett potential is defined for any $f \in L^p(\mathbb{R})$ by (see Flett [35])

$$F^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} Q_t(f)(x)dt,$$

where $Q_t(f)(x) = \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2+t^2} du$ is the classical Poisson–Cauchy singular integral.

It is known that the following convergence properties hold (see Sezer [131]):

(i) *If* $f \in L^p(\mathbb{R}) \cap C_0(\mathbb{R})$, *then* $\lim_{\alpha \to 0^+} F^{\alpha}(f) = f$ *uniformly on* $\mathbb{R}$ *.;*
(ii) *For* f *in some suitable Lipschitz-type classes, quantitative upper estimates of order* $O(\alpha)$ *are obtained.*

Remark. The form of the Flett potential suggests us to study the approximation properties as $\alpha \to 0^+$ of new potentials, as follows:

$$F_U^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} U_t(f)(x)dt,$$

where $U_t(f)(x)$ can be any from $P_t(f)(x) = \frac{1}{2t} \int_{-\infty}^{+\infty} f(x-u)e^{-|u|/t}\, du$ (the Picard singular integral), $R_t(f)(x) = \frac{2t^3}{\pi} \int_{-\infty}^{+\infty} \frac{f(x-u)}{(u^2+t^2)^2}\, du$ (a Poisson–Cauchy-type singular integral), and $W_t^*(f)(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x-u)e^{-u^2/t}\, du$ (the Gauss–Weierstrass singular integral). Also, in the form of the Flett potential, we could replace the Gamma function with other special function, for example, with the Beta function, so that we could study the approximation properties as $\alpha \to 0^+$, of new potentials of the form

$$G_U^{\alpha,\beta}(f)(x) = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} U_t(f)(x)dt,$$

where $\alpha, \beta > 0$, $\alpha + \beta \geq 1$ and $U_t(f)(x)$ are any of the above-mentioned singular integrals.

In what follows, first we study the approximation properties of the complex versions of the potentials $F_U^{\alpha}(f)(x)$ (that includes the Flett potential) and $G_U^{\alpha}(f)(x)$. The complexification is made in two directions:

1) The complex forms are obtained from their real versions by replacing the translation $x - y$ by the rotation ze^{-iy}, where $z = re^{ix} \in \mathbb{C}$, that is, in the convolution form

$$F_U^{\alpha}(f)(z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} U_t(f)(z)dt,$$

$$G_U^{\alpha,\beta}(f)(z) = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} U_t(f)(z)dt,$$

where $U_t(f)(z) = Q_t(f)(z) = \frac{t}{\pi}\int_{-\infty}^{\infty} \frac{f(ze^{-iu})}{u^2+t^2} du$ or $U_t(f)(z) = P_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-|u|/t}\, du$ or $U_t(f)(z) = R_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{(u^2+t^2)^2}\, du$ or $U_t(f)(z) = W_t^*(f)(z) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/t}\, du$.

2) The complex forms are obtained simply replacing in the form of any $U_t(f)(x)$, the real variable $x \in \mathbb{R}$ by $z \in S$, where $S \subset \mathbb{C}$ is a strip, case when in fact we obtain some overconvergence results of these potentials.

Then, we will study the approximation properties of the complex Bessel-type potential, obtained from its real version by replacing the translation $x - y$ by the rotation ze^{-iy}, where $z = re^{ix} \in \mathbb{C}$, that is,

$$B^{\alpha}(f)(z,t) = \frac{1}{\Gamma(\alpha/2)}\int_0^{\infty}\left[\int_{-\infty}^{\infty} \tau^{(\alpha/2)-1}e^{-\tau}W(y,\tau)f(ze^{-iy}, t-\tau)dy\right] d\tau.$$

Note that in order to exist $F_U^{\alpha}(f)(z)$ and $G_U^{\alpha,\beta}(f)(z)$ for all $|z| < R$, it is enough to suppose that the function $f(z)$ is analytic in $|z| < R$, with $R > 1$, while in order to exist $B^{\alpha}(f)(z,t)$, it is enough to suppose that the function $f(z,t)$ is in $L^p(\overline{\mathbb{D}_R} \times \mathbb{R})$, $1 \le p < \infty$, where $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$. For the approximation properties of $B^{\alpha}(f)(z,t)$, we will suppose, in addition, that $f(z,t)$ is analytic in $\mathbb{D}_R$, $R > 1$, for any fixed $t \in \mathbb{R}$.

For $R > 0$ let us denote $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$.

The first main result is the following.

Theorem 2.2.1 (Gal [56]). *Let us suppose that $\alpha > 0$ and that $f : \mathbb{D}_R \to \mathbb{C}$, with $R > 1$, is analytic in $\mathbb{D}_R$, that is, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, for all $z \in \mathbb{D}_R$.*

(i) For $U_t(f)(z) = \frac{t}{\pi}\int_{-\infty}^{\infty} \frac{f(ze^{-iu})}{u^2+t^2} du$, we have that $F_U^{\alpha}(f)(z)$ is analytic in $\mathbb{D}_R$ and we can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot \frac{1}{(k+1)^{\alpha}} \cdot z^k, z \in \mathbb{D}_R.$$

Also, if f is not constant for $q = 0$ and not a polynomial of degree $\le q-1$ for $q \in \mathbb{N}$, then for all $1 \le r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1]$, we have

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where $\|f\|_r = \sup\{|f(z)|; |z| \le r\}$ and the constants in the equivalence depend only on f, q, r, and r_1.

(ii) For $U_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-|u|/t}\, du$, we have that $F_U^{\alpha}(f)(z)$ is analytic in $\mathbb{D}_R$ and we can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k b_{k,\alpha} z^k, z \in \mathbb{D}_R,$$

where $b_{k,\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} \frac{t^{\alpha-1}e^{-t}}{1+t^2k^2} dt$.

Also, if f is not constant for $q=0$ and not a polynomial of degree $\le q-1$ for $q \in \mathbb{N}$, then for all $1 \le r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1]$, we have

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f, q, r, and r_1.

(iii) For $U_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{(u^2+t^2)^2}\,du$, we have that $F_U^{\alpha}(f)(z)$ is analytic in $\mathbb{D}_R$ and we can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \frac{1}{(k+1)^{\alpha+1}}[k(\alpha+1)+1]\cdot z^k, z \in \mathbb{D}_R.$$

Also, there exists $\alpha_0 \in (0,1]$ (absolute constant) such that if f is not constant for $q=0$ and not a polynomial of degree $\le q-1$ for $q \in \mathbb{N}$, then for all $1 \le r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,\alpha_0]$, we have

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f, q, r, and r_1.

(iv) For $U_t(f)(z) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/t}\,du$, we have that $F_U^{\alpha}(f)(z)$ is analytic in $\mathbb{D}_R$ and we can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot \frac{1}{(1+k^2/4)^{\alpha+1}} \cdot z^k, z \in \mathbb{D}_R.$$

Also, if f is not constant for $q=0$ and not a polynomial of degree $\le q-1$ for $q \in \mathbb{N}$, then for all $1 \le r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,1]$, we have

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f, q, r, and r_1.

Proof. (i) By Gal [49], p. 213, Theorem 3.2.5, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-kt} z^k, \text{ for all } |z| < R \text{ and } t \ge 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-kt} z^k| \le \sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k e^{-kt} z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-(k+1)t} dt,$$

where by making use of the change of variable $(k+1)t = s$, we easily get that $\int_0^\infty t^{\alpha-1}e^{-(k+1)t}dt = \frac{\Gamma(\alpha)}{(k+1)^\alpha}$.

In other order of ideas, we easily can write

$$F_U^\alpha(f)(z) - f(z) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty t^{\alpha-1}e^{-t}[U_t(f)(z) - f(z)]dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t$ in Gal [49], p. 213, Theorem 3.2.5, (iii), implies

$$|F_U^\alpha(f)(z) - f(z)| \leq \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty t^{\alpha-1}e^{-t}|U_t(f)(z) - f(z)|dt$$

$$\leq C_r(f)\frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty t^\alpha e^{-t}dt = C_r(f) \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. Denoting by γ the circle of radius r_1 and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$, by using Cauchy's formula, for all $|z| \leq r$ and $\alpha > 0$, we get

$$|[F_U^\alpha(f)]^{(q)}(z) - f^{(q)}(z)| = \frac{q!}{2\pi}\left|\int_\gamma \frac{F_U^\alpha(f)(z) - f(z)}{(v-z)^{q+1}}dv\right|$$

$$\leq C_{r_1}(f)\alpha \cdot \frac{q}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{q+1}},$$

which proves the upper estimate

$$\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r \leq C^*\alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 218–219 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)\ldots(p+1)r^p[1 - e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ and then integrating with respect to t, it follows

$$I := \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi \right\} t^{\alpha-1} e^{-t} dt$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t}[1-e^{-(q+p)t}]dt$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p \left[1 - \frac{1}{(q+p+1)^\alpha}\right],$$

because taking into account that by making use of the change of variable $(q+p+1)t = s$, we easily get that

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t}[1-e^{-(q+p)t}]dt = 1 - \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(q+p+1)t} dt$$

$$= 1 - \frac{1}{(q+p+1)^\alpha}.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[\frac{1}{\Gamma(\alpha)} \int_0^\infty [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1} e^{-t} dt \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p \left[1 - \frac{1}{(q+p+1)^\alpha}\right].$$

Since

$$\frac{1}{\Gamma(\alpha)} \int_0^\infty [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1} e^{-t} dt = f^{(q)}(z) - [F_U^\alpha(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[f^{(q)}(z) - (F_U^\alpha(f))^{(q)}(z) \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p \left[1 - \frac{1}{(q+p+1)^\alpha}\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p \left[1 - \frac{1}{(q+p+1)^\alpha}\right] \le \|f^{(q)} - (F_U^\alpha(f))^{(q)}\|_r.$$

First take $q = 0$. From the previous inequality, we immediately obtain

$$|a_p| r^p \left(1 - \frac{1}{(p+1)^\alpha}\right) \le \|f - F_U^\alpha(f)\|_r.$$

In what follows, denoting $V_\alpha = \inf_{p\geq 1}\left(1-\frac{1}{(p+1)^\alpha}\right)$, we clearly get $V_\alpha = 1-\frac{1}{2^\alpha}$.

Denoting $g(x)=2^{-x}$, by the mean value theorem, there exists $\xi \in (0,\alpha) \subset (0,1]$ such that

$$V_\alpha = g(0)-g(\alpha) = -\alpha g'(\xi) = \alpha \cdot 2^{-\xi} ln(2) \geq \alpha 2^{-\alpha} ln(2) \geq \alpha 2^{-1} ln(2),$$

which immediately implies

$$\alpha \cdot \frac{ln(2)}{2} \cdot r^p \cdot |a_p| \leq \|f - F_U^\alpha(f)\|_r,$$

that is,

$$\frac{ln(2)}{2} \cdot r^p \cdot |a_p| \leq \frac{\|f - F_U^\alpha(f)\|_r}{\alpha},$$

for all $p \geq 1$ and $\alpha \in (0,1]$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty} \alpha_k = 0$ and such that $\lim_{k\to\infty} \frac{\|F_U^\alpha(f)-f\|_r}{\alpha_k} = 0$, then $a_p = 0$ for all $p \geq 1$, that is, f is constant on $\overline{\mathbb{D}}_r$.

Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,1]} \frac{\|F_U^\alpha(f)-f\|_r}{\alpha} > 0$, which implies that there exists a constant $C_r(f)>0$ such that $\frac{\|F_U^\alpha(f)-f\|_r}{\alpha} \geq C_r(f)$, for all $\alpha \in (0,1]$, that is,

$$\|F_U^\alpha(f) - f\|_r \geq C_r(f)\alpha, \text{ for all } \alpha \in (0,1].$$

Now, consider $q \geq 1$ and denote $V_{q,\alpha} = \inf_{p\geq 0}(1-\frac{1}{(q+p+1)^\alpha})$. Evidently that we have $V_{q,\alpha} \geq \inf_{p\geq 1}(1-\frac{1}{(p+1)^\alpha}) \geq \alpha \cdot \frac{ln(2)}{2}$.

Reasoning as in the case of $q=0$ we obtain

$$\frac{\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r}{\alpha} \geq |a_{q+p}| \frac{(q+p)!}{p!} \cdot \frac{ln(2)}{2} \cdot r^p,$$

for all $p \geq 0$ and $\alpha \in (0,1]$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty} \alpha_k = 0$ and such that $\lim_{k\to\infty} \frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha_k} = 0$, then $a_{q+p} = 0$ for all $p \geq 0$, that is, f is a polynomial of degree $\leq q-1$ on $\overline{\mathbb{D}}_r$.

Therefore, because by hypothesis f is not a polynomial of degree $\leq q-1$, we obtain $\inf_{\alpha\in(0,1]} \frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha} > 0$, which implies that there exists a constant $C_{r,q}(f)>0$ such that $\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha} \geq C_{r,q}(f)$, for all $\alpha \in (0,1]$, that is,

$$\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r \geq C_{r,q}(f)\alpha, \text{ for all } \alpha \in (0,1].$$

(ii) By Gal [49], p. 206, Theorem 3.2.1, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} \frac{a_k}{1+t^2k^2} z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} \frac{a_k}{1+t^2k^2} z^k| \leq \sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} \frac{a_k}{1+t^2k^2} z^k$ is uniformly convergent on $[0, \infty)$, and therefore we immediately can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{t^{\alpha-1} e^{-t}}{1+t^2k^2} dt.$$

In other order of ideas, we easily can write

$$F_U^{\alpha}(f)(z) - f(z) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} t^{\alpha-1} e^{-t} [U_t(f)(z) - f(z)] dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f) t^2$ in Gal [49], p. 207, Theorem 3.2.1, (iv), implies

$$|F_U^{\alpha}(f)(z) - f(z)| \leq \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} t^{\alpha-1} e^{-t} |U_t(f)(z) - f(z)| dt$$

$$\leq C_r(f) \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} t^{\alpha+1} e^{-t} dt = C_r(f) \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = C_r(f)\alpha(\alpha+1) \leq 2C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. Denoting by γ the circle of radius r_1 and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$, by using Cauchy's formula, for all $|z| \leq r$ and $\alpha > 0$, we get

$$|[F_U^{\alpha}(f)]^{(q)}(z) - f^{(q)}(z)| = \frac{q!}{2\pi} \left| \int_{\gamma} \frac{F_U^{\alpha}(f)(z) - f(z)}{(v-z)^{q+1}} dv \right|$$

$$\leq 2C_{r_1}(f)\alpha \cdot \frac{q}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{q+1}},$$

which proves the upper estimate

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \leq C^* \alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.1, at pages 209–210 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$=a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p\cdot\frac{t^2(q+p)^2}{1+t^2(q+p)^2}.$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ and then integrating with respect to t, it follows

$$I:=\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}e^{-t}dt$$

$$=a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt\right]d\varphi\right|$$

$$=|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt=f^{(q)}(z)-[F_U^{\alpha}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z)-(F_U^{\alpha}(f))^{(q)}(z)\right]d\varphi\right|$$

$$=|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right]$$

$$\leq\|f^{(q)}-(F_U^{\alpha}(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality we immediately obtain

$$|a_p|r^p\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right)\leq\|f-F_U^{\alpha}(f)\|_r.$$

In what follows, denoting $V_\alpha = \inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right)$, we clearly get

$$V_\alpha = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[\frac{t^2}{1+t^2}\right]dt.$$

Taking into account that $1+t^2 \leq 2e^t$ for all $t\geq 0$, we obtain

$$V_\alpha \geq \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha+1}e^{-2t}dt = \frac{\Gamma(\alpha+2)}{2^{\alpha+2}\Gamma(\alpha)} = \frac{\alpha}{4}\cdot\frac{\alpha+1}{2^\alpha} \geq C\alpha,$$

since the function $f(x) = \frac{x+1}{2^x}$ is strictly positive and continuous in $[0,1]$.

This immediately implies

$$C\cdot r^p\cdot|a_p| \leq \frac{\|f - F_U^\alpha(f)\|_r}{\alpha},$$

for all $p\geq 1$ and $\alpha\in(0,1]$.

Now, if a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\alpha_k = 0$ would exist and such that $\lim_{k\to\infty}\frac{\|F_U^\alpha(f)-f\|_r}{\alpha_k} = 0$, then $a_p = 0$ for all $p\geq 1$, that is, f would be constant on $\overline{\mathbb{D}}_r$. Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,1]}\frac{\|F_U^\alpha(f)-f\|_r}{\alpha} > 0$, which implies that there exists a constant $C_r(f) > 0$ such that $\frac{\|F_U^\alpha(f)-f\|_r}{\alpha} \geq C_r(f)$, for all $\alpha\in(0,1]$, that is,

$$\|F_U^\alpha(f) - f\|_r \geq C_r(f)\alpha, \text{ for all } \alpha\in(0,1].$$

Now, consider $q\geq 1$ and denote

$$V_{q,\alpha} = \inf_{p\geq 0}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]\right).$$

Evidently that we have $V_{q,\alpha} \geq \inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right) \geq \alpha\cdot C$.

Reasoning as in the case of $q=0$, we obtain

$$\frac{\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r}{\alpha} \geq |a_{q+p}|\frac{(q+p)!}{p!}\cdot C\cdot r^p,$$

for all $p\geq 0$ and $\alpha\in(0,1]$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\alpha_k = 0$ and such that $\lim_{k\to\infty}\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha_k} = 0$, then $a_{q+p} = 0$ for all $p\geq 0$, that is, f is a polynomial of degree $\leq q-1$ on $\overline{\mathbb{D}}_r$.

Therefore, because by hypothesis f is not a polynomial of degree $\leq q-1$, we obtain $\inf_{\alpha\in(0,1]}\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha} > 0$, which implies that there exists a constant $C_{r,q}(f) > 0$ such that $\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha} \geq C_{r,q}(f)$, for all $\alpha\in(0,1]$, that is,

$$\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r \geq C_{r,q}(f)\alpha, \text{ for all } \alpha \in (0,1].$$

(iii) By Gal [49], p. 213, Theorem 3.2.5, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-kt}(1+kt)z^k| \leq 2\sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^\alpha(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(1+kt)e^{-(k+1)t}dt,$$

where by making use of the change of variable $(k+1)t = s$, we easily get that $\int_0^\infty t^{\alpha-1}e^{-(k+1)t}dt = \frac{\Gamma(\alpha)}{(k+1)^\alpha}$ and therefore we immediately obtain

$$F_U^\alpha(f)(z) = \sum_{k=0}^{\infty} a_k \frac{1}{(k+1)^{\alpha+1}}[k(\alpha+1)+1]z^k.$$

In other order of ideas, we easily can write

$$F_U^\alpha(f)(z) - f(z) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty t^{\alpha-1}e^{-t}[U_t(f)(z) - f(z)]dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t^2$ in Gal [49], p. 213–214, Theorem 3.2.5, (iv), implies

$$|F_U^\alpha(f)(z) - f(z)| \leq \frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty t^{\alpha-1}e^{-t}|U_t(f)(z) - f(z)|dt$$

$$\leq C_r(f)\frac{1}{\Gamma(\alpha)} \cdot \int_0^\infty t^{\alpha+1}e^{-t}dt = C_r(f) \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = C_r(f)\alpha(\alpha+1) \leq 2C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. Denoting by γ the circle of radius r_1 and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$, by using Cauchy's formula, for all $|z| \leq r$ and $\alpha > 0$, we get

$$|[F_U^\alpha(f)]^{(q)}(z) - f^{(q)}(z)| = \frac{q!}{2\pi}\left|\int_\gamma \frac{F_U^\alpha(f)(z) - f(z)}{(v-z)^{q+1}}dv\right|$$

$$\leq C_{r_1}(f)\alpha \cdot \frac{q}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{q+1}},$$

which proves the upper estimate

$$\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r \le C^*\alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 219–220 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p[1-(1+(q+p)t)e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ and then integrating with respect to t, it follows

$$I := \frac{1}{\Gamma(\alpha)}\cdot\int_0^\infty \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}e^{-t}dt$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{\Gamma(\alpha)}\int_0^\infty[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^\infty[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt = f^{(q)}(z) - [F_U^\alpha(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z) - (F_U^\alpha(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$
$$\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right] \le \|f^{(q)}-(F_U^{\alpha}(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality, we immediately obtain

$$|a_p|r^p\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+pt)e^{-pt}\right]dt\right) \le \|f-F_U^{\alpha}(f)\|_r.$$

In what follows, denoting

$$V_\alpha=\inf_{p\ge 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+pt)e^{-pt}\right]dt\right),$$

by simple calculation, we get

$$V_\alpha=\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+t)e^{-t}\right]dt=1-\frac{1}{2^\alpha}-\frac{\alpha}{2^{\alpha+1}}.$$

But there exists $\alpha_0\in(0,1]$, such that if C is an absolute constant with $0<C<ln(2)-\frac{1}{2}$, then we have

$$1-\frac{1}{2^\alpha}-\frac{\alpha}{2^{\alpha+1}}\ge C\alpha,\ \text{for all}\ \alpha\in[0,\alpha_0].$$

Indeed, denoting $g(\alpha)=1-\frac{1}{2^\alpha}-\frac{\alpha}{2^{\alpha+1}}-C\alpha$, we have $g(0)=0$ and $g'(\alpha)=2^{-\alpha}ln(2)-\frac{1}{2^{\alpha+1}}+\frac{\alpha(\alpha+1)ln(2)}{2^{\alpha+1}}-C$, which implies $g'(0)=ln(2)-\frac{1}{2}-C>0$. Since $g'(\alpha)$ obviously is continuous with respect to α, there exists $\alpha_0>0$ such that $g'(\alpha)>0$ for all $\alpha\in[0,\alpha_0]$, that is, $V_\alpha\ge C\alpha$, for all $\alpha\in[0,\alpha_0]$.

This immediately implies

$$C\cdot r^p\cdot|a_p|\le\frac{\|f-F_U^{\alpha}(f)\|_r}{\alpha},$$

for all $p\ge 1$ and $\alpha\in(0,\alpha_0]$.

Now, if a subsequence $(\alpha_k)_k$ in $(0,\alpha_0]$ with $\lim_{k\to\infty}\alpha_k=0$ would exist and such that $\lim_{k\to\infty}\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha_k}=0$, then $a_p=0$ for all $p\ge 1$, that is, f would be constant on $\overline{\mathbb{D}}_r$. Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,\alpha_0]}\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha}>0$, which implies that there exists a constant $C_r(f)>0$ such that $\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha}\ge C_r(f)$, for all $\alpha\in(0,\alpha_0]$, that is,

$$\|F_U^{\alpha}(f)-f\|_r\ge C_r(f)\alpha,\ \text{for all}\ \alpha\in(0,\alpha_0].$$

Now, consider $q\ge 1$ and denote

$$V_{q,\alpha} = \inf_{p\geq 0}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]\right).$$

Evidently that we have $V_{q,\alpha} \geq \inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+pt)e^{-pt}\right]\right.$ $\left. dt\right) \geq \alpha\cdot C$, for $C\in(0, ln(2)-1/2)$ and $\alpha\in[0,\alpha_0]$.

Reasoning exactly as in the case of $q=0$ and as in the previous case (ii), we easily obtain that because by hypothesis f is not a polynomial of degree $\leq q-1$, there exists a constant $C_{r,q}(f)>0$ such that

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \geq C_{r,q}(f)\alpha, \text{ for all } \alpha\in(0,\alpha_0].$$

(iv) By Gal [49], p. 223, Theorem 3.2.8, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-k^2t/4}z^k, \text{ for all } |z|<R \text{ and } t\geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-k^2t/4}z^k| \leq \sum_{k=0}^{\infty}|a_k|\cdot|z|^k<\infty$, this implies that for fixed $|z|<R$, the series in t, $\sum_{k=0}^{\infty} a_k e^{-k^2t/4}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-(1+k^2/4)t}dt,$$

where by making use of the change of variable $(1+k^2/4)t=s$, we easily get that $\int_0^{\infty} t^{\alpha-1}e^{-(1+k^2/4)t}dt = \frac{\Gamma(\alpha)}{(1+k^2/4)^{\alpha}}$ and therefore we immediately obtain

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \frac{1}{(1+k^2/4)^{\alpha+1}}z^k.$$

In other order of ideas, we easily can write

$$F_U^{\alpha}(f)(z) - f(z) = \frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty} t^{\alpha-1}e^{-t}[U_t(f)(z)-f(z)]dt,$$

which together with the estimate $|U_t(f)(z)-f(z)|\leq C_r(f)t$ in Gal [49], p. 224, Theorem 3.2.8, (iv), implies

$$|F_U^{\alpha}(f)(z) - f(z)| \leq \frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty} t^{\alpha-1}e^{-t}|U_t(f)(z)-f(z)|dt$$

$$\leq C_r(f)\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty} t^{\alpha}e^{-t}dt = C_r(f)\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. Denoting by γ the circle of radius r_1 and center 0, since for any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$, by using Cauchy's formula, for all $|z| \leq r$ and $\alpha > 0$, we get

$$|[F_U^{\alpha}(f)]^{(q)}(z) - f^{(q)}(z)| = \frac{q!}{2\pi} \left| \int_{\gamma} \frac{F_U^{\alpha}(f)(z) - f(z)}{(v-z)^{q+1}} dv \right|$$

$$\leq C_{r_1}(f)\alpha \cdot \frac{q}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{q+1}},$$

which proves the upper estimate

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \leq C^* \alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.8, at pages 227–228 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)\ldots(p+1)r^p[1 - e^{-(q+p)^2 t/4}].$$

Multiplying above with $\frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t}$ and then integrating with respect to t, it follows

$$I := \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi \right\} t^{\alpha-1} e^{-t} dt$$

$$= a_{q+p}(q+p)(q+p-1)\ldots(p+1)r^p$$

$$\cdot \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} \left[1 - e^{-(q+p)^2 t/4}\right] dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[\frac{1}{\Gamma(\alpha)} \int_0^{\infty} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1} e^{-t} dt \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\ldots(p+1)r^p$$

$$\cdot \left[\frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} \left[1 - e^{-(q+p)^2 t/4}\right] dt \right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt=f^{(q)}(z)-[F_U^{\alpha}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z)-(F_U^{\alpha}(f))^{(q)}(z)\right]d\varphi\right|\\&=|a_{q+p}|(q+p)(q+p-1)\ldots(p+1)r^p\\&\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[1-e^{-(q+p)^2t/4}\right]dt\right]\end{aligned}$$

and

$$\begin{aligned}&|a_{q+p}|(q+p)(q+p-1)\ldots(p+1)r^p\\&\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[1-e^{-(q+p)^2t/4}\right]dt\right]\le\|f^{(q)}-(F_U^{\alpha}(f))^{(q)}\|_r.\end{aligned}$$

First take $q=0$. From the previous inequality, we immediately obtain

$$|a_p|r^p\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[1-e^{-p^2t/4}\right]dt\right)\le\|f-F_U^{\alpha}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha}=\inf_{p\ge1}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[1-e^{-p^2t/4}\right]dt\right),$$

by simple calculation, we get

$$V_{\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[1-e^{-t/4}\right]dt=1-\left(\frac{4}{5}\right)^{\alpha}.$$

Denoting $g(x)=\left(\frac{4}{5}\right)^x$, by the mean value theorem, there exists $\xi\in(0,\alpha)\subset(0,1]$ such that

$$\begin{aligned}V_{\alpha}=g(0)-g(\alpha)=-\alpha g'(\xi)=\alpha\cdot\left(\frac{4}{5}\right)^{\xi}ln\left(\frac{4}{5}\right)\ge\alpha\left(\frac{4}{5}\right)^{\alpha}ln\left(\frac{4}{5}\right)\\\ge\alpha\left(\frac{4}{5}\right)ln\left(\frac{4}{5}\right),\end{aligned}$$

which immediately implies

$$\left(\frac{4}{5}\right)ln\left(\frac{4}{5}\right)\cdot r^p\cdot|a_p|\le\frac{\|f-F_U^{\alpha}(f)\|_r}{\alpha},$$

for all $p\ge1$ and $\alpha\in(0,1]$.

Reasoning now exactly as in the proof of the above point (i), we similarly get that if f is not a constant function, then there exists a constant $C_r(f) > 0$ such that

$$\|F_U^\alpha(f) - f\|_r \geq C_r(f)\alpha, \text{ for all } \alpha \in (0, 1].$$

Now, consider $q \geq 1$ and denote

$$V_{q,\alpha} = \inf_{p\geq 0}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1 - e^{-(q+p)^2 t/4}\right]\right).$$

Evidently that we have

$$V_{q,\alpha} \geq \inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1 - e^{-p^2 t/4}\right]dt\right) \geq \alpha \cdot C,$$

for all $\alpha \in [0, 1]$.

Reasoning in continuation exactly as in the case of $q = 0$ and as in the previous case (i), we easily obtain that because by hypothesis f is not a polynomial of degree $\leq q - 1$, there exists a constant $C_{r,q}(f) > 0$ such that

$$\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r \geq C_{r,q}(f)\alpha, \text{ for all } \alpha \in (0, 1].$$

The theorem is proved. □

Replacing now everywhere in Theorem 2.2.1 the $\Gamma(\alpha)$ function by the $Beta(\alpha, \beta)$ function and considering the construction of $G_U^{\alpha,\beta}(f)(z)$ defined in the direction 1) just before the statement of Theorem 2.4.1, we have the following result.

Theorem 2.2.2 (Gal [57]). *Let us suppose that* $0 < \alpha \leq \beta \leq 1$, $\alpha + \beta \geq 1$ *and that* $f : \mathbb{D}_R \to \mathbb{C}$, *with* $R > 1$, *is analytic in* $\mathbb{D}_R$, *that is,* $f(z) = \sum_{k=0}^\infty a_k z^k$, *for all* $z \in \mathbb{D}_R$.

(i) For $U_t(f)(z) = \frac{t}{\pi}\int_{-\infty}^\infty \frac{f(ze^{-iu})}{u^2+t^2}du$, *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^\infty a_k b_k(\alpha, \beta) \cdot z^k, z \in \mathbb{D}_R,$$

where

$$b_k(\alpha, \beta) = \frac{1}{Beta(\alpha, \beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-kt}dt.$$

Also, if f *is not constant for* $q = 0$ *and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$, *then for all* $1 \leq r < r_1 < R$, $q \in \mathbb{N}\cup\{0\}$, $\alpha \in (0, \beta]$, *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where $\|f\|_r = \sup\{|f(z)|; |z| \leq r\}$ *and the constants in the equivalence depend only on* f, q, r, r_1, β.

(ii) For $U_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-|u|/t}\,du$ *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k, z \in \mathbb{D}_R,$$

where $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{1+t^2k^2}dt$.
Also, if f *is not constant for* $q = 0$ *and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$, *then for all* $1 \leq r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,\beta]$, *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f, q, r, r_1, *and* β.

(iii) For $U_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{(u^2+t^2)^2}\,du$, *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k, z \in \mathbb{D}_R,$$

where $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1+kt)e^{-kt}dt$.
Also, if f *is not constant for* $q = 0$ *and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$, *then for all* $1 \leq r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,\beta]$, *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f, q, r, r_1, *and* β.

(iv) For $U_t(f)(z) = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(ze^{-iu})e^{-u^2/t}\,du$, *we have that* $G_U^{\alpha,\beta}(f)(z)$ *is analytic in* $\mathbb{D}_R$ *and we can write*

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) z^k, z \in \mathbb{D}_R,$$

where $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-(k^2/4)t}dt$.
Also, if f *is not constant for* $q = 0$ *and not a polynomial of degree* $\leq q-1$ *for* $q \in \mathbb{N}$, *then for all* $1 \leq r < r_1 < R$, $q \in \mathbb{N} \cup \{0\}$, $\alpha \in (0,\beta]$, *we have*

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \sim \alpha,$$

where the constants in the equivalence depend only on f, q, r, r_1, *and* β.

Proof. (i) By Gal [49], p. 213, Theorem 3.2.5, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-kt} z^k, \text{ for all } |z| < R \text{ and } t \ge 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-kt} z^k| \le \sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k e^{-kt} z^k$ is uniformly convergent on $[0, \infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k b_k(\alpha,\beta) z^k,$$

where

$$b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{-kt} dt.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z) - f(z) = \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z) - f(z)]dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \le C_r(f)t$ in Gal [49], p. 213, Theorem 3.2.5, (iii), implies

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \le \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z) - f(z)|dt$$

$$\le C_r(f)\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha}(1-t)^{\beta-1}dt = C_r(f) \cdot \frac{Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)}$$

$$= C_r(f) \cdot \frac{\alpha}{\alpha+\beta} \le C_r(f) \cdot \alpha,$$

for all $|z| \le r$, where $C_r(f) > 0$ is independent of z (and α, β) but depends on f and r. Here we used the well-known formula $\frac{Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)} = \frac{\alpha}{\alpha+\beta}$.

Now, for $q \in \mathbb{N} \cup \{0\}$ and $1 \le r < r_1 < R$, by using Cauchy's formula and the standard reasonings in the proof of Theorem 2.2.1, we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \le C^* \alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 218–219 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p[1 - e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ and then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi \right\} t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1}(1-t)^{\beta-1} dt \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt \right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)} \int_0^1 [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= f^{(q)}(z) - [G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[f^{(q)}(z) - (G_U^{\alpha,\beta}(f))^{(q)}(z) \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt \right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-(q+p)t}]dt \right]$$

$$\leq \|f^{(q)} - (G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q = 0$. In what follows, denoting

$$V_{\alpha,\beta} = \inf_{p\geq 1} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-pt}]dt \right),$$

we clearly get

$$V_{\alpha,\beta} = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1-e^{-t}]dt.$$

But denoting $g(t) = e^{-t}$, by the mean value theorem, there exists $\xi \in (0,1)$ such that $1 - e^{-t} = g(0) - g(t) = te^{-\xi} \geq \frac{t}{e}$, which immediately implies

$$V_{\alpha,\beta} \geq \frac{1}{e \cdot Beta(\alpha,\beta)} \int_0^1 t^{\alpha}(1-t)^{\beta-1}dt = \frac{Beta(\alpha+1,\beta)}{e \cdot Beta(\alpha,\beta)}$$

$$= \frac{1}{e} \cdot \frac{\alpha}{\alpha+\beta} \geq \frac{1}{e} \cdot \frac{\alpha}{2\beta} \geq \frac{\alpha}{2e}.$$

By following now for $q \geq 0$ the standard reasonings as in the proof of Theorem 2.2.1, we get the desired equivalence in the statement.

(ii) By Gal [49], p. 206, Theorem 3.2.1, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} \frac{a_k}{1+t^2k^2} z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} \frac{a_k}{1+t^2k^2} z^k| \leq \sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} \frac{a_k}{1+t^2k^2} z^k$ is uniformly convergent on $[0, \infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{Beta(\alpha,\beta)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{1+t^2k^2} dt.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z) - f(z) = \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z) - f(z)]dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t^2$ in Gal [49], p. 207, Theorem 3.2.1, (iv), implies

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \leq \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z) - f(z)|dt$$

$$\leq C_r(f) \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha+1}(1-t)^{\beta-1}dt = C_r(f) \cdot \frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)}$$

$$= C_r(f) \frac{\alpha+1}{\alpha+\beta+1} \cdot \frac{\alpha}{\alpha+\beta} \leq C_r(f) \frac{\alpha(\alpha+1)}{2} \leq C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α, β) but depends on f and r.

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. By standard reasonings and using Cauchy's formula as in the proof of Theorem 2.2.1, we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \leq C^* \alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.1, at pages 209–210 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p \cdot \frac{t^2(q+p)^2}{1+t^2(q+p)^2}.$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1}$ and then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] e^{-ip\varphi} d\varphi \right\} t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2} \right] dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ip\varphi} \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1}(1-t)^{\beta-1} dt \right] d\varphi \right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot \left[\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2} \right] dt \right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)} \int_0^1 [f^{(q)}(z) - [U_t(f)]^{(q)}(z)] t^{\alpha-1}(1-t)^{\beta-1} dt$$

$$= f^{(q)}(z) - [G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[f^{(q)}(z)-(G_U^{\alpha,\beta}(f))^{(q)}(z)\right]d\varphi\right|$$

$$=|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right]\le\|f^{(q)}-(G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality we immediately obtain

$$|a_p|r^p\left(\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right)$$

$$\le\|f-G_U^{\alpha,\beta}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha,\beta}=\inf_{p\ge 1}\left(\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right),$$

we clearly get

$$V_{\alpha,\beta}=\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[\frac{t^2}{1+t^2}\right]dt$$

$$=\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-\frac{1}{1+t^2}\right]dt.$$

But we have $1-\frac{1}{1+t^2}\ge\frac{t^2}{4}$, for all $t\in[0,1]$. Indeed, denoting $g(t)=1-\frac{1}{1+t^2}-\frac{t^2}{4}$, we get $g(0)=0$ and $g'(t)=\frac{2t}{(1+t^2)^2}-\frac{2t}{4}=2t\left(\frac{1}{(1+t^2)^2}-\frac{1}{4}\right)\ge 0$, for all $t\in[0,1]$. It follows that $g(t)$ is nondecreasing on $[0,1]$ and therefore $g(t)\ge 0$ for all $t\in[0,1]$.

In conclusion,

$$V_{\alpha,\beta}\ge\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\frac{t^2}{4}dt$$

$$=\frac{1}{4}\cdot\frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)}=\frac{1}{4}\cdot\frac{\alpha+1}{\alpha+\beta+1}\cdot\frac{\alpha}{\alpha+\beta}$$

$$\ge\frac{1}{4}\cdot\frac{\alpha(\alpha+1)}{2}\ge\frac{\alpha}{8}.$$

By following now for $q \geq 0$ the standard reasonings as in the proof of Theorem 2.2.1, we get the desired equivalence in the statement.

(iii) By Gal [49], p. 213, Theorem 3.2.5, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-kt}(1+kt)z^k| \leq 2\sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k$ is uniformly convergent on $[0, \infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1+kt)e^{-kt}dt,$$

where denoting $b_k(\alpha,\beta) = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1+kt)e^{-kt}dt$, we obtain

$$G_U^{\alpha,\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \cdot b_k(\alpha,\beta) \cdot z^k.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z) - f(z) = \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z) - f(z)]dt,$$

which together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t^2$ in Gal [49], p. 213–214, Theorem 3.2.5, (iv), implies

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \leq \frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z) - f(z)|dt$$

$$\leq C_r(f)\frac{1}{Beta(\alpha,\beta)} \cdot \int_0^1 t^{\alpha+1}(1-t)^{\beta-1}dt = C_r(f) \cdot \frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)} \leq C_r(f)\alpha,$$

for all $|z| \leq r$, where $C_r(f) > 0$ is independent of z (and α) but depends on f and r. We used here the estimate from the above point (ii).

Now, let $q \in \mathbb{N} \cup \{0\}$ and $1 \leq r < r_1 < R$. By standard reasonings and using Cauchy's formula as in the proof of Theorem 2.2.1, we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)} - f^{(q)}\|_r \leq C^*\alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 219–220 in the book of Gal [49], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p[1-(1+(q+p)t)e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ and then integrating with respect to t, it follows

$$I :=$$

$$\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt$$

$$= f^{(q)}(z)-[G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z)-(G_U^{\alpha,\beta}(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right]$$

$$\leq \|f^{(q)}-(G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q = 0$. From the previous inequality, we immediately obtain

$$|a_p|r^p \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+pt)e^{-pt}\right] dt \right)$$
$$\leq \|f - G_U^{\alpha,\beta}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha,\beta} = \inf_{p\geq 1} \left(\frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+pt)e^{-pt}\right] dt \right),$$

we immediately get

$$V_{\alpha,\beta} = \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \left[1-(1+t)e^{-t}\right] dt.$$

But we have $1-(1+t)e^{-t} \geq \frac{t^2}{e}$, for all $t \in [0,1]$. Indeed, denoting $g(t) = 1-(1+t)e^{-t} - \frac{t^2}{e}$, we have $g(0) = 0$ and $g'(t) = te^{-t} - \frac{t}{e} = t\left(\frac{1}{e^t} - \frac{1}{e}\right) \geq 0$ for all $t \in [0,1]$. This implies that $g(t)$ is nondecreasing on $[0,1]$ and therefore $g(t) \geq 0$ for all $t \in [0,1]$.

Therefore,

$$V_{\alpha,\beta} \geq \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \frac{t^2}{2e} dt$$
$$= \frac{Beta(\alpha+2,\beta)}{2e \cdot B(\alpha,\beta)} = \frac{1}{2e} \cdot \frac{\alpha+1}{\alpha+\beta+1} \cdot \frac{\alpha}{\alpha+\beta}$$
$$\geq \frac{1}{2e} \cdot \frac{\alpha(\alpha+1)}{2} \geq \frac{\alpha}{4e}.$$

By following now for $q \geq 0$ the standard reasonings as in the proof of Theorem 2.4.1, we get the desired equivalence in the statement.

(iv) By Gal [49], p. 223, Theorem 3.2.8, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k e^{-k^2 t/4} z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-k^2t/4} z^k| \leq \sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k e^{-k^2t/4} z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$G_U^{\alpha,\beta}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{Beta(\alpha,\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} e^{-(k^2/4)t} dt,$$

where denoting $b_k(\alpha,\beta)=\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}e^{-(k^2/4)t}dt$, we can write

$$G_U^{\alpha,\beta}(f)(z)=\sum_{k=0}^{\infty}a_k\cdot b_k(\alpha,\beta)\cdot z^k.$$

In other order of ideas, we easily can write

$$G_U^{\alpha,\beta}(f)(z)-f(z)=\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[U_t(f)(z)-f(z)]dt,$$

which together with the estimate $|U_t(f)(z)-f(z)|\le C_r(f)t$ in Gal [49], p. 224, Theorem 3.2.8, (iv), implies

$$|G_U^{\alpha,\beta}(f)(z)-f(z)|\le\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z)-f(z)|dt$$

$$\le C_r(f)\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1 t^{\alpha}(1-t)^{\beta-1}dt=C_r(f)\cdot\frac{Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)}\le C_r(f)\alpha,$$

for all $|z|\le r$, where $C_r(f)>0$ is independent of z (and α) but depends on f and r.

Now, let $q\in\mathbb{N}\cup\{0\}$ and $1\le r<r_1<R$. By standard reasonings and using Cauchy's formula as in the proof of Theorem 2.2.1, we get the upper estimate

$$\|[G_U^{\alpha,\beta}(f)]^{(q)}-f^{(q)}\|_r\le C^*\alpha,$$

with C^* depending only on f, q, r, and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.8, at pages 227–228 in the book of Gal [49], for $z=re^{i\varphi}$ and $p\in\mathbb{N}\cup\{0\}$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$=a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p[1-e^{-(q+p)^2t/4}].$$

Multiplying above with $\frac{1}{Beta(\alpha,\beta)}t^{\alpha-1}(1-t)^{\beta-1}$ and then integrating with respect to t, it follows

$$I:=$$

$$\frac{1}{Beta(\alpha,\beta)}\cdot\int_0^1\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1})1-t)^{\beta-1}dt$$

$$=a_{q+p}(q+p)(q+p-1)\dots(p+1)r^p$$

$$\cdot\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-e^{-(q+p)^2t/4}\right]dt.$$

Applying Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[\frac{1}{Beta(\alpha,\beta)}\int_0^{\infty}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt\right]d\varphi\right|$$
$$=|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$
$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}e^{-t}\left[1-e^{-(q+p)^2t/4}\right]dt\right].$$

Since

$$\frac{1}{Beta(\alpha,\beta)}\int_0^1[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}(1-t)^{\beta-1}dt$$
$$=f^{(q)}(z)-[G_U^{\alpha,\beta}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[f^{(q)}(z)-(G_U^{\alpha,\beta}(f))^{(q)}(z)\right]d\varphi\right|$$
$$=|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$
$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-e^{-(q+p)^2t/4}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)\dots(p+1)r^p$$
$$\cdot\left[\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-e^{-(q+p)^2t/4}\right]dt\right]$$
$$\le\|f^{(q)}-(G_U^{\alpha,\beta}(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality, we immediately obtain

$$|a_p|r^p\left(\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-e^{-p^2t/4}\right]dt\right)\le\|f-G_U^{\alpha,\beta}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha,\beta}=\inf_{p\ge1}\left(\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-e^{-p^2t/4}\right]dt\right),$$

by simple calculation, we get

$$V_{\alpha,\beta}=\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\left[1-e^{-t/4}\right]dt.$$

But denoting $g(t) = e^{-t/4}$, by the mean value theorem, there exists $\xi \in (0,1)$ such that $1-e^{-t/4} = g(0)-g(t) = t\frac{e^{-\xi/4}}{4} \geq \frac{t}{4e^{1/4}}$, which immediately implies

$$V_{\alpha,\beta} \geq \frac{1}{4e^{1/4} \cdot Beta(\alpha,\beta)} \int_0^1 t^{\alpha}(1-t)^{\beta-1}dt = \frac{Beta(\alpha+1,\beta)}{4e^{1/4} \cdot Beta(\alpha,\beta)}$$

$$= \frac{1}{4e^{1/4}} \cdot \frac{\alpha}{\alpha+\beta} \geq \frac{1}{4e^{1/4}} \cdot \frac{\alpha}{2\beta} \geq \frac{\alpha}{8e^{1/4}}.$$

By following now for $q \geq 0$ the standard reasonings as in the proof of Theorem 2.2.1, we get the desired equivalence in the statement.

The theorem is proved. □

Concerning the overconvergence phenomenon for the potentials of real variable x, $F_U^{\alpha}(f)(x)$ and $G_U^{\alpha,\beta}(f)(x)$, we can present the next two results.

Theorem 2.2.3. *Let $d > 0$ and suppose that $f : S_d \to \mathbb{C}$ is bounded and uniformly continuous in the strip $S_d = \{z = x+iy \in \mathbb{C};\ x \in \mathbb{R},\ |y| \leq d\}$.*

(i) Denoting $U_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(z+u)e^{-|u|/t}\,du$, for all $0 < \alpha \leq 1/2$ and $z \in S_d$, we have

$$|F_U^{\alpha}(f)(z) - f(z)| \leq 5\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where

$$\omega_2(f;\delta)_{S_d} = \sup\{|f(u+t)-2f(u)+f(u-t)|; u, u-t, u+t \in S_d, |t| \leq \delta\}.$$

(ii) Denoting $U_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(z+u)}{(u^2+t^2)^2}\,du$, for all $0 < \alpha \leq 1/2$ and $z \in S_d$, we have

$$|F_U^{\alpha}(f)(z) - f(z)| \leq C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ is independent of z, α, and f.

(iii) Denoting $U_t(f)(z) = \frac{1}{\sqrt{\pi t^2}}\int_{-\infty}^{+\infty} f(z+u)e^{-u^2/t^2}\,dt$, for all $0 < \alpha \leq 1/2$ and $z \in S_d$, we have

$$|F_U^{\alpha}(f)(z) - f(z)| \leq C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ is independent of z, α, and f.

Proof. (i) If $z \in S_d$ then clearly that for all $t \in \mathbb{R}$, we have $z+t \in S_d$, and since f is bounded in S_d (denote its bound by $M(f)$), it easily follows $|U_t(f)(z)| \leq 2M(f)$ for all $z \in S_d$. Therefore $U_t(f)(z)$ exists for all $z \in S_d$. Also, the uniform continuity of f on S_d implies that $0 \leq \lim_{t\to 0}\omega_2(f;t)_{S_d} \leq 2\lim_{t\to 0}\omega_1(f;t)_{S_d} = 0$.

For all $z \in S_d$, we have

$$|U_t(f)(z) - f(z)| = \left| \frac{1}{2t} \int_0^{\infty} [f(z+u) - 2f(z) + f(z-u)] e^{-|u|/t} du \right|$$

$$\leq \frac{1}{2t} \int_0^{\infty} \omega_2(f;(u/t)t)_{S_d} e^{-u/t} du$$

$$\leq \omega_2(f;t)_{S_d} \frac{1}{2t} \int_0^{\infty} [1+(u/t)]^2 e^{-u/t} du = \frac{5}{2} \omega_2(f;t)_{S_d}.$$

For the last equality, see Gal [48], pp. 252–253, proof of Theorem 5.2. We get

$$|F_U^{\alpha}(f)(z) - f(z)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} |U_t(f)(z) - f(z)| dt$$

$$\leq \frac{5}{2} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} \omega_2(f;t)_{S_d} dt = \frac{5}{2} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} \omega_2(f;\alpha(t/\alpha))_{S_d} dt$$

$$\leq \frac{5}{2} \cdot \omega_2(f;\alpha)_{S_d} \cdot \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} [1+(t/\alpha)]^2 dt$$

$$= \frac{5}{2} \cdot \omega_2(f;\alpha)_{S_d} \left[1 + \frac{2}{\Gamma(\alpha)\alpha} \int_0^{\infty} t^{\alpha} e^{-t} dt + \frac{1}{\Gamma(\alpha)\alpha^2} \int_0^{\infty} t^{\alpha+1} e^{-t} dt \right]$$

$$= \frac{5}{2} \cdot \omega_2(f;\alpha)_{S_d} \left[1 + \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha)\alpha} + \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\alpha^2} \right] = \frac{5}{2} \cdot \omega_2(f;\alpha)_{S_d} \left[3 + \frac{\alpha+1}{\alpha} \right]$$

$$\leq 5 \frac{\alpha+1}{\alpha} \omega_2(f;\alpha)_{S_d},$$

because $3 \leq \frac{\alpha+1}{\alpha}$.

(ii) We obtain

$$|U_t(f)(z) - f(z)| =$$

$$\left| \frac{2t^3}{\pi} \int_0^{\infty} \frac{[f(z+u) - 2f(z) + f(z-u)]}{(u^2+t^2)^2} du \right| \leq \frac{2t^3}{\pi} \int_0^{+\infty} \frac{\omega_2(f;(u/t)t)_{S_d}}{(u^2+t^2)^2} du$$

$$\leq \omega_2(f;t)_{S_d} \frac{2t^3}{\pi} \int_0^{\infty} \left[1 + \frac{u}{t} \right]^2 \cdot \frac{1}{(u^2+t^2)^2} du \leq C \omega_2(f;t)_{S_d},$$

since by easy calculation, we get that

$$\frac{2t^3}{\pi} \int_0^{\infty} \left[1 + \frac{u}{t} \right]^2 \cdot \frac{1}{(u^2+t^2)^2} du \leq C,$$

where $C > 0$ is independent of t, z, and f.
Reasoning exactly as at the above point (i), we obtain the estimate

$$|F_U^{\alpha}(f)(z) - f(z)| \le C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ is independent of f, z, and α.

(iii) We get

$$|U_t(f)(z) - f(z)| = \left|\frac{1}{\sqrt{\pi t}}\int_0^{\infty}[f(z+u) - 2f(z) + f(z-u)]e^{-u^2/t}du\right| \le$$

$$\frac{1}{\sqrt{\pi t}}\int_0^{+\infty}\omega_2(f;(u/\sqrt{t})\sqrt{t})_{S_d}e^{-u^2/t}du$$

$$\le \omega_2(f;\sqrt{t})_{S_d}\frac{1}{\sqrt{\pi t}}\int_0^{\infty}\left[\frac{u}{\sqrt{t}}+1\right]^2 e^{-u^2/t}du \le C\omega_2(f;\sqrt{t})_{S_d},$$

since

$$\frac{1}{\sqrt{\pi t}}\int_0^{\infty}\frac{u^2}{t}e^{-u^2/t}du = \frac{1}{\sqrt{\pi}}\int_0^{\infty}v^2e^{-v^2}dv < \infty$$

and

$$\frac{2}{\sqrt{\pi t}}\int_0^{\infty}\frac{u}{\sqrt{t}}e^{-u^2/t}du = \frac{2}{\sqrt{\pi t}}\sqrt{t}\int_0^{\infty}ve^{-v^2}dv = \frac{2}{\sqrt{\pi}}\int_0^{\infty}ve^{-v^2}dv < \infty.$$

Above, $C > 0$ is independent of t, z, and f.

Reasoning exactly as at the above point (i), we obtain the estimate

$$|F_U^{\alpha}(f)(z) - f(z)| \le C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ is independent of f, z, and α.

The theorem is proved. □

Remark. If f is such that its second derivative f'' is bounded in the strip S_d, then by the mean value theorem in complex analysis (see, e.g., Stancu [133], p. 258, Exercise 4.20), we get $\omega_2(f;\alpha)_{S_d} \le C\alpha^2$, and therefore it follows the upper estimate $\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d} \le C_1\alpha$, which proves the overconvergence phenomenon as $\alpha \to 0$ for $F^{\alpha}(f)(x)$ in all the three cases for $U_t(f)(z)$.

Theorem 2.2.4. *Let $d > 0$ and suppose that $f : S_d \to \mathbb{C}$ is bounded and uniformly continuous in the strip $S_d = \{z = x + iy \in \mathbb{C}; x \in \mathbb{R},\ |y| \le d\}$. Also, suppose that $0 < \alpha \le \beta \le 1$, $\alpha + \beta \ge 1$.*

(i) Denoting $U_t(f)(z) = \frac{1}{2t}\int_{-\infty}^{+\infty} f(z+u)e^{-|u|/t}\,du$, for all $0 < \alpha < 1$ and $z \in S_d$, we have

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \le 5\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where

$$\omega_2(f;\delta)_{S_d} = \sup\{|f(u+t) - 2f(u) + f(u-t)|; u, u-t, u+t \in S_d, |t| \le \delta\}.$$

(ii) Denoting $U_t(f)(z) = \frac{2t^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(z+u)}{(u^2+t^2)^2}\,du$*, for all* $0 < \alpha < 1$ *and* $z \in S_d$*, we have*

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \le C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ *is independent of* z*,* α*, and* f*.*

(iii) Denoting $U_t(f)(z) = \frac{1}{\sqrt{\pi t^2}}\int_{-\infty}^{+\infty} f(z+u)e^{-u^2/t^2}\,du$*, for all* $0 < \alpha < 1$ *and* $z \in S_d$*, we have*

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \le C\frac{\alpha+1}{\alpha}\cdot\omega_2(f;\sqrt{\alpha})_{S_d},$$

where $C > 0$ *is independent of* z*,* α*, and* f*.*

Proof. (i) If $z \in S_d$ then clearly that for all $t \in \mathbb{R}$, we have $z+t \in S_d$, and since f is bounded in S_d (denote its bound by $M(f)$), it easily follows $|U_t(f)(z)| \le 2M(f)$ for all $z \in S_d$. Therefore $U_t(f)(z)$ exists for all $z \in S_d$. Also, the uniform continuity of f on S_d implies that $0 \le \lim_{\xi\to 0}\omega_2(f;t)_{S_d} \le 2\lim_{t\to 0}\omega_1(f;t)_{S_d} = 0$.

As in the proof of the above Theorem 2.2.3, (i), we have

$$|U_t(f)(z) - f(z)| \le \frac{5}{2}\cdot\omega_2(f;t)_{S_d},\ \text{for all } z \in S_d.$$

We get

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \le \frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}|U_t(f)(z) - f(z)|dt$$

$$\le \frac{5}{2}\cdot\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\omega_2(f;t)_{S_d}dt$$

$$= \frac{5}{2}\cdot\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\omega_2(f;\alpha(t/\alpha))_{S_d}dt$$

$$\le \frac{5}{2}\cdot\omega_2(f;\alpha)_{S_d}\cdot\frac{1}{Beta(\alpha,\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1+(t/\alpha)]^2dt$$

$$= \frac{5}{2}\cdot\omega_2(f;\alpha)_{S_d}$$

$$\left[1 + \frac{2}{Beta(\alpha,\beta)\alpha}\int_0^1 t^{\alpha}(1-t)^{\beta-1}dt + \frac{1}{Beta(\alpha,\beta)\alpha^2}\int_0^1 t^{\alpha+1}(1-t)^{\beta-1}dt\right]$$

$$= \frac{5}{2}\cdot\omega_2(f;\alpha)_{S_d}\left[1 + \frac{2Beta(\alpha+1,\beta)}{Beta(\alpha,\beta)\alpha} + \frac{Beta(\alpha+2,\beta)}{Beta(\alpha,\beta)\alpha^2}\right]$$

$$= \frac{5}{2} \cdot \omega_2(f;\alpha)_{S_d} \left[1 + \frac{2}{\alpha} \cdot \frac{\alpha}{\alpha+\beta} + \frac{1}{\alpha^2} \cdot \frac{\alpha+1}{\alpha+\beta+1} \cdot \frac{\alpha}{\alpha+\beta}\right]$$

$$\leq \frac{5}{2} \cdot \omega_2(f;\alpha)_{S_d} \left[3 + \frac{\alpha+1}{2\alpha}\right] \leq 5\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

because $1 \leq \frac{\alpha+1}{2\alpha}$.

(ii) As in the proof of the above Theorem 2.2.3, (ii), we have

$$|U_t(f)(z) - f(z)| \leq C\omega_2(f;t)_{S_d}, \text{ for all } z \in S_d, t \geq 0,$$

where $C > 0$ is independent of t, z, and f.
Reasoning exactly as at the above point (i), we obtain the estimate

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \leq C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ is independent of f, z, and α.

(iii) As in the proof of the above Theorem 2.2.3, (iii), we have

$$|U_t(f)(z) - f(z)| \leq C\omega_2(f;t)_{S_d}, \text{ for all } z \in S_d, t \geq 0,$$

where $C > 0$ is independent of t, z, and f.

Reasoning exactly as at the above point (i), we obtain the estimate

$$|G_U^{\alpha,\beta}(f)(z) - f(z)| \leq C\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d},$$

where $C > 0$ is independent of f, z, and α.

The theorem is proved. □

Remarks. 1) If f is such that its second derivative f'' is bounded in the strip S_d, then by the mean value theorem in complex analysis (see, e.g., Stancu [133], p. 258, Exercise 4.20), we get $\omega_2(f;\alpha)_{S_d} \leq C\alpha^2$, and therefore it follows the upper estimate $\frac{\alpha+1}{\alpha}\omega_2(f;\alpha)_{S_d} \leq C_1\alpha$, which proves the overconvergence phenomenon as $\alpha \to 0$ for $G^{\alpha,\beta}(f)(x)$ in all the three cases for $U_t(f)(z)$.

2) Note that Theorem 2.2.1 differs from Theorem 2.2.3 and Theorem 2.2.2 differs from Theorem 2.2.4, by the different formulas for the corresponding $U_t(f)(z)$. Thus, in Theorems 2.2.1 and 2.2.2, these $U_t(f)(z)$ are complex convolution-type integrals, while in Theorems 2.2.3 and 2.2.4, they are obtained from their real correspondents, simply replacing the real variable x by the complex one z.

At the end of this section, we study the approximation properties of the complex Bessel-type potential

$$B^{\alpha}(f)(z,t)=\frac{1}{\Gamma(\alpha/2)}\int_0^{\infty}\left[\int_{-\infty}^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}W(y,\tau^2)f(ze^{-iy},t-\tau)dy\right]d\tau,$$

where $\Gamma(\alpha)$ is the Gamma function and

$$W(y,\tau^2)=\frac{1}{\sqrt{4\pi\tau^2}}e^{-y^2/(4\tau^2)}=\frac{1}{2\tau\sqrt{\pi}}e^{-y^2/(4\tau^2)}.$$

In this sense, we present the following:

Theorem 2.2.5. *Let us suppose that the function $f(z,t)$ is in $L^p(\overline{\mathbb{D}_1}\times\mathbb{R})$, $1\le p<\infty$, where $\mathbb{D}_1=\{z\in\mathbb{C};|z|<1\}$ and that $f(z,t)$ is analytic in $\mathbb{D}_1$ and continuous in $\overline{\mathbb{D}_1}$ for any fixed $t\in\mathbb{R}$. Then for all $|z|\le 1$, $t\in\mathbb{R}$ and $\alpha>0$, we have*

$$|B^{\alpha}(f)(z,t)-f(z,t)|\le\left(2+\frac{2}{\sqrt{\pi}}\right)\omega_1(f;\alpha/2,\alpha/2),$$

where

$$\omega_1(f;a,b)$$
$$=\sup\{|f(u,t)-f(v,s)|;|u-v|\le a;\ u,v\in\overline{\mathbb{D}_1},|t-s|\le b;\ t,s\in\mathbb{R}\}.$$

Proof. We have

$$B^{\alpha}(f)(z,t)-f(z,t)$$
$$=\frac{1}{\Gamma(\alpha/2)}\int_0^{\infty}\left[\int_{-\infty}^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}W(y,\tau^2)(f(ze^{-iy},t-\tau)-f(z,t))dy\right]d\tau$$
$$=\frac{1}{\Gamma(\alpha/2)}\int_0^{\infty}$$
$$\cdot\left[\int_0^{\infty}\tau^{(\alpha/2)-1}e^{-\tau}W(y,\tau^2)(f(ze^{-iy},t-\tau)-f(z,t)\right.$$
$$\left.+f(ze^{iy},t-\tau)-f(z,t))dy\right]d\tau.$$

In all what follows, for the simplicity of notations, denote $a=\alpha/2$. Passing to absolute value for $|z|\le 1$, we immediately get

$$|B^{\alpha}(f)(z,t)-f(z,t)|$$
$$\le\frac{1}{\Gamma(a)}\int_0^{\infty}\left[2\cdot\int_0^{\infty}\tau^{a-1}e^{-\tau}W(y,\tau^2)\omega_1(f;y,\tau)dy\right]d\tau$$
$$=\frac{1}{\Gamma(a)}\int_0^{\infty}\left[2\cdot\int_0^{\infty}\tau^{a-1}e^{-\tau}W(y,\tau^2)\omega_1(f;a(y/a),a(\tau/a))dy\right]d\tau$$

$$= \omega_1(f;a,a) \cdot \frac{1}{\Gamma(a)} \int_0^{\infty} \tau^{a-1} e^{-\tau} \left[2 \cdot \int_0^{\infty} \left[1 + \frac{y}{a} + \frac{\tau}{a} \right] W(y, \tau^2) dy \right] d\tau$$

$$= \omega_1(f;a,a) \cdot \frac{1}{\Gamma(a)} \int_0^{\infty} \tau^{a-1} e^{-\tau} \left[1 + \frac{\tau}{a} + \frac{2\tau}{\sqrt{\pi} a} \right] d\tau = \left(2 + \frac{2}{\sqrt{\pi}} \right) \omega_1(f;a,a).$$

We used here the simple formulas $\int_0^{\infty} yW(y, \tau^2) dy = \frac{\tau}{\sqrt{\pi}}$ and $\Gamma(a+1) = a\Gamma(a)$.

The theorem is proved. □

2.3 Notes

Note 2.3.1. Theorems 2.1.1, 2.2.3, 2.2.4, and 2.2.5 appear for the first time here.

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