Preface

It is a well-known fact that the concept of overconvergence in approximation theory may have several meanings. The most common, developed for the first time by Ostrovski and Walsh, is that given a sequence of functions approximating a given (analytic) function in a set (region), the convergence may hold not merely in that set, but in a larger one containing the first set in its interior.

A second meaning is the well-known Walsh's overconvergence phenomenon in interpolation of functions introduced by Walsh in [144], p. 153, intensively studied by many mathematicians (see, e.g., the recent research monograph by Jakimovski–Sharma–Szabados [88]), which briefly can be described as follows: given $r > 1$ and the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $|z| < r$, denoting by $L_{n-1}(f)(z)$ the Lagrange polynomial of degree $\leq n-1$ interpolating f in the n-th roots of unity, we have

$$\lim_{n\to\infty}\left(L_{n-1}(f)(z) - \sum_{k=0}^{n-1} a_k z^k\right) = 0, \text{ for all } z \in \mathbb{C} \text{ with } |z| < r^2,$$

the convergence being uniform and geometric in any compact subset of $\{z \in \mathbb{C}; |z| < r^2\}$. Moreover, the result is best possible, that is, it does not hold at any point $|z| = r^2$.

The third meaning refers to the overconvergence of power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ with the radius of convergence $R \geq 0$ and it can be described as follows (see, e.g., Bourion [22], Ilieff [84], Walsh [143], Luh [98], Kovacheva [92], Beise–Meyrath–Müller [14]): denoting $S_n(f, z) = \sum_{j=0}^{n} a_j z^j$, $n \in \mathbb{N}$, if there exist a subsequence $(S_{n_k})_{k\in\mathbb{N}}$ and a domain U containing the open disk $\mathbb{D}_R$ as a proper set, such that $S_{n_k}(f, z)$ converges inside of U to an $S \in \mathbb{C}$ as $k \to \infty$, then the power series is called overconvergent.

The overconvergence studied in this research monograph belongs to the first meaning of the phenomenon and consists in two directions, which briefly can be described as follows:

1) Let $I \subset \mathbb{R}$ be a subinterval, $C(I) = \{f : I \to \mathbb{R}; f \text{ is continuous on } I\}$ and $L_n : C(I) \to C(I)$, $n \in \mathbb{N}$, a given sequence of approximation operators with the property that for any $f \in C(I)$ we have $\lim_{n\to\infty} L_n(f)(x) = f(x)$ for all $x \in I$ (pointwise or uniformly).
   We say that the overconvergence phenomenon holds for the sequence $(L_n)_n$ if there exists $G \subset \mathbb{C}$ containing I (e.g., if I is a compact real interval, then G might be a compact disk containing I), such that for any function $f : G \to \mathbb{C}$ analytic on G we have $\lim_{n\to\infty} L_n(f)(z) = f(z)$, for all $z \in G$ (pointwise or uniformly).
   Note that instead of the space $C(I)$, more generally we can consider the space $C(I; X) = \{f : I \to X; f \text{ is continuous on } I\}$, where $(X, \|\cdot\|)$ is a complex Banach space.
2) For $f : \mathbb{R} \to \mathbb{R}$ and the kernel $K : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, let us consider the integral convolution operator $C_t(f)(x) = \int_{-\infty}^{+\infty} K(u,t)f(x-u)du$, $t \geq 0$, with the property that for any f in a subclass of continuous functions, we have $\lim_{t\to 0} C_t(f)(x) = f(x)$, for all $x \in \mathbb{R}$ (pointwise or uniformly). In this case, besides the overconvergence phenomenon defined at point 1, we can consider another one, as follows.

We say that for $(C_t(f))_{t\geq 0}$, the *convolution overconvergence phenomenon* holds if by replacing in the formula of the integral convolution $C_t(f)(x)$ the translation $x - u$ with the rotation ze^{iu}, then for any analytic function f in a compact disk or subset $G \subset \mathbb{C}$, we have $lim_{t\to 0} C_t^*(f)(z) = f(z)$, for all $z \in G$ (pointwise or uniformly), where at this time $C_t^*(f)(z) = \int_{-\infty}^{+\infty} K(u,t)f(ze^{iu})du$.

Evidently, besides the qualitative aspects just mentioned above, the overconvergence phenomenon presents quantitative aspects too regarding the order of approximation of f by $L_n(f)$ or by $C_t^*(f)$ on G. An important characteristic of all these results (which does not hold in the case of real approximation) is that the approximation orders obtained are exact.

The history of the overconvergence phenomenon in complex approximation by Bernstein-type operators goes back to the work of Wright [146], Kantorovich [89], Bernstein [16–18], Lorentz [96] (Chap. 4), and Tonne [139], who in the case of complex Bernstein operators defined by

$$B_n(f)(z) = \sum_{k=0}^{n} p_{n,k}(z) f\left(\frac{k}{n}\right), \quad p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}, |z| \leq r,$$

have given interesting qualitative results, but without giving quantitative estimates. Also, qualitative results without any quantitative estimates were obtained for the complex Favard–Szász–Mirakjan operators by Dressel–Gergen and Purcell [32] and for the complex Jakimovski–Leviatan operators by Wood [147]. We notice that the qualitative results are theoretically based on the "bridge" made by the classical result of Vitali (see Theorem 1.1.1), between the (well-established) approximation results for these Bernstein-type operators of real variable and those for the Bernstein-type operators of complex variable.

In the very recent book of Gal [49], a systematic study of the overconvergence phenomenon in complex approximation was made for the following important classes of Bernstein-type operators: Bernstein, Bernstein–Faber, Bernstein–Butzer, q-Bernstein with $0 < q \leq 1$, Bernstein–Stancu, Bernstein–Kantorovich, Favard–Szász–Mirakjan, Baskakov, and Balázs–Szabados.

Also, in the same book, the convolution overconvergence phenomenon in the sense of the above direction 2) was studied, for the following types of integral convolution operators: de la Vallée Poussin, Fejér, Riesz–Zygmund, Jackson, Rogosinski, Picard, Poison–Cauchy, Gauss–Weierstrass, q-Picard, q-Gauss–Weierstrass, Post–Widder, rotation invariant, Sikkema, and nonlinear.

The aim of this book is to continue these studies, naturally completing and generalizing the results in the previously mentioned book, as follows.

In the sense of the above-mentioned direction 1), we present here similar results for the Schurer–Faber operator, Beta operators of the first kind, Bernstein–Durrmeyer-type operators and Lorentz operator.

But unlike the previous book of Gal [49], here in six sections (Sects. 1.8–1.13) we also consider the approximation by several complex q-Bernstein-kind operators for $q > 1$, the case when they give the geometric order of approximation $O(1/q^n)$ (which is nearly to the best approximation). It is worth noting that the q-approximation problem with $q > 1$ is considered here not only in compact disks of $\mathbb{C}$ for various q-approximation operators but also in compact disks of the noncommutative field of quaternions for the q-Bernstein operator (see Sect. 1.12) and in compact subsets of the complex plane $\mathbb{C}$ for the so-called q-Bernstein–Faber polynomials (see Sect. 1.11) and q-Stancu–Faber polynomials (see Sect. 1.9). We emphasize that the results in Sect. 1.11 represent natural and strong extensions of the approximation results for the q-Bernstein polynomials in $[0, 1]$ to various compact subsets of the complex plane, for example, the compact disks, the circular lunes, the annulus sectors, the compact set bounded by the m-cusped hypocycloid H_m, $m = 2, 3, \ldots$, given by the parametric equation

$$z = e^{i\theta} + \frac{1}{m-1}e^{-(m-1)i\theta},\ \theta \in [0, 2\pi),$$

the regular m-star ($m = 2, 3, \ldots,$) given by

$$S_m = \{x\omega^k; 0 \leq x \leq 4^{1/m}, k = 0, 1, \ldots, m-1,\ \omega^m = 1\},$$

the m-leafed symmetric lemniscate, $m = 2, 3, \ldots$, with its boundary given by

$$L_m = \{z \in \mathbb{C}; |z^m - 1| = 1\},$$

and the semidisk

$$SD = \{z \in \mathbb{C}; |z| \leq 1 \text{ and } |Arg(z)| \leq \pi/2\}.$$

The approximation results obtained refer to exact estimates in approximation and in simultaneous approximation and to quantitative Voronovskaja-type results.

In the sense of the above-mentioned direction 2), quantitative overconvergence and convolution overconvergence results are presented here for the convolution potentials generated by the Beta and Gamma Euler's functions.

Finally, the overconvergence phenomenon in the sense of direction 1) for the most classical orthogonal expansions (of Chebyshev, Legendre, Hermite, Laguerre, and Gegenbauer kinds) attached to vector-valued functions is studied.

More detailed, the book can be described as follows.

The structure of Chap. 1 is the following:

– Section 1.1 contains the main results and concepts in complex analysis required for the proofs of the results in this book. For example, we mention here the following: the Vitali's theorem, Cauchy's formula, Bernstein's inequality, Faber polynomials associated with a domain in $\mathbb{C}$, Faber series, and Faber coefficients.
– In Sect. 1.2 the exact order in the generalized Voronovskaja's result for the derivatives of the complex Bernstein polynomials is obtained, thus generalizing the Voronovskaja's theorem for the Bernstein polynomials in Gal [49], pp. 36–42.
– In Sects. 1.3–1.7 we prove similar properties with those obtained for the complex Bernstein polynomials in Gal [49], Chap. 1, for the following classes of complex operators: Schurer–Faber polynomials, Beta operators of the first kind, genuine Bernstein–Durrmeyer polynomials, Bernstein–Durrmeyer polynomials with Jacobi weights, and Lorentz polynomials, respectively.
– In Sects. 1.8–1.12, error estimates of order $\frac{1}{q^n}$, $q > 1$, in approximation by complex q-Lorentz polynomials, q-Stancu and q-Stancu–Faber polynomials, q-Favard–Szász–Mirakjan operators, q-Bernstein–Faber polynomials, and q-Bernstein polynomials of quaternion variable, respectively, are obtained.
– Section 1.13 contains notes and open problems including the approximation by q-Lorentz–Faber, q-Bernstein–Kantorovich, q-Szàsz–Kantorovich, and q-Durrmeyer polynomials, with $q > 1$.

In Chap. 2 we present the overconvergence phenomenon in strips and the convolution overconvergence of the integral convolutions with trigonometric kernels including the Beatson kernel and its iterates and the approximation by complex potentials generated by the Euler-type functions.

Chapter 3 studies the overconvergence phenomenon in the sense of direction 1) with explicit quantitative estimates of geometric order, for the orthogonal expansions of Chebyshev and Legendre kinds attached to vector-valued functions. It is worth noting that in the last Sect. 3.4, one presents with details some interesting open problems concerning the possible application of the results obtained to the orthogonal systems of Hermite polynomials, Laguerre polynomials, and Gegenbauer polynomials.

Let us mention that most of the results presented here have been obtained by the author of this monograph in a series of papers, single or jointly written as can be seen in the bibliography. Also Theorems 1.2.7, 1.3.1, 1.7.7, 1.11.4, 2.1.1, 2.2.3, 2.2.4, 2.2.5, 3.2.4, 3.3.1, and 3.3.3 and Corollaries 1.11.5 and 3.2.5 appear for the first time in this book.

It is worth noting that this book suggests for future research, similar studies for other complex linear and nonlinear convolutions and for other Bernstein-type operators (including approximation in compact disks in $\mathbb{C}$, in compact disks in the field of quaternions and in compact subsets in $\mathbb{C}$ by their Faber-kind variants), like those of Meyer–König–Zeller type, Jakimovski–Leviatan type, Bleimann–Butzer–Hahn type, and Gamma type (including their q-variants with $q > 0$). For other examples of approximation operators to which the overconvergence theory could be applied, see Sect. 1.13.

The book mainly is addressed to researchers in the fields of the complex approximation of functions and its applications, mathematical analysis, and numerical analysis.

Also, since most of the proofs use elementary complex analysis, it is accessible to graduate students and suitable for graduate courses in the above domains.

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