Chapter 2
Heavy-Tailed and Long-Tailed Distributions

In this chapter we are interested in *(right-) tail properties* of distributions, i.e. in properties of a distribution which, for any x, depend only on the restriction of the distribution to (x,∞). More generally it is helpful to consider tail properties of functions.

Recall that for any distribution F on $\mathbb{R}$ we define the *tail function* $\overline{F}$ by

$$\overline{F}(x) = F(x,\infty), \quad x \in \mathbb{R}.$$

We start with characteristic properties of heavy-tailed distributions, i.e., of distributions all of whose positive exponential moments are infinite. The main result here concerns lower limits for convolution tails, see Sect. 2.3.

Following this we study different properties of long-tailed distributions, i.e., of distributions whose tails are asymptotically self-similar under shifting by a constant. Of particular interest are convolutions of long-tailed distributions. Our approach is based on a simple decomposition for such convolutions and on the concept of "h-insensitivity" for a long-tailed distribution with respect to some (slowly) increasing function h. In Sect. 2.8, we present useful characterisations of h-insensitive distributions.

2.1 Heavy-Tailed Distributions

The usage of the term "heavy-tailed distribution" varies according to the area of interest but is frequently taken to correspond to an absence of (positive) exponential moments. In the following definitions—which, for completeness here, repeat some of those made in the Introduction—we follow this tradition.

Definition 2.1. A distribution F on $\mathbb{R}$ is said to have *right-unbounded support* if $\overline{F}(x) > 0$ for all x.

Definition 2.2. We define a *distribution* F to be (right-) *heavy-tailed* if and only if

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$$\int_{\mathbb{R}} e^{\lambda x} F(dx) = \infty \quad \text{for all } \lambda > 0. \tag{2.1}$$

It will follow from Theorem 2.6 that to be heavy-tailed is indeed a tail property of a distribution. As a counterpart we give also the following definition.

Definition 2.3. A *distribution* F is called *light-tailed* if and only if

$$\int_{\mathbb{R}} e^{\lambda x} F(dx) < \infty \quad \text{for some } \lambda > 0, \tag{2.2}$$

i.e. if and only if it fails to be heavy-tailed.

Clearly, for any light-tailed distribution F on the positive half-line $\mathbb{R}^+ = [0,\infty)$, all moments are finite, i.e., $\int_0^\infty x^k F(dx) < \infty$ for all $k > 0$.

We shall say that a non-negative *function* (usually tending to zero) is *heavy-tailed* if it fails to be bounded by a decreasing exponential function. More precisely we make the following definition.

Definition 2.4. We define a *function* $f \geq 0$ to be *heavy-tailed* if and only if

$$\limsup_{x\to\infty} f(x)e^{\lambda x} = \infty \quad \text{for all } \lambda > 0. \tag{2.3}$$

For a function to be heavy-tailed is clearly a tail-property of that function. Theorem 2.6 shows in particular that a *distribution* is heavy-tailed if and only if its tail function is a heavy-tailed function. First we make the following definition.

Definition 2.5. For any distribution F, the function $R(x) := -\ln \overline{F}(x)$ is called the *hazard function* of the distribution. If the hazard function is differentiable, then its derivative $r(x) = R'(x)$ is called the *hazard rate*.

The hazard rate, when it exists, has the usual interpretation discussed in the Introduction.

Theorem 2.6. *For any distribution F the following assertions are equivalent:*
(i) *F is a heavy-tailed distribution.*
(ii) *The function $\overline{F}$ is heavy-tailed.*
(iii) *The corresponding hazard function R satisfies* $\liminf_{x\to\infty} R(x)/x = 0$.
(iv) *For some (any) fixed $T > 0$, the function $F(x, x+T]$ is heavy-tailed.*

Proof. (i)⇒(iv). Suppose that the function $F(x, x+T]$ is not heavy-tailed. Then

$$c := \sup_{x\in\mathbb{R}} F(x, x+T]e^{\lambda' x} < \infty \quad \text{for some } \lambda' > 0,$$

and, therefore, for all $\lambda < \lambda'$

$$\begin{aligned}\int_0^\infty e^{\lambda x} F(dx) &\leq \sum_{n=0}^\infty e^{\lambda(n+1)T} F(nT, nT+T] \\ &\leq c\sum_{n=0}^\infty e^{\lambda(n+1)T} e^{-\lambda' nT} = ce^{\lambda T}\sum_{n=0}^\infty e^{(\lambda-\lambda')nT} < \infty.\end{aligned}$$

It follows that the integral defined in (2.1) is finite for all $\lambda \in (0,\lambda')$, which implies that the distribution F cannot be heavy-tailed. The required implication now follows.

(iv)⇒(ii). This implication follows from the inequality $\overline{F}(x) \geq F(x,x+T]$.

(ii)⇒(iii). Suppose that, on the contrary, "lim inf" in (iii) is (strictly) positive. Then there exist $x_0 > 0$ and $\varepsilon > 0$ such that $R(x) \geq \varepsilon x$ for all $x \geq x_0$ which implies that $\overline{F}(x) \leq e^{-\varepsilon x}$ in contradiction of (ii).

(iii)⇒(i). Suppose that, on the contrary, F is light-tailed. It then follows from (2.2) (e.g., by the exponential Chebyshev inequality) that, for some $\lambda > 0$ and $c > 0$, we have $\overline{F}(x) \leq ce^{-\lambda x}$ for all x. This implies that $\liminf_{x\to\infty} R(x)/x \geq \lambda$ which contradicts (iii). □

Lemma 2.7. *Let the distribution F be absolutely continuous with density function f. Suppose that the distribution F is heavy-tailed. Then the function $f(x)$ is heavy-tailed also.*

Proof. Suppose that $f(x)$ is not heavy-tailed; then there exist $\lambda' > 0$ and x_0 such that

$$c := \sup_{x>x_0} f(x)e^{\lambda' x} < \infty,$$

and, therefore, for all $\lambda \in (0,\lambda')$

$$\int_{\mathbb{R}} e^{\lambda x} F(dx) \leq e^{\lambda x_0} + c\int_{x_0}^{\infty} e^{\lambda x} e^{-\lambda' x} dx < \infty.$$

It follows that the integral defined in (2.1) is finite for all λ such that $0 < \lambda < \lambda'$, which contradicts heavy-tailedness of the distribution F. □

We give an example to show that the converse assertion is not in general true. Consider the following piecewise continuous density function:

$$f(x) = \sum_{n=1}^{\infty} \mathbb{I}\{x \in [n, n+2^{-n}]\}.$$

We have $\limsup_{x\to\infty} f(x)e^{\lambda x} = \infty$ for all $\lambda > 0$, so that f is heavy-tailed. On the other hand, for all $\lambda \in (0, \ln 2)$,

$$\int_0^{\infty} e^{\lambda x} f(x)dx < \sum_{n=1}^{\infty} e^{\lambda(n+2^{-n})} 2^{-n} = \sum_{n=1}^{\infty} e^{\lambda(n+2^{-n})-n\ln 2} < \infty,$$

so that F is light-tailed.

For lattice distributions we have the following result.

Lemma 2.8. *Let F be a distribution on some lattice $\{a+hn,\ n \in \mathbb{Z}\}$, $a \in \mathbb{R}$, $h > 0$, with probabilities $F\{a+hn\} = p_n$. Then F is heavy-tailed if and only if the sequence $\{p_n\}$ is heavy-tailed, i.e.,*

$$\limsup_{n\to\infty} p_n e^{\lambda n} = \infty \quad \textit{for all } \lambda > 0. \tag{2.4}$$

Proof. The result follows from Theorem 2.6 with $T = h$. □

Examples of Heavy-Tailed Distributions

We conclude this section with a number of examples.

- The *Pareto distribution* on $\mathbb{R}^+$. This has tail function $\overline{F}$ given by

$$\overline{F}(x) = \left(\frac{\kappa}{x+\kappa}\right)^{\alpha}$$

  for some scale parameter $\kappa > 0$ and shape parameter $\alpha > 0$. Clearly we have $\overline{F}(x) \sim (x/\kappa)^{-\alpha}$ as $x \to \infty$, and for this reason the Pareto distributions are sometimes referred to as the *power law distributions*. The Pareto distribution has all moments of order $\gamma < \alpha$ finite, while all moments of order $\gamma \geq \alpha$ are infinite.
- The *Burr distribution* on $\mathbb{R}^+$. This has tail function $\overline{F}$ given by

$$\overline{F}(x) = \left(\frac{\kappa}{x^{\tau}+\kappa}\right)^{\alpha}$$

  for parameters $\alpha, \kappa, \tau > 0$. We have $\overline{F}(x) \sim \kappa^{\alpha} x^{-\tau\alpha}$ as $x \to \infty$; thus the Burr distribution is similar in its tail to the Pareto distribution, of which it is otherwise a generalisation. All moments of order $\gamma < \alpha\tau$ are finite, while those of order $\gamma \geq \alpha\tau$ are infinite.
- The *Cauchy distribution* on $\mathbb{R}$. This is most easily given by its density function f where

$$f(x) = \frac{\kappa}{\pi((x-a)^2+\kappa^2)}$$

  for some scale parameter $\kappa > 0$ and position parameter $a \in \mathbb{R}$. All moments of order $\gamma < 1$ are finite, while those of order $\gamma \geq 1$ are infinite.
- The *lognormal distribution* on $\mathbb{R}^+$. This is again most easily given by its density function f, where

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\log x-\mu)^2}{2\sigma^2}\right)$$

  for parameters μ and $\sigma > 0$. All moments of the lognormal distribution are finite. Note that a (positive) random variable ξ has a lognormal distribution with parameters μ and σ if and only if $\log \xi$ has a *normal distribution* with mean μ and variance σ^2. For this reason the distribution is natural in many applications.
- The *Weibull distribution* on $\mathbb{R}^+$. This has tail function $\overline{F}$ given by

$$\overline{F}(x) = e^{-(x/\kappa)^{\alpha}}$$

for some scale parameter $\kappa > 0$ and shape parameter $\alpha > 0$. This is a heavy-tailed distribution if and only if $\alpha < 1$. Note that in the case $\alpha = 1$ we have the *exponential distribution*. All moments of the Weibull distribution are finite.

Another useful class of heavy-tailed distributions is that of dominated-varying distributions. We say that F is a *dominated-varying distribution* (and write $F \in \mathcal{D}$) if there exists $c > 0$ such that

$$\overline{F}(2x) \geq c\overline{F}(x) \quad \text{for all } x.$$

Any intermediate regularly varying distribution (see Sect. 2.8) belongs to $\mathcal{D}$. Other examples may be constructed using the following scheme. Let G be a distribution with a regularly varying tail (again see Sect. 2.8). Then a distribution F belongs to the class $\mathcal{D}$, provided $c_1\overline{G}_1(x) \leq \overline{F}(x) \leq c_2\overline{G}(x)$ for some $0 < c_1 < c_2 < \infty$ and for all sufficiently large x.

2.2 Characterisation of Heavy-Tailed Distributions in Terms of Generalised Moments

A major objective of this and the succeeding section is to establish the important, if somewhat analytical, result referred to in the Introduction that for a heavy-tailed distribution F on $\mathbb{R}^+$ we have $\liminf_{x\to\infty} \overline{F*F}(x)/\overline{F}(x) = 2$. This is Theorem 2.12. As remarked earlier, it will then follow (see Chap. 3) that the subexponentiality of a distribution F on $\mathbb{R}^+$ is then *equivalent* to heavy-tailedness plus the reasonable regularity requirement that the limit as $x \to \infty$ of $\overline{F*F}(x)/\overline{F}(x)$ should exist.

In this section we therefore consider an important (and again quite analytical) characterisation of heavy-tailed distributions on $\mathbb{R}^+$, which is both of interest in itself and essential to the consideration of convolutions in the following section. In very approximate terms, for any such distribution we seek the existence of a monotone concave function h such that the function $e^{-h(\cdot)}$ characterises the tail of the distribution.

If a distribution F on the positive half-line $\mathbb{R}^+$ is such that not all of its moments are finite, i.e., $\int_0^\infty x^k F(dx) = \infty$ for some k, then F is heavy-tailed. In this case we can find such $k \geq 1$ that the kth moment is infinite, while the $(k-1)$th moment is finite. That is

$$\int_0^\infty xe^{(k-1)\ln x} F(dx) = \infty \quad \text{and} \quad \int_0^\infty e^{(k-1)\ln x} F(dx) < \infty. \tag{2.5}$$

Note that here the power of the exponent is a concave function. This observation can be generalised onto the whole class of heavy-tailed distributions as follows.

Theorem 2.9. *Let $\xi \geq 0$ be a random variable with a heavy-tailed distribution. Let the function $g(x)$ be such that $g(x) \to \infty$ as $x \to \infty$. Then there exists a monotone concave function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(x) = o(x)$ as $x \to \infty$, $\mathbb{E}e^{h(\xi)} < \infty$, and $\mathbb{E}e^{h(\xi)+g(\xi)} = \infty$.*

Now (2.5) is a particular example of the latter theorem with $g(x) = \ln x$. In this case, if not all moments of ξ are finite, the concave function $h(x)$ may be taken as $(k-1)\ln x$ for k as defined above. However, Theorem 2.9 is considerably sharper: it guarantees the existence of a concave function h for any function g (such that $g(x) \to \infty$ as $x \to \infty$), which may be taken as slowly increasing as we please.

As a further example, note that if ξ has a Weibull distribution with tail function $\overline{F}(x) = e^{-x^\alpha}$, $\alpha \in (0,1)$, and if $g(x) = \ln x$, then one can choose $h(x) = (x+c)^\alpha - \ln(x+c)$, with $c > 0$ sufficiently large.

Note also that Theorem 2.9 provides a characteristic property of heavy-tailed distributions; it fails for any light-tailed distribution. Indeed, consider any non-negative random variable ξ having a light-tailed distribution, i.e., $\mathbb{E}e^{\lambda\xi} < \infty$ for some $\lambda > 0$. Take $g(x) = \ln x$. If $h(x) = o(x)$ as $x \to \infty$, then $h(x) \leq c + \lambda x/2$ for some $c < \infty$ and, hence,

$$\mathbb{E}e^{h(\xi)+g(\xi)} \leq \mathbb{E}\xi e^{c+\lambda\xi/2} < \infty.$$

Proof (of Theorem 2.9). We will construct a piecewise linear function $h(x)$. To do so we construct two positive sequences $x_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$ as $n \to \infty$ and let

$$h(x) = h(x_{n-1}) + \varepsilon_n(x - x_{n-1}) \qquad \text{if } x \in (x_{n-1}, x_n],\ n \geq 1.$$

This function is monotone, since $\varepsilon_n > 0$. Moreover, this function is concave, due to the monotonicity of ε_n.

Put $x_0 = 0$ and $h(0) = 0$. Since ξ is heavy-tailed and $g(x) \to \infty$, we can choose x_1 sufficiently large that $e^{g(x)} \geq 2$ for all $x > x_1$ and

$$\mathbb{E}\{e^{\xi}; \xi \in (x_0, x_1]\} + e^{x_1}\overline{F}(x_1) > \overline{F}(x_0) + 1.$$

Choose $\varepsilon_1 > 0$ so that

$$\mathbb{E}\{e^{\varepsilon_1\xi}; \xi \in (x_0, x_1]\} + e^{\varepsilon_1 x_1}\overline{F}(x_1) = \overline{F}(0) + 1/2,$$

which is equivalent to

$$\mathbb{E}\{e^{h(\xi)}; \xi \in (x_0, x_1]\} + e^{h(x_1)}\overline{F}(x_1) = e^{h(x_0)}\overline{F}(0) + 1/2.$$

By induction we construct an increasing sequence x_n and a decreasing sequence $\varepsilon_n > 0$ such that $e^{g(x)} \geq 2^n$ for all $x > x_n$ and

$$\mathbb{E}\{e^{h(\xi)}; \xi \in (x_{n-1}, x_n]\} + e^{h(x_n)}\overline{F}(x_n) = e^{h(x_{n-1})}\overline{F}(x_{n-1}) + 1/2^n$$

for any $n \geq 2$. For $n = 1$ this is already done. Make the induction hypothesis for some $n \geq 2$. Due to the heavy-tailedness of ξ and to the convergence $g(x) \to \infty$, there exists x_{n+1} so large that $e^{g(x)} \geq 2^{n+1}$ for all $x > x_{n+1}$ and

$$\mathbb{E}\{e^{\varepsilon_n(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_n(x_{n+1} - x_n)}\overline{F}(x_{n+1}) > 2.$$

As a function of ε_{n+1}, the sum

$$\mathbb{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1})$$

is continuously decreasing to $\overline{F}(x_n)$ as $\varepsilon_{n+1} \downarrow 0$. Therefore, we can choose $\varepsilon_{n+1} \in (0, \varepsilon_n)$ so that

$$\mathbb{E}\{e^{\varepsilon_{n+1}(\xi - x_n)}; \xi \in (x_n, x_{n+1}]\} + e^{\varepsilon_{n+1}(x_{n+1} - x_n)}\overline{F}(x_{n+1}) = \overline{F}(x_n) + 1/(2^{n+1}e^{h(x_n)}).$$

By the definition of $h(x)$ this is equivalent to the following equality:

$$\mathbb{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1}) = e^{h(x_n)}\overline{F}(x_n) + 1/2^{n+1}.$$

Our induction hypothesis now holds with $n+1$ in place of n as required.

Next, for any N,

$$\begin{aligned}
\mathbb{E}\{e^{h(\xi)}; \xi \leq x_{N+1}\} &= \sum_{n=0}^{N} \mathbb{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} \\
&= \sum_{n=0}^{N} \left(e^{h(x_n)}\overline{F}(x_n) - e^{h(x_{n+1})}\overline{F}(x_{n+1}) + 1/2^{n+1}\right) \\
&\leq e^{h(x_0)}\overline{F}(x_0) + 1.
\end{aligned}$$

Hence, $\mathbb{E}e^{h(\xi)}$ is finite. On the other hand, since $e^{g(x)} \geq 2^n$ for all $x > x_n$,

$$\begin{aligned}
\mathbb{E}\{e^{h(\xi)+g(\xi)}; \xi > x_n\} &\geq 2^n \mathbb{E}\{e^{h(\xi)}; \xi > x_n\} \\
&\geq 2^n \left(\mathbb{E}\{e^{h(\xi)}; \xi \in (x_n, x_{n+1}]\} + e^{h(x_{n+1})}\overline{F}(x_{n+1})\right) \\
&= 2^n \left(e^{h(x_n)}\overline{F}(x_n) + 1/2^{n+1}\right).
\end{aligned}$$

Then $\mathbb{E}\{e^{h(\xi)+g(\xi)}; \xi > x_n\} \geq 1/2$ for any n, which implies $\mathbb{E}e^{h(\xi)+g(\xi)} = \infty$. Note also that necessarily $\lim_{n\to\infty} \varepsilon_n = 0$; otherwise $\liminf\limits_{x\to\infty} h(x)/x > 0$ and ξ is light tailed. $\square$

The latter theorem can be strengthened in the following way (for a proof see [20]):

Theorem 2.10. *Let $\xi \geq 0$ be a random variable with a heavy-tailed distribution. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a concave function such that $\mathbb{E}e^{f(\xi)} = \infty$. Let the function $g : \mathbb{R}^+ \to \mathbb{R}$ be such that $g(x) \to \infty$ as $x \to \infty$. Then there exists a concave function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h \leq f$, $\mathbb{E}e^{h(\xi)} < \infty$, and $\mathbb{E}e^{h(\xi)+g(\xi)} = \infty$.*

2.3 Lower Limit for Tails of Convolutions

Recall that the *convolution* $F * G$ of any two distributions F and G is given by, for any Borel set B,

$$(F * G)(B) = \int_{-\infty}^{\infty} F(B-y)G(dy) = \int_{-\infty}^{\infty} G(B-y)F(dy),$$

where $B - y = \{x - y : x \in B\}$. If, on some probability space with probability measure $\mathbb{P}$, ξ and η are independent random variables with respective distributions F and G, then $(F * G)(B) = \mathbb{P}\{\xi + \eta \in B\}$. The tail function of the convolution, the *convolution tail*, of F and G is then given by, for any $x \in \mathbb{R}$,

$$\overline{F * G}(x) = \mathbb{P}\{\xi + \eta > x\} = \int_{-\infty}^{\infty} \overline{F}(x-y)G(dy) = \int_{-\infty}^{\infty} \overline{G}(x-y)F(dy).$$

Now let F be a distribution on $\mathbb{R}^+$. In this section we discuss the following lower limit:

$$\liminf_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)},$$

in the case where F is heavy-tailed. We start with the following result, which generalises an observation in the Introduction.

Theorem 2.11. *Let $F_1, \ldots, F_n$ be distributions on $\mathbb{R}^+$ with unbounded supports. Then*

$$\liminf_{x\to\infty} \frac{\overline{F_1 * \ldots * F_n}(x)}{\overline{F}_1(x) + \ldots + \overline{F}_n(x)} \geq 1.$$

Proof. Let $\xi_1, \ldots, \xi_n$ be independent random variables with respective distributions $F_1, \ldots, F_n$. Since the events $\{\xi_k > x, \xi_j \in [0,x]$ for all $j \neq k\}$ are disjoint for different k, the convolution tail can be bounded from below in the following way:

$$\begin{aligned}
\overline{F_1 * \ldots * F_n}(x) &\geq \sum_{k=1}^{n} \mathbb{P}\{\xi_k > x, \xi_j \in [0,x] \text{ for all } j \neq k\} \\
&= \sum_{k=11}^{n} \overline{F}_k(x) \prod_{j\neq k} F_j(x) \\
&\sim \sum_{k=1}^{n} \overline{F}_k(x) \qquad \text{as } x \to \infty,
\end{aligned}$$

which implies the desired statement. □

Note that in the above proof we have heavily used the condition $F_k(\mathbb{R}^+) = 1$; for distributions on the whole real line $\mathbb{R}$ Theorem 2.11 in general fails.

It follows in particular that, for any distribution F on $\mathbb{R}^+$ with unbounded support and for any $n \geq 2$,

$$\liminf_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \geq n. \tag{2.6}$$

In particular,

$$\liminf_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} \geq 2. \tag{2.7}$$

As already discussed in the Introduction, in the light-tailed case the limit given by the left side of (2.7) is typically greater than 2. For example, for an exponential distribution it equals infinity. Thus we may ask under what conditions do we have equality in (2.7). We show that heavy-tailedness of F is sufficient.

Theorem 2.12. *Let F be a heavy-tailed distribution on $\mathbb{R}^+$. Then*

$$\liminf_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} = 2. \tag{2.8}$$

Proof. By the lower bound (2.7), it remains to prove the upper bound only, i.e.,

$$\liminf_{x\to\infty} \frac{\overline{F*F}(x)}{\overline{F}(x)} \leq 2.$$

Assume the contrary, i.e. there exist $\delta > 0$ and x_0 such that

$$\overline{F*F}(x) \geq (2+\delta)\overline{F}(x) \quad \text{for all } x > x_0. \tag{2.9}$$

Applying Theorem 2.9 with $g(x) = \ln x$, we can choose an increasing concave function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mathbb{E}e^{h(\xi)} < \infty$ and $\mathbb{E}\xi e^{h(\xi)} = \infty$. For any positive $b > 0$, consider the concave function

$$h_b(x) := \min(h(x), bx).$$

Since F is heavy-tailed, $h(x) = o(x)$ as $x \to \infty$; therefore, for any fixed b there exists x_1 such that $h_b(x) = h(x)$ for all $x > x_1$. Hence, $\mathbb{E}e^{h_b(\xi)} < \infty$ and $\mathbb{E}\xi e^{h_b(\xi)} = \infty$.

For any x, we have the convergence $h_b(x) \downarrow 0$ as $b \downarrow 0$. Then $\mathbb{E}e^{h_b(\xi_1)} \downarrow 1$ as $b \downarrow 0$. Thus there exists b such that

$$\mathbb{E}e^{h_b(\xi_1)} \leq 1 + \delta/4. \tag{2.10}$$

For any real a and t, put $a^{[t]} = \min(a, t)$. Then

$$\mathbb{E}(\xi_1^{[t]} + \xi_2^{[t]})e^{h_b(\xi_1+\xi_2)} = 2\mathbb{E}\xi_1^{[t]} e^{h_b(\xi_1+\xi_2)} \leq 2\mathbb{E}\xi_1^{[t]} e^{h_b(\xi_1)+h_b(\xi_2)},$$

by the concavity of the function h_b. Hence,

$$\frac{\mathbb{E}(\xi_1^{[t]}+\xi_2^{[t]})e^{h_b(\xi_1+\xi_2)}}{\mathbb{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \le 2\frac{\mathbb{E}\xi_1^{[t]}e^{h_b(\xi_1)}\mathbb{E}e^{h_b(\xi_2)}}{\mathbb{E}\xi_1^{[t]}e^{h_b(\xi_1)}}$$
$$= 2\mathbb{E}e^{h_b(\xi_2)} \le 2+\delta/2, \tag{2.11}$$

by (2.10). On the other hand, since $(\xi_1+\xi_2)^{[t]} \le \xi_1^{[t]}+\xi_2^{[t]}$,

$$\frac{\mathbb{E}(\xi_1^{[t]}+\xi_2^{[t]})e^{h_b(\xi_1+\xi_2)}}{\mathbb{E}\xi_1^{[t]}e^{h_b(\xi_1)}} \ge \frac{\mathbb{E}(\xi_1+\xi_2)^{[t]}e^{h_b(\xi_1+\xi_2)}}{\mathbb{E}\xi_1^{[t]}e^{h_b(\xi_1)}}$$
$$= \frac{\int_0^\infty x^{[t]}e^{h_b(x)}(F*F)(dx)}{\int_0^\infty x^{[t]}e^{h_b(x)}F(dx)}. \tag{2.12}$$

The right side, after integration by parts, is equal to

$$\frac{\int_0^\infty \overline{F*F}(x)d(x^{[t]}e^{h_b(x)})}{\int_0^\infty \overline{F}(x)d(x^{[t]}e^{h_b(x)})}.$$

Since $\mathbb{E}\xi_1 e^{h_b(\xi_1)} = \infty$, in the latter fraction both the integrals in the numerator and the denominator tend to infinity as $t\to\infty$. For the *increasing* function $h_b(x)$, together with the assumption (2.9) this implies that

$$\liminf_{t\to\infty}\frac{\int_0^\infty \overline{F*F}(x)d(x^{[t]}e^{h_b(x)})}{\int_0^\infty \overline{F}(x)d(x^{[t]}e^{h_b(x)})} \ge 2+\delta.$$

Substituting this into (2.12) we get a contradiction to (2.11) for sufficiently large t. $\square$

It turns out that the "liminf" given by the left side of (2.7) is equal to 2 not only for heavy-tailed but also for some light-tailed, distributions. Here is an example. Let F be an atomic distribution at the points x_n, $n = 0, 1, \ldots$, with masses p_n, i.e., $F\{x_n\} = p_n$. Suppose that $x_0 = 1$ and that $x_{n+1} > 2x_n$ for every n. Then the tail of the convolution $F*F$ at the point $x_n - 1$ is equal to

$$\overline{F*F}(x_n-1) = (F\times F)([x_n,\infty)\times\mathbb{R}^+) + (F\times F)([0,x_{n-1}]\times[x_n,\infty))$$
$$\sim 2\overline{F}(x_n-1) \quad \text{as } n\to\infty.$$

Hence,

$$\lim_{n\to\infty}\frac{\overline{F*F}(x_n-1)}{\overline{F}(x_n-1)} = 2.$$

From this equality and from (2.7),

$$\liminf_{x\to\infty}\frac{\overline{F*F}(x)}{\overline{F}(x)} = 2. \tag{2.13}$$

Take now $x_n = 3^n$, $n = 0, 1, \ldots$, and $p_n = ce^{-3^n}$, where c is the normalising constant. Then F is a light-tailed distribution satisfying the relation (2.13).

We conclude this section with the following result for convolutions of non-identical distributions.

Theorem 2.13. *Let F_1 and F_2 be two distributions on $\mathbb{R}^+$ and let the distribution F_1 be heavy-tailed. Then*

$$\liminf_{x\to\infty} \frac{\overline{F_1 * F_2}(x)}{\overline{F}_1(x) + \overline{F}_2(x)} = 1. \tag{2.14}$$

Proof. By Theorem 2.11, the left side of (2.14) is at least 1. Assume now that it is strictly greater than 1. Then there exists $\varepsilon > 0$ such that, for all sufficiently large x,

$$\frac{\overline{F_1 * F_2}(x)}{\overline{F}_1(x) + \overline{F}_2(x)} \geq 1 + 2\varepsilon. \tag{2.15}$$

Consider the distribution $G = (F_1 + F_2)/2$. This distribution is heavy-tailed. By Theorem 2.12 we get

$$\liminf_{x\to\infty} \frac{\overline{G * G}(x)}{\overline{G}(x)} = 2. \tag{2.16}$$

On the other hand, (2.15) and Theorem 2.11 imply that, for all sufficiently large x,

$$\begin{aligned} \overline{G * G}(x) &= \frac{\overline{F_1 * F_1}(x) + \overline{F_2 * F_2}(x) + 2\overline{F_1 * F_2}(x)}{4} \\ &\geq \frac{2(1-\varepsilon)\overline{F}_1(x) + 2(1-\varepsilon)\overline{F}_2(x) + 2(1+2\varepsilon)(\overline{F}_1(x) + \overline{F}_2(x))}{4} \\ &= 2(1+\varepsilon/2)\overline{G}(x), \end{aligned}$$

which contradicts (2.16). □

2.4 Long-Tailed Functions and Their Properties

Our plan is to introduce and to study the subclass of heavy-tailed distributions which are *long-tailed*. Later on we will study also long-tailedness properties of other characteristics of distributions. Therefore, we find it reasonable to start with a discussion of some generic properties of long-tailed functions.

Definition 2.14. An ultimately positive function f is *long-tailed* if and only if

$$\lim_{x\to\infty} \frac{f(x+y)}{f(x)} = 1, \quad \text{for all } y > 0. \tag{2.17}$$

Clearly if f is long-tailed, then we may also replace y by $-y$ in (2.17).

The following result makes a useful connection.

Lemma 2.15. *The function f is long-tailed if and only if $g(x) := f(\log x)$ (defined for positive x) is slowly varying at infinity, i.e., for any fixed $a > 0$,*

$$\frac{g(ax)}{g(x)} \to 1 \quad \text{as } x \to \infty.$$

Proof. The proof is immediate from the definition of g since

$$\frac{g(ax)}{g(x)} = \frac{f(\log x + \log a)}{f(\log x)}. \qquad \square$$

If f is long-tailed, then we also have uniform convergence in (2.17) over y in compact intervals. This is obvious for monotone functions, but in the general case the result follows from the Uniform Convergence Theorem for functions slowly varying at infinity, see Theorem 1.2.1 in [9]. Thus, for any $a > 0$, we have

$$\sup_{|y|\le a} |f(x) - f(x+y)| = o(f(x)) \quad \text{as } x \to \infty. \tag{2.18}$$

We give some quite basic closure properties for the class of long-tailed functions. We shall make frequent use of these—usually without further comment.

Lemma 2.16. *Suppose that the functions $f_1, \ldots, f_n$ are all long-tailed. Then*
(i) *For constants c_1 and c_2 where $c_2 > 0$, the function $f_1(c_1 + c_2 x)$ is long-tailed.*
(ii) *If $f \sim \sum_{k=1}^n c_k f_k$ where $c_1, \ldots, c_n > 0$, then f is long-tailed.*
(iii) *The product function $f_1 \cdots f_n$ is long-tailed.*
(iv) *The function $\min(f_1, \ldots, f_n)$ is long-tailed.*
(v) *The function $\max(f_1, \ldots, f_n)$ is long-tailed.*

Proof. The proofs of (i)–(iii) are routine from the definition of long-tailedness.
For (iv) observe that, for any $a > 0$ and any x, we have

$$\begin{aligned}\min\left(\frac{f_1(x+a)}{f_1(x)}, \frac{f_2(x+a)}{f_2(x)}\right) &\le \frac{\min(f_1(x+a), f_2(x+a))}{\min(f_1(x), f_2(x))} \\ &\le \max\left(\frac{f_1(x+a)}{f_1(x)}, \frac{f_2(x+a)}{f_2(x)}\right).\end{aligned}$$

Since f_1, f_2 are long-tailed the result now follows for the case $n = 2$. The result for general n follows by induction.
For (v) observe that, analogously to the argument for (iv) above, for any $a > 0$ and any x, we have

$$\begin{aligned}\min\left(\frac{f_1(x+a)}{f_1(x)}, \frac{f_2(x+a)}{f_2(x)}\right) &\le \frac{\max(f_1(x+a), f_2(x+a))}{\max(f_1(x), f_2(x))} \\ &\le \max\left(\frac{f_1(x+a)}{f_1(x)}, \frac{f_2(x+a)}{f_2(x)}\right),\end{aligned}$$

and the result now follows as before. $\square$

We now have the following result.

Lemma 2.17. *Let f be a long-tailed function. Then f is heavy-tailed and, moreover, satisfies the following relation: for every $\lambda > 0$,*

$$\lim_{x\to\infty} f(x)e^{\lambda x} = \infty.$$

Proof. Fix $\lambda > 0$. Since f is long-tailed, $f(x+y) \sim f(x)$ as $x \to \infty$ uniformly in $y \in [0,1]$. Hence, there exists x_0 such that, for all $x \geq x_0$ and $y \in [0,1]$,

$$f(x+y) \geq f(x)e^{-\lambda/2}.$$

Then $f(x_0+n+y) \geq f(x_0)e^{-\lambda(n+1)/2}$ for all $n \geq 1$ and $y \in [0,1]$, and, therefore,

$$\liminf_{x\to\infty} f(x)e^{\lambda x} \geq f(x_0) \lim_{n\to\infty} e^{-\lambda(n+1)/2}e^{\lambda n} = \infty,$$

so that the lemma now follows. □

However, it is not difficult to construct a heavy-tailed function f which fails to be sufficiently smooth so as to be long-tailed. Put

$$f(x) = \sum_{n=1}^{\infty} 2^{-n}\mathbb{I}\{2^{n-1} < x \leq 2^n\}.$$

Then, for any $\lambda > 0$,

$$\limsup_{x\to\infty} f(x)e^{\lambda x} \geq \limsup_{n\to\infty} 2^{-n}e^{\lambda 2^n} = \infty,$$

so that f is heavy-tailed. On the other hand,

$$\liminf_{x\to\infty} \frac{f(x+1)}{f(x)} \leq \liminf_{n\to\infty} \frac{f(2^n+1)}{f(2^n)} = \frac{1}{2},$$

which shows that f is not long-tailed.

h-Insensitivity

We now introduce a very important concept of which we shall make frequent subsequent use.

Definition 2.18. Given a strictly positive non-decreasing function h, an ultimately positive function f is called *h-insensitive* (or *h-flat*) if

$$\sup_{|y|\leq h(x)} |f(x+y) - f(x)| = o(f(x)) \quad \text{as } x \to \infty \text{, uniformly in } |y| \leq h(x). \tag{2.19}$$

It is clear that the relation (2.19) implies that the function f is long-tailed and conversely that any long-tailed function is h-insensitive for any constant function h. The following lemma gives a strong converse result, which we shall use repeatedly in Sect. 2.7 and subsequently throughout the monograph.

Lemma 2.19. *Suppose that the function f is long-tailed. Then there exists a function h such that $h(x) \to \infty$ as $x \to \infty$ and f is h-insensitive.*

Proof. For any integer $n \geq 1$, by (2.18), we can choose x_n such that

$$\sup_{|y| \leq n} |f(x+y) - f(x)| \leq f(x)/n \qquad \text{for all } x > x_n.$$

Without loss of generality we may assume that the sequence $\{x_n\}$ is increasing to infinity. Put $h(x) = n$ for $x \in [x_n, x_{n+1}]$. Since $x_n \to \infty$ as $n \to \infty$, we have $h(x) \to \infty$ as $x \to \infty$. By the construction we have

$$\sup_{|y| \leq h(x)} |f(x+y) - f(x)| \leq f(x)/n$$

for all $x > x_n$, which completes the proof. □

One important use of h-insensitivity is the following. The "natural" definition of long-tailedness of a function f is that of h-insensitivity with respect to any constant function $h(x) = a$ for all x and some $a > 0$. The use of this property in this form would then require that both the statements and the proofs of many results would involve a double limiting operation in which first x was allowed to tend to infinity, with the use of the relation (2.18), and following which a was allowed to tend to infinity. The replacement of the constant a by a function h itself increasing to infinity, but sufficiently slowly that the long-tailed function f is h-insensitive, not only enables two limiting operations to be replaced with a single one in proofs, but also permits simpler, cleaner, and more insightful presentations of many results (a typical example is the all-important Lemma 2.34 in Sect. 2.7).

Now observe that if a long-tailed function f is h-insensitive for some function h and if a further positive non-decreasing function $\hat{h}$ is such that $\hat{h}(x) \leq h(x)$ for all x, then (by definition) f is also $\hat{h}$-insensitive. Two trivial, but important (and frequently used), consequences of the combination of this observation with Lemma 2.19 are given by the following proposition.

Proposition 2.20. (i) *Given a finite collection of long-tailed functions $f_1, \ldots, f_n$, we may choose a* single *function h, increasing to infinity, with respect to which each of the functions f_i is h-insensitive.*

(ii) *Given any long-tailed function f and any positive non-decreasing function $\hat{h}$, we may choose a function h such that $h(x) \leq \hat{h}(x)$ for all x and f is h-insensitive.*

Proof. For (i), note that for each i we may choose a function h_i, increasing to infinity, such that f_i is h_i-insensitive, and then define h by $h(x) = \min_i h_i(x)$.

For (ii), note that we may take $h(x) = \min(\hat{h}(x), \bar{h}(x))$ where $\bar{h}$ is such that f is $\bar{h}$-insensitive. □

Finally we note that a further important use of h-insensitivity is the following. For any given positive function h, increasing to infinity, we may consider the class of those distributions whose (necessarily long-tailed) tail functions are h-insensitive. For varying h, this gives a powerful method for the classification of such distributions, which we explore in detail in Sect. 2.8.

2.5 Long-Tailed Distributions

As discussed in the Introduction, all heavy-tailed distributions likely to be encountered in practical applications are sufficiently regular as to be long-tailed, and it is the latter property, as applied to distributions, which we study in this section.

First, for any distribution F on $\mathbb{R}$, recall that we denote by R the hazard function $R(x) := -\ln \overline{F}(x)$. By definition, R is always a non-decreasing function and

$$R(x+1) - R(x) = -\ln \frac{\overline{F}(x+1)}{\overline{F}(x)}.$$

Definition 2.21. A distribution F on $\mathbb{R}$ is called *long-tailed* if $\overline{F}(x) > 0$ for all x and, for any fixed $y > 0$,

$$\overline{F}(x+y) \sim \overline{F}(x) \quad \text{as } x \to \infty. \tag{2.20}$$

That is, the distribution F is long-tailed if and only if its tail function $\overline{F}$ is a long-tailed function. Note that in (2.20) we may again replace y by $-y$. Further, for a distribution F to be long-tailed it is sufficient to require (2.20) to hold for any one non-zero value of y. Note also that the convergence in (2.20) is again uniform over y in compact intervals.

We shall write $\mathcal{L}$ for the class of long-tailed distributions on $\mathbb{R}$. Clearly $F \in \mathcal{L}$ is a *tail property* of the distribution F, since it depends only on $\{\overline{F}(x) : x \geq x_0\}$ for any finite x_0. Further, it follows from Lemma 2.17 that if the distribution F is long-tailed ($F \in \mathcal{L}$) then $\overline{F}$ is a heavy-tailed function, and so, by Theorem 2.6, F is also a heavy-tailed distribution. However, as the example following Lemma 2.17 shows, a heavy-tailed distribution need not be long-tailed.

The following lemma gives some readily verified equivalent characterisations of long-tailedness.

Lemma 2.22. *Let F be a distribution on $\mathbb{R}$ with right-unbounded support, and let ξ be a random variable with distribution F. Then the following are equivalent:*

(i) *The distribution F is long-tailed ($F \in \mathcal{L}$).*
(ii) *For any fixed $y > 0$, $F(x, x+y] = o(\overline{F}(x))$ as $x \to \infty$.*
(iii) *For any fixed $y > 0$, $\mathbb{P}\{\xi > x+y \mid \xi > x\} \to 1$ as $x \to \infty$.*
(iv) *The hazard function $R(x)$ satisfies $R(x+1) - R(x) \to 0$ as $x \to \infty$.*

Analogously to Lemma 2.16 we further have the following result.

Lemma 2.23. *Suppose that the distributions $F_1, \dots, F_n$ are all long-tailed (i.e. belong to the class $\mathcal{L}$) and that $\xi_1,\dots,\xi_n$ are independent random variables with distributions $F_1, \dots, F_n$, respectively. Then*
(i) *For any constants c_1 and $c_2 > 0$, the distribution of $c_2\xi_1 + c_1$ is long-tailed.*
(ii) *If $\overline{F}(x) \sim \sum_{k=1}^{n} c_k \overline{F}_k(x)$ where $c_1, \dots, c_n > 0$, then F is long-tailed.*
(iii) *If $F(x) = \min(F_1(x),\dots,F_n(x))$, then F is long-tailed.*
(iv) *If $F(x) = \max(F_1(x),\dots,F_n(x))$, then F is long-tailed.*
(v) *The distribution of $\min(\xi_1,\dots,\xi_n)$ is long-tailed.*
(vi) *The distribution of $\max(\xi_1,\dots,\xi_n)$ is long-tailed.*

Proof. The proofs follow from the application of Lemma 2.16 to the corresponding tail functions. In particular (v) and (vi) follow from (i) and (iii) of Lemma 2.16. □

2.6 Long-Tailed Distributions and Integrated Tails

In the study of random walks in particular, a key role is played by the integrated tail distribution, the fundamental properties of which we introduce in this section.

Definition 2.24. For any distribution F on $\mathbb{R}$ such that

$$\int_0^\infty \overline{F}(y)\,dy < \infty, \tag{2.21}$$

(and hence $\int_x^\infty \overline{F}(y)\,dy < \infty$ for any finite x) we define the *integrated tail distribution* F_I via its tail function by

$$\overline{F}_I(x) = \min\left(1, \int_x^\infty \overline{F}(y)dy\right). \tag{2.22}$$

Note that if ξ is a random variable with distribution F, then

$$\int_x^\infty \overline{F}(y)dy = \mathbb{E}\{\xi;\xi > x\} - x\mathbb{P}\{\xi > x\} = \mathbb{E}\{\xi - x;\xi > x\}. \tag{2.23}$$

An associated concept (in renewal theory and in queueing) is the *residual distribution F_r* which is defined for any distribution F on $\mathbb{R}^+$ with finite mean a by:

$$F_r(B) = \frac{1}{a}\int_B \overline{F}(y)dy, \quad B \in \mathcal{B}(\mathbb{R}^+).$$

The integrated tail and residual distributions satisfy the equality $\overline{F}_r(x) = \overline{F}_I(x)/a$ for all sufficiently large x.

The following characterisation will frequently be useful.

Lemma 2.25. *Suppose that the distribution F is such that (2.21) holds. Then F_I is long-tailed if and only if $\overline{F}(x) = o(\overline{F}_I(x))$ as $x \to \infty$.*

Proof. The integrated tail distribution F_I is long-tailed ($F_I \in \mathcal{L}$) if and only if $\overline{F}_I(x) - \overline{F}_I(x+1) = o(\overline{F}_I(x))$, or, equivalently, $\overline{F}_I(x) - \overline{F}_I(x+1) = o(\overline{F}_I(x+1))$. The required result now follows from the inequalities

$$\overline{F}(x+1) \leq \overline{F}_I(x) - \overline{F}_I(x+1) \leq \overline{F}(x),$$

valid for all sufficiently large x. □

Lemma 2.26. *Suppose that the distribution F is long-tailed ($F \in \mathcal{L}$) and such that (2.21) holds. Then F_I is long-tailed as well ($F_I \in \mathcal{L}$) and $\overline{F}(x) = o(\overline{F}_I(x))$ as $x \to \infty$.*

Proof. The long-tailedness of F_I follows from the relations, as $x \to \infty$,

$$\overline{F}_I(x+t) = \int_x^\infty \overline{F}(x+t+y)dy \sim \int_x^\infty \overline{F}(x+y)dy = \overline{F}_I(x),$$

for any fixed t. That $\overline{F}(x) = o(\overline{F}_I(x))$ as $x \to \infty$ now follows from Lemma 2.25. □

The converse assertion, i.e., that long-tailedness of F_I implies long-tailedness of F is not in general true. This is illustrated by the following example.

Example 2.27. Let the distribution F be such that $\overline{F}(x) = 2^{-2n}$ for $x \in [2^n, 2^{n+1})$. Then F is not long-tailed since $\overline{F}(2^n - 1)/\overline{F}(2^n) = 4$ for any n, so that $\overline{F}(x - 1)/\overline{F}(x) \not\to 1$ as $x \to \infty$. But we have $x^{-2} \leq \overline{F}(x) \leq 4x^{-2}$ for any $x > 0$. In particular, $\overline{F}_I(x) \geq x^{-1}$ and thus $\overline{F}(x) = o(\overline{F}_I(x))$ as $x \to \infty$. Thus, by Lemma 2.25, F_I is long-tailed.

We now formulate a more general result which will be needed in the theory of random walks with heavy-tailed increments and is also of some interest in its own right. Let F be a distribution on $\mathbb{R}$ and μ a non-negative measure on $\mathbb{R}^+$ such that

$$\int_0^\infty \overline{F}(t)\mu(dt) < \infty. \tag{2.24}$$

We may then define the distribution F_μ on $\mathbb{R}^+$ given by

$$\overline{F}_\mu(x) := \min\left(1, \int_0^\infty \overline{F}(x+t)\mu(dt)\right), \quad x \geq 0. \tag{2.25}$$

If μ is Lebesgue measure, then F_μ is the integrated tail distribution. We can formulate the same question as for F_I: what type of conditions on F imply long-tailedness of F_μ? The answer is given by the following theorem.

Theorem 2.28. *Let F be a long-tailed distribution. Then F_μ is a long-tailed distribution and, for any fixed $y > 0$,*

$$\overline{F}_\mu(x+y) \sim \overline{F}_\mu(x)$$

as $x \to \infty$ uniformly in all μ satisfying (2.24), i.e.,

$$\inf_{\mu} \inf_{x > x_0} \frac{\overline{F}_{\mu}(x+y)}{\overline{F}_{\mu}(x)} \to 1 \quad \text{as } x_0 \to \infty. \tag{2.26}$$

If, in addition, $\overline{F}(x+h(x)) \sim \overline{F}(x)$ as $x \to \infty$, for some positive function h, then (2.26) holds with $h(x)$ in place of y.

Proof. Fix $\varepsilon > 0$. Since $\overline{F}(x+y+u) \sim \overline{F}(x+u)$ as $x \to \infty$ uniformly in $u \geq 0$, there exists x_0 such that (2.24),

$$\overline{F}(x+y+u) \geq (1-\varepsilon)\overline{F}(x+u) \quad \text{for all } x > x_0.$$

Then, for all $x > x_0$ and μ,

$$\overline{F_{\mu}}(x+y) = \int_0^{\infty} \overline{F}(x+y+u)\mu(dy) \geq (1-\varepsilon)\int_0^{\infty} \overline{F}(x+u)\mu(du) = (1-\varepsilon)\overline{F_{\mu}}(x).$$

Letting $\varepsilon \to 0$ we obtain the desired result. The same argument holds when y is replaced by $h(x)$. □

2.7 Convolutions of Long-Tailed Distributions

We know from Theorem 2.11 that for any distributions F and G on the positive half-line $\mathbb{R}^+$

$$\liminf_{x \to \infty} \frac{\overline{F * G}(x)}{\overline{F}(x) + \overline{G}(x)} \geq 1. \tag{2.27}$$

In order to get an analogous result for distributions on the entire real line $\mathbb{R}$, we assume some of those involved to be long-tailed. The assumption of the theorem below seems to be the weakest possible in the absence of conditions on left tails.

Theorem 2.29. *Let the distributions $F_1, \ldots, F_n$ on $\mathbb{R}$ be such that the function $\overline{F}_1(x) + \ldots + \overline{F}_n(x)$ is long-tailed. Then,*

$$\liminf_{x \to \infty} \frac{\overline{F_1 * \ldots * F_n}(x)}{\overline{F}_1(x) + \ldots + \overline{F}_n(x)} \geq 1. \tag{2.28}$$

In particular (2.28) holds whenever each of the distributions F_i is long-tailed.

Proof (cf Theorem 2.11). Let $\xi_1, \ldots, \xi_n$ be independent random variables with respective distributions $F_1, \ldots, F_n$. For any fixed $a > 0$, we have the following lower bound:

$$\overline{F_1 * \ldots * F_n}(x) \geq \sum_{k=1}^{n} \mathbb{P}\{\xi_k > x + (n-1)a, \xi_j \in (-a, x] \text{ for all } j \neq k\}$$
$$= \sum_{k=1}^{n} \overline{F}_k(x + (n-1)a) \prod_{j \neq k} F_j(-a, x]. \tag{2.29}$$

For every $\varepsilon > 0$ there exists a such that $F_j(-a, a] \geq 1 - \varepsilon$ for all j. Thus, for all $x > a$,

$$\overline{F_1 * \ldots * F_n}(x) \geq (1-\varepsilon)^{n-1} \sum_{k=1}^{n} \overline{F}_k(x + (n-1)a).$$

Since the function $\overline{F}_1 + \ldots + \overline{F}_n$ is long-tailed,

$$\liminf_{x \to \infty} \frac{\overline{F_1 * \ldots * F_n}(x)}{\overline{F}_1(x) + \ldots + \overline{F}_n(x)} \geq (1-\varepsilon)^{n-1}.$$

The required result (2.28) now follows by letting $\varepsilon \to 0$. □

For identical distributions, Theorem 2.29 yields the following corollary.

Corollary 2.30. *Let the distribution F on $\mathbb{R}$ be long-tailed ($F \in \mathcal{L}$). Then, for any $n \geq 2$,*

$$\liminf_{x \to \infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \geq n.$$

We also have the following result for the convolution of a long-tailed distribution F with an arbitrary distribution G, the proof of which is similar in spirit to that of Theorem 2.29.

Theorem 2.31. *Let the distributions F and G on $\mathbb{R}$ be such that F is long-tailed ($F \in \mathcal{L}$). Then,*

$$\liminf_{x \to \infty} \frac{\overline{F * G}(x)}{\overline{F}(x)} \geq 1. \tag{2.30}$$

Proof. Let ξ and η be independent random variables with respective distributions F and G. For any fixed a,

$$\overline{F * G}(x) \geq \mathbb{P}\{\xi > x - a, \eta > a\}$$
$$= \overline{F}(x-a)\overline{G}(a). \tag{2.31}$$

For every $\varepsilon > 0$ there exists a such that $\overline{G}(a) \geq 1 - \varepsilon$. Thus, for all x,

$$\overline{F * G}(x) \geq (1-\varepsilon)\overline{F}(x-a).$$

Since the distribution F is long-tailed, it now follows that

$$\liminf_{x\to\infty} \frac{\overline{F*G}(x)}{\overline{F}(x)} \geq 1-\varepsilon$$

and the required result (2.30) once more followed by letting $\varepsilon \to 0$. □

We now have the following corollary.

Corollary 2.32. *Let the distribution F on $\mathbb{R}$ be such that F is long-tailed ($F \in \mathcal{L}$) and let the distribution G be such that $\overline{G}(a) = 0$ for some a. Then $\overline{F*G}(x) \sim \overline{F}(x)$ as $x \to \infty$.*

Proof. Since $\overline{G}(a) = 0$, we have $\overline{F*G}(x) \leq \overline{F}(x-a)$. Thus since F is long-tailed we have

$$\limsup_{x\to\infty} \frac{\overline{F*G}(x)}{\overline{F}(x)} \leq 1.$$

Combining this result with the lower bound of Theorem 2.31, we obtain the desired equivalence. □

In order to further study the convolutions of long-tailed distributions, we make repeated use of two fundamental decompositions. Let $h > 0$ and let ξ and η be independent random variables with distributions F and G, respectively. Then the tail function of the convolution of F and G possesses the following decomposition: for $x > 0$,

$$\overline{F*G}(x) = \mathbb{P}\{\xi+\eta > x, \xi \leq h\} + \mathbb{P}\{\xi+\eta > x, \xi > h\}. \tag{2.32}$$

If in addition $h \leq x/2$, then

$$\begin{aligned}\overline{F*G}(x)& \\ = \mathbb{P}\{\xi+\eta > x, \xi \leq h\} + \mathbb{P}\{\xi+\eta > x, \eta \leq h\} + \mathbb{P}\{\xi+\eta > x, \xi > h, \eta > h\},&\end{aligned} \tag{2.33}$$

since if $\xi \leq h$ and $\eta \leq h$ then $\xi+\eta \leq 2h \leq x$.

Note that

$$\mathbb{P}\{\xi+\eta > x, \xi \leq h\} = \int_{-\infty}^{h} \overline{G}(x-y)F(dy), \tag{2.34}$$

while the probability of the event $\{\xi+\eta>x, \xi>h, \eta>h\}$ is symmetric in F and G, and

$$\begin{aligned}\mathbb{P}\{\xi+\eta > x, \xi > h, \eta > h\} &= \int_{h}^{\infty} \overline{F}(\max(h, x-y))G(dy) \\ &= \int_{h}^{\infty} \overline{G}(\max(h, x-y))F(dy).\end{aligned} \tag{2.35}$$

Definition 2.33. Given a strictly positive non-decreasing function h, a distribution F on $\mathbb{R}$ is called *h-insensitive*(or *h-flat*) if its tail function $\overline{F}$ is an h-insensitive function (see Definition 2.18). Since $\overline{F}$ is monotone, this reduces to the requirement that $\overline{F}(x \pm h(x)) \sim \overline{F}(x)$ as $x \to \infty$.

Recall from the results for h-insensitive functions that a distribution F is long-tailed if and only if there exists a function h as above with respect to which F is h-insensitive.

For long-tailed distributions F and G we shall now make particular use of the decomposition (2.33) in which the constant h is replaced by a function h increasing to infinity (with $h(x) < x/2$ for all x) and such that both F and G are h-insensitive.

The following three lemmas are the keys to everything that follows later in this section.

Lemma 2.34. *Suppose that the distribution G on $\mathbb{R}$ is long-tailed ($G \in \mathcal{L}$) and that the positive function h is such that $h(x) \to \infty$ as $x \to \infty$ and G is h-insensitive. Then, for any distribution F, as $x \to \infty$,*

$$\int_{-\infty}^{h(x)} \overline{G}(x-y)F(dy) \sim \overline{G}(x),$$

$$\int_{x-h(x)}^{\infty} \overline{F}(x-y)G(dy) \sim \overline{G}(x).$$

Proof. The existence of the function h is guaranteed by Lemma 2.19. We now have

$$\int_{-\infty}^{h(x)} \overline{G}(x-y)F(dy) \le \overline{G}(x-h(x)).$$

On the other hand we also have,

$$\begin{aligned} \int_{-\infty}^{h(x)} \overline{G}(x-y)F(dy) &\ge \int_{-h(x)}^{h(x)} \overline{G}(x-y)F(dy) \\ &\ge F(-h(x),h(x)]\overline{G}(x+h(x)) \\ &\sim \overline{G}(x+h(x)) \quad \text{ as } x \to \infty, \end{aligned}$$

where the last equivalence follows since $h(x) \to \infty$ as $x \to \infty$. The first result now follows from the choice of the function h. The second result follows similarly: the integral is again bounded from above by $\overline{G}(x-h(x))$ and from below by $\overline{F}(-h(x))\overline{G}(x+h(x))$ and the result follows as previously. □

Remark 2.35. Note the crucial role played by the monotonicity of the tail function $\overline{G}$ in the proof of Lemma 2.34—something which is not available to us in considering, e.g., densities in Chap. 4.

We now prove a version of Lemma 2.34 which is symmetric in the distributions F and G, and which allows us to get many important results for convolutions—see the further discussion below.

Lemma 2.36. *Suppose that the distributions F and G on $\mathbb{R}$ are such that the sum $\overline{F}+\overline{G}$ of their tail functions is a long-tailed function (equivalently the measure $F+G$ is long-tailed in the obvious sense) and that the positive function h is such that $h(x)\to\infty$ as $x\to\infty$ and $\overline{F}+\overline{G}$ is h-insensitive. Then*

$$\int_{-\infty}^{h(x)} \overline{G}(x-y)F(dy)+\int_{-\infty}^{h(x)} \overline{F}(x-y)G(dy) \sim \overline{G}(x)+\overline{F}(x) \quad \text{as } x\to\infty.$$

Proof. The proof is simply a two-sided version of that for the first assertion of Lemma 2.34. The existence of the function h is again guaranteed by Lemma 2.19. Now note first that, as in the earlier proof,

$$\int_{-\infty}^{h(x)} \overline{G}(x-y)F(dy)+\int_{-\infty}^{h(x)} \overline{F}(x-y)G(dy) \le \overline{G}(x-h(x))+\overline{F}(x-h(x)),$$

and second that

$$\begin{aligned}\int_{-\infty}^{h(x)} \overline{G}(x-y)F(dy)&+\int_{-\infty}^{h(x)} \overline{F}(x-y)G(dy)\\ &\ge \int_{-h(x)}^{h(x)} \overline{G}(x-y)F(dy)+\int_{-h(x)}^{h(x)} \overline{F}(x-y)G(dy)\\ &\ge F(-h(x),h(x)]\overline{G}(x+h(x))+G(-h(x),h(x)]\overline{F}(x+h(x))\\ &\sim \overline{G}(x+h(x))+\overline{F}(x+h(x)) \quad \text{as } x\to\infty,\end{aligned}$$

where the last equivalence follows since $h(x)\to\infty$ as $x\to\infty$. The required result now follows from the choice of the function h. $\square$

Note that special cases under which $\overline{F}+\overline{G}$ is long-tailed are (a) F and G are both long-tailed—in which case Lemma 2.36 (almost) follows from 2.34, and (b) F is long-tailed and $\overline{G}(x)=o(\overline{F}(x))$ as $x\to\infty$.

In various calculations we need to estimate the "internal" part of the convolution. The following result will be useful.

Lemma 2.37. *Let h be any increasing function on $\mathbb{R}^+$ such that $h(x)\to\infty$. Then, for any distributions F_1, F_2, G_1, and G_2 on $\mathbb{R}$,*

$$\limsup_{x\to\infty}\frac{\mathbb{P}\{\xi_1+\eta_1>x,\xi_1>h(x),\eta_1>h(x)\}}{\mathbb{P}\{\xi_2+\eta_2>x,\xi_2>h(x),\eta_2>h(x)\}} \le \limsup_{x\to\infty}\frac{\overline{F_1}(x)}{\overline{F_2}(x)}\cdot\limsup_{x\to\infty}\frac{\overline{G_1}(x)}{\overline{G_2}(x)},$$

where ξ_1, ξ_2, η_1, and η_2 are independent random variables with respective distributions F_1, F_2, G_1 and G_2.

In particular, in the case where the limits of the ratios $\overline{F_1}(x)/\overline{F_2}(x)$ and $\overline{G_1}(x)/\overline{G_2}(x)$ exist, we have

$$\lim_{x\to\infty}\frac{\mathbb{P}\{\xi_1+\eta_1>x,\xi_1>h(x),\eta_1>h(x)\}}{\mathbb{P}\{\xi_2+\eta_2>x,\xi_2>h(x),\eta_2>h(x)\}} = \lim_{x\to\infty}\frac{\overline{F_1}(x)}{\overline{F_2}(x)}\cdot\lim_{x\to\infty}\frac{\overline{G_1}(x)}{\overline{G_2}(x)}.$$

Proof. It follows from (2.35) that

$$\begin{aligned}
&\mathbb{P}\{\xi_1+\eta_1>x,\xi_1>h(x),\eta_1>h(x)\}\\
&\qquad\le \sup_{z>h(x)}\frac{\overline{F_1}(z)}{\overline{F_2}(z)}\int_{h(x)}^{\infty}\overline{F}_2(\max(h(x),x-y))G_1(dy)\\
&\qquad= \sup_{z>h(x)}\frac{\overline{F_1}(z)}{\overline{F_2}(z)}\int_{h(x)}^{\infty}\overline{G}_1(\max(h(x),x-y))F_2(dy).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{h(x)}^{\infty}\overline{G}_1(\max(h(x),x-y))F_2(dy)\\
&\qquad\le \sup_{z>h(x)}\frac{\overline{G_1}(z)}{\overline{G_2}(z)}\int_{h(x)}^{\infty}\overline{G}_2(\max(h(x),x-y))F_2(dy)\\
&\qquad= \sup_{z>h(x)}\frac{\overline{G_1}(z)}{\overline{G_2}(z)}\mathbb{P}\{\xi_2+\eta_2>x,\xi_2>h(x),\eta_2>h(x)\}.
\end{aligned}$$

Combining these results and recalling that $h(x)\to\infty$ as $x\to\infty$, we obtain the desired conclusion. □

Definition 2.38. Two distributions F and G with right-unbounded supports are said to be *tail-equivalent* if $\overline{F}(x)\sim\overline{G}(x)$ as $x\to\infty$ (i.e. $\lim_{x\to\infty}\overline{F}(x)/\overline{G}(x)=1$).

In the next two theorems we provide conditions under which a random shifting preserves tail equivalence.

Theorem 2.39. *Suppose that F_1, F_2 and G are distributions on $\mathbb{R}$ such that $\overline{F}_1(x)\sim\overline{F}_2(x)$ as $x\to\infty$. Suppose further that G is long-tailed. Then $\overline{F_1*G}(x)\sim\overline{F_2*G}(x)$ as $x\to\infty$.*

Proof. By Lemma 2.19 we can find a function h such that $h(x)\to\infty$ and

$$\overline{G}(x\pm h(x))\sim\overline{G}(x)\qquad\text{as } x\to\infty,$$

i.e. G is h-insensitive. We use the following decomposition: for $k=1, 2$,

$$\overline{F_k*G}(x)=\left(\int_{-\infty}^{x-h(x)}+\int_{x-h(x)}^{\infty}\right)\overline{F}_k(x-y)G(dy). \tag{2.36}$$

It follows from the tail equivalence of F_1 and F_2 that $\overline{F}_1(x-y)\sim\overline{F}_2(x-y)$ as $x\to\infty$ uniformly in $y<x-h(x)$. Thus,

$$\int_{-\infty}^{x-h(x)}\overline{F}_1(x-y)G(dy)\sim\int_{-\infty}^{x-h(x)}\overline{F}_2(x-y)G(dy) \tag{2.37}$$

as $x \to \infty$. Next, by Lemma 2.34, for $k = 1, 2$,

$$\int_{x-h(x)}^{\infty} \overline{F}_k(x-y)G(dy) \sim \overline{G}(x) \quad \text{as } x \to \infty. \tag{2.38}$$

Substituting (2.37) and (2.38) into (2.36) we obtain the required equivalence $\overline{F_1 * G}(x) \sim \overline{F_2 * G}(x)$. □

Theorem 2.40. *Suppose that F_1, F_2, G_1 and G_2 are distributions on $\mathbb{R}$ such that $\overline{F}_1(x) \sim \overline{F}_2(x)$ and $\overline{G}_1(x) \sim \overline{G}_2(x)$ as $x \to \infty$. Suppose further that the function $\overline{F}_1 + \overline{G}_1$ is long-tailed. Then $\overline{F_1 * G_1}(x) \sim \overline{F_2 * G_2}(x)$ as $x \to \infty$.*

Proof. The conditions of the theorem imply that the function $\overline{F}_2 + \overline{G}_2$ is similarly long-tailed. By Lemma 2.19 and the following remark we can choose a function h such that $h(x) \to \infty$ as $x \to \infty$, $h(x) \le x/2$ and, for $k = 1, 2$,

$$\overline{F}_k(x \pm h(x)) + \overline{G}_k(x \pm h(x)) \sim \overline{F}_k(x) + \overline{G}_k(x) \quad \text{as } x \to \infty,$$

i.e. $\overline{F}_k + \overline{G}_k$ is h-insensitive. We use the following decomposition which follows from (2.33) to (2.35):

$$\begin{aligned}\overline{F_k * G_k}(x) &= \int_{-\infty}^{h(x)} \overline{F}_k(x-y)G_k(dy) + \int_{-\infty}^{h(x)} \overline{G}_k(x-y)F_k(dy) \\ &\quad + \int_{h(x)}^{\infty} \overline{F}_k(\max(h(x), x-y))G_k(dy).\end{aligned} \tag{2.39}$$

Since F_1 and F_2 are tail equivalent and G_1 and G_2 are tail equivalent, it follows from Lemma 2.37 that, as $x \to \infty$,

$$\int_{h(x)}^{\infty} \overline{F}_1(\max(h(x), x-y))G_1(dy) \sim \int_{h(x)}^{\infty} \overline{F}_2(\max(h(x), x-y))G_2(dy). \tag{2.40}$$

Further, by Lemma 2.36, for $k = 1, 2$ and as $x \to \infty$,

$$\int_{-\infty}^{h(x)} \overline{F}_k(x-y)G_k(dy) + \int_{-\infty}^{h(x)} \overline{G}_k(x-y)F_k(dy) \sim \overline{F}_k(x) + \overline{G}_k(x). \tag{2.41}$$

Substituting (2.40) and (2.41) into (2.39) we obtain the required equivalence $\overline{F_1 * G_1}(x) \sim \overline{F_2 * G_2}(x)$. □

We now use Theorem 2.40 to show that the class $\mathcal{L}$ is closed under convolutions. This is a corollary of the following result.

Theorem 2.41. *Suppose that the distributions F and G are such that F is long-tailed and the measure $F + G$ is also long-tailed (i.e. the sum $\overline{F} + \overline{G}$ of the tail functions of the two distributions is long-tailed). Then the convolution $F * G$ is also long-tailed.*

Proof. Fix $y > 0$. Take $F_1 = F$ and F_2 to be equal to F shifted by $-y$, i.e., $\overline{F}_2(x) = \overline{F}(x+y)$. Then $F_2 * G$ is equal to $F * G$ shifted by $-y$. Since F is long-tailed, $\overline{F}_1(x) \sim \overline{F}_2(x)$. Since also $\overline{F}_1 + \overline{G}$ is long-tailed, it follows from Theorem 2.40 with $G_1 = G_2 = G$ that $\overline{F_1 * G}(x) \sim \overline{F_2 * G}(x)$. Hence $\overline{F * G}(x) \sim \overline{F * G}(x+y)$ as $x \to \infty$. □

Both the following corollaries are now immediate from Theorem 2.41 since in each case the measure $F+G$ is long-tailed.

Corollary 2.42. *Let the distributions F and G be long-tailed. Then the convolution $F*G$ is also long-tailed.*

Corollary 2.43. *Suppose that F and G are distributions and that F is long-tailed. Suppose also that $\overline{G}(x) = o(\overline{F}(x))$ as $x \to \infty$. Then $F*G$ is long-tailed.*

Finally in this section we have the following converse result.

Lemma 2.44. *Let F and G be two distributions on $\mathbb{R}^+$ such that F has unbounded support and G is non-degenerate at 0. Suppose that $\overline{G}(x) \le c\overline{F}(x)$ for some $c < \infty$ and*

$$\limsup_{x\to\infty} \frac{\overline{F*G}(x)}{\overline{F}(x)+\overline{G}(x)} \le 1. \tag{2.42}$$

Then F is long-tailed.

Proof. Take any a such that $G(a,\infty) > 0$ which is possible because G is not concentrated at 0. Since for any two distributions on $\mathbb{R}^+$

$$\overline{F*G}(x) = \int_0^x \overline{F}(x-y)G(dy) + \overline{G}(x),$$

it follows from the condition (2.42) that

$$\begin{aligned}\int_0^x \overline{F}(x-y)G(dy) &\le \overline{F}(x) + o(\overline{F}(x)+\overline{G}(x)) \\ &= \overline{F}(x) + o(\overline{F}(x)) \quad \text{ as } x\to\infty,\end{aligned}$$

due to the condition $\overline{G}(x) \le c\overline{F}(x)$. This implies that

$$\begin{aligned}\int_0^x F(x-y,x]G(dy) &= \int_0^x (\overline{F}(x-y)-\overline{F}(x))G(dy) \\ &= o(\overline{F}(x)) \quad \text{ as } x\to\infty.\end{aligned}$$

For $x \ge a$, the left side is not less than $F(x-a,x]G(a,x]$, hence $F(x-a,x] = o(\overline{F}(x))$ as $x\to\infty$. The latter relation is equivalent to $\overline{F}(x-a) \sim \overline{F}(x)$ which completes the proof. □

2.8 *h*-Insensitive Distributions

Let F be a long-tailed distribution ($F \in \mathcal{L}$), i.e. a distribution whose tail function $\overline{F}$ is such that for some (and hence for all) non-zero y, we have $\overline{F}(x+y) \sim \overline{F}(x)$ as $x\to\infty$. We saw in Lemma 2.19 that we can then find a non-decreasing positive function h such that $h(x)\to\infty$ as $x\to\infty$ and

$$\overline{F}(x+y) \sim \overline{F}(x) \quad \text{uniformly in } |y| \le h(x), \tag{2.43}$$

i.e. such that the distribution F is h-insensitive (see Definition 2.33).

In this section we turn this process around: we fix a positive function h which is increasing to infinity and seek to identify those long-tailed distributions which are h-insensitive. By varying the choice of h, we then have an important technique for classifying long-tailed distributions according to the heaviness of their tails and for establishing characteristic properties of various classes of these distributions.

Slowly Varying Distributions

As a first example, consider the function h given by $h(x) = \varepsilon x$ for some $\varepsilon > 0$; then the class of h-insensitive distributions coincides with the class of distributions whose tails are *slowly varying at infinity*, i.e., for any $\varepsilon > 0$,

$$\frac{\overline{F}((1+\varepsilon)x)}{\overline{F}(x)} \to 1 \quad \text{as } x \to \infty. \tag{2.44}$$

These distributions are extremely heavy; in particular they do not possess any finite positive moments, i.e., $\int x^\gamma F(dx) = \infty$ for any $\gamma > 0$. Examples are given by distributions F with the following tail functions:

$$\overline{F}(x) \sim 1/\ln^\gamma x, \;\; \overline{F}(x) \sim 1/(\ln\ln x)^\gamma \;\text{ as } x \to \infty, \quad \gamma > 0.$$

Regularly Varying Distributions

We introduce here the well-known class of regularly varying distributions and consider their insensitivity properties.

Recall that an ultimately positive function f is called *regularly varying at infinity with index* $\alpha \in \mathbb{R}$ if, for any fixed $c > 0$,

$$f(cx) \sim c^\alpha f(x) \quad \text{as } x \to \infty. \tag{2.45}$$

A distribution F on $\mathbb{R}$ is called *regularly varying at infinity with index* $-\alpha < 0$ if $\overline{F}(cx) \sim c^{-\alpha}\overline{F}(x)$ as $x \to \infty$, i.e., $\overline{F}(x)$ is regularly varying at infinity with index $-\alpha < 0$.

Particular examples of regularly varying distributions which were introduced in Sect. 2.1 are the *Pareto*, *Burr* and *Cauchy* distributions.

If a distribution F on $\mathbb{R}^+$ is regularly varying at infinity with index $-\alpha < 0$, then all moments of order $\gamma < \alpha$ are finite, while all moments of order $\gamma > \alpha$ are infinite. The moment of order $\gamma = \alpha$ may be finite or infinite depending on the particular behaviour of the corresponding slowly varying function (see below).

If a function f is regularly varying at infinity with index α, then we have $f(x) = x^\alpha l(x)$ for some slowly varying function l. Hence it follows from the discussion of Sect. 2.4 that, for any positive function h such that $h(x) = o(x)$ as $x \to \infty$, we have $f(x+y) \sim f(x)$ as $x \to \infty$ uniformly in $|y| \le h(x)$; we shall then say that f is $o(x)$*-insensitive*. Similarly we shall say that a distribution F is $o(x)$*-insensitive* if its tail function $\overline{F}$ is $o(x)$-insensitive. Thus distributions which are regularly varying at infinity are $o(x)$-insensitive.

It turns out that integration preserves regular variation of distribution; this result is formulated next and is known as Karamata's theorem for distribution functions.

Theorem 2.45. *A distribution F is regularly varying with index $-\alpha < -1$ if and only if the integrated tail distribution F_I is regularly varying with index $-\alpha+1 < 0$. If any holds, then $\overline{F}(x) \sim (\alpha-1)\overline{F_I}(x)/x$ as $x \to \infty$.*

Proof. Due to monotonicity of $\overline{F}(y)$, for any $c_1 > 1$,

$$\overline{F_I}(x) - \overline{F_I}(c_1 x) = \int_x^{c_1 x} \overline{F}(y)dy \le (c_1 x - x)\overline{F}(x) \tag{2.46}$$

and, for any $c_2 < 1$,

$$\overline{F}_I(c_2 x) - \overline{F_I}(x) \ge (x - c_2 x)\overline{F}(x). \tag{2.47}$$

Assume that F_I is regularly varying with index $-\alpha+1$. Then, from (2.46) and by regular variation of F_I, for any $c_1 > 1$,

$$\liminf_{x\to\infty} \frac{x\overline{F}(x)}{\overline{F_I}(x)} \ge \frac{1-c_1^{1-\alpha}}{c_1 - 1}.$$

Letting $c_1 \downarrow 1$, we get

$$\liminf_{x\to\infty} \frac{x\overline{F}(x)}{\overline{F_I}(x)} \ge \alpha - 1. \tag{2.48}$$

Similarly, from (2.47) we get, for any $c_2 < 1$,

$$\limsup_{x\to\infty} \frac{x\overline{F}(x)}{\overline{F_I}(x)} \le \frac{c_2^{1-\alpha} - 1}{1 - c_2},$$

Letting $c_2 \uparrow 1$, we get

$$\limsup_{x\to\infty} \frac{x\overline{F}(x)}{\overline{F_I}(x)} \le \alpha - 1. \tag{2.49}$$

Then (2.48) and (2.49) lead to $x\overline{F}(x) \sim (\alpha-1)\overline{F_I}(x)$ as $x \to \infty$, which implies regular variation of F with index $-\alpha$.

Assume now that F is a regularly varying distribution with index $-\alpha$. Then, for every $c > 0$,

$$
\begin{aligned}
\overline{F_I}(cx) &= \int_{cx}^{\infty} \overline{F}(y)dy \\
&= c^{-1} \int_{x}^{\infty} \overline{F}(z/c)dz \\
&\sim c^{\alpha-1} \int_{x}^{\infty} \overline{F}(z)dz = c^{\alpha-1}\overline{F_I}(x) \quad \text{as } x \to \infty,
\end{aligned}
$$

which means the regular variation of F_I with index $1-\alpha$. □

Intermediate Regularly Varying Distributions

It turns out that the property of $o(x)$-insensitivity characterises a slightly wider class of distributions than that of distributions whose tails are regularly varying, and we now discuss this.

Definition 2.46. A distribution F on $\mathbb{R}$ is called *intermediate regularly varying* if

$$\lim_{\varepsilon \downarrow 0} \liminf_{x\to\infty} \frac{\overline{F}(x(1+\varepsilon))}{\overline{F}(x)} = 1. \tag{2.50}$$

Any regularly varying distribution is intermediate regularly varying. But the latter class is richer. We provide first a simple example. Take any density function g which is regularly varying at infinity with index $-\alpha < -1$. Then, by Karamata's Theorem, the corresponding distribution G will be regularly varying with index $-\alpha+1<0$. Now consider any density function f such that $c_1 g(x) \le f(x) \le c_2 g(x)$, for some $0 < c_1 < c_2 < \infty$ and for all x. The corresponding distribution F is intermediate regularly varying because

$$F(x, x(1+\varepsilon)] \le c_2 G(x, x(1+\varepsilon)] \quad \text{and} \quad \overline{F}(x) \ge c_1 \overline{G}(x).$$

On the other hand, F is not necessarily a regularly varying distribution. We now have the following characterisation result.

Theorem 2.47. *A distribution F on $\mathbb{R}$ is intermediate regularly varying if and only if, for any positive function h such that $h(x) = o(x)$ as $x \to \infty$,*

$$\overline{F}(x + h(x)) \sim \overline{F}(x), \tag{2.51}$$

i.e. if and only if F is $o(x)$-insensitive.

Proof. It is straightforward that if F is intermediate regularly varying, then it is $o(x)$-insensitive. Hence it only remains to prove the reverse implication. Assume, on the contrary, that this implication fails. Thus let F be a distribution which is $o(x)$-insensitive but which fails to be intermediate regularly varying. The function

$$l(\varepsilon) := \liminf_{x\to\infty} \frac{\overline{F}(x(1+\varepsilon))}{\overline{F}(x)}$$

decreases in $\varepsilon > 0$, due to the monotonicity of $\overline{F}$. Therefore, the failure of (2.50) implies that there exists a positive δ such that $l(\varepsilon) \leq 1 - 2\delta$ for any $\varepsilon > 0$. Hence, for any positive integer n, we can find x_n such that

$$\overline{F}(x_n(1+1/n)) \leq (1-\delta)\overline{F}(x_n)$$

Without loss of generality we may assume the sequence $\{x_n\}$ to be increasing. Now put $h(x) = x/n$ for $x \in [x_n, x_{n+1})$. Then $h(x) = o(x)$ as $x \to \infty$. However,

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\overline{F}(x+h(x))}{\overline{F}(x)} &\leq \liminf_{n\to\infty} \frac{\overline{F}(x_n+h(x_n))}{\overline{F}(x_n)} \\
&= \liminf_{n\to\infty} \frac{\overline{F}(x_n(1+1/n))}{\overline{F}(x_n)} \\
&\leq 1-\delta,
\end{aligned}$$

which contradicts the $o(x)$-insensitivity of F. □

We now give an attractive probabilistic characterisation of intermediate regularly varying distributions.

Theorem 2.48. *A distribution F on $\mathbb{R}$ is intermediate regularly varying if and only if, for any sequence of independent identically distributed random variables ξ_1, ξ_2, ... with finite positive mean,*

$$\frac{\overline{F}(S_n)}{\overline{F}(n\mathbb{E}\xi_1)} \to 1 \quad \textit{as } n \to \infty \tag{2.52}$$

with probability 1*, where $S_n = \xi_1 + \ldots + \xi_n$.*

Proof. We suppose first that F is intermediate regularly varying; let $\xi_1, \xi_2, \ldots$ be any sequence of independent identically distributed random variables with finite positive mean, and, for each n, let $S_n = \xi_1 + \ldots + \xi_n$; we show that then the relation (2.52) holds. Let $a = \mathbb{E}\xi_1$. Fix any $\varepsilon > 0$. It follows from the definition of intermediate regular variation that there is n_0 and a $\delta > 0$ such that

$$\sup_{n\geq n_0}\left|\frac{\overline{F}(n(a\pm\delta))}{\overline{F}(na)} - 1\right| \leq \varepsilon.$$

By the Strong Law of Large Numbers, with probability 1, there exists a random number N such that $|S_n - na| \leq n\delta$ for all $n \geq N$. Then, for $n \geq \max\{N, n_0\}$,

$$\left|\frac{\overline{F}(S_n)}{\overline{F}(na)} - 1\right| \leq \sup_{n\geq n_0}\left|\frac{\overline{F}(n(a\pm\delta))}{\overline{F}(na)} - 1\right| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this implies the convergence (2.52).

We now prove the converse implication. Assume that the distribution F is not intermediate regularly varying. It is sufficient to construct a sequence of independent identically distributed random variables $\xi_1, \xi_2, \ldots$ with mean 1, such that the relation (2.52) fails to hold (where again $S_n = \xi_1 + \ldots + \xi_n$). By Theorem 2.47 F fails to be $o(x)$-insensitive, and so there exists an $\varepsilon > 0$, an increasing sequence n_k and an increasing function h with $h(x) = o(x)$ such that

$$\overline{F}(n_k + h(n_k)) \le (1-\varepsilon)\overline{F}(n_k) \quad \text{for all } k. \tag{2.53}$$

Since $h(x)/x \to 0$, we can choose an increasing subsequence n_{k_m} such that

$$\sum_{m=1}^{\infty} \frac{h(n_{k_m})}{n_{k_m}} < \infty. \tag{2.54}$$

Since h is increasing it follows also that $\sum_{m=1}^{\infty} n_{k_m}^{-1} < \infty$, and so we can define a random variable ξ taking values on $\{1 \pm h(n_{k_m}), m = 1, 2, \ldots\}$ with probabilities

$$\mathbb{P}\{\xi = 1 - h(n_{k_m})\} = \mathbb{P}\{\xi = 1 + h(n_{k_m})\} = c/n_{k_m}$$

(where c is the appropriate normalising constant). It further follows from (2.54) that the random variable ξ has a finite mean; moreover, this mean equals 1. Define the sequence of independent random variables ξ_1, ξ_2, ... to each have the same distribution as ξ. We shall show that

$$\liminf_{m\to\infty} \mathbb{P}\{S_{n_{k_m}} \ge n_{k_m} + h(n_{k_m})\} > 0. \tag{2.55}$$

From this and from (2.53), and since also $\overline{F}$ is non-increasing, it will follow that

$$\liminf_{m\to\infty} \mathbb{P}\{\overline{F}(S_{n_{k_m}}) \le (1-\varepsilon)\overline{F}(n_k)\} > 0,$$

so that (2.52) cannot hold.

To show (2.55), fix m and consider the events

$$A_j = \bigcap_{i \le n_{k_m}, i \ne j} \{\xi_i \ne 1 \pm h(n_{k_m})\}, \qquad j = 1, \ldots n_{k_m}.$$

Then the events $A_j \cap \{\xi_j = 1 + h(n_{k_m})\}$ are disjoint. Therefore,

$$\begin{aligned}
&\mathbb{P}\{S_{n_{k_m}} \ge n_{k_m} + h(n_{k_m})\} \\
&\ge \sum_{j=1}^{n_{k_m}} \mathbb{P}\{S_{n_{k_m}} \ge n_{k_m} + h(n_{k_m}) \,|\, A_j, \xi_j = 1 + h(n_{k_m})\} \mathbb{P}\{A_j, \xi_j = 1 + h(n_{k_m})\} \\
&= n_{k_m} \mathbb{P}\{S_{n_{k_m}} - n_{k_m} \ge h(n_{k_m}) \,|\, A_1, \xi_1 - 1 = h(n_{k_m})\} \mathbb{P}\{A_1\} \mathbb{P}\{\xi_1 = 1 + h(n_{k_m})\} \\
&= c\mathbb{P}\{S_{n_{k_m}} - n_{k_m} \ge h(n_{k_m}) \,|\, A_1, \xi_1 - 1 = h(n_{k_m})\} \mathbb{P}\{A_1\},
\end{aligned}$$

where the final equality follows from the definition of the distribution of ξ_1. Using again the independence of the random variables ξ_i, we have

$$\begin{aligned}\mathbb{P}\{S_{n_{k_m}} - n_{k_m} \geq h(n_{k_m})\,|\,A_1, \xi_1 - 1 = h(n_{k_m})\}\\ = \mathbb{P}\{S_{n_{k_m}} - n_{k_m} - (\xi_1 - 1) \geq 0\,|\,A_1\} \geq 1/2,\end{aligned}$$

where the final inequality follows from the symmetry about 1 of the common distribution of the random variables ξ_i. In addition,

$$\begin{aligned}\mathbb{P}\{A_1\} &= \left(\mathbb{P}\{\xi_i \neq 1 \pm h(n_{k_m})\}\right)^{n_{k_m}-1}\\ &= (1 - 2c/n_{k_m})^{n_{k_m}-1} \to e^{-2c} \quad \text{as } m \to \infty.\end{aligned}$$

We thus finally obtain that

$$\liminf_{m\to\infty} \mathbb{P}\{S_{n_{k_m}} \geq n_{k_m} + h(n_{k_m})\} \geq ce^{-2c}/2,$$

so that (2.55) follows. □

Other Heavy-Tailed Distributions

We proceed now to heavy-tailed distributions with thinner tails. For the *lognormal distribution*, one can take $h(x) = o(x/\ln x)$ in order to have h-insensitivity. For the *Weibull distribution* with parameter $\alpha \in (0,1)$, one can take $h(x) = o(x^{1-\alpha})$.

In many practical situations, the class of so-called *$\sqrt{x}$-insensitive distributions*—those which are h-insensitive for the function $h(x) = x^{1/2}$—is of special interest. Among these are intermediate regularly varying distributions (in particular regularly varying distributions), lognormal distributions and Weibull distributions with shape parameter $\alpha < 1/2$. The reason for interest in this quite broad class is explained by the following theorem, which should be compared with Theorem 2.48.

Theorem 2.49. *For any distribution F on $\mathbb{R}$, the following assertions are equivalent:*

(i) *F is $\sqrt{x}$-insensitive.*

(ii) *For some (any) sequence of independent identically distributed random variables ξ_1, ξ_2, ... with positive mean and with finite positive variance,*

$$\frac{\overline{F}(S_n)}{\overline{F}(n\mathbb{E}\xi_1)} \to 1 \quad \text{as } n \to \infty \tag{2.56}$$

in probability, where $S_n = \xi_1 + \ldots + \xi_n$.

Proof. To show (i)⇒(ii) suppose that the distribution F is $\sqrt{x}$-insensitive and that the independent identically distributed random variables ξ_1, ξ_2, ... have common mean $a > 0$ and finite variance. Fix $\varepsilon > 0$. By the Central Limit Theorem, there exist N and A such that $\mathbb{P}\{|S_n - na| \leq A\sqrt{n}\} \geq 1 - \varepsilon$ for all $n \geq N$. It follows from the definition of $\sqrt{x}$-insensitivity that there is n_0 such that

$$\left| \frac{\overline{F}(na \pm A\sqrt{n}))}{\overline{F}(na)} - 1 \right| \leq \varepsilon \qquad \text{for all } n \geq n_0.$$

Then, for $n \geq \max\{N, n_0\}$,

$$\mathbb{P}\left\{ \left| \frac{\overline{F}(S_n)}{\overline{F}(na)} - 1 \right| \leq \varepsilon \right\} \geq \mathbb{P}\{|S_n - na| \leq A\sqrt{n}\} \geq 1 - \varepsilon,$$

which establishes (2.56).

To show (ii)⇒(i) assume that the independent identically distributed random variables ξ_1, ξ_2, ... have common mean $a > 0$ and finite variance $\sigma^2 > 0$, but that, on the contrary, the distribution F fails to be $\sqrt{x}$-insensitive. Then there exists $\varepsilon > 0$ and an increasing sequence n_k such that, for all k,

$$\overline{F}(n_k a + \sqrt{n_k \sigma^2}) \leq (1 - \varepsilon)\overline{F}(n_k a).$$

Therefore,

$$\mathbb{P}\left\{ \left| \frac{\overline{F}(S_{n_k})}{\overline{F}(n_k a)} - 1 \right| \geq \varepsilon \right\} \geq \mathbb{P}\{S_{n_k} - n_k a \geq \sqrt{n_k \sigma^2}\} \to \int_1^\infty \frac{e^{-u^2/2}}{\sqrt{2\pi}} du > 0,$$

which contradicts (2.56). □

We finish this section by observing that the exponential distribution, while itself light-tailed, is, in an obvious sense, on the boundary of the class of such distributions. We may construct examples of long-tailed (and hence heavy-tailed) distributions on $\mathbb{R}^+$, say, whose tails are, in a sense, arbitrarily close to that of the exponential distribution. For example, the distribution with tail function

$$\overline{F}(x) = e^{-cx/\ln^\alpha x}, \ \alpha > 0, \ c > 0,$$

is very close to the exponential distribution but is still long-tailed; indeed one can take the function h of Lemma 2.19 to be any such that $h(x) = o(\ln^\alpha x)$ as $x \to \infty$. Further, if we replace the logarithmic function by the mth iterated logarithm, we obtain again a long-tailed distribution.

2.9 Comments

The lower bound (2.7) may be found in Chistyakov [13] and in Pakes [44].

Theorem 2.12 was proved by Foss and Korshunov in [27]. For earlier results see Rudin [48] and Rogozin [46]. Some generalisations may be found in the papers [19, 20] by Denisov, Foss and Korshunov.

The class of long-tailed distributions (but not the term itself) was introduced by Chistyakov in [13], in the context of branching processes.

Theorem 2.40 generalises a result of Cline [16] where the case $F_1, F_2, G_1, G_2 \in \mathcal{L}$ was considered.

Corollary 2.42 is well known from Embrechts and Goldie [21].

A comprehensive study of the theory of regularly varying functions may be found in Seneta [50] and in Bingham, Goldie and Teugels [9].

2.10 Problems

2.1. Let distribution F on $\mathbb{R}^+$ has a regularly varying tail with index α, i.e., let $\overline{F}(x) = L(x)/x^\alpha$ where function $L(x)$ is slowly varying at infinity. Prove that:

(i) Any power moment of distribution F of order $\gamma < \alpha$ is finite.
(ii) Any power moment of order $\gamma > \alpha$ is infinite.

Show by examples that the moment of order $\gamma = \alpha$ may either exist or not, depending on the tail behaviour of slowly varying function $L(x)$.

2.2. Let distribution F on $\mathbb{R}^+$ have a regularly varying tail with index $\alpha > 0$. Prove the distribution of $\log \xi$ is light-tailed.

2.3. Let $\xi > 0$ be a random variable. Prove that the distribution of $\log \xi$ is light-tailed if and only if ξ has a finite power moment of order α, for some $\alpha > 0$.

2.4. Let distribution F on $\mathbb{R}^+$ have an infinite moment of order $\gamma > 0$. Prove that F is heavy-tailed.

2.5. Let random variable $\xi \geq 0$ be such that $\mathbb{E}e^{\xi^\alpha} = \infty$ for some $\alpha < 1$. Prove that the distribution of ξ is heavy-tailed.

2.6. Let random variable ξ has

(i) exponential; (ii) normal

distribution. Prove that the distribution of $e^{\alpha\xi}$ is both heavy- and long-tailed, for every $\alpha > 0$.

2.7. *Student's t-distribution.* Assume we do not know the exact formula for its density. By estimating the moments, prove that the distribution of the ratio

$$\frac{\xi}{\sqrt{(\xi_1^2 + \ldots + \xi_n^2)/n}}$$

is heavy-tailed where the independent random variables ξ, ξ_1, ..., ξ_n are sampled from the standard normal distribution. Moreover, prove that this distribution is regularly varying at infinity.

Hint: Show that the denominator has a positive density function in the neighbourhood of zero.

2.8. Let $\eta_1, \ldots, \eta_n$ be n positive random variables (we do not assume their independence, in general). Prove that the distribution of $\eta_1 + \ldots + \eta_n$ is heavy-tailed if and only if the distribution of at least one of the summands is heavy-tailed.

2.9. Let $\xi > 0$ and $\eta > 0$ be two random variables with heavy-tailed distributions. Can the minimum $\min(\xi, \eta)$ have a light-tailed distribution?

2.10. Suppose that $\xi_1, \ldots, \xi_n$ are independent random variables with a common distribution F and that

$$\xi_{(1)} \le \xi_{(2)} \le \ldots \le \xi_{(n)}$$

are the order statistics.

(i) For $k \le n$, prove that the distribution of $\xi_{(k)}$ is heavy-tailed if and only if F is heavy-tailed.
(ii) For $k \le n-1$, prove that the distribution of $\xi_{(k+1)} - \xi_{(k)}$ is heavy-tailed if and only if F is heavy-tailed.
(iii) Based on (ii) and on Problem 8, prove that $\xi_{(k)} - \xi_{(l)}$ has a heavy-tailed distribution if and only if F is heavy-tailed.

2.11. Let ξ and η be two positive independent random variables. Prove that the distribution of $\xi - \eta$ is heavy-tailed if and only if the distribution of ξ is heavy-tailed.

2.12. Let ξ_n, $n = 1, 2, \ldots$, be independent identically distributed random variables on $\mathbb{R}^+$. Let $\nu \ge 1$ be an independent counting random variable. Let both ξ_1 and ν have light-tailed distributions. Prove that the distribution of random sum $\xi_1 + \xi_2 + \ldots + \xi_\nu$ is light-tailed too.

2.13. Let ξ_n, $n = 1, 2, \ldots$, be independent identically distributed random variables on $\mathbb{R}^+$ such that $\mathbb{P}\{\xi_1 > 0\} > 0$. Let $\nu \ge 1$ be an independent counting random variable. Let ν have heavy-tailed distribution. Prove that the distribution of random sum $\xi_1 + \xi_2 + \ldots + \xi_\nu$ is heavy-tailed.

2.14. Find a light-tailed distribution F such that the distribution of the product $\xi_1\xi_2$ is heavy-tailed where ξ_1 and ξ_2 are two independent random variables with distribution F.

2.15. Let non-negative random variable ξ has distribution F. Consider a family of distributions $F_x(B) := \mathbb{P}\{\xi \in x + B | \xi > x\}$, $B \in \mathcal{B}(\mathbb{R}^+)$.

(i) Prove F is long-tailed if and only if $F_x \Rightarrow \infty$, as $x \to \infty$.
(ii) Prove F is h-insensitive if and only if $\xi_x / h(x) \Rightarrow \infty$ as $x \to \infty$ where ξ_x is a random variable with distribution F_x.

2.16. We say that $H(x)$ is a *boundary function* for a long-tailed distribution F if the following condition holds: F is h-insensitive if and only if $h(x) = o(H(x))$ as $x \to \infty$. Find any boundary function for:

(i) A regularly varying distribution with index $\alpha > 0$.
(ii) A standard log-normal distribution.
(iii) A Weibull distribution with tail $\overline{F}(x) = e^{-x^\beta}$ where $0 < \beta < 1$.
(iv) A distribution with tail $\overline{F}(x) = e^{-x/\log(1+x)}$, $x \ge 0$.

2.17. Prove that a distribution whose tail is slowly varying at infinity does not have a boundary function.

2.18. Let a random variable ξ has the standard normal distribution.

(i) Find all values of $\alpha > 0$ such that the power $|\xi|^\alpha$ has a heavy-tailed distribution.
(ii) Prove the power $|\xi|^\alpha$ has a heavy- and long-tailed distribution for every $\alpha < 0$.

2.19. Let independent random variables $\xi_1, \dots, \xi_n$ have the standard normal distribution. Find all n such that the product $\xi_1 \cdot \ldots \cdot \xi_n$ is heavy-tailed. For those n, is the product also long-tailed?

2.20. Let independent non-negative random variables $\xi_1, \dots, \xi_n$ have Weibull distribution with the tail $\overline{F}(x) = e^{-x^\beta}$, $\beta > 0$. Find the values of β, for which the product $\xi_1 \cdot \ldots \cdot \xi_n$ has a heavy-tailed distribution.

2.21. *Perpetuity*. Suppose $\xi_1, \xi_2, \ldots$ are independent identically distributed random variables with common uniform distribution in the interval $[-2, 1]$. Let $S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$ and

$$Z = \sum_{n=0}^{\infty} e^{S_n}.$$

(i) Prove Z is finite with probability 1 and that Z has a heavy-tailed distribution. *Hint*: Show that $\mathbb{E}Z^\gamma = \infty$ for some $\gamma > 0$.
(ii) How can the result of (i) be generalised to other distributions of ξ's?

2.22. Let F and G be two distributions on $\mathbb{R}^+$ with finite means a_F and a_G. Prove that, for all sufficiently large x,

$$\begin{aligned}\overline{(F*G)_I}(x) &= \overline{F_I}(x) + \overline{F*G_I}(x) + (a_G - 1)\overline{F}(x)\\ &= \overline{G_I}(x) + \overline{F_I*G}(x) + (a_F - 1)\overline{G}(x).\end{aligned}$$

2.23. Prove the distribution of a random variable $\xi \ge 0$ is $\sqrt{x}$-insensitive if and only if the distribution of $\sqrt{\xi}$ is long-tailed. More generally, prove that the distribution of $\xi \ge 0$ is x^α-insensitive with some $0 < \alpha < 1$ if and only if the distribution of $\xi^{1-\alpha}$ is long-tailed.

2.24. Suppose X_n is a time-homogeneous Markov chain with state space $\mathbb{Z}^+$ and transition probabilities p_{ij}. Let X_n be a skip-free Markov chain, i.e., only the transition probabilities $p_{i,i-1}$, $p_{i,i}$ and $p_{i,i+1}$ are non-zero. Let X_n be positive recurrent with invariant probabilities π_i. Show that

$$\pi_i = \prod_{j=1}^{i} \frac{p_{j-1,j}}{p_{j,j-1}}.$$

Further, assume that $\limsup_{i\to\infty} p_{ii} < 1$.

(i) Prove if $\limsup_{i\to\infty}(p_{i,i+1} - p_{i,i-1}) < 0$, then the invariant distribution is light-tailed.
(ii) Prove if $\limsup_{i\to\infty}(p_{i,i+1} - p_{i,i-1}) = 0$, then the invariant distribution is heavy-tailed.

2.25. Suppose X_n is a time-homogeneous irreducible aperiodic Markov chain with state space $\mathbb{Z}^+$. Let X_n be positive recurrent. Let there exist a state i_0 such that the distribution of the jump from this state is heavy-tailed, i.e.,

$$\mathbb{E}\{e^{\lambda X_1} \mid X_0 = i_0\} = \infty,$$

for every $\lambda > 0$. Prove that the invariant distribution of the Markov chain is also heavy-tailed.

2.26. *Excess process*. Suppose X_n is a time-homogeneous irreducible non-periodic Markov chain with state space $\{1,2,3,\ldots\}$. Let $\mathbb{P}\{X_1 = i-1 \mid X_0 = i\} = 1$ for every $i \ge 2$. Denote by F the distribution of the jump from the state 1, i.e., for every $i \in \mathbb{Z}^+$,

$$F\{i\} = \mathbb{P}\{X_1 = i+1 \mid X_0 = 1\}.$$

(i) Prove this Markov chain is positive recurrent if and only if F has finite mean. Find the corresponding invariant distribution.
(ii) Prove the invariant distribution of this chain is heavy-tailed if and only if F is heavy-tailed.
(iii) Prove the invariant distribution of this chain is long-tailed if and only if the integrated tail distribution F_I is long-tailed.

2.27. Suppose X_n is a time-homogeneous irreducible aperiodic Markov chain with state space $\{1,2,3,\ldots\}$. Let there exist $i_0 \ge 1$ such that $\mathbb{P}\{X_1 = i-1 \mid X_0 = i\} = 1$ for every $i \ge i_0+1$. Denote by F_i the distribution of the jump from the state i, $i \in \{1,\ldots,i_0\}$, i.e., for every $j \in \mathbb{Z}^+$,

$$F_i\{j\} = \mathbb{P}\{X_1 = j+i \mid X_0 = i\}.$$

(i) Prove this Markov chain is positive recurrent if and only if all F_i, $i \in \{1,\ldots,i_0\}$, have finite mean. Prove that the invariant probabilities $\{\pi_j\}$ satisfy the equations, for $j \ge i_0+1$,

$$\pi_j = \sum_{i=1}^{i_0} \pi_i F_i[j-i,\infty).$$

(ii) Prove the invariant distribution of this chain is heavy-tailed if and only if at least one of F_i, $i \in \{1,\ldots,i_0\}$, is heavy-tailed.
(iii) Prove the invariant distribution of this chain is long-tailed if all the integrated tail distributions $F_{i,I}$ are long-tailed.

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