

Chapter 2

Mathematical Tools for Modeling Social Complex Systems

Abstract This chapter deals with the derivation of mathematical structures suitable for constructing models of phenomena of interest in social sciences. The reference framework is the approach of the Kinetic Theory for Active Particles (KTAP), which uses distribution functions over the microscopic states of the individuals composing the system under consideration. Modeling includes: the strategic behavior of active particles from a stochastic game perspective; a Darwinian-like evolution of the particles, which learn from past experience and evolve their strategy in time; and hints about small-network dynamics, in particular particle interactions within and among the nodes of the network. A critical analysis is finally proposed in order to assess the consistency of the mathematical tools with the main features of complexity.

2.1 Introduction

This chapter presents the concepts underlying the KTAP approach, which has been selected as the mathematical tool for deriving specific models of socio-economic phenomena. The contents of the chapter focus, in particular, on the assumptions needed for interpreting complexity and deriving consistent mathematical structures, with the aim of reducing both the actual complexity of the real world systems and the technical ones of its mathematical description.

The KTAP approach was originally proposed to model biological systems, in particular competition between immune systems and pathogens [44]. Next it was revisited in the framework of social cooperation and competition dynamics [48, 49], and afterward extended to include more advanced modeling structures [6–8]. However, the approach requires further improvements in order to match the complexity features discussed in Chap. 1. In particular, we recall nonlinear interactions and network topology, the latter also implying communication and possibly transitions from node to node of the network. The literature on this topic is constantly growing, as documented in [10, 13, 28, 74, 148, 158], among other sources.

Additionally, it is worth mentioning that the assumption of a constant total number of individuals, which usually is tacitly made in the modeling of socio-economic systems, may be valid only for short time periods. In the long run, birth and death processes, as well as inlets from the outer environment, have to be taken into account, which requires a further evolution of modeling structures.

Before tackling technical issues it is worth discussing in some detail the proper role played by *mathematical structures* in the derivation of particular models. It is known [119] that the modeling of living systems cannot take advantage of field theories, as in the case of inert matter. This concept was well presented in the celebrated book by Schrödinger [141]. Therefore heuristic approaches are generally adopted, relying mainly on personal intuitions of the modelers. A more rigorous approach can be developed by grounding models on the preliminary derivation of abstract mathematical structures consistent with the complexity features presented in Chap. 1, which, as a matter of fact, can serve as mathematical/theoretical guidelines.

The advantage of such an approach over the various heuristic strategies discussed in the literature is that if, at some point, a sound field theory becomes available then it can be implemented, at the proper level, in the mathematical structures, thereby guiding the deduction of more targeted models. In other words, the mathematical structures provide modelers with a rigorous conceptual framework virtually independent of (overly) specific heuristic intuitions. As such, they act as a mathematical theory for the construction of models, which can receive (with due control) external phenomenological insights while not being forced to pursue them.

In general, a modeling approach aiming at identifying proper background mathematical structures for complex systems (such as those treated in this monograph) should cope with the following issues:

- Understanding the links between system dynamics and their complexity features.
- Deriving a general mathematical structure that offers a conceptual framework for the derivation of specific models.
- Designing specific models, corresponding to specific classes of systems, by complementing the mathematical structure with suitable models of individual-based interactions according to a detailed interpretation of the dynamics at the microscale.
- Validating models by comparison of their predictions with empirical data.

Apparently, these are standard issues in all fields of applied mathematics. Nevertheless, it is worth pointing out that in the case of inert matter a background field theory is available. For instance, Newtonian mechanics establishes balance equations for mass, momentum, and energy. When physical conditions reveal an inadequacy of classical mechanical laws, more refined theories, such as relativistic and quantum mechanics, can improve the aforesaid field theories thereby contributing to more targeted mathematical models.

More substantial differences arise when dealing with living matter, for no background theory that generally supports the derivation of models exists yet. Moreover, the great heterogeneity that characterizes living systems induces stochastic features

that cannot be neglected, nor simply replaced by noise. Rather, they should be inserted in the mathematical structures as hallmarks of the modeling approach.

In this context, the so-called Kinetic Theory for Active Particles (KTAP) has been developed in the last decade to model large systems of interacting entities. Generally they are *living* entities; hence they are termed *active* particles. The conceptual guidelines that inspire the KTAP approach can be listed in detail as follows:

- The system is separated into *functional subsystems* constituted by active particles featuring a microstate called *activity*, which is collectively expressed in each subsystem. It may represent a behavioral characteristic of the particles to be selected according to the specific microscopic dynamics considered in the system, and particularly within that subsystem.
- The state of each functional subsystem is expressed by a time-dependent distribution function over the microscopic states of the active particles.
- The time evolution of the distribution function is triggered by *microscopic interactions* of active particles within and among subsystems. Interactions are modeled by *stochastic games* whose payoffs are probabilistic.
- An equation for the time evolution of the distribution function is obtained by a balance of particles within elementary volumes of the space of microstates, inflow and outflow dynamics of particles being related to interactions at the microscopic scale.

Readers who are more interested in applications should be patient with the cold language of equations characterizing this chapter. Applications will appear soon, starting with the next chapter. On the other hand, as the discussion above should have highlighted, this preliminary path through mathematical tools is a necessary step toward a more rigorous approach to the interplay between mathematical sciences and living systems [40].

The ideas just outlined are organized in the chapter's following sections. In detail, Sect. 2.2 deals with introductory issues concerning the representation of socio-economic systems, which possibly also apply to the inner structure of each node of a small social network. Section 2.3 then yields the mathematical structures at the core of the KTAP approach, focusing on *closed* systems; i.e., systems that do not interact with the outer environment. Subsequently, the structures are extended in Sect. 2.4 to *open* systems, confining attention to the modeling of known external actions. Section 2.5 concerns a technical modification of the mathematical structures, however relevant for modeling purposes, oriented to systems with *discrete* microscopic states (viz., activity). Section 2.6 proposes a general analysis and various hints toward the modeling of interactions at the microscale. The contents also include detailed insight into different sources of nonlinearity that might characterize the interaction dynamics. Section 2.7 outlines some general ideas about the search for solutions to the mathematical models previously derived, also from an approximate numerical point of view, and shows how numerical solutions can provide useful feedback for a deeper understanding of the phenomenology of complex systems. Finally, Sect. 2.8 presents a critical analysis centered around both the ability of the mathematical tools to capture the complexity features presented in Chap. 1 and the general problem of reducing and handling the real world complexity via mathematical approaches.

2.2 Complexity Reduction and Mathematical Representation

As already mentioned, socio-economic systems are understood as ensembles of interacting active particles, which can be either individuals or aggregates clustered by common organization and aims and are, in any case, the atomic (viz., minimal) entities of the system. Their microscopic state (the activity) which also identifies the behavioral strategy to be used in game-type interactions, will be denoted by a variable u belonging to a domain D_u .

Let us consider, at first, particles at a specific node of a social network. That is, we focus on the inner representation of a single node, without explicitly considering links with other nodes. Therefore, in the following discussion, expressions such as “the (global) system” will refer to the node as the universe. In general, it is reasonable to assume that the system is composed of different types of active particles, which express different activities. Aiming at a complexity reduction, it is then convenient to decompose the system into *functional subsystems* composed of active particles that collectively express the same activity.

In principle, the activity therefore identifies a different microscopic feature of active particles in each subsystem, although it is always denoted by the same letter u . Nevertheless, the above decomposition can also be applied if the activity is the same but each subsystem has a different way to express it.

As we will see in the following chapters, u can usually be considered to be a scalar variable. As an example, it may be (an indicator of) the wealth of the active particles. For this reason, it is convenient to think of its domain as a subset of the real line; $D_u \subseteq \mathbb{R}$. More precisely, if we agree to use the letter p to label the various functional subsystems, then in view of the above discussion, we need to distinguish among different domains $D_u^p \subseteq \mathbb{R}$, where p runs from 1 to the total number of functional subsystems, say $m \geq 1$.

The expression of the activity is, in general, heterogeneously distributed within the functional subsystems. Accordingly, the large scale (viz., collective) representation of each of the latter is provided by a *distribution function*:

$$f^p = f^p(t, u) : [0, T_{\max}] \times D_u^p \rightarrow [0, +\infty), \quad p = 1, \dots, m,$$

$T_{\max} > 0$ being a certain final time, possibly $+\infty$. The meaning of the distribution function is that $f^p(t, u) du$ is the (infinitesimal) number of active particles of the p -th subsystem which, at time t , are expressing an activity located in the infinitesimal volume du centered at u . Under suitable integrability assumptions, the number of active particles of the p -th subsystem at time t is therefore given by:

$$N^p[f^p](t) := \int_{D_u^p} f^p(t, u) du,$$

where square brackets are used, henceforth throughout the monograph, when it is necessary to stress that there is a functional dependence of some quantities on the distribution function f^p (here, for instance, N^p is a linear operator over f^p).

If the N^p 's are constant in time then each f^p can be normalized with respect to the corresponding N^p at $t = 0$ and understood as a probability density:

$$\int_{D_u^p} f^p(t, u) du = 1, \quad \forall p = 1, \dots, m, \quad \forall t \in [0, T_{\max}].$$

In this case, ℓ -th order moments of the probability distribution f^p can be defined as follows:

$$\mathbb{E}_\ell^p[f^p](t) := \int_{D_u^p} u^\ell f^p(t, u) du, \quad \ell = 0, 1, 2, \dots;$$

notice in particular that $N^p(t) = \mathbb{E}_0^p(t)$.

The variance of the distribution is:

$$\text{Var}^p[f^p](t) := \int_{D_u^p} |u - \mathbb{E}_1^p[f^p](t)|^2 f^p(t, u) du,$$

which provides a measure of the local microscopic oscillations of the system with respect to an average macroscopic description.

Alternatively, if particle transitions among functional subsystems occur, then each f^p is no longer a probability distribution. Nonetheless, due to the hypothesis that the system is closed, which entails that birth/death processes are disregarded, we have:

$$\sum_{p=1}^m \int_{D_u^p} f^p(t, u) du = \text{constant in } t,$$

hence the normalization can be performed with respect to the total number of active particles of the system. The expression for the moments of each distribution function is technically modified as follows:

$$\mathbb{E}_\ell^p[f^p](t) := \frac{1}{N^p[f^p](t)} \int_{D_u^p} u^\ell f^p(t, u) du,$$

which also applies when N^p varies in time because of birth/death events within the p -th subsystem.

It is worth mentioning that this representation does not include any variable related to space among those charged to describe the microscopic state of the active particles. This is because we consider either spatially homogeneous systems or systems in which active particles communicate independently of their localization; e.g., via media. However, when a node is embedded in a social network structure of, say, $M \geq 2$ nodes, additional notation is needed for representing the global system, which now coincides with the network. For instance, the distribution functions of each node can be labeled with two indices:

$$f^{pq} = f^{pq}(t, u) : [0, T_{\max}] \times D_u^{pq} \rightarrow [0, +\infty),$$

where $q = 1, \dots, M$ identifies the node. In principle, the number of functional subsystems may vary from node to node: $m = m(q)$. Also notice that the domain of the activity has been doubly labeled, in order to account for the fact that the variable u may identify different microscopic characteristics of the active particles in each node and each subsystem of a node.

At this point, it is useful to consider a few examples which show how the above representation works in practice. The same examples will be used later on to show the application of the approach at a more practical level.

Example 2.1 (Secessionist trends rising up on a regional basis in a given country [7, 8]). Proceeding from the outermost to the innermost level of description:

- the country can be viewed as a network of M regions;
- each region can be understood as a node q of the network;
- within each region, two functional subsystems $p = 1, 2$ can be identified, corresponding to individuals (viz., active particles) in favor of secession, or against secession, respectively;
- the activity u of the former individuals can represent their inclination for secession, and can be conventionally assumed to be positive. Hence $D_u^{1q} = [0, +\infty)$ for all $q = 1, \dots, M$;
- the activity, still denoted by u , of the latter individuals can represent instead their aversion to secession, and can be conventionally assumed to be negative. Hence $D_u^{2q} = (-\infty, 0]$ for all $q = 1, \dots, M$.

It can be questioned whether partitioning each node into two functional subsystems is really useful. In fact, a single distribution function, defined over $D_u = \mathbb{R}$, may be sufficient to capture the population opinion in each region, by agreeing that the more positive (respectively, negative) the value of u , the more in favor of (respectively, against) secession the population is. However, this simplification cannot be adopted as a general rule, because the detail of representation needed also depends on the specific interaction rules that are established within and among the nodes.

Example 2.2 (Process of democratization of a dictatorship in a given country [3, 11, 146]). In this case, a possible representation of the system is as follows:

- the country itself can be viewed as a network of $M = 4$ components of the society;
- the components of the society, namely the dictator ($q = 1$), the ministers ($q = 2$), the parliament ($q = 3$), and the citizens ($q = 4$), are the nodes of the network;
- each node is not further subdivided into functional subsystems; thus, for each of them, a single distribution function defined over $D_u = \mathbb{R}$ is sufficient to describe the trends of the corresponding component of the society. In particular, it can be decided that $u > 0$ corresponds to support for the dictatorship, whereas $u < 0$ corresponds to dissent.

Different representations are possible, some of which are actually equivalent to the proposed one. For instance, the country can also be viewed as an isolated node and the various social components as its functional subsystems, thereby not emphasizing the network structure.

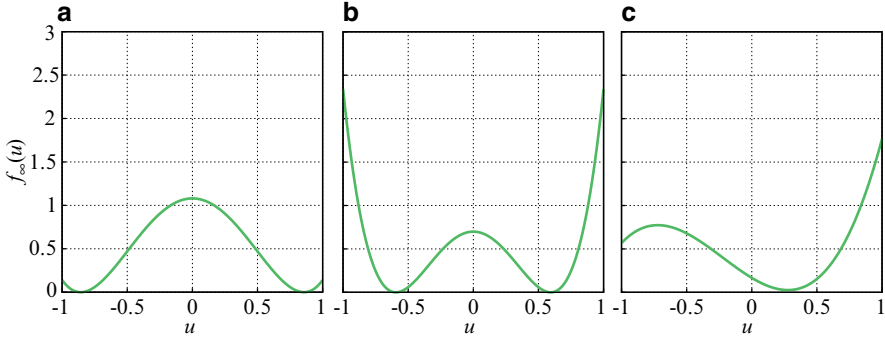


Fig. 2.1 Wealth distribution functions corresponding to various possible social profiles. **(a)** Society with a balanced welfare where middle classes (about $u = 0$) dominate. **(b)** Society strongly radicalized into poor (about $u = -1$) and wealthy (about $u = 1$) classes. **(c)** Asymmetric social profile indicating that most of the population is distributed in the poor classes ($-1 \leq u < 0$) with however a non-negligible presence of very wealthy ones (about $u = 1$)

It is worth stressing that the use of a distribution function over the activity variable improves the representation of the global state of the system with respect to a purely deterministic average representation. In fact, the use of random dependent variables not only allows one to compute moments, but also to detail more precisely how active particles are distributed over the possible microscopic states. For instance, in problems in which the activity represents the wealth status of the particles, such a representation can provide both mean and variance, as deterministic representations might also do, and also additional descriptions such as clustering on certain social states.

Let us be more precise about this issue. Looking ahead at the contents of the next chapter, let us consider dynamics of wealth redistribution assuming that $u \in D_u = [-1, 1]$ is the wealth variable. Negative values close to -1 correspond to poor social classes, and positive values close to 1 correspond to wealthy social classes. Middle classes are located about $u = 0$. Moreover, let us suppose that a certain mathematical model can provide the asymptotic distribution function $f_{\infty} = f_{\infty}(u)$ representing the stationary wealth distribution reached for large times (ideally, $t \rightarrow +\infty$). Several configurations are possible, such as those shown in Fig. 2.1a, b. Both distributions are symmetric with respect to $u = 0$, and thus they have the same zero mean. However, the former depicts a society with a significant presence of the middle class whereas the latter depicts a society radicalized into poor and wealthy classes, with a very limited presence of middle classes. Such a detail would not be caught by purely average deterministic representations.

The symmetry of these asymptotic distributions can be broken, for instance, by either political choices (to be regarded as external actions on the system) or by asymmetric initial conditions; see Fig. 2.1c.

2.3 Mathematical Structures Toward Modeling

In this section we consider the derivation of the evolution equations for the distribution functions introduced in Sect. 2.2. From this section on, to fix the ideas, we disregard a possible network structure of the system; i.e., we concentrate on the equations valid for each isolated node. The extension to networks is worth being developed with reference to specific applications, which can allow the cumbersome notation required in the general case to be conveniently reduced. We feel confident that readers will be able to draw inspiration from the contents of the following sections to also model interactions among active particles belonging to different interconnected nodes.

As already stressed, the approach is based on the assumption that the evolution of the activity distribution depends on interactions among the active particles taking place at the microscopic scale, the effects of which are specified probabilistically (*stochastic games*). The evolution equation for the distribution function of the p -th functional subsystem is obtained by a balance of particles playing games in the space of microscopic states. It can be expressed in the following general form:

$$\frac{\partial f^p}{\partial t} = J^p[\mathbf{f}], \quad (2.1)$$

where $\mathbf{f} = (f^1, \dots, f^m)$ and J^p is the p -th *interaction operator*, which depends in principle on all distribution functions because interactions can involve particles of both the same and of different functional subsystems.

The operator J^p will include, in general, both *conservative* and *non-conservative* interactions. The former do not change the total number of particles of the system, whereas the latter account for possible birth/death of particles in the various subsystems. Accordingly:

$$J^p[\mathbf{f}](t, u) = C^p[\mathbf{f}](t, u) + P^p[\mathbf{f}](t, u), \quad (2.2)$$

C standing for *conservative* and P for *proliferative* (agreeing that one may also have *negative* proliferation). Let us now detail the two types of interactions.

2.3.1 Conservative Interactions

Roughly speaking, Eq. (2.1) describes the evolution in time of the distribution of the microscopic state u . Reasoning microscopically, imagine freezing u and asking for the net number of active particles per unit time that reach state u . Such a number is determined by two concomitant facts:

- It is *increased* whenever an active particle, approaching a game-type interaction with a strategy different from u , changes the latter to u in consequence of the interaction.

- It is *decreased* whenever an active particle, approaching a game-type interaction with strategy u , obtains as a payoff a new strategy different from u .

Summarizing, the conservative operator C^p includes both a *gain* and a *loss* contribution of active particles with state u . For this reason, it is technically written as:

$$\begin{aligned}
 C^p[\mathbf{f}](t, u) = & \sum_{p_*, p^*=1}^m \int_{D_{u^*}^{p_*} \times D_u^{p^*}} \eta^{p_* p^*}(u_*, u^*) \mathcal{C}^{p_* p^*}(u_* \rightarrow u; p | u_*, u^*) \\
 & \times f^{p_*}(t, u_*) f^{p^*}(t, u^*) du_* du^* \\
 & - f^p(t, u) \sum_{p^*=1}^m \int_{D_u^{p^*}} \eta^{p p^*}(u, u^*) f^{p^*}(t, u^*) du^*, \quad (2.3)
 \end{aligned}$$

where:

- $\eta^{p_* p^*}(u_*, u^*)$ is the *interaction rate*, namely the frequency of the interactions (or encounters) among particles of the p_* -th subsystem with state $u_* \in D_{u^*}^{p_*}$ and particles of the p^* -th subsystem with state $u^* \in D_u^{p^*}$.
- $\mathcal{C}^{p_* p^*}(u_* \rightarrow u; p | u_*, u^*)$ is the probability density that a particle of the p_* -th subsystem, playing a game with strategy $u_* \in D_{u^*}^{p_*}$ against a particle of the p^* -th subsystem with strategy u^* , shifts to the p -th subsystem getting simultaneously the new strategy $u \in D_u^p$. The collection $\{\mathcal{C}^{p_* p^*}\}$ of all such transition probabilities is called the *table of games*.
- The following probability density property holds true:

$$\sum_{p=1}^m \int_{D_u^p} \mathcal{C}^{p_* p^*}(u_* \rightarrow u; p | u_*, u^*) du = 1, \quad \begin{aligned} & \forall u_* \in D_{u^*}^{p_*}, u^* \in D_u^{p^*} \\ & \forall p_*, p^* = 1, \dots, m, \end{aligned} \quad (2.4)$$

which basically means that every game event produces a payoff within the set D_u^p of admissible strategies for the subsystem p .

The structure (2.3) of the conservative interaction operator takes into account possible transitions of particles across the subsystems. In the particular case that they are not allowed, the same expression of the operator is still formally valid with:

$$\mathcal{C}^{p_* p^*}(u_* \rightarrow u; p | u_*, u^*) = \mathcal{C}^{p p^*}(u_* \rightarrow u | u_*, u^*) \delta_{p_* p},$$

where: $\mathcal{C}^{p p^*}(u_* \rightarrow u | u_*, u^*)$ is now the probability that a particle already belonging to the target subsystem p , with pre-interaction strategy $u_* \in D_{u^*}^{p^*}$, changes its state to $u \in D_u^p$ because of an interaction with a particle of the p^* -th subsystem playing with strategy $u^* \in D_u^{p^*}$; and $\delta_{p_* p} = 1$ if $p_* = p$, $\delta_{p_* p} = 0$ otherwise is the Kronecker's delta.

A straightforward calculation shows that property (2.4) is the one ensuring the conservation of the total number of particles of the system under the interactions described by the operators C^p . Indeed, it follows from Eq. (2.3) that:

$$\sum_{p=1}^m \int_{D_u^p} C^p[\mathbf{f}](t, u) du = 0, \quad \forall t \in [0, T_{\max}],$$

whence Eq. (2.1), in the absence of the operator P^p in J^p , yields:

$$\frac{d}{dt} \sum_{p=1}^m \int_{D_u^p} f^p(t, u) du = \frac{d}{dt} \sum_{p=1}^m N^p(t) = 0.$$

If, moreover, particle transitions among subsystems are not possible, then a similar calculation shows that the operators C^p keep the number of particles within each subsystem constant.

Before concluding the discussion about conservative interactions, it is worth recalling the following terminology, which is very customary in the KTAP approach:

- A particle possessing the target state u is called a *test* particle, and is taken to be representative of a generic entity of the system.
- A particle with pre-interaction strategy u_* , which may obtain the target state after a game event, is called a *candidate* particle.
- A particle with strategy u^* , which triggers the interaction, is called a *field* particle.

Hence the structure of conservative interactions can be rephrased by saying that a candidate particle can obtain, with a certain probability, the state of a test particle, whereas the latter can lose it with a certain probability, because of stochastic games with field particles.

2.3.2 Non-conservative Interactions

The non-conservative operator P^p of the p -th subsystem is written following reasoning formally similar to that which led to the gain term of the conservative operator:

$$\begin{aligned} P^p[\mathbf{f}](t, u) = & \sum_{p_*, p^*=1}^m \int_{D_{u_*}^{p_*} \times D_{u^*}^{p^*}} \eta^{p_* p^*}(u_*, u^*) \mathcal{P}^{p_* p^*}(u_*, u^*; u, p) \\ & \times f^{p_*}(t, u_*) f^{p^*}(t, u^*) du_* du^*, \end{aligned} \quad (2.5)$$

where:

- $\mathcal{P}^{p_* p^*}(u_*, u^*; u, p)$ is the net *birth/death rate* of a test particle in the p -th subsystem due to an interaction between a candidate particle in the p_* -th subsystem and a field particle in the p^* -th subsystem. In particular, a birth event occurs when $\mathcal{P}^{p_* p^*}(u_*, u^*; u, p) > 0$ while a death event occurs when $\mathcal{P}^{p_* p^*}(u_*, u^*; u, p) < 0$

The birth/death rate is required to satisfy

$$\mathcal{P}^{p_* p^*}(u_*, u^*; u, p) \geq 0 \quad \forall p_* \neq p, \forall u_* \neq u,$$

so that death events can at most take place in the same functional subsystem and state of the candidate particle.

The framework depicted by Eqs. (2.1)–(2.3), and (2.5) offers the mathematical environment for the derivation of models of socio-economic systems. Specific ones are obtained by particularization of the terms η , \mathcal{C} , and \mathcal{P} , which describe particle dynamics at the microscopic scale. This is indeed the purpose of the following chapters, which will further clarify the concepts just introduced by referring specifically to the modeling of welfare distribution dynamics.

Before concluding this section, we further discuss the table of games \mathcal{C} . It is worth stressing that this term of the equations models the game that active particles play at each interaction. As observed in [66], the table of games can be inspired by classical game theory applied to socio-economic systems [122] but it can also include more recent developments addressing evolutionary features of games [79, 88, 100, 123, 126, 138]. In addition, it can account for qualitative and quantitative differences of game rules in different functional subsystems.

A final definition, heuristically anticipated in Chap. 1, which can now be made more precise, is that of linearly vs. nonlinearly additive interactions concerning the terms η , \mathcal{C} , and \mathcal{P} . As Eqs. (2.3), (2.5) clearly show, pairwise interaction events among candidate and field particles add up to give rise to a variation of the distribution of active particles over the values of the activity. This is indeed the meaning of the integrations over $D_u^{p_*} \times D_u^{p^*}$ in the previous equations. However, if the interaction rate, the table of games, and the birth/death rates depend only on the pre-interaction states of the particles, so that each interaction event is not affected by the presence of field particles other than the one the candidate particle is interacting with, then the final output is the linear superposition of the actions individually applied by the field particles. In this case, hence, interactions are said to be *linearly additive*.

If, on the contrary, each interaction event also depends on field particles other than the one the candidate is interacting with, possibly in an aggregate manner, then the final output is not the linear superposition of the individual actions applied by the field particles. Interactions are therefore said to be *nonlinearly additive*. In practice, in the nonlinearly additive case the terms η , \mathcal{C} , and \mathcal{P} do not only depend on the state of the interacting pairs but also on the distribution functions. Recent papers [39] suggest, for instance, using low order moments. This amounts to assuming that each individual does not only feel the state of the interacting companion but also an aggregate effect induced by suitable moments of the particle distribution. This topic will be further discussed in Sect. 2.6.

2.4 Open Systems

We now generalize the mathematical structures presented in Sect. 2.3 to the case of open systems; that is, systems whose particles interact also with the outer environment. More specifically, the latter are understood as members of an additional category of active particles, called *field agents*, that express an activity $w \in D_w \subseteq \mathbb{R}$ by which they can affect the expression of the activity u in the inner system. For the sake of generality, it is convenient to assume that field agents are also organized in a certain number $A \geq 1$ of functional subsystems labeled by $\alpha = 1, \dots, A$. The activity distribution of the field agents in the α -th functional subsystem is given by the distribution function:

$$g^\alpha = g^\alpha(t, w) : [0, T_{\max}] \times D_w^\alpha \rightarrow [0, +\infty), \quad (2.6)$$

which can be thought of as normalized to a probability density if the number of field agents does not change in time. In the following we assume that the g^α 's are known functions, which means that the state of the outer system is prescribed *a priori*.

When applying their action on the inner system, field agents are assumed to trigger conservative interactions with active particles. Consequently, the evolution equation (2.1) for the f^p 's can be rewritten as follows:

$$\frac{\partial f^p}{\partial t} = J^p[\mathbf{f}] + Q^p[\mathbf{f}, \mathbf{g}], \quad (2.7)$$

where $\mathbf{g} = (g^1, \dots, g^A)$ and Q^p is the p -th *inner–outer conservative operator*. The formal expression of the latter recalls that of the conservative component C^p of J^p :

$$\begin{aligned} Q^p[\mathbf{f}, \mathbf{g}](t, u) = & \sum_{\alpha=1}^A \sum_{p_*=1}^m \int_{D_u^{p_*} \times D_w^\alpha} \hat{\eta}^{p_*\alpha}(u_*, w) \mathcal{Q}^{p_*\alpha}(u_* \rightarrow u; p|u_*, w) \\ & \times f^{p_*}(t, u_*) g^\alpha(t, w) du_* dw \\ & - f^p(t, u) \sum_{\alpha=1}^A \int_{D_w^\alpha} \hat{\eta}^{p\alpha}(u, w) g^\alpha(t, w) dw, \end{aligned} \quad (2.8)$$

where:

- $\hat{\eta}^{p_*\alpha}(u_*, w)$ is the *inner–outer interaction rate*; that is, the frequency at which candidate particles of the inner system interact with field agents of the outer environment.
- $\mathcal{Q}^{p_*\alpha}(u_* \rightarrow u; p|u_*, w)$ is the probability that candidate particles get the test state, shifting simultaneously to the p -th test functional subsystem, upon playing games with field agents. The collection $\{\mathcal{Q}^{p_*\alpha}\}$ of all such transition probabilities is the

inner–outer table of games, which is required to satisfy the probability density property:

$$\sum_{p=1}^m \int_{D_u^p} \mathcal{Q}^{p* \alpha}(u_* \rightarrow u; p|u_*, w) du = 1, \quad \begin{aligned} &\forall u_* \in D_u^{p*}, w \in D_w^\alpha \\ &\forall p_* = 1, \dots, m, \\ &\forall \alpha = 1, \dots, A, \end{aligned}$$

whence:

$$\sum_{p=1}^m \int_{D_u^p} \mathcal{Q}^p[\mathbf{f}, \mathbf{g}](t, u) du = 0, \quad \forall t \in [0, T_{\max}],$$

which expresses conservation of inner–outer interactions. Generalizing to the non-conservative case is straightforward, based on properties of the operator P^p discussed earlier; hence this is left as an exercise for the reader.

The mathematical framework (2.7) and (2.8) is derived under the assumption that the action of the outer environment on the inner system is applied through stochastic games. However, deterministic actions applied directly to the test particle can also be considered, which produce a stream effect in the space of the microscopic states. The mathematical translation of this idea is a conservative advection term in the evolution equation for f^p :

$$\frac{\partial f^p}{\partial t} + \frac{\partial}{\partial u}(K^p[\mathbf{g}]f^p) = J^p[\mathbf{f}], \quad (2.9)$$

where $K^p[\mathbf{g}]$ is the global microscopic action applied by field agents on the test particle. Usually, it takes the form of a *mean field action*:

$$K^p[\mathbf{g}](t, u) = \sum_{\alpha=1}^A \int_{D_w^\alpha} \mathcal{K}^{p\alpha}(u, w) g^\alpha(t, w) dw, \quad (2.10)$$

$\mathcal{K}^{p\alpha} : D_u^p \times D_w^\alpha \rightarrow \mathbb{R}$ being the *interaction kernel* for pairs of test particles and field agents belonging to the p -th and α -th inner and outer functional subsystem, respectively. Notice that, in this case, individuals do not adopt any stochastic game strategy for deciding the output of their interaction. The latter is indeed given deterministically by the interaction kernel, once the states of the interacting subjects are known.

In principle, both stochastic and deterministic inner–outer interactions can be taken into account. The corresponding mathematical structure is then:

$$\frac{\partial f^p}{\partial t} + \frac{\partial}{\partial u}(K^p[\mathbf{g}]f^p) = J^p[\mathbf{f}] + \mathcal{Q}^p[\mathbf{f}, \mathbf{g}]. \quad (2.11)$$

2.5 Systems with Discrete States

In applications to living complex systems, it is not always practical, or even possible, to identify a continuous distribution of microscopic states of the active particles. Indeed, the activity variable often refers to originally non-numerical and qualitative characteristics, such as e.g., individual opinions, political preferences, wealth status, which need to be transformed to quantitative information in the mathematical approach. In these cases, it might be convenient to reason in terms of *activity classes* roughly representative of the social structure of the system. Technically, this amounts to assuming that u is, in each functional subsystem, a discrete variable with only a finite number of possible values in a lattice:

$$I_u^p = \{u_1^p, u_2^p, \dots, u_n^p\} \subset D_u^p, \quad p = 1, \dots, m.$$

Each u_i^p is called an activity class of the p -th functional subsystem. As in the continuous case set forth in the preceding sections, it identifies a possible strategy by which active particles can approach game-type interaction events. The difference is that now the number of admissible strategies is finite, as is usually the case in classical game theory.

The corresponding formal expression of the distribution function is

$$f^p(t, u) = \sum_{i=1}^n f_i^p(t) \delta_{u_i^p}(u), \quad (2.12)$$

where $\delta_{u_i^p}$ is the Dirac distribution centered at u_i^p and $f_i^p(t) = f^p(t, u_i^p)$ in the sense of distributions. The function $f_i^p : [0, T_{\max}] \rightarrow [0, +\infty)$, for $i = 1, \dots, n$ and $p = 1, \dots, m$, is the distribution function of the i -th activity class in the p -th functional subsystem.

Using the representation (2.12), the formulas given in Sect. 2.2 for the moments of the distribution can be straightforwardly restated in the discrete activity case. For instance, the number of particles of the p -th subsystem (zeroth-order moment) is given by:

$$N^p[\mathbf{f}^p](t) = \sum_{i=1}^n f_i^p(t),$$

whereas ℓ -th order moments, $\ell \geq 1$, are computed as:

$$\mathbb{E}_\ell^p[\mathbf{f}^p](t) = \sum_{i=1}^n (u_i^p)^\ell f_i^p(t).$$

Analogously, the evolution equations for the f_i^p 's are technically deduced by substituting the representation (2.12) into Eqs. (2.2), (2.3), (2.5), (2.7), (2.8), after reinterpreting the latter in the sense of distributions. This yields:

$$\frac{df_i^p}{dt} = J_i^p[\mathbf{F}] + Q_i^p[\mathbf{F}, \mathbf{G}] = C_i^p[\mathbf{F}] + P_i^p[\mathbf{F}] + Q_i^p[\mathbf{F}, \mathbf{G}], \quad (2.13)$$

where now

$$\mathbf{F} = \{f_i^p\}_{i=1, \dots, n, p=1, \dots, m}, \quad \text{and} \quad \mathbf{G} = \{g_i^\alpha\}_{i=1, \dots, v, \alpha=1, \dots, A}$$

and the operators on the right side of Eq. 2.13 are:

$$\begin{aligned} C_i^p[\mathbf{F}] &= \sum_{p_*, p^*=1}^m \sum_{h, k=1}^n \eta_{hk}^{p_* p^*} \mathcal{C}_{hk}^{p_* p^*}(i, p) f_h^{p_*} f_k^{p^*} - f_i^p \sum_{p^*=1}^m \sum_{k=1}^n \eta_{ik}^{p p^*} f_k^{p^*} \\ P_i^p[\mathbf{F}] &= \sum_{p_*, p^*=1}^m \sum_{h, k=1}^n \eta_{hk}^{p_* p^*} \mathcal{P}_{hk}^{p_* p^*}(i, p) f_h^{p_*} f_k^{p^*} \\ Q_i^p[\mathbf{F}, \mathbf{G}] &= \sum_{\alpha=1}^A \sum_{p_*=1}^m \sum_{h=1}^n \sum_{j=1}^v \hat{\eta}_{hj}^{p_* \alpha} \mathcal{Q}_{hj}^{p_* \alpha}(i, p) f_h^{p_*} g_j^\alpha - f_i^p \sum_{\alpha=1}^A \sum_{j=1}^v \hat{\eta}_{ij}^{p \alpha} g_j^\alpha, \end{aligned}$$

The symbols above have an intuitive meaning, that we explicitly detail, however, for the sake of clarity:

- $\eta_{hk}^{p_* p^*}$ is the *interaction rate* between a candidate particle in the h -th activity class of the p_* -th subsystem and a field particle in the k -th activity class of the p^* -th subsystem. An analogous interpretation holds for $\hat{\eta}_{ij}^{p \alpha}$, which refers to inner-outer interactions among candidate particles and field agents.
- $\mathcal{C}_{hk}^{p_* p^*}(i, p)$ is the probability that a candidate particle in the h -th activity class of the p_* -th subsystem shifts to the test activity class i of the test subsystem p after playing a game with a field particle in the k -th activity class of the p^* -th subsystem. The set of all such transition probabilities forms the *table of games* $\{\mathcal{C}_{hk}^{p_* p^*}(i, p)\}$, which models the game played by active particles. It satisfies the probability density property:

$$\sum_{p=1}^m \sum_{i=1}^n \mathcal{C}_{hk}^{p_* p^*}(i, p) = 1, \quad \forall h, k = 1, \dots, n, \forall p_*, p^* = 1, \dots, m,$$

which guarantees the conservation of inner interactions described by the operator C_i^p .

- $\mathcal{P}_{hk}^{p_* p^*}(i, p)$ is the *net birth/death rate* of active particles for proliferative inner interactions. In order for deaths to occur only in the functional subsystem and activity class of candidate particles, the following assumption is made:

$$\mathcal{P}_{hk}^{p_* p^*}(i, p) \geq 0 \quad \forall h \neq i, \forall p_* \neq p.$$

- $\mathcal{Q}_{hj}^{p*\alpha}(i, p)$ is the probability that a candidate particle in the h -th activity class of the p_* -th functional subsystem shifts to the test activity class i of the test subsystem p when playing a game with a field agent in the j -th activity class of the α -th outer functional subsystem. It is also assumed that field agents are grouped into a finite number $v \geq 1$ of activity classes over the lattice

$$I_w^\alpha = \{w_1^\alpha, w_2^\alpha, \dots, w_v^\alpha\} \subset D_w^\alpha$$

and represented by a set of $A \times v$ known distribution functions $g_j^\alpha = g_j^\alpha(t) : [0, T_{\max}] \rightarrow [0, +\infty)$ such that the distribution function (2.6) is recovered, for each $\alpha = 1, \dots, A$, as:

$$g^\alpha(t, w) = \sum_{j=1}^v g_j^\alpha(t) \delta_{w_j^\alpha}(w),$$

with $g_j^\alpha(t) = g^\alpha(t, w_j^\alpha)$ in the sense of distributions. The set of all of the above inner-outer transition probabilities constitutes the *inner-outer table of games* $\{\mathcal{Q}_{hj}^{p*\alpha}(i, p)\}$, which is required to satisfy the probability density property:

$$\sum_{p=1}^m \sum_{i=1}^n \mathcal{Q}_{hj}^{p*\alpha}(i, p) = 1, \quad \forall h = 1, \dots, n, \forall j = 1, \dots, v \quad (2.14)$$

$$\forall p_* = 1, \dots, m, \forall \alpha = 1, \dots, A$$

in order for the interactions described by the operator \mathcal{Q}_i^p to be conservative.

2.6 Microscopic Interactions and Sources of Nonlinearity

Mathematical models can be derived from the mathematical structures presented in Sects. 2.3–2.5 by devising descriptions of the interactions at the microscale, for instance at the levels of the interaction rate and of the table of games.

As already mentioned, theoretical tools from game theory can be used. However, the literature does not yet offer a unified systematic approach. Hence, heuristic methods are generally applied for each specific case. Actually, recent contributions [38, 43] give some hints that will be revisited in this section.

An important issue is nonlinear interactions, when particles are not simply subject to the superposition of binary actions but are also affected by the aggregate state of neighboring individuals. In this case, the interaction terms have to be viewed as operators over the distribution function, whereby further nonlinearities are introduced in the right-hand sides of the relevant equations.

For instance, the interaction rate may be assumed to decay with the distance between the states of the interacting particles as well as with the distance between the distribution functions of the subsystems they belong to. Such an assumption

would imply a higher interaction frequency for similar particles belonging to similarly distributed subsystems. In other cases, dissimilar particles carrying out different functions may be more likely to interact with high frequency, as happens, for example, in hiding-learning processes [72] such as chasing and/or escaping dynamics involving criminals and detectives [143].

Another source of nonlinearity in the interaction dynamics can be the actual activity subdomain where interactions are effective, which may not coincide with the whole D_u^p . In other words, candidate particles may interact only with some field particles selected on a distribution-dependent basis (one then speaks of *topological* interactions). This idea originates from conjectures made by physicists about the dynamics of swarms [25], which can be interestingly transferred to social sciences for addressing swarming behaviors such as those treated in [142, 143, 165].

Finally, a great source of nonlinearities in the evolution equations is definitely the table of games. Indeed, as already pointed out, candidate particles can also modify their interaction strategy depending on some (local) aggregate state of the field particles they interact with, which is duly described by suitable moments of the distribution function.

Some technical arguments about these topics are presented in the following paragraphs, without claiming to be exhaustive. For expository purposes the focus will be on closed systems. Technical generalizations to open systems are left to interested readers.

2.6.1 Interaction Rate

The frequency of the interactions among candidate and field particles belonging to the p_* -th and p^* -th functional subsystems, respectively, is modeled by the term $\eta^{p_*p^*}$. Following [38], the latter can be assumed to decay with the distance between the interacting particles. In the linear case, such a distance depends only on the microstates of the interacting pairs, hence it is given by $|u_* - u^*|$. Conversely, in the nonlinear case it can also depend on the distribution functions f^{p_*} and f^{p^*} , particularly on $\|f^{p_*} - f^{p^*}\|$. Here, $\|\cdot\|$ is a suitable norm such as, for example, the uniform L^∞ -norm or the mean L^1 -norm, also depending on the physics of the system under consideration (which might suggest appropriate ways of measuring the distance between different configurations of the system). This concept is based on the idea that two systems with close distributions are *affine*, hence they tend to interact with higher frequency.

The interaction rate can also include a dependence on the actual interaction domain. If particles interact only within a certain distance or with a predefined number of other particles then $\eta^{p_*p^*}$ is zero outside the respective bounds. In turn, the latter can be linked to the distribution functions f^{p_*} , f^{p^*} themselves.

2.6.2 Table of Games

The table of games \mathcal{C}^{P*P*} can be modeled by relating the payoffs of the interactions to:

- *Cooperative/competitive* games, in which candidate particles try to profit from the state of field particles in order to consolidate their well-being or to fairly redistribute wealth.
- *Hiding-learning* dynamics, in which attempts by candidate particles to improve their state are balanced by a tendency to reduce social distances thus produced (learning process).

In general, it may be argued that the occurrence of either type of game depends, once again, on a microstate-based distance between the interacting particles. Moreover, if the distance between the configurations of the subsystems that particles belong to is involved, further nonlinearities are brought into the problem.

2.6.3 Inner Reorganization of Functional Subsystems

Some social systems, for instance political parties, are characterized by a (small) *critical size for survival*. That is, if the size $N^P[f^P]$ of a certain functional subsystem falls below a critical threshold then particles may prefer to migrate to other functional subsystem or to aggregate in brand new ones. Similarly, a (large) critical size can exist such that, when it is overcome, particles are again induced to migrate to other subsystems for avoiding a kind of depersonalization due to “overcrowding” of their original subsystem.

In both cases, the two sizes are generally not constant but depend on the global state of the system. Therefore disappearance, splitting, or creation of functional subsystems can be additional sources of nonlinearity in the equations.

The previous arguments drop a hint that interactions at the microscale can be strongly characterized by various types of nonlinearities. Consequently, in most cases the evolution equations feature further nonlinearities besides the standard quadratic one due to the product of the distribution functions in the terms C^P and P^P .

The mathematical structures proposed in the preceding sections are still valid but the following notations are worth being introduced for stressing such *constitutive* nonlinearities:

$$\eta^{P*P*}[\mathbf{f}](u_*, u^*), \quad \mathcal{C}^{P*P*}[\mathbf{f}](u_* \rightarrow u; p|u_*, u^*), \quad \text{and} \quad \mathcal{P}^{P*P*}[\mathbf{f}](u_*, u^*; u, p).$$

This notation should also be extended to the domain of interaction, when it depends on the distribution functions: $D_u = D_u[\mathbf{f}]$. See [43] for a detailed analysis of the related convolution problems.

According to our perspective as advanced in this monograph, the modeling approach cannot be oversimplified by neglecting nonlinearly additive interactions with the only aim of pursuing analytical results. On the contrary, no matter how difficult the analytical treatment of the resulting equations may appear, most of the existing literature should probably be revisited under this new perspective.

2.7 On the Solution of Mathematical Problems

The application of the mathematical structures presented in the previous sections to real social phenomena generates *mathematical problems*. The latter can be of essentially two types:

- *Initial-value problems*, namely those generated by Eqs. (2.1), (2.7), and (2.13) linked to *initial conditions*:

$$f^p(0, u) = f_0^p(u), \quad u \in D_u^p, \quad p = 1, \dots, m \quad \text{for Eqs. (2.1), (2.7)}$$

or

$$f_i^p(0) = f_{0i}^p, \quad i = 1, \dots, n, \quad p = 1, \dots, m \quad \text{for Eq. (2.13),}$$

where $f_0^p : D_u^p \rightarrow [0, +\infty)$, $f_{0i}^p \in [0, +\infty)$ are, respectively, known functions and numerical values prescribed for describing the distribution of the active particles over the activity u or the activity classes u_i^p at the initial time $t = 0$.

- *Initial/boundary-value problems*, namely those generated by Eqs. (2.9) and (2.11) linked to initial conditions analogous to those discussed above and, in addition, to conditions at the boundary of D_u^p due to the flux term $\partial_u(K^p[\mathbf{g}]f^p)$. These are of the form:

$$f^p(t, u \in \partial D_u^p) = \varphi^p(t), \quad p = 1, \dots, m,$$

$\varphi^p : [0, T_{\max}] \rightarrow [0, +\infty)$ being known functions. These conditions provide the values of the distribution functions f^p on the boundary of D_u^p at all times. They do not necessarily have to be prescribed on the whole boundary of D_u^p but possibly only on subsets of ∂D_u^p , depending on the structure of the advection speed $K^p[\mathbf{g}]$. If D_u^p is unbounded then boundary conditions are replaced by suitable properties of decay to zero of the corresponding p -th distribution function at infinity for ensuring its integrability with respect to u .

In all cases, the mathematical problem consists of computing the solution, being either

$$\{f^p(t, u)\}_{p=1}^m, \quad \forall u \in D_u^p, \quad \forall t \in [0, T_{\max}],$$

or

$$\{f_i^p(t)\}_{\substack{i=1,\dots,n \\ p=1,\dots,m}}, \quad \forall t \in [0, T_{\max}],$$

starting from the input data required by the model.

The numerical solution of such problems is generally not a difficult task, at least in the absence of mean-field fluxes K^p . In fact, collocation methods can be used to transform activity-continuous integro-differential equations into ordinary differential equations in time, which can then be discretized by the most appropriate computational schemes, with consideration of stability and accuracy. The technique is described in [41], where it is also shown how boundary conditions can be implemented directly into the system of ODEs. If the activity variable is discrete, models immediately take the structure of a system of ordinary differential equations.

Simulations should be supported by a qualitative analysis of the mathematical problems. Existence and uniqueness of solutions to initial-value problems in the absence of non-conservative interactions have been studied in [19] by extending a previous study about linearly additive interactions [16] to nonlinearly additive interactions. Proliferative terms require additional studies, as they may even induce bifurcation phenomena. This issue has been carefully addressed in [44] referring specifically to models of competition between the immune system and pathogens. Analogous investigations in the case of social systems are not yet available; see Chap. 5 for a critical analysis of this issue.

The use of models for simulating real systems also motivates further challenging analytical investigations not necessarily limited to a qualitative analysis of the solutions to mathematical problems. Other challenging issues are, for instance, understanding the links between the KTAP approach and Markov processes [112] or classical kinetic theories [26]. These aspects are quite well mastered in the case of models with linearly additive interactions [17], whereas they are still an open problem in the case of nonlinearly additive interactions.

2.8 Critical Analysis

In this section we analyze how far the mathematical structures proposed in this chapter are able to capture the complexity features of social systems discussed in Chap. 1. Readers should be aware that, for the moment, only preliminary considerations can be made. A more exhaustive analysis has to be deferred to the next chapters, after seeing specific models in action on well defined applications. In the following, we proceed by keywords reminiscent of crucial complexity aspects dealt with so far.

- *Emerging behaviors and validation.* Mathematical models provide the time evolution of the distribution function, which in some cases is a probability

density (when the number of active particles is constant in time), over the activity variable. This output potentially depicts both microscopic details and average macroscopic quantities. Validation of models should be based on their ability to describe actually observed emerging behaviors. A successful model may also be expected to predict, under special circumstances, events which have never been observed before.

- *Strategy, organization ability, and heterogeneity.* All of these features are variously linked to the activity variable u , which expresses the (non-mechanical) state of the active particles, namely the behavioral strategy they apply when interacting in a game-type fashion with other particles or with external agents. The heterogeneous distribution of such a variable among the active particles is expressed, within each functional subsystem, by the distribution function f^p , which evolves in time. The activity can be thought of, at the microscopic scale, as a *random variable* attached to each active particle. The KTAP approach then studies the evolution in time of its distribution at the mesoscopic scale. Reference to random variables is necessary in order to model the partly irrational (viz., stochastic) as opposed to rational (viz., deterministic) behavior of the active particles, which may not react in the same way even if placed in similar conditions.
- *Interactions by stochastic games.* Microscopic interactions among active particles are modeled by the terms η , \mathcal{C} , and \mathcal{P} , namely the interaction rate, the table of games, and the net birth/death rate, respectively. Particularly important is the table of games, which describes interactions as stochastic games. In more detail, it encodes, in probabilistic terms, the (conservative) interaction rules, hence the game played by active particles. The distribution function then changes as a result of a change in the activity of the particles after the game they play when interacting.
- *Reducing the complexity generated by a large variety of components.* Splitting the whole system into functional subsystems is a way to reduce the technical complexity induced by a large number of variables. Such a decomposition can possibly also be viewed as associated with a secondary activity variable, say v , which takes only m discrete values v_1, v_2, \dots, v_m . According to this interpretation, the p -th functional subsystem would thus group all active particles which collectively express the value v_p of the activity v . This point of view is conceptually convenient when the primary activity variable u has the same meaning in all subsystems but the rules at the basis of the microscopic interactions depend on the specific function v carried out by different groups of active particles. Alternatively, the decomposition in functional subsystems can be thought of as a way to have a scalar activity variable u in each of them rather than a vector-valued one $\mathbf{u} = (u_1, u_2, \dots, u_m)$ when the various components u_p , $p = 1, \dots, m$, have different meanings in each subsystem.
- *Mutations and selections.* These events are respectively related to transitions across functional subsystems, accounted for by the table of games, and to proliferative interactions, modeled by the operators P^p . Their modeling requires,

in general, that the influence of the outer environment on the conservative interactions be carefully analyzed.

- *Multiscale essence.* The modeling of aforementioned terms η , \mathcal{C} , and \mathcal{P} is generally obtained by a mainly phenomenological approach at the *microscopic* scale. Due to the behavioral individuality of the active particles composing the system, this approach is probably better than trying to directly model the collective behavior of groups of particles at the *macroscopic* scale by means of *constitutive relationships*. Next, ensemble dynamics are predicted by the model and can be studied *a posteriori*, once the evolution of the distribution function is available at the *mesoscopic* scale. Phenomenological guidelines for modeling the terms η , \mathcal{C} , and \mathcal{P} cannot however be given in full generality. Specific applications can instead indicate, each time, some reasonable ones, thereby also suggesting possible technical developments of the mathematical structures.

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