Chapter 2
Preliminaries

Where there is matter, there is geometry.
Johannes Kepler (1571–1630)

2.1 Logic

2.1.1 Basic Concepts of Logic

Let us consider A to be a non-empty set of mathematical objects. One may construct various expressions using these objects. An expression is called a *proposition* if it can be characterized as "true" or "false."

Example

1. "The number $\sqrt{2}$ is irrational" is a true proposition.
2. "An isosceles triangle has all three sides mutually unequal" is a false proposition.
3. "The median and the altitude of an equilateral triangle have different lengths" is false.
4. "The diagonals of a parallelogram intersect at their midpoints" is true.

A proposition is called *compound* if it is the juxtaposition of propositions connected to one another by means of *logical connectives*. The truth values of compound propositions are determined by the truth values of their constituting propositions and by the behavior of logical connectives involved in the expression. The set of propositions equipped with the operations defined by the logical connectives becomes the *algebra of propositions*. Therefore, it is important to understand the behavior of logical connectives.

The logical connectives used in the algebra of propositions are the following:

$$\wedge(\text{and}) \qquad \vee(\text{or}) \qquad \Rightarrow (\text{if} \ldots \text{then})$$

$$\Leftrightarrow (\text{if and only if}) \quad \text{and} \quad \neg(\text{not}).$$

The mathematical behavior of the connectives is described in the *truth tables*, seen in Tables 2.1, 2.2, 2.3, 2.4, and 2.5.

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Table 2.1 Truth table for $\wedge$

a	b	$a \wedge b$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2 Truth table for $\vee$

a	b	$a \vee b$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.3 Truth table for $\Rightarrow$

a	b	$a \Rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.4 Truth table for $\Leftrightarrow$

a	b	$a \Leftrightarrow b$
T	T	T
T	F	F
F	T	F
F	F	T

Table 2.5 Truth table for $\neg$

a	$\neg a$
T	F
F	T

In the case when

$$a \Rightarrow b \quad \text{and} \quad b \Rightarrow a \tag{2.1}$$

are simultaneously true, we say that a and b are "equivalent" or that "a if and only if b" or that a is a necessary and sufficient condition for b.

Let us now focus on mathematical problems. A mathematical problem is made up from the hypothesis and the conclusion. The hypothesis is a proposition assumed to be true in the context of the problem. The conclusion is a proposition whose truth one is asked to show. Finally, the solution consists of a sequence of logical implications

$$a \Rightarrow b \Rightarrow c \Rightarrow \cdots . \tag{2.2}$$

Mathematical propositions are categorized in the following way: *Axioms, theorems, corollaries, problems*.

Axioms are propositions considered to be true without requiring a proof. Another class of propositions are the *lemmata*, which are auxiliary propositions; the proof of a lemma is a step in the proof of a theorem.

In Euclidean Geometry, we have three basic axioms concerning comparison of figures:

1. Two figures, A and B, are said to be *congruent* if and only if there exists a translation, or a rotation, or a symmetry, or a composition of these transformations such that the image of figure A coincides with figure B.
2. Two figures which are congruent to a third figure are congruent to each other.
3. A part of a figure is a subset of the entire figure.

2.1.2 On Related Propositions

Consider the proposition

$$p : a \Rightarrow b.$$

Then:

1. The *converse* of proposition p is the proposition
$$q : b \Rightarrow a. \tag{2.3}$$
2. The *inverse* of proposition p is the proposition
$$r : \neg a \Rightarrow \neg b. \tag{2.4}$$
3. The *contrapositive* of proposition p is the proposition
$$s : \neg b \Rightarrow \neg a. \tag{2.5}$$

Example Consider the proposition

p: *If a convex quadrilateral is a parallelogram then its diagonals bisect each other.*

1. The *converse* proposition of p is q:

   If the diagonals of a converse quadrilateral bisect each other, then it is a parallelogram.

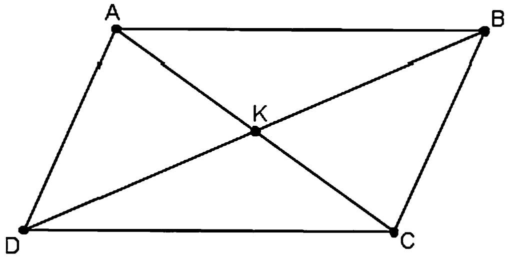

Fig. 2.1 Example 2.1.1

2. The *inverse* proposition of p is r:
   If a convex quadrilateral is not a parallelogram, then its diagonals do not bisect each other.
3. The *contrapositive* proposition of p is s:
   If the diagonals of a convex quadrilateral do not bisect each other, then it is not a parallelogram.

2.1.3 On Necessary and Sufficient Conditions

Proofs of propositions are based on proofs of the type

$$a \Rightarrow b,$$

where a is the set of hypotheses and b the set of conclusions. In this setup, we say that condition a is sufficient for b and that b is necessary for a. Similarly, in the case of the converse proposition

$$q : b \Rightarrow a, \tag{2.6}$$

condition b is sufficient for a and condition a is necessary for b.

In the case where both

$$a \Rightarrow b \quad \text{and} \quad b \Rightarrow a \tag{2.7}$$

are true, we have

$$a \Leftrightarrow b, \tag{2.8}$$

which means that a is a necessary and sufficient condition for b.

Example 2.1.1 A necessary and sufficient condition for a convex quadrilateral to be a parallelogram is that its diagonals bisect.

Proof Firstly, we assume that the quadrilateral $ABCD$ is a parallelogram (see Fig. 2.1). Let K be the point of intersection of its diagonals. We use the property that the opposite sides of a parallelogram are parallel and equal and we have that

$$AB = DC \tag{2.9}$$

and

$$\widehat{KAB} = \widehat{KCD} \tag{2.10}$$

since the last pair of angles are alternate interior. Also, we have

$$\widehat{ABK} = \widehat{CDK} \tag{2.11}$$

as alternate interior angles. Therefore, the triangles KAB and KDC are equal, and hence

$$KB = DK \quad \text{and} \quad AK = KC. \tag{2.12}$$

For the converse, we assume now that the diagonals of a convex quadrilateral $ABCD$ bisect each other. Then, if K is their intersection, we have that

$$KA = CK \quad \text{and} \quad KB = DK. \tag{2.13}$$

Furthermore, we have

$$\widehat{BKA} = \widehat{DKC} \tag{2.14}$$

because they are corresponding angles. Hence, the triangles KAB and KDC are equal. We conclude that

$$AB = DC \tag{2.15}$$

and also that

$$AB \parallel DC \tag{2.16}$$

since

$$\widehat{KAB} = \widehat{KCD}. \tag{2.17}$$

Therefore, the quadrilateral $ABCD$ is a parallelogram. □

2.2 Methods of Proof

We now present the fundamental methods used in geometric proofs.

2.2.1 Proof by Analysis

Suppose we need to show that

$$a \Rightarrow b. \tag{2.18}$$

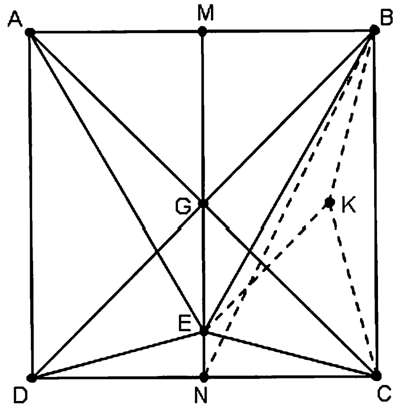

Fig. 2.2 Proof by analysis (Example 2.2.1)

We first find a condition b_1 whose truth guarantees the truth of b, i.e., a sufficient condition for b. Subsequently, we find a condition b_2 which is sufficient for b_1. Going *backwards* in this way, we construct a chain of conditions

$$b_n \Rightarrow b_{n-1} \Rightarrow \cdots \Rightarrow b_1 \Rightarrow b,$$

with the property that b_n is true by virtue of a being true. This completes the proof.

Example 2.2.1 Consider the square $ABCD$. From the vertices C and D we consider the half-lines that intersect in the interior of $ABCD$ at the point E and such that

$$\widehat{CDE} = \widehat{ECD} = 15°.$$

Show that the triangle EAB is equilateral (see Fig. 2.2).

Proof We observe that

$$AD = BC, \tag{2.19}$$

since they are sides of a square. Furthermore, we have (see Fig. 2.2)

$$\widehat{CDE} = \widehat{ECD} = 15° \tag{2.20}$$

hence

$$\widehat{EDA} = \widehat{BCE} = 75°$$

and

$$ED = EC, \tag{2.21}$$

since the triangle EDC is isosceles. Therefore, the triangles ADE and BEC are equal and thus

$$EA = EB. \tag{2.22}$$

Therefore, the triangle EAB is isosceles. In order to show that the triangle EAB is, in fact, equilateral, it is enough to prove that

$$EB = BC = AB. \tag{2.23}$$

In other words, it is sufficient to show the existence of a point K such that the triangles KBE and KCB are equal. In order to use our hypotheses, we can choose K in such a way that the triangles KBC and EDC are equal. This will work as long as K is an interior point of the square.

Let G be the center of the square $ABCD$. Then, if we consider a point K such that the triangles KCB and EDC are equal, we have the following:

$$GN < GB, \tag{2.24}$$

and hence

$$\widehat{GBN} < \widehat{GNB} = \widehat{NBC}. \tag{2.25}$$

Therefore,

$$2\widehat{GBN} < 45°, \tag{2.26}$$

so

$$\widehat{GBN} < 22.5°, \tag{2.27}$$

and hence

$$\widehat{NBC} > 22.5°. \tag{2.28}$$

Therefore,

$$\widehat{NBC} > \widehat{KBC}, \tag{2.29}$$

where M, N are the midpoints of the sides AB, DC, respectively. Therefore, the point K lies in the interior of the angle $\widehat{EBC}$. We observe that

$$\widehat{KCE} = 90° - 15° - 15° = 60°, \tag{2.30}$$

with

$$KC = CE. \tag{2.31}$$

Therefore, the isosceles triangle CKE has an angle of $60°$ and thus it is equilateral, implying that

$$EK = KC = CE \tag{2.32}$$

and

$$\widehat{BKE} = 360° - 60° - 150° = 150°. \tag{2.33}$$

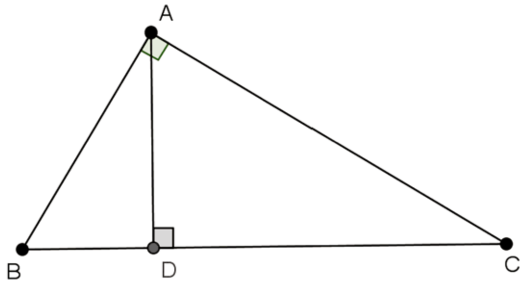

Fig. 2.3 Proof by synthesis (Example 2.2.2)

Therefore, the triangles KCB and KBE are equal, and hence

$$EB = BC. \tag{2.34}$$

□

2.2.2 *Proof by Synthesis*

Suppose we need to show that

$$a \Rightarrow b. \tag{2.35}$$

The method we are going to follow consists of combining proposition a with a number of suitable true propositions and creating a sequence of necessary conditions leading to b.

Example 2.2.2 Let ABC be a right triangle with $\widehat{BAC} = 90°$ and let AD be the corresponding height. Show that

$$\frac{1}{AD^2} = \frac{1}{AB^2} + \frac{1}{AC^2}.$$

Proof First, consider triangles ABD and CAB (see Fig. 2.3). Since they are both right triangles and

$$\widehat{BAD} = \widehat{ACB},$$

they are similar. Then we have (see Fig. 2.3)

$$\frac{AB}{BD} = \frac{BC}{AB},$$

and therefore,

$$AB^2 = BD \cdot BC.$$

Similarly,

$$AC^2 = DC \cdot BC$$

and

$$AD^2 = BD \cdot DC.$$

Hence

$$\begin{aligned}\frac{1}{AB^2} + \frac{1}{AC^2} &= \frac{1}{BD \cdot BC} + \frac{1}{DC \cdot BC} \\ &= \frac{BC}{BD \cdot BC \cdot DC} \\ &= \frac{1}{AD^2}.\end{aligned}$$

□

2.2.3 Proof by Contradiction

Suppose that we need to show

$$a \Rightarrow b. \tag{2.36}$$

We assume that the negation of proposition $a \Rightarrow b$ is true. Observe that

$$\neg(a \Rightarrow b) = a \wedge (\neg b). \tag{2.37}$$

In other words, we assume that given a, proposition b does not hold. If with this assumption we reach a false proposition, then we have established that

$$a \Rightarrow b \tag{2.38}$$

is true.

Example 2.2.3 Let ABC be a triangle and let D, E, Z be three points in its interior such that

$$3S_{DBC} < S_{ABC}, \tag{2.39}$$

$$3S_{EAC} < S_{ABC}, \tag{2.40}$$

$$3S_{ZAB} < S_{ABC}, \tag{2.41}$$

where S_{ABC} denotes the area of the triangle ABC and so on. Prove that the points D, E, Z cannot coincide.

Proof Suppose that inequalities (2.39), (2.40), and (2.41) hold true and let P be the point where D, E, Z coincide, that is,

$$D \equiv E \equiv Z \equiv P.$$

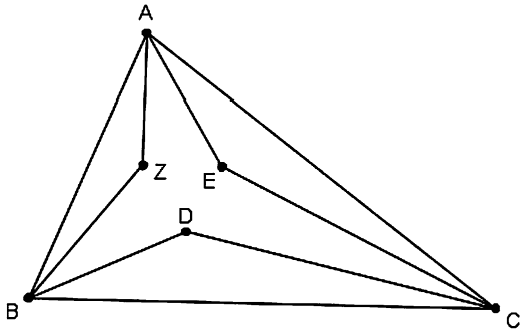

Fig. 2.4 Proof by contradiction (Example 2.2.3)

Then (see Fig. 2.4),

$$3[S_{PBC} + S_{PAC} + S_{PAB}] < 3S_{ABC} \tag{2.42}$$

and thus

$$S_{ABC} < S_{ABC}, \tag{2.43}$$

which is a contradiction. Therefore, when relations (2.39), (2.40), and (2.41) are satisfied, the three points D, E, Z cannot coincide. □

Now, we consider Example 2.2.1 from a different point of view.

Example 2.2.4 Let $ABCD$ be a square. From the vertices C and D we consider the half-lines that intersect in the interior of $ABCD$ at the point E and such that

$$\widehat{CDE} = \widehat{ECD} = 15°.$$

Show that the triangle EBA is equilateral.

Proof We first note that the triangle EBA is isosceles. Indeed, since by assumption

$$\widehat{CDE} = \widehat{ECD} = 15°$$

one has

$$\widehat{EDA} = \widehat{BCE} = 75° \Rightarrow ED = EC. \tag{2.44}$$

We also have that $AD = BC$, and therefore the triangles ADE and BEC are equal and thus $EA = EB$, which means that the point E belongs to the common perpendicular bisector MN of the sides AB, DC of the square $ABCD$.

Let us assume that the triangle EBA is not equilateral. Then, there exists a point Z on the straight line segment MN, Z different from E, such that the triangle ZAB is equilateral. Indeed, by choosing the point Z on the half straight line MN such that

$$MZ = \frac{AB\sqrt{3}}{2} < AB = AD = MN,$$

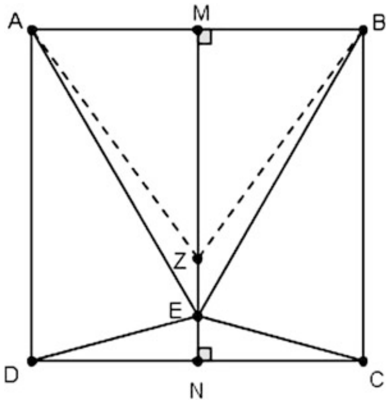

Fig. 2.5 Proof by contradiction (Example 2.2.4)

the point Z is an interior point of the straight line segment MN and the equilaterality of the triangle ZBA shall be an obvious consequence (see Fig. 2.5).

We observe that

$$\widehat{DAZ} = \widehat{ZBC} = 30^\circ \quad \text{and} \quad DA = ZA = AB = BZ = BC.$$

Thus

$$2\widehat{ZDA} = 180^\circ - 30^\circ \Rightarrow \widehat{ZDA} = 75^\circ,$$

which implies that

$$\widehat{CDZ} = 90^\circ - 75^\circ = 15^\circ. \tag{2.45}$$

Hence the points E, Z coincide, which is a contradiction. Therefore, the triangle EBA is equilateral. □

2.2.4 *Mathematical Induction*

This is a method that can be applied to propositions which depend on natural numbers. In other words, propositions of the form

$$p(n), \quad n \in \mathbb{N}. \tag{2.46}$$

The proof of proposition (2.46) is given in three steps.

1. One shows that $p(1)$ is true.
2. One assumes proposition $p(n)$ to be true.
3. One shows that $p(n+1)$ is true.

Remarks

- If instead of proposition (2.46) one needs to verify the proposition

$$p(n), \quad \forall n \in \mathbb{N} \setminus \{1, 2, \ldots, N\}, \tag{2.47}$$

  then the first step of the process is modified as follows:
  Instead of showing $p(1)$ to be true, one shows $p(N+1)$ to be true. After that, we assume proposition $p(n)$ to be true and we prove that $p(n+1)$ is true.
- Suppose $p(n)$ is of the form

$$p(n): \quad k(n) \geq q(n), \quad \forall n \geq N, \tag{2.48}$$

  where $n, N \in \mathbb{N}$.
  Suppose that we have proved

$$k(N) = q(N). \tag{2.49}$$

  We must examine the existence of at least one natural number $m > N$ for which $k(m) > q(m)$.

This is demonstrated in the following example.

Example 2.2.5 Let ABC be a right triangle with $\widehat{BAC} = 90°$, with lengths of sides $BC = a$, $AC = b$, and $AB = c$. Prove that

$$a^n \geq b^n + c^n, \quad \forall n \in \mathbb{N}\setminus\{1\}. \tag{2.50}$$

Proof Applying the induction method we have.

- Evidently, for $n = 2$, the Pythagorean Theorem states that Eq. (2.50) holds true and is, in fact, an equality. We shall see that for $n = 3$ it holds

$$a^3 > b^3 + c^3.$$

  In order to prove this, it suffices to show

$$a\left(b^2 + c^2\right) > b^3 + c^3. \tag{2.51}$$

  To show (2.51), it is enough to show

$$ab^2 + ac^2 - b^3 - c^3 > 0, \tag{2.52}$$

  for which it is sufficient to show

$$b^2(a-b) + c^2(a-c) > 0. \tag{2.53}$$

  This inequality holds because the left hand is strictly positive, since $a > c$ and $a > b$.

- We assume that

$$a^n > b^n + c^n \tag{2.54}$$

for $n \in \mathbb{N} \setminus \{1, 2\}$.

- We shall prove that

$$a^{n+1} > b^{n+1} + c^{n+1}. \tag{2.55}$$

For (2.55) it suffices to show that

$$a\big(b^n + c^n\big) - b^{n+1} - c^{n+1} > 0, \tag{2.56}$$

for which, in turn, it is enough to show that

$$b^n(a - b) + c^n(a - c) > 0. \tag{2.57}$$

Again, in the last inequality the left hand side term is greater than 0, since $a > c$ and $a > b$. Therefore, (2.50) is true. □

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