Chapter 2
Classical combinatorial concepts

Abstract In this chapter we begin by defining partially ordered sets, linear extensions, the dual of a poset, and the disjoint union of two posets. We then define further combinatorial objects we will need including compositions, partitions, diagrams and Young tableaux, reverse tableaux, Young's lattice and Schensted insertion.

2.1 Partially ordered sets

A useful notion for us throughout this book will be that of a partially ordered set.

Definition 2.1.1. A *partially ordered set*, or simply *poset*, is a pair $(P,\leqslant)$ consisting of a set P and a binary relation $\leqslant$ on P that is reflexive, antisymmetric and transitive, that is, for all $p,q,r \in P$,

1. $p \leqslant p$
2. $p \leqslant q$ and $q \leqslant p$ implies $p = q$
3. $p \leqslant q$ and $q \leqslant r$ implies $p \leqslant r$.

The relation $\leqslant$ is called a *partial order* on or *partial ordering* of P.

We write $p < q$ if $p \leqslant q$ and $p \neq q$, $p \geqslant q$ if $q \leqslant p$, and $p > q$ if $q < p$. Elements $p,q \in P$ are called *comparable* if $p \leqslant q$ or $q \leqslant p$.

If $p \leqslant q$, then we define the *closed interval*

$$[p,q] = \{r \in P | p \leqslant r \leqslant q\}$$

and the *open interval*

$$(p,q) = \{r \in P | p < r < q\}.$$

An element q *covers* an element p if $p < q$ and $(p,q) = \emptyset$. If q covers p, then we write $p \lessdot q$.

K. Luoto et al., *An Introduction to Quasisymmetric Schur Functions*,
SpringerBriefs in Mathematics, DOI 10.1007/978-1-4614-7300-8_2,

A *chain* is a poset in which any two elements are comparable. The order here is called a *total* or *linear* order. A *saturated chain* of *length* $n-1$ in a poset is a subset with order $q_1 \lessdot \cdots \lessdot q_n$. For our purposes we say a poset is *graded* if it has a unique minimal element $\hat{0}$ and every saturated chain between $\hat{0}$ and a poset element x has the same length, called the *rank* of x.

Example 2.1.2. Familiar posets include $(\mathbb{Z}, \leqslant)$ where $\leqslant$ is the usual relation *less than or equal to* on the integers, and $(\mathscr{P}(A), \subseteq)$ where $\mathscr{P}(A)$ is the collection of all subsets of a set A. In addition, the poset $(\mathbb{Z}, \leqslant)$ is a chain.

We shall often abuse notation and give both a poset and its underlying set the same name. Thus a *poset* P shall mean, unless otherwise specified, a set P together with a partial order on P. The partial order will usually be denoted by the symbol $\leqslant$, with the words 'in P', or subscript P, added if necessary to distinguish it from the partial order on a different poset.

We will be interested in chains that contain a given poset in the following sense.

Definition 2.1.3. A *linear extension* of a poset P is a chain w consisting of the set P with a total order that satisfies

$$p < q \text{ in } P \text{ implies } p < q \text{ in } w.$$

When P is finite, we can let this total order be $w_1 < w_2 < \cdots$ and restate the last condition as

$$w_i < w_j \text{ in } P \text{ implies } i < j.$$

The set of all linear extensions of P is denoted by $\mathscr{L}(P)$.

Example 2.1.4. There are two linear extensions of $(\mathscr{P}(\{1,2\}), \subseteq)$, namely

$$\emptyset < \{1\} < \{2\} < \{1,2\}$$

and

$$\emptyset < \{2\} < \{1\} < \{1,2\}.$$

Finally, we introduce two operations on posets. The *dual* of a poset P is the poset P^* consisting of the set P with partial order defined by

$$p \leqslant q \text{ in } P^* \text{ if } q \leqslant p \text{ in } P.$$

If P and Q are posets with disjoint underlying sets P and Q, then the *disjoint union* $P+Q$ is the poset consisting of the set $P \cup Q$ with partial order defined by

$$p \leqslant q \text{ in } P+Q \text{ if } p \leqslant q \text{ in } P \text{ or } p \leqslant q \text{ in } Q.$$

Since P and Q are disjoint, $p \leqslant q$ in $P+Q$ is possible only if $p,q \in P$ or $p,q \in Q$.

2.2 Compositions and partitions

Compositions and partitions will be the foundation for the indexing sets of the functions we will be studying.

Definition 2.2.1. A *composition* is a finite ordered list of positive integers. A *partition* is a finite unordered list of positive integers that we write in weakly decreasing order when read from left to right. In both cases we call the integers the *parts* of the composition or partition.

The *underlying partition* of a composition α, denoted by $\tilde{\alpha}$, is the partition obtained by sorting the parts of α into weakly decreasing order.

Given a composition or partition $\alpha = (\alpha_1, \dots, \alpha_k)$, we define its *weight* or *size* to be $|\alpha| = \alpha_1 + \cdots + \alpha_k$ and its *length* to be $\ell(\alpha) = k$. When $\alpha_{j+1} = \cdots = \alpha_{j+m} = i$ we often abbreviate this sublist to i^m. If α is a composition with $|\alpha| = n$, then we write $\alpha \vDash n$ and say α is a composition of n. If λ is a partition with $|\lambda| = n$, then we write $\lambda \vdash n$ and say λ is a partition of n. For convenience we denote by $\emptyset$ the unique composition or partition of weight and length 0, called the *empty* composition or partition.

Example 2.2.2. The compositions of 4 are

$$(4),(3,1),(1,3),(2,2),(2,1,1),(1,2,1),(1,1,2),(1,1,1,1).$$

The partitions of 4 are

$$(4),(3,1),(2,2),(2,1,1),(1,1,1,1).$$

If $\alpha = (1,4,1,2)$, then $\tilde{\alpha} = (4,2,1,1)$.

Let $[n] = \{1,2,\dots,n\}$. There is a natural one-to-one correspondence between compositions of n and subsets of $[n-1]$, given by the following.

Definition 2.2.3. Let n be a nonnegative integer.

1. If $\alpha = (\alpha_1, \dots, \alpha_k) \vDash n$, then we define
$$\mathrm{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\} \subseteq [n-1].$$
2. If $A = \{a_1, \dots, a_\ell\} \subseteq [n-1]$ where $a_1 < \cdots < a_\ell$, then we define
$$\mathrm{comp}(A) = (a_1, a_2 - a_1, \dots, a_\ell - a_{\ell-1}, n - a_\ell) \vDash n.$$

In particular, the empty set corresponds to the composition $\emptyset$ if $n = 0$, and to (n) if $n > 0$.

Example 2.2.4. Let $\alpha = (1,1,3,1,2) \vDash 8$. Then

$$\operatorname{set}(\alpha) = \{1, 1+1, 1+1+3, 1+1+3+1\} = \{1,2,5,6\} \subseteq [7].$$

Conversely, if $A = \{1,2,5,6\} \subseteq [7]$, then

$$\operatorname{comp}(A) = (1, 2-1, 5-2, 6-5, 8-6) = (1,1,3,1,2).$$

For every composition $\alpha \vDash n$ there exist three closely related compositions: its reversal, its complement, and its transpose. Firstly, the *reversal* of α, denoted by α^r, is obtained by writing the parts of α in the reverse order. Secondly, the *complement* of α, denoted by α^c, is given by

$$\alpha^c = \operatorname{comp}(\operatorname{set}(\alpha)^c),$$

that is, α^c is the composition that corresponds to the complement of the set that corresponds to α. Lastly, the *transpose* (also known as the *conjugate*) of α, denoted by α^t, is defined to be $\alpha^t = (\alpha^r)^c = (\alpha^c)^r$.

Example 2.2.5. If $\alpha = (1,4,1,2) \vDash 8$, then $\operatorname{set}(\alpha) = \{1,5,6\} \subseteq [7]$, and hence $\alpha^r = (2,1,4,1)$, $\alpha^c = (2,1,1,3,1)$, $\alpha^t = (1,3,1,1,2)$.

Pictorially, we can view a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ as a row consisting of α_1 dots, then a bar followed by α_2 dots, then a bar followed by α_3 dots, and so on. We can use the picture of α to create the pictures of $\alpha^r, \alpha^c, \alpha^t$ as follows.

To create the picture of α^r, reflect the picture of α in a vertical axis. To create the picture of α^c, place a bar between two dots if there is no bar between the corresponding dots in the picture of α. Finally, create the picture of α^t by performing one of these actions, then using the resulting picture to perform the other action.

Example 2.2.6. Repeating our previous example, if $\alpha = (1,4,1,2)$ then the picture of α is

$$\bullet \mid \bullet\bullet\bullet\bullet \mid \bullet \mid \bullet\bullet$$

so to compute $\alpha^r, \alpha^c, \alpha^t$ we draw the pictures

$$\bullet\bullet \mid \bullet \mid \bullet\bullet\bullet\bullet \mid \bullet \quad , \quad \bullet\bullet \mid \bullet \mid \bullet \mid \bullet\bullet\bullet \mid \bullet \quad , \quad \bullet \mid \bullet\bullet\bullet \mid \bullet \mid \bullet \mid \bullet\bullet$$

to obtain $\alpha^r = (2,1,4,1)$, $\alpha^c = (2,1,1,3,1)$, $\alpha^t = (1,3,1,1,2)$.

Given a pair of compositions, there are also two operations that can be performed. The *concatenation* of $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$ is

$$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$$

while the *near concatenation* is

$$\alpha \odot \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \ldots, \beta_\ell).$$

For example, $(1,4,1,2)\cdot(3,1,1) = (1,4,1,2,3,1,1)$ while $(1,4,1,2)\odot(3,1,1) = (1,4,1,5,1,1)$.

Given compositions α,β, we say that α is a *coarsening* of β (or equivalently β is a *refinement* of α), denoted by $\alpha \succcurlyeq \beta$, if we can obtain the parts of α in order by adding together adjacent parts of β in order. For example, $(1,4,1,2) \succcurlyeq (1,1,3,1,2)$.

We end this section with the following result on refinement, which is straightforward to verify, and is illustrated by Examples 2.2.4 and 2.2.5 and the definition of refinement.

Proposition 2.2.7. *Let α and β be compositions of the same weight. Then*

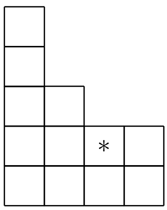
$$\alpha \preccurlyeq \beta \ \textit{ if and only if } \ \mathrm{set}(\beta) \subseteq \mathrm{set}(\alpha).$$

2.3 Partition diagrams

We now associate compositions and partitions with diagrams.

Definition 2.3.1. Given a partition $\lambda = (\lambda_1,\ldots,\lambda_{\ell(\lambda)}) \vdash n$, we say the *Young diagram* of λ, also denoted by λ, is the left-justified array of n cells with λ_i cells in the i-th row. We follow the Cartesian or French convention, which means that we number the rows from bottom to top, and the columns from left to right. The cell in the i-th row and j-th column is denoted by the pair (i,j).

Example 2.3.2. The cell filled with a $*$ is the cell $(2,3)$.



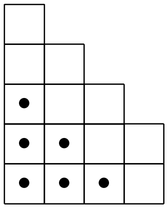
$$\lambda = (4,4,2,1,1)$$

Let λ,μ be two Young diagrams. We say μ is *contained* in λ, denoted by $\mu \subseteq \lambda$, if $\ell(\mu) \leqslant \ell(\lambda)$ and $\mu_i \leqslant \lambda_i$ for $1 \leqslant i \leqslant \mu_{\ell(\mu)}$. If $\mu \subseteq \lambda$, then we define the *skew shape* λ/μ to be the array of cells

$$\lambda/\mu = \{(i,j) \mid (i,j) \in \lambda \text{ and } (i,j) \notin \mu\}.$$

For convenience, we refer to μ as the *inner shape* and to λ as the *outer shape*. The *size* of λ/μ is $|\lambda/\mu| = |\lambda| - |\mu|$. Note that the skew shape $\lambda/\emptyset$ is the same as the Young diagram λ. Consequently, we write λ instead of $\lambda/\emptyset$. Such a skew shape is said to be of *straight shape*.

Example 2.3.3. In this example the inner shape is denoted by cells filled with a $\bullet$, although often these cells are not drawn.



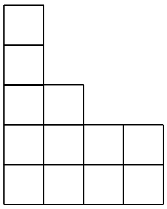
$$\lambda/\mu = (4,4,3,2,1)/(3,2,1)$$

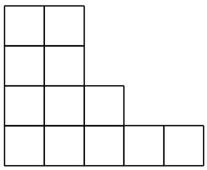
The *transpose* of a Young diagram λ, denoted by λ^t, is the array of cells

$$\lambda^t = \{(j,i)|(i,j) \in \lambda\}.$$

Note that this defines the transpose of a partition.

Example 2.3.4.



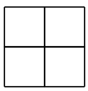


$\lambda = (4,4,2,1,1)$ $\qquad$ $\lambda^t = (5,3,2,2)$

We extend the definition of transpose to skew shapes by

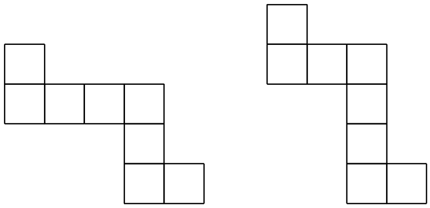
$$(\lambda/\mu)^t = \{(j,i)|(i,j) \in \lambda \text{ and } (i,j) \notin \mu\} = \lambda^t/\mu^t.$$

Three skew shapes of particular note are horizontal strips, vertical strips and ribbons. We say a skew shape is a *horizontal strip* if no two cells lie in the same column, and a *vertical strip* if no two cells lie in the same row. A skew shape is *connected* if for every cell d with another cell strictly below or to the right of it, there exists a cell edge-adjacent to d either below or to the right. We say a connected skew shape is a *ribbon* if the following subarray of four cells does not occur in it.



It follows that a ribbon is an array of cells in which, if we number rows from top to bottom, the leftmost cell of row $i+1$ lies immediately below the rightmost cell of row i. Consequently, a ribbon can be efficiently indexed by the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$, where α_i is the number of cells in row i. This indexing, which follows [34] and involves a deviation from the Cartesian convention for numbering rows, will simplify our discussion of duality later. It also ensures that the definitions of transpose of a ribbon and transpose of a composition agree, as illustrated in Examples 2.2.5 and 2.3.5.

Example 2.3.5.



$\alpha = (1,4,1,2) \qquad \alpha^t = (1,3,1,1,2)$

2.4 Young tableaux and Young's lattice

We now take the diagrams of the previous section and fill their cells with positive integers to form tableaux.

Definition 2.4.1. Given a skew shape λ/μ, we define a *semistandard Young tableau* (abbreviated to *SSYT*) T of *shape* $sh(T) = \lambda/\mu$ to be a filling

$$T : \lambda/\mu \to \mathbb{Z}^+$$

of the cells of λ/μ such that

1. the entries in each row are weakly increasing when read from left to right
2. the entries in each column are strictly increasing when read from bottom to top.

A *standard Young tableau* (abbreviated to *SYT*) is an SSYT in which the filling is a bijection $T : \lambda/\mu \to [\,|\lambda/\mu|\,]$, that is, each of the numbers $1, 2, \ldots, |\lambda/\mu|$ appears exactly once. Sometimes we will abuse notation and use SSYTs and SYTs to denote the set of all such tableaux.

Example 2.4.2. An SSYT and SYT, respectively, are shown below.

7			
5	5		
•	2	3	
•	•	2	2
•	•	•	1

7			
6	9		
2	5		
1	3	4	8

Given an SSYT T, we define the *content* of T, denoted by $\mathrm{cont}(T)$, to be the list of nonnegative integers

$$\mathrm{cont}(T) = (c_1, c_2, \ldots, c_{max})$$

where c_i is the number of times i appears in T, and *max* is the largest integer appearing in T. Furthermore, given variables $x_1, x_2, \ldots$, we define the *monomial of* T to be

$$x^T = x_1^{c_1} x_2^{c_2} \cdots x_{max}^{c_{max}}.$$

Given an SYT T, its *column reading word*, denoted by $w_{col}(T)$, is obtained by listing the entries from the leftmost column in *decreasing* order, followed by the entries from the second leftmost column, again in decreasing order, and so on.

The *descent set* of an SYT T of size n, denoted by $\mathrm{Des}(T)$, is the subset of $[n-1]$ consisting of all entries i of T such that $i+1$ appears in the same column or a column to the left, that is,

$$\mathrm{Des}(T) = \{i | i+1 \text{ appears weakly left of } i\} \subseteq [n-1]$$

and the corresponding *descent composition* of T is

$$\mathrm{comp}(T) = \mathrm{comp}(\mathrm{Des}(T)).$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, the *canonical* SYT V_λ is the unique SYT satisfying $sh(V_\lambda) = \lambda$ and $\mathrm{comp}(V_\lambda) = (\lambda_1, \ldots, \lambda_k)$. In V_λ the first row is filled with $1, 2, \ldots, \lambda_1$ and row i for $2 \leqslant i \leqslant \ell(\lambda)$ is filled with

$$x+1, x+2, \ldots, x+\lambda_i$$

where $x = \lambda_1 + \cdots + \lambda_{i-1}$.

Example 2.4.3.

$$T = \begin{array}{|c|cccc}\hline 7 & \multicolumn{1}{|c}{} & & & \\ \hline 6 & \multicolumn{1}{|c|}{9} & & & \\ \hline 2 & \multicolumn{1}{|c|}{5} & & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{8} \\ \hline \end{array} \qquad V_{(4,2,2,1)} = \begin{array}{|c|cccc}\hline 9 & \multicolumn{1}{|c}{} & & & \\ \hline 7 & \multicolumn{1}{|c|}{8} & & & \\ \hline 5 & \multicolumn{1}{|c|}{6} & & & \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

$$\begin{aligned} \mathrm{Des}(T) &= \{1,4,5,6,8\} \\ \mathrm{comp}(T) &= (1,3,1,1,2,1) \\ w_{col}(T) &= 7621\,953\,4\,8 \end{aligned}$$

Definition 2.4.4. *Young's lattice* $\mathscr{L}_Y$ is the poset consisting of all partitions with the partial order $\subseteq$ of containment of the corresponding diagrams or, equivalently, the partial order in which $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is covered by

1. $(\lambda_1, \ldots, \lambda_\ell, 1)$, that is, the partition obtained by suffixing a part of size 1 to λ.
2. $(\lambda_1, \ldots, \lambda_k + 1, \ldots, \lambda_\ell)$, provided that $\lambda_i \neq \lambda_k$ for all $i < k$, that is, the partition obtained by adding 1 to a part of λ as long as that part is the leftmost part of that size.

Example 2.4.5. A saturated chain in $\mathscr{L}_Y$ is

$$(3,1,1) \lessdot_Y (3,2,1) \lessdot_Y (4,2,1) \lessdot_Y (4,2,1,1).$$

To any cover relation $\mu \lessdot_Y \lambda$ in $\mathscr{L}_Y$ we can associate the column number $\mathrm{col}(\mu \lessdot_Y \lambda)$ of the cell that is in the diagram λ but not μ. For example, we have $\mathrm{col}((3,1,1) \lessdot_Y (3,2,1)) = 2$ and $\mathrm{col}((4,3) \lessdot_Y (4,3,1)) = 1$. We extend this notion to the *column sequence* of a saturated chain, which is the sequence of column numbers of the successive cover relations in the chain, that is,

$$\mathrm{col}(\lambda^1 \lessdot_Y \cdots \lessdot_Y \lambda^k) = \mathrm{col}(\lambda^1 \lessdot_Y \lambda^2), \mathrm{col}(\lambda^2 \lessdot_Y \lambda^3), \ldots, \mathrm{col}(\lambda^{k-1} \lessdot_Y \lambda^k).$$

For example,

$$\mathrm{col}(\ (3,1,1) \lessdot_Y (3,2,1) \lessdot_Y (4,2,1) \lessdot_Y (4,2,1,1)\ )\ =\ 2,4,1.$$

Staying with saturated chains, we end this subsection with a well-known bijection between SYTs and saturated chains in $\mathscr{L}_Y$ implicit in [81, 7.10.3 Proposition].

Proposition 2.4.6. *A one-to-one correspondence between saturated chains in $\mathscr{L}_Y$ and SYTs is given by*

$$\lambda^0 \lessdot_Y \lambda^1 \lessdot_Y \lambda^2 \lessdot_Y \cdots \lessdot_Y \lambda^n \ \leftrightarrow\ T$$

where T is the SYT of shape λ^n/λ^0 such that the number i appears in the cell in T that exists in λ^i but not λ^{i-1}.

Example 2.4.7. The saturated chain in $\mathscr{L}_Y$

$$\begin{aligned}\emptyset \lessdot_Y (1) \lessdot_Y (1,1) \lessdot_Y (2,1) \lessdot_Y (3,1) \lessdot_Y (3,2)& \\ \lessdot_Y (3,2,1) \lessdot_Y (3,2,1,1) \lessdot_Y (4,2,1,1) \lessdot_Y (4,2,2,1)&\end{aligned}$$

corresponds to the following SYT.

$$\begin{array}{|c|c|c|c|}\cline{1-1} 7 \\ \cline{1-2} 6 & 9 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 4 & 8 \\ \hline\end{array}$$

2.5 Reverse tableaux

Closely related to the tableaux of the previous section are reverse tableaux, which we introduce now.

Definition 2.5.1. Given a skew shape λ/μ, we define a *semistandard reverse tableau* (abbreviated to *SSRT*) $\check{T}$ of *shape* $sh(\check{T}) = \lambda/\mu$ to be a filling

$$\check{T} : \lambda/\mu \to \mathbb{Z}^{+}$$

of the cells of λ/μ such that

1. the entries in each row are weakly decreasing when read from left to right
2. the entries in each column are strictly decreasing when read from bottom to top.

A *standard reverse tableau* (abbreviated to *SRT*) is an SSRT in which the filling is a bijection $\check{T} : \lambda/\mu \to [\,|\lambda/\mu|\,]$, that is, each of the numbers $1, 2, \ldots, |\lambda/\mu|$ appears exactly once. Sometimes we will abuse notation and use SSRTs and SRTs to denote the set of all such tableaux.

Example 2.5.2. An SSRT and SRT, respectively, are shown below.

$$\begin{array}{cccc} 1 & & & \\ 3 & 3 & & \\ \bullet & 6 & 5 & \\ \bullet & \bullet & 6 & 6 \\ \bullet & \bullet & \bullet & 7 \end{array} \qquad\qquad \begin{array}{cccc} 3 & & & \\ 4 & 1 & & \\ 8 & 5 & & \\ 9 & 7 & 6 & 2 \end{array}$$

Exactly as with SSYTs, given an SSRT $\check{T}$, we define the *content* of $\check{T}$, denoted by $\mathrm{cont}(\check{T})$, to be the list of nonnegative integers

$$\mathrm{cont}(\check{T}) = (c_1, c_2, \ldots, c_{max})$$

where c_i is the number of times i appears in $\check{T}$, and *max* is the largest integer appearing in $\check{T}$. Given variables $x_1, x_2, \ldots$, we define the *monomial of* $\check{T}$ to be

$$x^{\check{T}} = x_1^{c_1} x_2^{c_2} \cdots x_{max}^{c_{max}}.$$

Given an SRT $\check{T}$, its *column reading word*, denoted by $w_{col}(\check{T})$, is obtained by listing the entries from the leftmost column in *increasing* order, followed by the entries from the second leftmost column, again in increasing order, and so on.

The *descent set* of an SRT $\check{T}$ of size n, denoted by $\mathrm{Des}(\check{T})$, is the subset of $[n-1]$ consisting of all entries i of $\check{T}$ such that $i+1$ appears in the same column or a column to the right, that is,

$$\mathrm{Des}(\check{T}) = \{i \mid i+1 \text{ appears weakly right of } i\} \subseteq [n-1]$$

and the corresponding *descent composition* of $\check{T}$ is

$$\operatorname{comp}(\check{T}) = \operatorname{comp}(\operatorname{Des}(\check{T})).$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, the *canonical* SRT $\check{V}_\lambda$ is the unique SRT satisfying $sh(\check{V}_\lambda) = \lambda$ and $\operatorname{comp}(\check{V}_\lambda) = (\lambda_k, \ldots, \lambda_1)$. In $\check{V}_\lambda$ the first row is filled with $n, n-1, \ldots, n-\lambda_1+1$ and row i for $2 \leqslant i \leqslant \ell(\lambda)$ is filled with

$$x, x-1, \ldots, x-\lambda_i+1$$

where $x = n - (\lambda_1 + \cdots + \lambda_{i-1})$.

Example 2.5.3.

$$\check{T} = \begin{array}{|c|c|c|c|} \cline{1-1} 3 \\ \cline{1-2} 4 & 1 \\ \cline{1-2} 8 & 5 \\ \hline 9 & 7 & 6 & 2 \\ \hline \end{array} \qquad \check{V}_{(4,2,2,1)} = \begin{array}{|c|c|c|c|} \cline{1-1} 1 \\ \cline{1-2} 3 & 2 \\ \cline{1-2} 5 & 4 \\ \hline 9 & 8 & 7 & 6 \\ \hline \end{array}$$

$$\begin{aligned} \operatorname{Des}(\check{T}) &= \{1,3,4,5,8\} \\ \operatorname{comp}(\check{T}) &= (1,2,1,1,3,1) \\ w_{col}(\check{T}) &= 3489\ 157\ 6\ 2 \end{aligned}$$

There is a natural shape-preserving bijection

$$\check{\Gamma} : SYTs \to SRTs$$

where for an SYT T with n cells we replace each entry i by the entry $n-i+1$, obtaining an SRT $\check{T} = \check{\Gamma}(T)$ of the same skew shape.

Therefore, we have an analogue to Proposition 2.4.6 for SRTs.

Proposition 2.5.4. *A one-to-one correspondence between saturated chains in $\mathcal{L}_Y$ and SRTs is given by*

$$\lambda^0 \lessdot_Y \lambda^1 \lessdot_Y \lambda^2 \lessdot_Y \cdots \lessdot_Y \lambda^n \ \leftrightarrow\ \check{T}$$

where $\check{T}$ is the SRT of shape λ^n/λ^0 such that the number $n-i+1$ appears in the cell in $\check{T}$ that exists in λ^i but not λ^{i-1}.

Example 2.5.5. The saturated chain in $\mathcal{L}_Y$

$$\begin{aligned} \emptyset \lessdot_Y (1) \lessdot_Y (1,1) \lessdot_Y (2,1) \lessdot_Y (3,1) \lessdot_Y (3,2) \\ \lessdot_Y (3,2,1) \lessdot_Y (3,2,1,1) \lessdot_Y (4,2,1,1) \lessdot_Y (4,2,2,1) \end{aligned}$$

corresponds to the following SRT.

$$\begin{array}{|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-2} 4 & 1 \\ \cline{1-2} 8 & 5 \\ \hline 9 & 7 & 6 & 2 \\ \hline\end{array}$$

Additionally, there is a simple relationship between descent compositions of SYTs and SRTs. We include its proof as the equivalent statements are useful to know.

Proposition 2.5.6. *Given an SYT T, we have* $\mathrm{comp}(\check{\Gamma}(T))) = \mathrm{comp}(T)^r$.

Proof. Suppose T is an SYT with n cells. The following statements are equivalent.

1. $i \in \mathrm{Des}(T)$.
2. $i+1$ is weakly to the left of i in T.
3. $n-i$ is weakly to the left of $n-i+1$ in $\check{\Gamma}(T)$.
4. $n-i+1$ is weakly to the right of $n-i$ in $\check{\Gamma}(T)$.
5. $n-i \in \mathrm{Des}(\check{\Gamma}(T))$.

This establishes the claim. □

2.6 Schensted insertion

Schensted insertion is an algorithm with many interesting combinatorial properties and applications to representation theory. For further details see [31,72,81]. We will also use this algorithm and the variation below in Chapter 5.

In particular, *Schensted insertion* inserts a positive integer k_1 into a semistandard or standard Young tableau T and is denoted by $T \leftarrow k_1$.

1. If k_1 is greater than or equal to the last entry in row 1, place it at the end of the row, else
2. find the leftmost entry in that row strictly larger than k_1, say k_2, then
3. replace k_2 by k_1, that is, k_1 *bumps* k_2.
4. Repeat the previous steps with k_2 and row 2, k_3 and row 3, etc.

The set of cells whose values are modified by the insertion, including the final cell added, is called the *insertion path*, and the final cell is called the *new cell*.

Example 2.6.1. If we insert 5, then we have

$$\begin{array}{|c|c|c|c|}\cline{1-2} 7 & 7 \\ \cline{1-3} 5 & 6 & 6 \\ \cline{1-3} 2 & 4 & 5 \\ \hline 1 & 3 & 4 & 6 \\ \hline\end{array} \quad \leftarrow 5 \quad = \quad \begin{array}{|c|c|c|c|}\cline{1-2} 7 & 7 \\ \cline{1-3} 5 & 6 & 6 \\ \hline 2 & 4 & 5 & \mathbf{6} \\ \hline 1 & 3 & 4 & \mathbf{5} \\ \hline\end{array}$$

where the bold cells indicate the insertion path. Meanwhile, if we insert 3, then we have

$$\begin{ytableau} 7 & 7 \\ 5 & 6 & 6 \\ 2 & 4 & 5 \\ 1 & 3 & 4 & 6 \end{ytableau} \quad \leftarrow 3 \quad = \quad \begin{ytableau} \mathbf{7} \\ \mathbf{6} & 7 \\ 5 & \mathbf{5} & 6 \\ 2 & 4 & \mathbf{4} \\ 1 & 3 & \mathbf{3} & 6 \end{ytableau}$$

where the bold cells again indicate the insertion path.

Similarly we have *Schensted insertion for reverse tableaux* [40], which inserts a positive integer k_1 into a semistandard or standard reverse tableau $\check{T}$ and is denoted by $\check{T} \leftarrow k_1$.

1. If k_1 is less than or equal to the last entry in row 1, place it at the end of the row, else
2. find the leftmost entry in that row strictly smaller than k_1, say k_2, then
3. replace k_2 by k_1, that is, k_1 *bumps* k_2.
4. Repeat the previous steps with k_2 and row 2, k_3 and row 3, etc.

As before, the set of cells whose values are modified by the insertion, including the final cell added, is called the *insertion path*, and the final cell is called the *new cell*.

Example 2.6.2. If we insert 5, then we have

$$\begin{ytableau} 1 & 1 \\ 3 & 2 & 2 \\ 6 & 4 & 3 \\ 7 & 5 & 4 & 2 \end{ytableau} \quad \leftarrow 5 \quad = \quad \begin{ytableau} \mathbf{1} \\ \mathbf{2} & 1 \\ 3 & \mathbf{3} & 2 \\ 6 & 4 & \mathbf{4} \\ 7 & 5 & \mathbf{5} & 2 \end{ytableau}$$

where the bold cells indicate the insertion path.

Given a type of insertion and list of positive integers $\sigma = \sigma_1 \cdots \sigma_n$, we define the P-*tableau*, or *insertion tableau*, or *rectification* of σ, denoted by $\mathrm{P}(\sigma)$, to be

$$(\cdots((\emptyset \leftarrow \sigma_1) \leftarrow \sigma_2)\cdots) \leftarrow \sigma_n.$$

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