Chapter 2
Well-posedness of Optimal Control Problems Without Convexity Assumptions

In this chapter we prove generic existence results for classes of optimal control problems in which constraint maps are also subject to variations as well as the cost functions. These results were obtained in [87, 90]. More precisely, we establish generic existence results for classes of optimal control problems (with the same system of differential equations, the same boundary conditions and without convexity assumptions) which are identified with the corresponding complete metric spaces of pairs (f, U) (where f is an integrand satisfying a certain growth condition and U is a constraint map) endowed with some natural topology. We will show that for a generic pair (f, U) the corresponding optimal control problem has a unique solution.

In the theory developed here topologies on spaces of integrands and on spaces of integrand–map pairs are of great importance. Actually one space of integrand–map pairs, say $\mathcal{A}$, considered here is a topological product of a space of integrands and a space of multivalued maps. The values of these maps are elements of the space of all nonempty convex closed subsets of a finite-dimensional Euclidean space endowed with the Hausdorff distance. In the space of multivalued maps we consider the topology of uniform convergence. For the space of integrands we consider weak and strong topologies which induce weak and strong topologies on the space $\mathcal{A}$. We will prove the existence of a set $\mathcal{A}' \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets such that for each $(f, U) \in \mathcal{A}'$ the corresponding optimal control problem has a unique solution. In fact we will establish our result for various spaces of integrands: the space of the so-called $\mathcal{L} \bigotimes \mathcal{B}$-measurable integrands, the space of lower semicontinuous integrands and the space of continuous integrands, as well as their subspaces consisting of integrands $f(t, x, u)$ differentiable in u and subspaces consisting of integrands $f(t, x, u)$ differentiable in x and u. All these spaces are endowed with same weak topology. Their strong topology is always stronger then the topology of uniform convergence.

If we say that a function (set) is measurable we mean that it is Lebesgue measurable.

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2.1 Optimal Control Problems with Cesari Growth Condition

We use the following notations and definitions. Let $k \geq 1$ be an integer. We denote by $\mathrm{mes}(E)$ the Lebesgue measure of a measurable set $E \subset R^k$, by $|\cdot|$ the Euclidean norm in R^k, and by $< \cdot, \cdot >$ the scalar product in R^k. We use the convention that $\infty - \infty = 0$. For any $f \in C^q(R^k)$ we set

$$||f||_{C^q} = ||f||_{C^q(R^k)} = \sup_{z \in R^k} \{|\partial^{|\alpha|} f(z)/\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}| : \tag{2.1}$$

$$\alpha_i \geq 0 \text{ is an integer}, \; i = 1, \dots, k, \; |\alpha| \leq q\},$$

where $|\alpha| = \sum_{i=1}^k \alpha_i$.

For each function $f : X \to [-\infty, \infty]$ where X is nonempty, we set $\inf(f) = \inf\{f(x) : \; x \in X\}$. For each set-valued mapping $U : X \to 2^Y \setminus \{\emptyset\}$ where X and Y are nonempty, we set

$$\mathrm{graph}(U) = \{(x, y) \in X \times Y : \; y \in U(x)\}. \tag{2.2}$$

Let $m, n, N \geq 1$ be integers. We assume that Ω is a fixed bounded domain in R^m, $H(t, x, u)$ is a fixed continuous function defined on $\Omega \times R^n \times R^N$ with values in R^{mn} such that $H(t, x, u) = (H_i)_{i=1}^n$ and $H_i = (H_{ij})_{j=1}^m$, $i = 1, \dots n$, B_1 and B_2 are fixed nonempty closed subsets of R^n and $\theta^* = (\theta_i^*)_{i=1}^n \in (W^{1,1}(\Omega))^n$ is also fixed. Here

$$W^{1,1}(\Omega) = \{u \in L^1(\Omega) : \; \partial u/\partial x_j \in L^1(\Omega), \; j = 1, \dots m\}$$

and $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,1}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of smooth functions $u : \; \Omega \to R^1$ with compact support in Ω [108].

If $m = 1$, then we assume that $\Omega = (T_1, T_2)$, where T_1 and T_2 are fixed real numbers for which $T_1 < T_2$.

For a function $u = (u_1, \dots u_n)$, where $u_i \in W^{1,1}(\Omega)$, $i = 1, \dots n$, we set

$$\nabla u_i = (\partial u_i/\partial x_j)_{j=1}^m, \; i = 1, \dots n, \; \nabla u = (\nabla u_i)_{i=1}^n.$$

Define set-valued mappings $\tilde{A} : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and $\tilde{U} : \Omega \times R^n \to 2^{R^N} \setminus \{\emptyset\}$ by

$$\tilde{A}(t) = R^n, \, t \in \Omega, \; \tilde{U}(t, x) = R^N, \; (t, x) \in \Omega \times R^n. \tag{2.3}$$

For each $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and each $U : \mathrm{graph}(A) \to 2^{R^N} \setminus \{\emptyset\}$ for which $\mathrm{graph}(U)$ is a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology, we denote by $X(A, U)$ the set of all pairs of functions (x, u), where

$x = (x_1, \ldots, x_n) \in (W^{1,1}(\Omega))^n$, $u = (u_1, \ldots u_N) : \Omega \to R^N$ is Lebesgue measurable and the following relations hold:

$$x(t) \in A(t),\ t \in \Omega \text{ (a.e.)},\ u(t) \in U(t, x(t)),\ t \in \Omega \text{ (a.e.)}, \tag{2.4a}$$

$$\nabla x(t) = H(t, x(t), u(t)),\ t \in \Omega \text{ (a.e.)}, \tag{2.4b}$$

$$\text{if } m = 1,\ \text{then } x(T_i) \in B_i,\ i = 1, 2, \tag{2.4c}$$

$$\text{if } m > 1 \text{ then } x - \theta^* \in (W_0^{1,1}(\Omega))^n. \tag{2.4d}$$

Note that in the definition of the space $X(A, U)$ we use the boundary condition (2.4c) in the case $m = 1$ while in the case $m > 1$ we use the boundary condition (2.4d). Both of them are common in the literature [12, 13, 17, 21].

We do this to provide a unified treatment for both cases. Note that we prove our generic result in the case $m = 1$ for a class of Bolza problems (with the same boundary condition (2.4c)) while in the case $m > 1$ it will be established for a class of Lagrange problems (with the same boundary condition (2.4d)).

To be more precise, we have to define elements of $X(A, U)$ as classes of pairs equivalent in the sense that (x_1, u_1) and (x_2, u_2) are equivalent if and only if $x_2(t) = x_1(t),\ u_2(t) = u_1(t)$, $t \in \Omega$ (a.e.) If $m = 1$, then by an appropriate choice of representatives, $W^{1,1}(T_1, T_2)$ can be identified with the set of absolutely continuous functions $x : [T_1, T_2] \to R^1$, and we will henceforth assume that this has been done.

Let $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, $U : \text{graph}(A) \to 2^{R^N} \setminus \{\emptyset\}$ and let $\text{graph}(U)$ be a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology.

For the set $X(A, U)$ defined above we consider the uniformity which is determined by the following base:

$$E_X(\epsilon) = \{((x_1, u_1), (x_2, u_2)) \in X(A, U) \times X(A, U) : \tag{2.5}$$
$$\text{mes}\{t \in \Omega :\ |x_1(t) - x_2(t)| + |u_1(t) - u_2(t)| \geq \epsilon\} \leq \epsilon\},$$

where $\epsilon > 0$. It is easy to see that the uniform space $X(A, U)$ is metrizable (by a metric ρ) (see [44]). In the space $X(A, U)$ we consider the topology induced by the metric ρ.

Next we define spaces of integrands associated with the maps A and U. By $\mathcal{M}(A, U)$ we denote the set of all functions $f : \text{graph}(U) \to R^1 \cup \{\infty\}$ with the following properties:

(i) f is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of Ω and Borel subsets of $R^n \times R^N$.
(ii) $f(t, \cdot, \cdot)$ is lower semicontinuous for almost every $t \in \Omega$.
(iii) For each $\epsilon > 0$ there exists an integrable scalar function $\psi_\epsilon(t) \geq 0,\ t \in \Omega$, such that $|H(t, x, u)| \leq \psi_\epsilon(t) + \epsilon f(t, x, u)$ for all $(t, x, u) \in$ graph (U).

The growth condition in (iii) was proposed by Cesari (see [21]) and its equivalents and modifications are rather common in the literature. Due to property (i) for every $f \in \mathcal{M}(A, U)$ and every $(x, u) \in X(A, U)$ the function $f(t, x(t), u(t))$, $t \in \Omega$ is measurable.

Denote by $\mathcal{M}^l(A, U)$ (respectively, $\mathcal{M}^c(A, U)$) the set of all lower semicontinuous (respectively, finite-valued continuous) functions $f : \mathrm{graph}(U) \to R^1 \cup \{\infty\}$ in $\mathcal{M}(A, U)$. Now we equip the set $\mathcal{M}(A, U)$ with the strong and weak topologies. For the space $\mathcal{M}(A, U)$ we consider the uniformity determined by the following base:

$$E_{\mathcal{M}}(\epsilon) = \{(f, g) \in \mathcal{M}(A, U) \times \mathcal{M}(A, U) : \quad (2.6)$$
$$|f(t, x, u) - g(t, x, u)| \le \epsilon, \ (t, x, u) \in \ \mathrm{graph}(U)\},$$

where $\epsilon > 0$. It is easy to see that the uniform space $\mathcal{M}(A, U)$ with this uniformity is metrizable (by a metric $d_{\mathcal{M}}$) and complete. This uniformity generates in $\mathcal{M}(A, U)$ the strong topology. Clearly $\mathcal{M}^l(A, U)$ and $\mathcal{M}^c(A, U)$ are closed subsets of $\mathcal{M}(A, U)$ with this topology.

For each $\epsilon > 0$ we set

$$E_{\mathcal{M}w}(\epsilon) = \{(f, g) \in \mathcal{M}(A, U) \times \mathcal{M}(A, U) : \ \text{there exists a nonnegative} \quad (2.7)$$
$$\phi \in L^1(\Omega) \text{ such that } \int_{\Omega} \phi(t) dt \le 1, \text{ and for almost every } t \in \Omega,$$
$$|f(t, x, u) - g(t, x, u)| < \epsilon + \epsilon \max\{|f(t, x, u)|, |g(t, x, u)|\} + \epsilon\phi(t)$$
$$\text{for each } x \in A(t) \text{ and each } u \in U(t, x)\}.$$

Using the following simple lemma [87] we can easily show that for the set $\mathcal{M}(A, U)$ there exists the uniformity which is determined by the base $E_{\mathcal{M}w}(\epsilon)$, $\epsilon > 0$. This uniformity induces in $\mathcal{M}(A, U)$ the weak topology.

Lemma 2.1. *Let $a, b \in R^1$, $\epsilon \in (0, 1)$, $\Delta \ge 0$, and*

$$|a - b| < (1 + \Delta)\epsilon + \epsilon \max\{|a|, |b|\}.$$

Then

$$|a - b| < (1 + \Delta)(\epsilon + \epsilon^2(1 - \epsilon)^{-1}) + \epsilon(1 - \epsilon)^{-1} \min\{|a|, |b|\}.$$

Denote by $C_l(B_1 \times B_2)$ the set of all lower semicontinuous functions $\xi : B_1 \times B_2 \to R^1 \cup \{\infty\}$ bounded from below. We also equip the set $C_l(B_1 \times B_2)$ with strong and weak topologies. For the set $C_l(B_1 \times B_2)$ we consider the uniformity determined by the following base:

$$E_c(\epsilon) = \{(\xi, h) \in C_l(B_1 \times B_2) \times C_l(B_1 \times B_2) : \ |\xi(z) - h(z)| \le \epsilon, \ z \in B_1 \times B_2\}, \quad (2.8)$$

where $\epsilon > 0$. It is easy to see that the uniform space $C_l(B_1 \times B_2)$ is metrizable (by a metric d_c) and complete. This metric induces in $C_l(B_1 \times B_2)$ the strong topology. We do not write down the explicit expressions for the metrics $d_{\mathcal{M}}$ and d_c because we are not going to use them in the sequel.

For any $\epsilon > 0$ we set

$$E_{cw}(\epsilon) = \{(\xi, h) \in C_l(B_1 \times B_2) \times C_l(B_1 \times B_2) : \ |\xi(z) - h(z)| \\ < \epsilon + \epsilon \max\{|\xi(z)|, |h(z)|\}, \ z \in B_1 \times B_2\}, \tag{2.9}$$

where $\epsilon > 0$. By using Lemma 2.1 we can easily show that for the set $C_l(B_1 \times B_2)$ there exists a uniformity which is determined by the base $E_{cw}(\epsilon)$, $\epsilon > 0$. This uniformity induces in $C_l(B_1 \times B_2)$ the weak topology. Denote by $C(B_1 \times B_2)$ the set of all finite-valued continuous functions h in $C_l(B_1 \times B_2)$. Clearly it is a closed subset of $C_l(B_1 \times B_2)$ with the weak topology.

In the case $m > 1$ for each $f \in \mathcal{M}(A, U)$ we define $I^{(f)} : \ X(A, U) \to R^1 \cup \{\infty\}$ by

$$I^{(f)}(x, u) = \int_{\Omega} f(t, x(t), u(t))dt, \ (x, u) \in X(A, U). \tag{2.10}$$

In the case $m = 1$ for each $f \in \mathcal{M}(A, U)$ and each $\xi \in C_l(B_1 \times B_2)$ we define $I^{(f,\xi)} : \ X(A, U) \to R^1 \cup \{\infty\}$ by

$$I^{(f,\xi)}(x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt + \xi(x(T_1), x(T_2)), \ (x, u) \in X(A, U). \tag{2.11}$$

We will show (see Propositions 2.6 and 2.7) that in both cases (2.10) and (2.11) define lower semicontinuous functionals on $X(A, U)$.

From now on in this section we consider a fixed set-valued mapping $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ for which graph(A) is a closed subset of the space $\Omega \times R^n$ with the product topology. Denote by $\tilde{U}_A$ the restriction of $\tilde{U}$ (see (2.3)) to the graph(A). Namely

$$\tilde{U}_A : \ \text{graph}(A) \to 2^{R^N}, \ \tilde{U}(t, x) = R^N, \ (t, x) \in \ \text{graph}(A). \tag{2.12}$$

We consider functionals $I^{(f,\xi)}$ with $(f, \xi) \in \mathcal{M}(A, \tilde{U}_A) \times C_l(B_1 \times B_2)$ (in the case $m = 1$) and functionals $I^{(f)}$ with $f \in \mathcal{M}(A, \tilde{U}_A)$ (in the case $m > 1$) defined on the space $X(A, \tilde{U}_A)$ (see (2.4)). Our generic existence result will be established for several classes of optimal control problems with different corresponding spaces of the integrands which are subsets of the space $\mathcal{M}(A, \tilde{U}_A)$. The subspaces of lower semicontinuous and continuous integrands ($\mathcal{M}^l(A, \tilde{U}_A)$ and $\mathcal{M}^c(A, \tilde{U}_A)$) have already been defined. Now we define subspaces of $\mathcal{M}(A, \tilde{U}_A)$ which consist of integrands differentiable with respect to the control variable u.

Let $k \geq 1$ be an integer. Denote by $\mathcal{M}_k(A, \tilde{U}_A)$ the set of all finite-valued $f \in \mathcal{M}(A, \tilde{U}_A)$ such that for each $(t, x) \in$ graph(A) the function $f(t, x, \cdot) \in C^k(R^N)$.

We consider the topological subspace $\mathcal{M}_k(A, \tilde{U}_A) \subset \mathcal{M}(A, \tilde{U}_A)$ with the relative weak topology. The strong topology on $\mathcal{M}_k(A, \tilde{U}_A)$ is induced by the uniformity which is determined by the following base:

$$E_{\mathcal{M}k}(\epsilon) = \{(f, g) \in \mathcal{M}_k(A, \tilde{U}_A) \times \mathcal{M}_k(A, \tilde{U}_A) : \; |f(t, x, u) - g(t, x, u)| \leq \epsilon \tag{2.13}$$

$$\text{for all } (t, x, u) \in \text{ graph}(A) \times R^N \text{ and}$$

$$||f(t, x, \cdot) - g(t, x, \cdot)||_{C^k(R^N)} \leq \epsilon \text{ for all } (t, x) \in \text{graph}(A)\},$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}_k(A, \tilde{U}_A)$ with this uniformity is metrizable (by a metric $d_{\mathcal{M},k}$) and complete. Define

$$\mathcal{M}_k^l(A, \tilde{U}_A) = \mathcal{M}_k(A, \tilde{U}_A) \cap \mathcal{M}^l(A, \tilde{U}_A),$$
$$\mathcal{M}_k^c(A, \tilde{U}_A) = \mathcal{M}_k(A, \tilde{U}_A) \cap \mathcal{M}^c(A, \tilde{U}_A). \tag{2.14}$$

Clearly $\mathcal{M}_k^l(A, \tilde{U}_A)$ and $\mathcal{M}_k^c(A, \tilde{U}_A)$ are closed sets in $\mathcal{M}_k(A, \tilde{U}_A)$ with the strong topology.

Finally we define subspaces of $\mathcal{M}(\tilde{A}, \tilde{U})$ which consist of integrands differentiable with respect to the state variable x and the control variable u. Denote by $\mathcal{M}_k^*(\tilde{A}, \tilde{U})$ the set of all $f : \Omega \times R^n \times R^N \to R^1$ in $\mathcal{M}(\tilde{A}, \tilde{U})$ (see (2.3)) such that for each $t \in \Omega$ the function $f(t, \cdot, \cdot) \in C^k(R^n \times R^N)$. We consider the topological subspace $\mathcal{M}_k^*(\tilde{A}, \tilde{U}) \subset \mathcal{M}(\tilde{A}, \tilde{U})$ with the relative weak topology. The strong topology in $\mathcal{M}_k^*(\tilde{A}, \tilde{U})$ is induced by the uniformity which is determined by the following base:

$$E^*_{\mathcal{M}k}(\epsilon) = \{(f, g) \in \mathcal{M}_k^*(\tilde{A}, \tilde{U}) \times \mathcal{M}_k^*(\tilde{A}, \tilde{U}) : \tag{2.15}$$

$$|f(t, x, u) - g(t, x, u)| \leq \epsilon \text{ for all } (t, x, u) \in \Omega \times R^n \times R^N \text{ and}$$

$$||f(t, \cdot, \cdot) - g(t, \cdot, \cdot)||_{C^k(R^{n+N})} \leq \epsilon \text{ for all } t \in \Omega\},$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{M}_k^*(\tilde{A}, \tilde{U})$ with this uniformity is metrizable (by a metric $d^*_{\mathcal{M},k}$) and complete. Define

$$\mathcal{M}_k^{*l}(\tilde{A}, \tilde{U}) = \mathcal{M}_k^*(\tilde{A}, \tilde{U}) \cap \mathcal{M}^l(\tilde{A}, \tilde{U}), \; \mathcal{M}_k^{*c}(\tilde{A}, \tilde{U}) = \mathcal{M}_k^*(\tilde{A}, \tilde{U}) \cap \mathcal{M}^c(\tilde{A}, \tilde{U}). \tag{2.16}$$

Clearly $\mathcal{M}_k^{*l}(\tilde{A}, \tilde{U})$ and $\mathcal{M}_k^{*c}(\tilde{A}, \tilde{U})$ are closed sets in $\mathcal{M}_k^*(\tilde{A}, \tilde{U})$ with the strong topology.

Thus we have defined all the spaces of integrands for which we will prove our generic existence result. Now we will define a space of constraint maps $\mathcal{P}_A$. Denote by $S(R^N)$ the set of all nonempty convex closed subsets of R^N. For each $x \in R^N$ and each $E \subset R^N$, set $d_H(x, E) = \inf_{y \in E} |x - y|$. For each pair of sets $C_1, C_2 \subset R^N$,

$$d_H(C_1, C_2) = \max\left\{ \sup_{y \in C_1} d_H(y, C_2),\ \sup_{x \in C_2} d_H(x, C_1) \right\}$$

is the Hausdorff distance between C_1 and C_2. For the space $S(R^N)$ we consider the uniformity determined by the following base:

$$E_{R^N}(\epsilon) = \{(C_1, C_2) \in S(R^N) \times S(R^N) :\ d_H(C_1, C_2) \le \epsilon\}, \tag{2.17}$$

where $\epsilon > 0$. It is well known that the space $S(R^N)$ with this uniformity is metrizable and complete. Denote by $\mathcal{P}_A$ the set of all set-valued mappings $U : \mathrm{graph}(A) \to S(R^N)$ such that $\mathrm{graph}(U)$ is a closed subset of the space $\mathrm{graph}(A) \times R^N$ with the product topology. For the space $\mathcal{P}_A$ we consider the uniformity determined by the following base:

$$E_{\mathcal{P}_A}(\epsilon) = \{(U_1, U_2) \in \mathcal{P}_A \times \mathcal{P}_A :\ d_H(U_1(t, x), U_2(t, x)) \le \epsilon \tag{2.18}$$
$$\text{for all } (t, x) \in \mathrm{graph}(A)\},$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{P}_A$ with this uniformity is metrizable and complete.

We consider the space $X(A, \tilde{U}_A)$ with the metric ρ (see (2.5)). For each $U \in \mathcal{P}_A$ define

$$S_U = X(A, U) = \{(x, u) \in X(A, \tilde{U}_A) :\ u(t) \in U(t, x(t)),\ t \in \Omega \text{ (a.e.)}\}. \tag{2.19}$$

In the case $m = 1$ for each $U \in \mathcal{P}_A$ and each $(f, \xi) \in \mathcal{M}(A, \tilde{U}_A) \times C_l(B_1 \times B_2)$ we consider the optimal control problem

$$I^{(f,\xi)}(x, u) \to \min,\ (x, u) \in X(A, U)$$

and in the case $m > 1$ for each $U \in \mathcal{P}_A$ and each $f \in \mathcal{M}(A, \tilde{U}_A)$ we consider the optimal control problem

$$I^{(f)}(x, u) \to \min,\ (x, u) \in X(A, U).$$

We will state our existence result, Theorem 2.2, in such a manner that it will be applicable to the Bolza problem in case $m = 1$ and to the Lagrange problem in case $m > 1$, and also applicable for all the spaces of integrands defined above.

To meet this goal we set $\mathcal{A}_2 = \mathcal{P}_A$ and define a space $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1 = \mathcal{A}_{11} \times \mathcal{A}_{12} \text{ if } m = 1 \text{ and } \mathcal{A}_1 = \mathcal{A}_{11} \text{ if } m > 1,$$

where $\mathcal{A}_{12}$ is either $C_l(B_1 \times B_2)$ or $C(B_1 \times B_2)$ or a singleton $\{\xi\} \subset C_l(B_1 \times B_2)$, and $\mathcal{A}_{11}$ is one of the following spaces:

$$\mathcal{M}(A, \tilde{U}_A);\ \mathcal{M}^l(A, \tilde{U}_A);\ \mathcal{M}^c(A, \tilde{U}_A);$$

$$\mathcal{M}_k(A, \tilde{U}_A);\ \mathcal{M}_k^l(A, \tilde{U}_A);\ \mathcal{M}_k^c(A, \tilde{U}_A) \ \text{(here } k \ge 1 \text{ is an integer);}$$

$\mathcal{M}_k^*(\tilde{A}, \tilde{U})$; $\mathcal{M}_k^{*l}(\tilde{A}, \tilde{U})$; $\mathcal{M}_k^{*c}(\tilde{A}, \tilde{U})$ (here $k \geq 1$ is an integer and $A = \tilde{A}$).

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define $J_a : X(A, \tilde{U}_A) \to R^1 \cup \{\infty\}$ by

$$J_a(x, u) = I^{(a_1)}(x, u),\ (x, u) \in S_{a_2},\ J_a(x, u) = \infty,\ (x, u) \in X(A, \tilde{U}_A) \setminus S_{a_2}.$$

We will show that J_a is lower semicontinuous for all $a \in \mathcal{A}_1 \times \mathcal{A}_2$. Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 :\ \inf(J_a) < \infty\}$ in the space $\mathcal{A}_1 \times A_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty. The following theorem is the main result of this chapter.

Theorem 2.2. *There exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the following assertions hold:*

(1) $\inf(J_a)$ *is finite and attained at a unique pair* $(\bar{x}, \bar{u}) \in X(A, \tilde{U}_A)$.
(2) For each $\epsilon > 0$ there are a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(J_b)$ *is finite and if $(z, w) \in X(A, \tilde{U}_A)$ satisfies $J_b(z, w) \leq \inf(J_b) + \delta$, then $\rho((\bar{x}, \bar{u}), (z, w)) \leq \epsilon$ and $|J_b(z, w) - J_a(\bar{x}, \bar{u})| \leq \epsilon$.*

Theorem 2.2 was obtained in [87].

2.2 A Generic Variational Principle

We will obtain Theorem 2.2 as a realization of a variational principle which was introduced in [87]. This variational principle is a modification of the variational principle in [42].

We consider a metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology. (Note that these topologies can coincide.) We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$. In our study we use the following basic hypotheses about the functions.

(H1) For any $a \in \mathcal{A}$, any $\epsilon > 0$, and any $\gamma > 0$ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$, and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A} :\ d(a, b) < \epsilon\} \neq \emptyset$$

and for any $b \in \mathcal{W}$

(i) $\inf(f_b)$ is finite.
(ii) If $z \in X$ is such that $f_b(z) \le \inf(f_b) + \eta$, then $\rho(z, x) \le \gamma$ and $|f_b(z) - \alpha| \le \gamma$.

(H2) If $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in X.

We will show (see Theorem 2.3) that if (H1) and (H2) hold, then for a generic $a \in \mathcal{A}$ the minimization problem $f_a(x) \to \min$, $x \in X$, has a unique solution. This result generalizes the variational principle in [42] which was obtained for the complete domain space (X, ρ). Note that if (X, ρ) is complete, the weak and strong topologies on $\mathcal{A}$ coincide, and for any $a \in \mathcal{A}$ the function f_a is not identically ∞, then the variational principles in [42] and in this section are equivalent.

For the classes of optimal control problems considered in this chapter the domain space is usually the space $X(A, \tilde{U}_A)$ with the metric ρ (see (2.5)) which is not complete. Since the variational principle in [42] was established only for complete domain spaces it cannot be applied to these classes of optimal control problems. Fortunately, instead of the completeness assumption we can use (H2) and this hypothesis holds for spaces of integrands (integrand–map pairs) which satisfy the Cesari growth condition.

Theorem 2.3. *Assume that (H1) and (H2) hold. Then there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the following assertions hold:*

(1) $\inf(f_a)$ *is finite and attained at a unique point* $\bar{x} \in X$.
(2) For each $\epsilon > 0$ *there are a neighborhood* $\mathcal{V}$ *of* a *in* $\mathcal{A}$ *with the weak topology and* $\delta > 0$ *such that for each* $b \in \mathcal{V}$, $\inf(f_b)$ *is finite and if* $z \in X$ *satisfies* $f_b(z) \le \inf(f_b) + \delta$, *then* $\rho(\bar{x}, z) \le \epsilon$ *and* $|f_b(z) - f_a(\bar{x})| \le \epsilon$.

Following the tradition, we can summarize the theorem by saying that under the assumptions (H1) and (H2) the minimization problem for f_a on (X, ρ) is generically strongly well posed with respect to $\mathcal{A}$.

Proof. Let $a \in \mathcal{A}$. By (H1) for any natural $n = 1, 2, \ldots$ there are a nonempty open set $\mathcal{U}(a, n)$ in $\mathcal{A}$ with the weak topology, $x(a, n) \in X$, $\alpha(a, n) \in R^1$, and $\eta(a, n) > 0$ such that

$$\mathcal{U}(a, n) \cap \{b \in \mathcal{A} : \ d(a, b) < 1/n\} \neq \emptyset$$

and for any $b \in \mathcal{U}(a, n)$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \eta(a, n)$, then

$$\rho(z, x(a, n)) \le 1/n, \ |f_b(z) - \alpha(a, n)| \le 1/n.$$

Define $\mathcal{B}_n = \cup\{\mathcal{U}(a, m) : \ a \in \mathcal{A}, \ m \ge n\}$ for $n = 1, 2, \ldots$. Clearly for each integer $n \ge 1$ the set $\mathcal{B}_n$ is open in the weak topology and everywhere dense in the strong topology. Set $\mathcal{B} = \cap_{n=1}^{\infty} \mathcal{B}_n$. Since for each integer $n \geq 1$ the set $\mathcal{B}_n$ is also open in the strong topology generated by the complete metric d we conclude that $\mathcal{B}$ is everywhere dense in the strong topology.

Let $b \in \mathcal{B}$. Evidently $\inf(f_b)$ is finite. There are a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$ and a strictly increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that $b \in \mathcal{U}(a_n, k_n)$, $n = 1, 2, \ldots$ Assume that $\{z_n\}_{n=1}^{\infty} \subset X$ and $\lim_{n\to\infty} f_b(z_n) = \inf(f_b)$.

Let $m \geq 1$ be an integer. Clearly for all large enough n the inequality $f_b(z_n) < \inf(f_b) + \eta(a_m, k_m)$ is true and it follows from the definition of $\mathcal{U}(a_m, k_m)$ that

$$\rho(z_n, x(a_m, k_m)) \leq k_m^{-1}, \ |f_b(z_n) - \alpha(a_m, k_m)| \leq k_m^{-1} \tag{2.20}$$

for all large enough n. Since m is an arbitrary natural number we conclude that $\{z_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence. By (H2) there is $\bar{x} = \lim_{n\to\infty} z_n$. As f_b is lower semicontinuous, we have $f_b(\bar{x}) = \inf(f_b)$. Clearly f_b does not have another minimizer for otherwise we would be able to construct a nonconvergent sequence $\{z_n\}_{n=1}^{\infty}$. This proves the first part of the theorem. We further note that by doing $n \to \infty$ in (2.20)

$$\rho(\bar{x}, x(a_m, k_m)) \leq k_m^{-1}, \ |f_b(\bar{x}) - \alpha(a_m, k_m)| \leq k_m^{-1}, \ m = 1, 2, \ldots \tag{2.21}$$

We turn now to the second assertion. Let $\epsilon > 0$. Choose a natural number m for which $4k_m^{-1} < \epsilon$. Let $a \in \mathcal{U}(a_m, k_m)$. Clearly $\inf(f_a)$ is finite. Let $z \in X$ and $f_a(z) \leq \inf(f_a) + \eta(a_m, k_m)$. By the definition of $\mathcal{U}(a_m, k_m)$,

$$\rho(z, x(a_m, k_m)) \leq k_m^{-1}, \ |f_a(z) - \alpha(a_m, k_m)| \leq k_m^{-1}.$$

Together with (2.21) this implies that

$$\rho(z, \bar{x}) \leq 2k_m^{-1}, \ |f_b(\bar{x}) - f_a(z)| \leq 2k_m^{-1} < \epsilon.$$

The second assertion is proved. □

2.3 Concretization of the Hypothesis (H1)

The proof of Theorem 2.2 consists in verifying that the hypotheses (H1) and (H2) hold for the space of integrand–map pairs introduced in Sect. 2.1. (H2) will follow from Proposition 2.7 which will be proved in Sect. 2.4. The verification of (H1) is more complicated. Recall that our space of integrand–map pairs is a product of the space of integrands and the space of maps. Therefore we should seek the set $\mathcal{W}$ (see (H1)) in the form $\mathcal{V} \times \mathcal{U}$ where $\mathcal{V}$ is an open set in the space of integrands and $\mathcal{U}$ is an open set in the space of maps. To simplify the verification of (H1) in this section we introduce new assumptions (A1)–(A4) and show that they imply (H1) (see Proposition 2.4). Using (A1)–(A4) we can construct the set $\mathcal{W} = \mathcal{V} \times \mathcal{U}$ step by step, roughly speaking. Namely, using (A4) we construct the set $\mathcal{U}$, using (A3) we find an integrand $\bar{a}_1$ and then using (A2) we construct the set $\mathcal{V}$ which is an open neighborhood of $\bar{a}_1$. Thus to verify (H1) we need to show that the assumptions (A1)–(A4) are valid. In fact this approach allows us to simplify the problem because each of (A2)–(A4) concerns either the space of integrands or the space of maps while it is not difficult to verify (A1).

Let (X, ρ) be a metric space with the topology generated by the metric ρ and let $(\mathcal{A}_1, d_1)$, $(\mathcal{A}_2, d_2)$ be metric spaces. For the space $\mathcal{A}_i$ $(i = 1, 2)$ we consider the topology generated by the metric d_i. This topology is called the strong topology. In addition to the strong topology we consider a weak topology on $\mathcal{A}_i$, $i = 1, 2$.

Assume that with every $a \in \mathcal{A}_1$ a lower semicontinuous function $\phi_a : X \to R^1 \cup \{\infty\}$ is associated and with every $a \in \mathcal{A}_2$ a set $S_a \subset X$ is associated. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \text{ for all } x \in S_{a_2}, \;\; f_a(x) = \infty \text{ for all } x \in X \setminus S_{a_2}. \tag{2.22}$$

Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 : \; \inf(f_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty.

We use the following hypotheses:

(A1) For each $a_1 \in \mathcal{A}_1$, $\inf(\phi_{a_1}) > -\infty$ and for each $a \in \mathcal{A}_1 \times \mathcal{A}_2$ the function f_a is lower semicontinuous.

(A2) For each $a \in \mathcal{A}_1$ and each $D, \epsilon > 0$ there is a neighborhood $\mathcal{U}$ of a in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{\phi_a(x), \phi_b(x)\} \le D$ the relation $|\phi_a(x) - \phi_b(x)| \le \epsilon$ holds.

(A3) For each $\gamma \in (0, 1)$ there exist positive numbers $\epsilon(\gamma)$ and $\delta(\gamma)$ such that $\epsilon(\gamma), \delta(\gamma) \to 0$ as $\gamma \to 0$ and the following property holds:

For each $\gamma \in (0, 1)$, each $a \in \mathcal{A}_1$, each nonempty set $Y \subset X$, and each $\bar{x} \in Y$ for which

$$\phi_a(\bar{x}) \le \inf\{\phi_a(z) : \; z \in Y\} + \delta(\gamma) < \infty \tag{2.23}$$

there is $\bar{a} \in \mathcal{A}_1$ such that the following conditions hold:

$$d_1(a, \bar{a}) \le \epsilon(\gamma), \;\; \phi_{\bar{a}}(z) \ge \phi_a(z), \;\; z \in X, \;\; \phi_{\bar{a}}(\bar{x}) \le \phi_a(\bar{x}) + \delta(\gamma); \tag{2.24}$$

for each $y \in Y$ satisfying

$$\phi_{\bar{a}}(y) \le \inf\{\phi_{\bar{a}}(z) : \; z \in Y\} + 2\delta(\gamma) \tag{2.25}$$

the inequality $\rho(y, \bar{x}) \le \gamma$ is valid.

(A4) For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying $\inf(f_a) < \infty$ and each $\epsilon, \delta > 0$ there exist $\bar{a}_2 \in \mathcal{A}_2$, $\bar{x} \in S_{\bar{a}_2}$, and an open set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology such that

$$d_2(a_2, \bar{a}_2) < \epsilon, \;\; \mathcal{U} \cap \{b \in \mathcal{A}_2 : \; d_2(b, a_2) < \epsilon\} \neq \emptyset, \tag{2.26}$$

$$\phi_{a_1}(\bar{x}) \le \inf\{\phi_{a_1}(z) : \; z \in S_{\bar{a}_2}\} + \delta < \infty, \tag{2.27}$$

and

$$\bar{x} \in S_b \subset S_{\bar{a}_2} \text{ for all } b \in \mathcal{U}. \tag{2.28}$$

Assume that (A3) holds. We show that the numbers $\epsilon(\gamma)$ and $\delta(\gamma)$ can be chosen such that $0 < \delta(\gamma) \le \epsilon(\gamma) \le \gamma$.

Let $\epsilon(\gamma)$ and $\delta(\gamma)$, $\gamma \in (0,1)$ be as guaranteed by (A3). Assume that $\gamma \in (0,1)$. Since $\lim_{t\to 0}\epsilon(t) = 0$ and $\lim_{t\to 0}\delta(t) = 0$ there exist $\gamma_1 \in (0,\gamma)$ and $\gamma_0 \in (0,\gamma_1)$ such that $\epsilon(\gamma_1) < \gamma$ and $\epsilon(\gamma_0), \delta(\gamma_0) < \epsilon(\gamma_1)$. Set $\bar{\epsilon}(\gamma) = \epsilon(\gamma_1)$ and $\bar{\delta}(\gamma) = \delta(\gamma_0)$. Clearly $\bar{\delta}(\gamma) < \bar{\epsilon}(\gamma) < \gamma$.

Assume that $a \in \mathcal{A}_1$, Y is a nonempty subset of X and $\bar{x} \in Y$ satisfies $\phi_a(\bar{x}) \le \inf\{\phi_a(z) : z \in Y\} + \bar{\delta}(\gamma) < \infty$. By (A3) and the equality $\bar{\delta}(\gamma) = \delta(\gamma_0)$ there exists $\bar{a} \in \mathcal{A}_1$ such that the following conditions hold:

$$d_1(a,\bar{a}) \le \epsilon(\gamma_0) < \epsilon(\gamma_1) = \bar{\epsilon}(\gamma),\ \phi_{\bar{a}}(z) \ge \phi_a(z),\ z \in X,$$

$$\phi_{\bar{a}}(\bar{x}) \le \phi_a(\bar{x}) + \delta(\gamma_0) = \phi_a(\bar{x}) + \bar{\delta}(\gamma);$$

for each $y \in Y$ satisfying

$$\phi_{\bar{a}}(y) \le \inf\{\phi_{\bar{a}}(z) : z \in Y\} + 2\delta(\gamma_0)$$

the inequality $\rho(y,\bar{x}) \le \gamma_0 \le \gamma$ is valid. Therefore (A3) holds with $\epsilon(\gamma) = \bar{\epsilon}(\gamma)$ and $\delta(\gamma) = \bar{\delta}(\gamma)$.

Proposition 2.4. *Assume that (A1)–(A4) hold. Then (H1) holds for the space $\mathcal{A}$.*

Proof. Let $a = (a_1,a_2) \in \mathcal{A}$ and let $\epsilon, \gamma > 0$. We may assume that $\inf(f_a) < \infty$. Choose a positive number

$$\gamma_0 < 8^{-1}\min\{1,\epsilon,\gamma\}. \tag{2.29}$$

Let $\epsilon(\gamma_0)$, $\delta(\gamma_0) > 0$ be as guaranteed by (A3) (namely (A3) is true with $\gamma = \gamma_0$, $\epsilon(\gamma) = \epsilon(\gamma_0)$, $\delta(\gamma) = \delta(\gamma_0)$). Choose

$$\delta_1 \in (0, 4^{-1}\delta(\gamma_0)). \tag{2.30}$$

By (A4) there are $\bar{a}_2 \in \mathcal{A}_2$, $\bar{x} \in S_{\bar{a}_2}$, and an open nonempty set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology such that (2.28) holds:

$$d_2(a_2,\bar{a}_2) < \epsilon(\gamma_0),\ \mathcal{U} \cap \{b \in \mathcal{A}_2 : d_2(b,a_2) < \epsilon(\gamma_0)\} \ne \emptyset \tag{2.31}$$

and

$$\phi_{a_1}(\bar{x}) \le \inf\{\phi_{a_1}(z) : z \in S_{\bar{a}_2}\} + \delta_1 < \infty. \tag{2.32}$$

It follows from the definition of $\epsilon(\gamma_0)$ and $\delta(\gamma_0)$, (A3) (with $a_1 = a$ and $Y = S_{\bar{a}_2}$), and (2.32) that there is $\bar{a}_1 \in \mathcal{A}_1$ such that

$$d_1(a_1, \bar{a}_1) \le \epsilon(\gamma_0), \ \phi_{\bar{a}_1}(z) \ge \phi_{a_1}(z), \ z \in X, \tag{2.33}$$
$$\phi_{\bar{a}_1}(\bar{x}) \le \phi_{a_1}(\bar{x}) + \delta(\gamma_0)$$

and the following property holds:

(Pi) For each $y \in S_{\bar{a}_2}$ satisfying

$$\phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) : \ z \in S_{\bar{a}_2}\} + 2\delta(\gamma_0) \tag{2.34}$$

the relation $\rho(y, \bar{x}) \le \gamma_0$ is valid.

Let $b \in \mathcal{U}$. Then by the definition of $\mathcal{U}$, (2.28), and (2.32)

$$\bar{x} \in S_b \subset S_{\bar{a}_2}, \ \inf\{\phi_{a_1}(z) : \ z \in S_b\} \le \phi_{a_1}(\bar{x}) < \infty. \tag{2.35}$$

We will show that the following property holds:

(Pii) If $y \in S_b$ satisfies

$$\phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) : \ z \in S_b\} + \delta_1, \tag{2.36}$$

then

$$\rho(y, \bar{x}) \le \gamma_0 \text{ and } |\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1 + \delta(\gamma_0). \tag{2.37}$$

It follows from (2.32), (2.35), and (2.33) that

$$\phi_{a_1}(\bar{x}) - \delta_1 \le \inf\{\phi_{a_1}(z) : \ z \in S_{\bar{a}_2}\} \le \inf\{\phi_{a_1}(z) : \ z \in S_b\} \tag{2.38}$$
$$\le \inf\{\phi_{\bar{a}_1}(z) : \ z \in S_b\} \le \phi_{\bar{a}_1}(\bar{x}) \le \phi_{a_1}(\bar{x}) + \delta(\gamma_0)$$
$$\le \inf\{\phi_{a_1}(z) : \ z \in S_{\bar{a}_2}\} + \delta_1 + \delta(\gamma_0).$$

Assume that $y \in S_b$ and (2.36) is true. It follows from (2.35), (2.36), (2.38), (2.33), and (2.30) that

$$y \in S_{\bar{a}_2}, \ \phi_{\bar{a}_1}(y) \le \inf\{\phi_{a_1}(z) : \ z \in S_{\bar{a}_2}\} + \delta(\gamma_0) + 2\delta_1$$
$$< \inf\{\phi_{\bar{a}_1}(z) : \ z \in S_{\bar{a}_2}\} + 2\delta(\gamma_0).$$

By these relations and property (Pi), $\rho(y, \bar{x}) \le \gamma_0$. (2.36), (2.38), (2.32), (2.35), and (2.33) imply that

$$|\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1 + \delta(\gamma_0). \tag{2.39}$$

Thus, (2.37) is valid. Therefore we have shown that for each $b \in \mathcal{U}$ relation (2.35) and property (Pii) hold. Choose a number

$$D > |\inf(\phi_{\bar{a}_1})| + 1 + |\phi_{\bar{a}_1}(\bar{x})|. \tag{2.40}$$

By (A2) there exists an open neighborhood $\mathcal{V}$ of $\bar{a}_1$ in $\mathcal{A}_1$ with the weak topology such that the following property holds:

(Piii) For each $b \in \mathcal{V}$ and each $x \in X$ for which $\min\{\phi_b(x), \phi_{\bar{a}_1}(x)\} \le D + 2$ the relation $|\phi_{\bar{a}_1}(x) - \phi_b(x)| \le 4^{-1}\delta_1$ is true.

Property (Piii) and (2.40) imply that for each $b \in \mathcal{V}$

$$|\phi_b(\bar{x}) - \phi_{\bar{a}_1}(\bar{x})| \le 4^{-1}\delta_1, \ \inf(\phi_b) \le \phi_b(\bar{x}) \le D. \tag{2.41}$$

Now we will show that (H1) is true with the open set $\mathcal{W} = \mathcal{V} \times \mathcal{U}$, $x = \bar{x}$, $\alpha = \phi_{\bar{a}_1}(\bar{x})$, and $\eta = 4^{-1}\delta_1$.

Assume that $b = (b_1, b_2) \in \mathcal{V} \times \mathcal{U}$. By (2.41) and (2.35)

$$\bar{x} \in S_{b_2}, \ \inf(f_b) = \inf\{\phi_{b_1}(z) : \ z \in S_{b_2}\} \le \phi_{b_1}(\bar{x}) < \infty. \tag{2.42}$$

Assume now that $z \in X$ and $f_b(z) \le \inf(f_b) + 4^{-1}\delta_1$. Then

$$z \in S_{b_2}, \ \phi_{b_1}(z) \le \inf\{\phi_{b_1}(y) : \ y \in S_{b_2}\} + 4^{-1}\delta_1. \tag{2.43}$$

By (2.42), (2.41), and (2.40),

$$\inf\{\phi_{b_1}(y) : \ y \in S_{b_2}\} \le \phi_{b_1}(\bar{x}) \le D, \ \inf\{\phi_{\bar{a}_1}(y) : \ y \in S_{b_2}\} \le \phi_{\bar{a}_1}(\bar{x}) \le D.$$

These inequalities imply that

$$\inf\{\phi_{b_1}(y) : \ y \in S_{b_2}\} = \inf\{\phi_{b_1}(y) : \ y \in S_{b_2} \text{ and } \phi_{b_1}(y) \le D + 1\}$$

and

$$\inf\{\phi_{\bar{a}_1}(y) : \ y \in S_{b_2}\} = \inf\{\phi_{\bar{a}_1}(y) : \ y \in S_{b_2} \text{ and } \phi_{\bar{a}_1}(y) \le D + 1\}.$$

It follows from these two relations and property (Piii) that

$$|\inf\{\phi_{b_1}(y) : \ y \in S_{b_2}\} - \inf\{\phi_{\bar{a}_1}(y) : \ y \in S_{b_2}\}| \le 4^{-1}\delta_1. \tag{2.44}$$

Equations (2.44), (2.43), (2.42), (2.40), and property (Piii) imply that

$$|\phi_{\bar{a}_1}(z) - \phi_{b_1}(z)| \le 4^{-1}\delta_1, \tag{2.45}$$

$$\phi_{\bar{a}_1}(z) \le \inf\{\phi_{\bar{a}_1}(y) : \ y \in S_{b_2}\} + \delta_1. \tag{2.46}$$

It follows from (2.46), (2.43), and property (Pii) that

$$\rho(z, \bar{x}) \le \gamma_0 \text{ and } |\phi_{\bar{a}_1}(z) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1 + \delta(\gamma_0).$$

Together with (2.45), (2.30), and the definition of $\delta(\gamma_0)$ this implies that

$$|\phi_{b_1}(z) - \phi_{\bar{a}_1}(\bar{x})| \le 2\delta(\gamma_0) \le 2\gamma_0 < \gamma.$$

This completes the proof of the proposition. □

Remark 2.5. In the proof of Proposition 2.4 for any $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying $\inf(f_a) < \infty$ and any $\epsilon > 0$ we constructed an open set $\mathcal{V}$ in $\mathcal{A}_1$ with the weak topology and an open set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology which satisfy

$$\mathcal{V} \cap \{b \in \mathcal{A}_1 : \ d_1(b, a_1) < \epsilon\} \neq \emptyset \text{ and } \mathcal{U} \cap \{b \in \mathcal{A}_2 : \ d_2(b, a_2) < \epsilon\} \neq \emptyset$$

and such that $\inf(f_b) < \infty$ for each $b = (b_1, b_2) \in \mathcal{V} \times \mathcal{U}$. This implies that there exists an open set $\mathcal{F}$ in $\mathcal{A}_1 \times \mathcal{A}_2$ with the weak topology such that $\inf(f_a) < \infty$ for all $a \in \mathcal{F}$ and $\mathcal{A}$ is the closure of $\mathcal{F}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology.

2.4 Preliminary Results for Hypotheses (A2) and (H2)

Assume that $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, U : graph$(A) \to 2^{R^N} \setminus \{\emptyset\}$ and that graph(U) is a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology. Consider the spaces $X(A, U)$, $\mathcal{M}(A, U)$, and $C_l(B_1 \times B_2)$ introduced in Sect. 2.1.

Proposition 2.6. *Let $f \in \mathcal{M}(A, U)$, $(x, u) \in X(A, U)$, $\{(x_i, u_i)\}_{i=1}^{\infty} \subset X(A, U)$ and let $\rho((x_i, u_i), (x, u)) \to 0$ as $i \to \infty$. Then*

$$\int_{\Omega} f(t, x(t), u(t))dt \le \liminf_{i \to \infty} \int_{\Omega} f(t, x_i(t), u_i(t))dt.$$

Proof. We may assume that there is a finite $\lim_{i \to \infty} \int_{\Omega} f(t, x_i(t), u_i(t))dt$. There is a subsequence $\{(x_{i_k}, u_{i_k})\}_{k=1}^{\infty}$ such that

$$(x_{i_k}(t), u_{i_k}(t)) \to (x(t), u(t)) \text{ as } k \to \infty, \ \ t \in \Omega \text{ (a.e.)}$$

(see p. 68 of [33]). By property (ii) (see the definition of $\mathcal{M}(A, U)$) for almost every $t \in \Omega$

$$\liminf_{k \to \infty} f(t, x_{i_k}(t), u_{i_k}(t)) \ge f(t, x(t), u(t)).$$

The proposition now follows from property (iii) (see the definition of $\mathcal{M}(A, U)$) and Fatou's lemma. □

The following proposition is an auxiliary result for the hypothesis (H2).

Proposition 2.7. *Assume that $f \in \mathcal{M}(A, U)$, $\{(x_i, u_i)\}_{i=1}^{\infty} \subset X(A, U)$ is a Cauchy sequence, and the sequence $\{\int_{\Omega} f(t, x_i(t), u_i(t))dt\}_{i=1}^{\infty}$ is bounded. Then* *there is* $(x_*, u_*) \in X(A, U)$ *such that* (x_i, u_i) *converges to* (x_*, u_*) *as* $i \to \infty$ *in* $X(A, U)$ *and moreover, if* $m = 1$*, then* $x_i(t) \to x_*(t)$ *as* $i \to \infty$ *uniformly on* $[T_1, T_2]$.

Proof. To prove the proposition it is sufficient to show that there exists a subsequence $\{(x_{i_k}, u_{i_k})\}_{k=1}^{\infty}$ and $(x_*, u_*) \in X(A, U)$ such that $(x_{i_k}, u_{i_k}) \to (x_*, u_*)$ as $k \to \infty$ in $X(A, U)$ and if $m = 1$, then $x_{i_k}(t) \to x_*(t)$ as $k \to \infty$ uniformly on $[T_1, T_2]$. (In the case $m = 1$ this implies that each subsequence of $\{x_i\}_{i=1}^{\infty}$ has a subsequence which converges to x_* uniformly on $[T_1, T_2]$. This proves that $\{x_i\}_{i=1}^{\infty}$ converges to x_* uniformly on $[T_1, T_2]$.)

Since $\{(x_i, u_i)\}_{i=1}^{\infty}$ is a Cauchy sequence there is a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ and a sequence of measurable sets $D_k \subset \Omega$, $k = 1, 2 \ldots$ such that for all $k = 1, 2, \ldots,$

$$\mathrm{mes}(D_k) \leq 2^{-k},\ |x_{i_{k+1}}(t) - x_{i_k}(t)| \leq 2^{-k}, \tag{2.47}$$
$$|u_{i_{k+1}}(t) - u_{i_k}(t)| \leq 2^{-k},\ t \in \Omega \setminus D_k.$$

Set $C_k = \cup_{i=k}^{\infty} D_i$, $k = 1, 2, \ldots$ By (2.47) there exist measurable functions $u_* : \Omega \to R^N$ and $x_* : \Omega \to R^n$ such that

$$\lim_{k\to\infty} x_{i_k}(t) = x_*(t),\ \lim_{k\to\infty} u_{i_k}(t) = u_*(t),\ t \in \Omega \setminus \cap_{k=1}^{\infty} C_k. \tag{2.48}$$

Since the function $f(t, \cdot, \cdot)$ is lower semicontinuous for $t \in \Omega$ (a.e.) (see the definition of $\mathcal{M}(A, U)$, property (ii)) it follows from (2.48) that

$$f(t, x_*(t), u_*(t)) \leq \liminf_{k\to\infty} f(t, x_{i_k}(t), u_{i_k}(t)),\ t \in \Omega \text{ (a.e.)}. \tag{2.49}$$

Clearly the function $f(t, x_*(t), u_*(t))$, $t \in \Omega$ is measurable. By (2.49), Fatou's lemma, and property (iii), $\int_{\Omega} f(t, x_*(t), u_*(t))dt$ is finite. It follows from property (iii) and the boundedness of the sequence $\{\int_{\Omega} f(t, x_i(t), u_i(t))dt\}_{i=1}^{\infty}$ that the family of functions

$$\mathcal{E} = \{|H(t, x_*(t), u_*(t))|,\ t \in \Omega,\ |H(t, x_{i_k}(t), u_{i_k}(t))|,\ t \in \Omega,\ k = 1, 2, \ldots\}$$

is uniformly integrable (see p. 74 of [32]). Namely for each $\epsilon > 0$ there exists $\delta > 0$ such that for each measurable set $e \subset \Omega$ satisfying $\mathrm{mes}(e) \leq \delta$ the following relations hold:

$$\int_e |H(t, x_*(t), u_*(t))|dt \leq \epsilon,\ \int_e |H(t, x_{i_k}(t), u_{i_k}(t))|dt \leq \epsilon,\ k = 1, 2, \ldots$$

It follows from this property, the continuity of H, (2.47), (2.48), and Egorov's theorem that for each measurable set $e \subset \Omega$

$$\int_e H(t, x_{i_k}(t), u_{i_k}(t))dt \to \int_e H(t, x_*(t), u_*(t))dt \text{ as } k \to \infty. \tag{2.50}$$

Now we consider the case with $m = 1$. Since the set $\mathcal{E}$ is uniformly integrable it follows from (2.4b), (2.48), and Ascoli's compactness theorem that a subsequence of the sequence $\{x_{i_k}\}_{k=1}^{\infty}$ converges to a continuous function $y : [T_1, T_2] \to R^n$ uniformly on $[T_1, T_2]$. By (2.48) we may assume that $x_*(t) = y(t)$, $t \in [T_1, T_2]$ (a.e.). Thus $x_* : \Omega \to R^n$ is continuous and some subsequence of $\{x_{i_k}\}_{k=1}^{\infty}$ converges to x_* uniformly on $[T_1, T_2]$. Together with (2.50) this implies that $(x_*, u_*) \in X(A, U)$. Since $\text{mes}(\cap_{k=1}^{\infty} C_k) = 0$ (see (2.47)) it follows from (2.48) that $(x_{i_k}, u_{i_k}) \to (x_*, u_*)$ as $k \to \infty$ in $X(A, U)$. Therefore the proposition is true in the case with $m = 1$.

We turn now to the case with $m > 1$. Since the set $\mathcal{E}$ is uniformly integrable it is easy to verify that

$$H(\cdot, x_*(\cdot), u_*(\cdot)) \in L^1(\Omega),\ H(\cdot, x_{i_k}(\cdot), u_{i_k}(\cdot)) \in L^1(\Omega),\ k = 1, 2, \dots, \tag{2.51}$$
$$H(\cdot, x_{i_k}(\cdot), u_{i_k}(\cdot)) \to H(\cdot, x_*(\cdot), u_*(\cdot)) \text{ as } k \to \infty \text{ in } L^1(\Omega).$$

Note that $x_{i_k} - \theta^* \in (W_0^{1,1}(\Omega))^n$, $k = 1, 2, \dots$ (see (2.4)). By Theorem 2.4.1 in [108] there is a constant $c > 0$ such that $||h||_{L^1(\Omega)} \le c||\nabla h||_{L^1(\Omega)}$ for all $h \in W_0^{1,1}(\Omega)$. Together with (2.51) and (2.48) this implies that $x_{i_k} \to x_*$ as $k \to \infty$ in $L^1(\Omega; R^n)$, $x_* \in (W^{1,1}(\Omega))^n$, $\nabla x_* = H(\cdot, x_*(\cdot), u_*(\cdot))$, and $(x_*, u_*) \in X(A, U)$. Analogously to the previous case we obtain that $(x_{i_k}, u_{i_k}) \to (x_*, u_*)$ as $k \to \infty$ in $X(A, U)$. Thus in the case $m > 1$ the proposition is proved. □

Proposition 2.8. *Let $h \in C_l(B_1 \times B_2)$ and $\epsilon, D > 0$. Then there exists a neighborhood $\mathcal{V}$ of h in $C_l(B_1 \times B_2)$ with the weak topology such that for each $\xi \in \mathcal{V}$ and each $x \in B_1 \times B_2$ which satisfies $\min\{\xi(x), h(x)\} \le D$ the relation $|\xi(x) - h(x)| \le \epsilon$ holds.*

Proof. There is $c_0 > 0$ such that $h(x) \ge -c_0$ for all $x \in B_1 \times B_2$. Choose a positive number $\epsilon_1 < 1$ for which

$$\epsilon_1 + \epsilon_1(1 - \epsilon_1)^{-1}(2 + D + c_0) < \epsilon$$

and define $\mathcal{V} = \{\xi \in C_l(B_1 \times B_2) : (\xi, h) \in E_{cw}(\epsilon_1)\}$ (see (2.9)). Assume that $\xi \in \mathcal{V}$, $x \in B_1 \times B_2$, and $\min\{\xi(x), h(x)\} \le D$. It follows from the definition of $\mathcal{V}$ and ϵ_1, (2.9) and Lemma 2.1, that $\xi(x), h(x)$ are finite and

$$|\xi(x) - h(x)| < \epsilon_1 + \epsilon_1^2(1 - \epsilon_1)^{-1} + \epsilon_1(1 - \epsilon_1)^{-1}\min\{|\xi(x)|, |h(x)|\}$$
$$< \epsilon_1 + \epsilon_1^2(1 - \epsilon_1)^{-1} + \epsilon_1(1 - \epsilon_1)^{-1}(D + c_0) < \epsilon.$$

The proposition is proved. □

Corollary 2.9. *Let $h \in C_l(B_1 \times B_2)$ and $\epsilon > 0$. Then there is a neighborhood $\mathcal{V}$ of h in $C_l(B_1 \times B_2)$ with the weak topology such that for each $\xi \in \mathcal{V}$ the inequality $|\inf(\xi) - \inf(h)| \le \epsilon$ holds.*

Proof. We may assume that $\inf(h)$ is finite and $\epsilon < 1$. By Proposition 2.8 there exists a neighborhood $\mathcal{V}$ of h in $C_l(B_1 \times B_2)$ with the weak topology such that for each $\xi \in \mathcal{V}$ and each $x \in B_1 \times B_2$ which satisfies $\min\{\xi(x), h(x)\} \le \inf(h) + 2$ the relation $|\xi(x) - h(x)| \le 2^{-1}\epsilon$ holds.

Assume that $\xi \in \mathcal{V}$. It follows from the definition of $\mathcal{V}$ that for each $x \in B_1 \times B_2$ satisfying $h(x) \le \inf(h) + 2$ the relation $|\xi(x) - h(x)| \le 2^{-1}\epsilon$ is true. Choose $y \in X$ such that $h(y) \le \inf(h) + 2^{-1}\epsilon$. Then

$$\inf(\xi) \le \xi(y) \le h(y) + 2^{-1}\epsilon \le \inf(h) + \epsilon \le \inf(h) + 1.$$

It follows from this inequality and the definition of $\mathcal{V}$ that for each $x \in B_1 \times B_2$ satisfying $\xi(x) \le \inf(\xi) + 1$ the relation $|\xi(x) - h(x)| \le 2^{-1}\epsilon$ holds. Choose $z \in X$ such that $\xi(z) \le \inf(\xi) + 2^{-1}\epsilon$. Then

$$\inf(h) \le h(z) \le \xi(z) + 2^{-1}\epsilon \le \inf(\xi) + \epsilon.$$

The corollary is proved. □

The following proposition is an auxiliary result for the assumption (A2).

Proposition 2.10. *Let $f \in \mathcal{M}(A, U)$ and $\epsilon \in (0, 1)$, $D > 0$. Then there exists a neighborhood $\mathcal{V}$ of f in $\mathcal{M}(A, U)$ with the weak topology such that for each $g \in \mathcal{V}$ and each $(x, u) \in X(A, U)$ satisfying*

$$\min\left\{\int_\Omega f(t, x(t), u(t))dt,\ \int_\Omega g(t, x(t), u(t))dt\right\} \le D \tag{2.52}$$

the following relation holds:

$$\left|\int_\Omega f(t, x(t), u(t))dt - \int_\Omega g(t, x(t), u(t))dt\right| \le \epsilon. \tag{2.53}$$

Proof. There is an integrable function $\phi_0(t) \ge 0$, $t \in \Omega$ such that

$$f(t, x, u) \ge -\phi_0(t) \text{ for all } (t, x, u) \in \text{ graph}(U). \tag{2.54}$$

Choose a positive number ϵ_1 for which

$$\epsilon_1 \left(2\text{mes}(\Omega) + 2 + \int_\Omega \phi_0(t)dt + D\right) < \epsilon \tag{2.55}$$

and a positive number ϵ_0 which satisfies

$$\epsilon_0 + \epsilon_0(1 - \epsilon_0)^{-1} < 4^{-1}\epsilon_1. \tag{2.56}$$

Define

$$\mathcal{V} = \{g \in \mathcal{M}(A, U) : (g, f) \in E_{\mathcal{M}w}(\epsilon_0)\} \text{ (see (2.7))}. \tag{2.57}$$

Assume that $g \in \mathcal{V}$, $(x, u) \in X(A, U)$, and (2.52) are valid. By (2.57) and (2.7) there is a nonnegative function $\phi \in L^1(\Omega)$ such that $\int_\Omega \phi(t)dt \leq 1$ and for almost every $t \in \Omega$ the inequality

$$|f(t, y, v) - g(t, y, v)| < \epsilon_0 + \epsilon_0\phi(t) + \epsilon_0 \max\{|f(t, y, v)|, |g(t, y, v)|\}$$

is true for each $y \in A(t)$ and each $v \in U(t, y)$. It follows from this inequality, Lemma 2.1, and (2.56) that for almost every $t \in \Omega$ the relation

$$\begin{aligned} |f(t, y, v) - g(t, y, v)| &< \epsilon_0 + \epsilon_0^2(1 - \epsilon_0)^{-1} + \phi(t)(\epsilon_0^2(1 - \epsilon_0)^{-1} + \epsilon_0) \\ &+\epsilon_0(1 - \epsilon_0)^{-1} \min\{|f(t, y, v)|, |g(t, y, v)|\} \\ &< 4^{-1}\epsilon_1 + 4^{-1}\epsilon_1\phi(t) + 4^{-1}\epsilon_1 \min\{|f(t, y, v)|, |g(t, y, v)|\} \end{aligned} \tag{2.58}$$

is valid for each $y \in A(t)$ and each $v \in U(t, y)$. Equations (2.58) and (2.54) imply that for almost every $t \in \Omega$ the inequality

$$\begin{aligned} g(t, y, v) &\geq f(t, y, v) - 4^{-1}\epsilon_1 - 4^{-1}\epsilon_1\phi(t) - 4^{-1}\epsilon_1|f(t, y, v)| \\ &\geq -4^{-1}\epsilon_1\phi(t) - 2\phi_0(t) - 4^{-1}\epsilon_1 \end{aligned} \tag{2.59}$$

holds for each $y \in A(t)$ and each $v \in U(t, y)$. Set

$$\lambda(t) = \min\{f(t, x(t), u(t)), g(t, x(t), u(t))\}, \ t \in \Omega. \tag{2.60}$$

It follows from (2.58), (2.54), (2.59), and (2.60) that for almost every $t \in \Omega$

$$\begin{aligned} &|f(t, x(t), u(t)) - g(t, x(t), u(t))| < 4^{-1}\epsilon_1 + 4^{-1}\epsilon_1\phi(t) + \\ &4^{-1}\epsilon_1 \min\{f(t, x(t), u(t)) + 2\phi_0(t), \ g(t, x(t), u(t)) + \phi(t) + 4\phi_0(t) + 2\} \\ &\leq 4^{-1}\epsilon_1 + 4^{-1}\epsilon_1\phi(t) + 4^{-1}\epsilon_1(\phi(t) + 4\phi_0(t) + 2) + 4^{-1}\epsilon_1\lambda(t). \end{aligned}$$

By this relation, (2.52) and (2.55),

$$\begin{aligned} \int_\Omega |f(t, x(t), u(t)) - g(t, x(t), u(t))|dt &\leq 4^{-1}\epsilon_1 \mathrm{mes}(\Omega) + 4^{-1}\epsilon_1 \int_\Omega \phi(t)dt \\ +4^{-1}\epsilon_1 \int_\Omega \phi(t)dt + \epsilon_1 \int_\Omega \phi_0(t)dt &+ \epsilon_1 \mathrm{mes}(\Omega) + 4^{-1}\epsilon_1 D < \epsilon. \end{aligned}$$

This completes the proof of the proposition. □

Analogously to the proof of Corollary 2.9 we can show that Proposition 2.10 implies the following corollary.

Corollary 2.11. *Let $f \in \mathcal{M}(A,U)$ and $\epsilon > 0$. Then there exists a neighborhood $\mathcal{V}$ of f in $\mathcal{M}(A,U)$ with the weak topology such that for all $g \in \mathcal{V}$*

$$\left|\inf\left\{\int_\Omega f(t,x(t),u(t))dt\, :\, (x,u) \in X(A,U)\right\} - \inf\left\{\int_\Omega g(t,x(t),u(t))dt\, :\, (x,u) \in X(A,U)\right\}\right| < \epsilon.$$

Proposition 2.12. *Let $m = 1$, $f \in \mathcal{M}(A,U)$ $h \in C_l(B_1 \times B_2)$ and $\epsilon \in (0,1)$, $D > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}(A,U)$ with the weak topology and a neighborhood $\mathcal{V}$ of h in $C_l(B_1 \times B_2)$ with the weak topology such that for each $(\xi, g) \in \mathcal{V} \times \mathcal{U}$ and each $(x,u) \in X(A,U)$ which satisfies*

$$\min\{I^{(f,h)}(x,u),\ I^{(g,\xi)}(x,u)\} \le D \tag{2.61}$$

the following relations are valid:

$$|h(x(T_1),x(T_2)) - \xi(x(T_1),x(T_2))| \le \epsilon, \tag{2.62}$$

$$\left|\int_{T_1}^{T_2} [f(t,x(t),u(t)) - g(t,x(t),u(t))]dt\right| \le \epsilon. \tag{2.63}$$

Proof. We may assume that $\inf(h)$ and

$$\inf\left\{\int_{T_1}^{T_2} f(t,x(t),u(t))dt\, :\, (x,u) \in X(A,U)\right\}$$

are finite. Choose a number

$$c_0 > 4 + |\inf(h)| + \left|\inf\left\{\int_{T_1}^{T_2} f(t,x(t),u(t))dt\, :\, (x,u) \in X(A,U)\right\}\right|.$$

By Corollaries 2.9 and 2.11 there exist a neighborhood $\mathcal{V}_1$ of $h \in C_l(B_1 \times B_2)$ with the weak topology such that

$$|\inf(\xi)| < c_0 \text{ for all } \xi \in \mathcal{V}_1 \tag{2.64}$$

and a neighborhood $\mathcal{U}_1$ of f in $\mathcal{M}(A,U)$ with the weak topology such that

$$\left|\inf\left\{\int_{T_1}^{T_2} g(t,x(t),u(t))dt\, :\, (x,u) \in X(A,U)\right\}\right| < c_0 \text{ for all } g \in \mathcal{U}_1. \tag{2.65}$$

By Proposition 2.8 there exists a neighborhood $\mathcal{V}$ of h in $C_l(B_1 \times B_2)$ with the weak topology such that $\mathcal{V} \subset \mathcal{V}_1$ and that for each $\xi \in \mathcal{V}$ and each $z \in B_1 \times B_2$ which satisfies $\min\{\xi(z), h(z)\} \le D + c_0 + 2$ the relation $|\xi(z) - h(z)| \le \epsilon$ holds. By Proposition 2.10 there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}(A, U)$ with the weak topology such that $\mathcal{U} \subset \mathcal{U}_1$ and that for each $g \in \mathcal{U}$ and each $(x, u) \in X(A, U)$ satisfying

$$\min\left\{\int_{T_1}^{T_2} f(t, x(t), u(t))dt,\ \int_{T_1}^{T_2} g(t, x(t), u(t))dt\right\} \le D + c_0 + 2$$

the inequality (2.63) holds.

Now assume that $(\xi, g) \in \mathcal{V} \times \mathcal{U}$ and $(x, u) \in X(A, U)$ satisfy (2.61). It follows from (2.61), (2.64), and (2.65) that

$$\min\{\xi(x(T_1), x(T_2)), h(x(T_1), x(T_2))\} - c_0$$
$$\le \min\{I^{(f,h)}(x, u),\ I^{(g,\xi)}(x, u)\} \le D$$

and

$$\min\left\{\int_{T_1}^{T_2} f(t, x(t), u(t))dt,\ \int_{T_1}^{T_2} g(t, x(t), u(t))dt\right\} - c_0$$
$$\le \min\{I^{(f,h)}(x, u),\ I^{(g,\xi)}(x, u)\} \le D.$$

By these inequalities and the definition of $\mathcal{U}$ and $\mathcal{V}$, the inequalities (2.62) and (2.63) are valid. The proposition is proved. □

2.5 A Preliminary Lemma for Hypothesis (A3)

Fix a number $d_0 \in (0, 1)$. There is a C^∞-function $\phi_0 : R^1 \to [0, 1]$ such that $\phi_0(t) = 1$ if $|t| \le d_0$, $1 > \phi_0(t) > 0$ if $d_0 < |t| < 1$, and $\phi_0(t) = 0$ if $|t| \ge 1$. Define a C^∞-function $\bar{\phi} : R^1 \to R^1$ by $\bar{\phi}(x) = \int_0^x \phi_0(t)dt,\ x \in R^1$. Clearly $\bar{\phi}$ is monotone increasing, $\bar{\phi}(x) = x$ if $|x| \le d_0$ and

$$\bar{\phi}(x) = \bar{\phi}(1) \text{ if } x \ge 1,\ \bar{\phi}(x) = \bar{\phi}(-1) \text{ if } x \le -1, \tag{2.66}$$

$$d_0 = \bar{\phi}(d_0) \le \bar{\phi}(x) \le \bar{\phi}(1) \le 1 \text{ for all } x \in (d_0, 1). \tag{2.67}$$

Now we define a set $\mathcal{L} \subset C_l(B_1 \times B_2)$. In the case $m = 1$ we set $\mathcal{L} = C_l(B_1 \times B_2)$ and in the case $m > 1$ denote by $\mathcal{L}$ a singleton $\{0\}$ where 0 is a function in $C_l(B_1 \times B_2)$ which is identical to zero. In the case $m > 1$ for each $(f, \xi) \in \mathcal{M}(A, U) \times \mathcal{L}$ and each $(x, u) \in X(A, U)$ we set

$$I^{(f,\xi)}(x, u) = I^{(f)}(x, u) \tag{2.68}$$

(see (2.10) and (2.11)). For each measurable set $E \subset R^m$, each measurable set $E_0 \subset E$, and each $h \in L^1(E)$ we set

$$||h||_{L^1(E_0)} = \int_{E_0} |h(t)|dt. \tag{2.69}$$

Fix an integer $k \geq 1$. It is easy to verify that all partial derivatives of the functions $(x, y) \to \bar{\phi}(|x - y|^2)$, $(x, y) \in R^q \times R^q$ with $q = n, N$ up to the order k are bounded (by some $\bar{d} > 0$).

For each $\gamma \in (0, 1)$ choose $\epsilon_0(\gamma) \in (0, \gamma)$ such that

$$E_X(8\epsilon_0(\gamma)) \subset \{((x_1, u_1), (x_2, u_2)) \in X(A, U) \times X(A, U) : \rho((x_1, u_1), (x_2, u_2)) \leq \gamma\} \tag{2.70}$$

(see (2.5)) and

$$\epsilon_0(\gamma) < 4^{-1}\gamma(\bar{d} + 2)^{-1} \tag{2.71}$$

and choose

$$\epsilon_1(\gamma) \in (0, d_0\epsilon_0(\gamma)), \tag{2.72}$$

$$\delta(\gamma) \in (0, 16^{-1}\epsilon_1(\gamma)^4). \tag{2.73}$$

Lemma 2.13. *Let $\gamma \in (0, 1)$, $f \in \mathcal{M}(A, U)$, $\xi \in \mathcal{L}$, and let $Y \subset X(A, U)$, $(\bar{x}, \bar{u}) \in Y$,*

$$I^{(f,\xi)}(\bar{x}, \bar{u}) \leq \inf\{I^{(f,\xi)}(x, u) : (x, u) \in Y\} + \delta(\gamma) < \infty. \tag{2.74}$$

Then there is $g : R^m \times R^n \times R^N \to R^1$ in $C^k(R^{m+n+N})$ which satisfies

$$0 \leq g(t, x, u) \leq \gamma \text{ for all } (t, x, u) \in R^m \times R^n \times R^N, \tag{2.75}$$
$$||g(t, \cdot, \cdot)||_{C^k(R^n \times R^N)} \leq \gamma \text{ for all } t \in R^m$$

such that for a function $\bar{f} \in \mathcal{M}(A, U)$ defined by

$$\bar{f}(t, x, u) = f(t, x, u) + g(t, x, u),\ (t, x, u) \in graph(U) \tag{2.76}$$

the following properties hold:

$$I^{(\bar{f},\xi)}(\bar{x}, \bar{u}) \leq I^{(f,\xi)}(\bar{x}, \bar{u}) + \delta(\gamma); \tag{2.77}$$

for each $(y, v) \in Y$ satisfying

$$I^{(\bar{f},\xi)}(y, v) \leq \inf\{I^{(\bar{f},\xi)}(z, w) : (z, w) \in Y\} + 2\delta(\gamma) \tag{2.78}$$

the relation $\rho((y, v), (\bar{x}, \bar{u})) \leq \gamma$ is valid.

Moreover the function g is the sum of two functions, one of them depending only on (t, x) while the other depending only on (t, u).

Proof. Choose a positive number ϵ_2 for which

$$\epsilon_2 < (\text{mes}(\Omega) + 1)^{-1} 8^{-1} \delta(\gamma) d_0 (\bar{d} + 1)^{-1}. \tag{2.79}$$

There is a measurable set $E_0 \subset \Omega$ such that

$$\text{mes}(\Omega \setminus E_0) < 2^{-1} \epsilon_2 \tag{2.80}$$

and the functions $\bar{x}$ and $\bar{u}$ are bounded on E_0. There exist sequences of functions $\{\bar{x}_i\}_{i=1}^{\infty} \in C^{\infty}(R^m; R^n)$ and $\{\bar{u}_i\}_{i=1}^{\infty} \subset C^{\infty}(R^m; R^N)$ such that

$$||\bar{u}_i - \bar{u}||_{L^1(E_0)}, \ ||\bar{x}_i - \bar{x}||_{L^1(E_0)} \to 0 \text{ as } i \to \infty \tag{2.81}$$

(p. 13 of [57]). We may assume without loss of generality that $\bar{u}_i(t) \to \bar{u}(t)$, $\bar{x}_i(t) \to \bar{x}(t)$ as $i \to \infty$, $t \in E_0$ (a.e.) By Egorov's theorem there is a measurable set $E_1 \subset E_0$ such that

$$\text{mes}(E_0 \setminus E_1) < 2^{-1} \epsilon_2 \tag{2.82}$$

and

$$\bar{u}_i(t) \to \bar{u}(t) \text{ and } \bar{x}_i(t) \to \bar{x}(t) \text{ uniformly in } E_1 \text{ as } i \to \infty. \tag{2.83}$$

There is an integer $s \geq 1$ such that

$$\max\{|\bar{u}_s(t) - \bar{u}(t)|, \ |\bar{x}_s(t) - \bar{x}(t)|\} \leq 4^{-1} \epsilon_2 (\text{mes}(\Omega) + 1)^{-1}, \ t \in E_1. \tag{2.84}$$

Define a function $g : R^m \times R^n \times R^N \to R^1$ by

$$g(t, x, u) = \epsilon_0(\gamma) \bar{\phi}(|x - \bar{x}_s(t)|^2) + \epsilon_0(\gamma) \bar{\phi}(|u - \bar{u}_s(t)|^2), \ (t, x, u) \in R^m \times R^n \times R^N. \tag{2.85}$$

Clearly $g \in C^{\infty}(R^m \times R^n \times R^N)$. Define

$$\bar{f}(t, x, u) = f(t, x, u) + g(t, x, u), \ (t, x, u) \in \text{ graph}(U). \tag{2.86}$$

Evidently $\bar{f} \in \mathcal{M}(A, U)$. It follows from (2.85), the definition of $\bar{d}$, (2.66), (2.67), and (2.71) that (2.75) is true. We will show that (2.77) is true. By (2.86), (2.85), (2.84), (2.66), and (2.67),

$$I^{(\bar{f}, \xi)}(\bar{x}, \bar{u}) = I^{(f, \xi)}(\bar{x}, \bar{u}) + \epsilon_0(\gamma) \int_{\Omega} \bar{\phi}(|\bar{x}(t) - \bar{x}_s(t)|^2) dt$$

$$+ \epsilon_0(\gamma) \int_{\Omega} \bar{\phi}(|\bar{u}(t) - \bar{u}_s(t)|^2) dt = I^{(f, \xi)}(\bar{x}, \bar{u}) + \epsilon_0(\gamma) \int_{E_1} \bar{\phi}(|\bar{x}(t) - \bar{x}_s(t)|^2 dt$$

$$+\epsilon_0(\gamma)\int_{\Omega\setminus E_1}\bar{\phi}(|\bar{x}(t)-\bar{x}_s(t)|^2)dt+\epsilon_0(\gamma)\int_{E_1}\bar{\phi}(|\bar{u}(t)-\bar{u}_s(t)|^2)dt$$
$$+\epsilon_0(\gamma)\int_{\Omega\setminus E_1}\bar{\phi}(|\bar{u}(t)-\bar{u}_s(t)|^2)dt \le I^{(f,\xi)}(\bar{x},\bar{u})$$
$$+2(\text{mes}(\Omega))\epsilon_0(\gamma)\bar{\phi}((4^{-1}\epsilon_2)^2)+2\epsilon_0(\gamma)\text{ mes }(\Omega\setminus E_1).$$

It follows from this relation, (2.79), (2.66), (2.67), (2.80), and (2.82) that

$$I^{(\bar{f},\xi)}(\bar{x},\bar{u}) \le I^{(f,\xi)}(\bar{x},\bar{u})+2\text{mes}(\Omega)\epsilon_0(\gamma)(4^{-1}\epsilon_2)^2+2\epsilon_0(\gamma)\epsilon_2$$
$$\le I^{(f,\xi)}(\bar{x},\bar{u})+4\epsilon_0(\gamma)\epsilon_2 \le I^{(f,\xi)}(\bar{x},\bar{u})+\delta(\gamma).$$

Thus (2.77) is valid. Now assume that $(y,v)\in Y$ satisfies (2.78). It follows from (2.78), (2.86), (2.85), and (2.74) that

$$I^{(f,\xi)}(y,v)+\epsilon_0(\gamma)\int_{\Omega}\bar{\phi}(|\bar{x}_s(t)-y(t)|^2)dt+\epsilon_0(\gamma)\int_{\Omega}\bar{\phi}(|v(t)-\bar{u}_s(t)|^2)dt$$
$$=I^{(\bar{f},\xi)}(y,v)\le 2\delta(\gamma)+I^{(\bar{f},\xi)}(\bar{x},\bar{u})$$
$$\le 3\delta(\gamma)+I^{(f,\xi)}(\bar{x},\bar{u})\le I^{(f,\xi)}(y,v)+4\delta(\gamma).$$

This implies that

$$\int_{\Omega}\bar{\phi}(|\bar{x}_s(t)-y(t)|^2)dt+\int_{\Omega}\bar{\phi}(|\bar{u}_s(t)-v(t)|^2)dt\le 4\delta(\gamma)(\epsilon_0(\gamma))^{-1}. \quad (2.87)$$

Set

$$E_2=\{t\in\Omega:\ |y(t)-\bar{x}_s(t)|\ge 2^{-1}\epsilon_1(\gamma)\},$$
$$E_3=\{t\in\Omega:\ |v(t)-\bar{u}_s(t)|\ge 2^{-1}\epsilon_1(\gamma)\}. \quad (2.88)$$

Then by (2.88), (2.87), (2.71), (2.72), (2.66), (2.67), and (2.73)

$$\text{mes}(E_2)+\text{mes}(E_3)\le 4\epsilon_1(\gamma)^{-2}\Big[\int_{E_2}\bar{\phi}(|\bar{x}_s(t)-y(t)|^2)dt$$
$$+\int_{E_3}\bar{\phi}(|\bar{u}_s(t)-v(t)|^2)dt\Big] \quad (2.89)$$
$$\le 16\epsilon_1(\gamma)^{-2}\delta(\gamma)(\epsilon_0(\gamma))^{-1}<\epsilon_1(\gamma).$$

It follows from (2.89), (2.88), (2.84), (2.79), (2.80), and (2.82) that

$$\text{mes}\{t\in\Omega:\ |y(t)-\bar{x}(t)|\ge\epsilon_1(\gamma)\}\le\ \text{mes}(\Omega\setminus E_1)$$
$$+\text{mes}(\{t\in\Omega:\ |y(t)-\bar{x}_s(t)|\ge 2^{-1}\epsilon_1(\gamma)\}\le\epsilon_2+\epsilon_1(\gamma)\le 2\epsilon_1(\gamma)$$

and

$$\mathrm{mes}\{t \in \Omega : |v(t) - \bar{u}(t)| \geq \epsilon_1(\gamma)\} \leq \mathrm{mes}(\Omega \setminus E_1)$$

$$+\mathrm{mes}\{t \in \Omega : |v(t) - \bar{u}_s(t)| \geq 2^{-1}\epsilon_1(\gamma)\}$$

$$\leq \epsilon_2 + \epsilon_1(\gamma) \leq 2\epsilon_1(\gamma).$$

These relations and (2.70) imply that

$$((y, v), (\bar{x}, \bar{u})) \in E_X(4\epsilon_1(\gamma)),$$

$$\rho((y, v), (\bar{x}, \bar{u})) \leq \gamma.$$

This completes the proof of the lemma. □

2.6 An Auxiliary Result

Let $p \geq 1$ be an integer and let $e_1 = (1, 0, \dots 0), \dots, e_p = (0, \dots, 0, 1)$ be the standard basis in R^p. For each set $E \subset R^p$ denote by $\mathrm{conv}(E)$ its convex hull.

Proposition 2.14. *Let a finite set* $E = \{h_{ij} : i = 1, 2, \dots p, \; j = 1, 2\} \subset R^p$ *satisfy*

$$|h_{i1} - e_i|, \; |h_{i2} + e_i| \leq (2p)^{-1}, \; i = 1, \dots p.$$

Then the relation $0 \in conv(E)$ *holds.*

Proof. Let us assume the converse. Then $0 \notin \mathrm{conv}(E)$ and there is $\xi = (\xi_1, \dots, \xi_p) \in R^p \setminus \{0\}$ such that $\inf\{< g, \xi >: \; g \in \mathrm{conv}(E)\} > 0$. We may assume that $|\xi_1| \geq |\xi_i|, i = 1, \dots, p$. There are two cases: $\xi_1 > 0$; $\xi_1 < 0$. Consider the case with $\xi_1 > 0$. Then $0 < \xi, h_{12} >=< \xi, -e_1 > + < \xi, h_{12} + e_1 >\leq -\xi_1 + (2p)^{-1} p|\xi_1| < 0$, a contradiction. Analogously we obtain a contradiction in the second case. The proposition is proved. □

2.7 An Auxiliary Lemma for Hypothesis (A4)

Assume that $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and $\mathrm{graph}(A)$ is a closed subset of the space $\Omega \times R^n$ with the product topology. Let $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots 0), \dots, e_N = (0, 0, \dots 1)$ be a standard basis in R^N. Now we define a set $\mathcal{L} \subset C_l(B_1 \times B_2)$. In the case $m = 1$ we set $\mathcal{L} = C_l(B_1 \times B_2)$ and in the case $m > 1$ we denote by $\mathcal{L}$ a singleton $\{0\}$ where 0 is a function in $C_l(B_1 \times B_2)$ which is identical zero. In the case $m > 1$ for each $(f, \xi) \in \mathcal{M}(A, \tilde{U}_A) \times \mathcal{L}$ and each $(x, u) \in X(A, \tilde{U}_A)$ we set

$$I^{(f,\xi)}(x,u) = I^{(f)}(x,u)$$

(see (2.10), (2.11), and (2.12)).

Lemma 2.15. *Let* $f \in \mathcal{M}(A, \tilde{U}_A)$, $\xi \in \mathcal{L}$, $U \in \mathcal{P}_A$,

$$\{(x,u) \in X(A,U) : \; I^{(f,\xi)}(x,u) < \infty\} \neq \emptyset, \tag{2.90}$$

and let $\epsilon, \delta > 0$. *Then there are* $U_* \in \mathcal{P}_A$, $(\bar{x}, \bar{u}) \in X(A, U_*)$, *and an open set* $\mathcal{W}$ *in* $\mathcal{P}_A$ *such that*

$$(U_*, U) \in E_{\mathcal{P}_A}(\epsilon), \; \mathcal{W} \cap \{V \in \mathcal{P}_A : \; (U,V) \in E_{\mathcal{P}_A}(\epsilon)\} \neq \emptyset, \tag{2.91}$$

$$I^{(f,\xi)}(\bar{x},\bar{u}) \le \inf\{I^{(f,\xi)}(x,u) : \; (x,u) \in X(A,U_*)\} + \delta < \infty \tag{2.92}$$

and for all $V \in \mathcal{W}$,

$$(\bar{x},\bar{u}) \in X(A,V) \subset X(A,U_*). \tag{2.93}$$

Proof. For each $r \in [0,1]$ define $U_r \in \mathcal{P}_A$ by

$$U_r(t,x) = \{u \in R^N : \; d_H(u, U(t,x)) \le r\}, \; (t,x) \in \mathrm{graph}(A) \tag{2.94}$$

and define

$$\mu(r) = \inf\{I^{(f,\xi)}(x,u) : \; (x,u) \in X(A,U_r)\}. \tag{2.95}$$

Clearly $\mu(r)$ is finite for all $r \in [0,1]$ and the function μ is monotone decreasing. There is $r_0 \in (0, 8^{-1}\epsilon)$ such that μ is continuous at r_0. Choose $r_1 \in (0, r_0)$ such that

$$|\mu(r_1) - \mu(r_0)| < 16^{-1}\delta. \tag{2.96}$$

There is

$$(\bar{x},\bar{u}) \in X(A, U_{r_1}) \tag{2.97}$$

such that

$$I^{(f,\xi)}(\bar{x},\bar{u}) \le \mu(r_1) + 16^{-1}\delta. \tag{2.98}$$

Equations (2.96) and (2.98) imply that

$$I^{(f,\xi)}(\bar{x},\bar{u}) \le \mu(r_0) + 8^{-1}\delta. \tag{2.99}$$

Set

$$r_2 = 2^{-1}(r_0 + r_1). \tag{2.100}$$

Clearly

$$(U_{r_i}, U) \in E_{\mathcal{P}_A}(\epsilon), \; i = 0,1,2. \tag{2.101}$$

Choose a positive number γ for which

$$\gamma < \min\{4^{-1}\delta,\ (16N)^{-1}(r_0 - r_1)\} \tag{2.102}$$

and define

$$\mathcal{W} = \{V \in \mathcal{P}_A :\ (U_{r_2}, V) \in E_{\mathcal{P}_A}(\gamma)\},\ U_* = U_{r_0}. \tag{2.103}$$

It follows from (2.101), (2.103), (2.99), and (2.95) that (2.91) and (2.92) are true.

Assume that $V \in \mathcal{W}$. Then by (2.103), (2.102), and (2.100) for each $(t, x) \in$ graph(A)

$$V(t, x) \subset \{z \in R^N :\ d_H(z, U_{r_2}(t, x)) \le \gamma\}$$
$$\subset \{z \in R^N :\ d_H(z, U(t, x)) \le r_0\} = U_{r_0}(t, x).$$

Therefore

$$X(A, V) \subset X(A, U_{r_0}). \tag{2.104}$$

We will show that $(\bar{x}, \bar{u}) \in X(A, V)$. It is sufficient to show that

$$\bar{u}(t) \in V(t, \bar{x}(t)) \text{ for almost every } t \in \Omega. \tag{2.105}$$

By (2.97) for almost every $t \in \Omega$

$$\bar{u}(t) \in U_{r_1}(t, \bar{x}(t)). \tag{2.106}$$

Assume that $t \in \Omega$ and (2.106) is true. By (2.106), (2.100), (2.94), and (2.103) for $i = 1, \dots, N$

$$\bar{u}(t) + 2^{-1}(r_0 - r_1)e_i,\ \bar{u}(t) - 2^{-1}(r_0 - r_1)e_i \in U_{r_2}(t, \bar{x}(t))$$

and there are $z_{i1}, z_{i2} \in R^N$ such that

$$\bar{u}(t) + z_{i1},\ \bar{u}(t) + z_{i2} \in V(t, \bar{x}(t)),\ |z_{i1} - 2^{-1}(r_0 - r_1)e_i|,\ |z_{i2} + 2^{-1}(r_0 - r_1)e_i| \le \gamma. \tag{2.107}$$

Since the set $V(t, \bar{x}(t))$ is convex it follows from (2.107), (2.102), and Proposition 2.14 that

$$0 \in \text{conv}\{z_{ij} :\ i = 1, \dots N,\ j = 1, 2\},\ \bar{u}(t) \in V(t, \bar{x}(t)).$$

This implies that $(\bar{x}, \bar{u}) \in X(A, V)$. The lemma is proved. □

2.8 Proof of Theorem 2.2 and Its Extensions

Proof. By Propositions 2.6 and 2.7 (A1) holds and J_a is lower semicontinuous for all $a \in \mathcal{A}_1 \times \mathcal{A}_2$. By Theorem 2.3 we need to verify that (H1) and (H2) are valid. (H2) follows from Proposition 2.7. Therefore it is sufficient to show that (H1) holds. By Proposition 2.4 it is sufficient to show that (A2), (A3), and (A4) are valid. (A2) follows from Propositions 2.10 and 2.12. By Lemma 2.13, (A3) holds. (A4) follows from Lemma 2.15. This completes the proof of the theorem. □

As we mentioned in Sect. 2.1 we proved Theorem 2.2 in such a manner that it is applicable for all the spaces of integrands introduced there. All the spaces of integrands are subspaces of $\mathcal{M}(A,U)$. Since (H2), (A1), (A2), and (A4) hold for the class of optimal control problems with the space of integrands $\mathcal{M}(A,U)$ they are also valid for all its subclasses considered here. On the other hand (A3) follows from Lemma 2.13 which establishes that $f+g$ and f belong to the same subspaces of integrands. This implies that (A3) holds for all classes of optimal control problems introduced in Sect. 2.1.

As seen from the proof of Lemma 2.13 the perturbation g of the integrand f is chosen as the sum of two functions, one of them depending only on (t,x) while the other depending only on (t,u). Therefore Theorem 2.2 can be easily extended to subclasses of the classes of optimal control problems introduced in Sect. 2.1 in which integrands are sums of two finite-valued functions, one of them, depending only on (t,x), is defined on graph(A) while the other, depending only on (t,u), is defined on $\Omega \times R^N$.

In this section we present the extension of the generic existence and uniqueness result established in [86, 88] for the space of lower semicontinuous integrands f : graph$(U) \to R^1$. This generalization holds for all the spaces of integrands defined in Sect. 2.1 and it is obtained as a realization of the generic variational principle established in Sect. 2.2.

Assume that $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, U : graph$(A) \to 2^{R^N} \setminus \{\emptyset\}$ and graph(U) is a closed subset of $\Omega \times R^n \times R^N$ with the product topology. We consider the metric space $X(A,U)$ with the metric ρ (see (2.5)).

Now we define $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1 = \mathcal{A}_{11} \times \mathcal{A}_{12} \text{ if } m = 1 \text{ and } \mathcal{A}_1 = \mathcal{A}_{11} \text{ if } m > 1,$$

where $\mathcal{A}_{12}$ is either $C_l(B_1 \times B_2)$ or $C(B_1 \times B_2)$ or a singleton $\{\xi\} \subset C_l(B_1 \times B_2)$, and $\mathcal{A}_{11}$ is one of the following spaces:

$$\mathcal{M}(A,U);\ \mathcal{M}^l(A,U);\ \mathcal{M}^c(A,U);$$

$$\mathcal{M}_k(A,\tilde{U}_A);\ \mathcal{M}_k^l(A,\tilde{U}_A);\ \mathcal{M}_k^c(A,\tilde{U}_A) \text{ (here } k \ge 1 \text{ is an integer, } U = \tilde{U}_A$$

$$\text{and graph}(A) \text{ is a closed subset of the space } \Omega \times R^n$$

$$\text{with the product topology);}$$

$$\mathcal{M}_k^*(\tilde{A}, \tilde{U});\ \mathcal{M}_k^{*l}(\tilde{A}, \tilde{U});$$

$$\mathcal{M}_k^{*c}(\tilde{A}, \tilde{U}) \text{ (here } k \geq 1 \text{ is an integer and } A = \tilde{A},\ U = \tilde{U}).$$

Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 :\ \inf(I^{(a)}) < \infty\}$ in the space $\mathcal{A}_1$ with the strong topology. We assume that $\mathcal{A}$ is nonempty. The following result is proved analogously to Theorem 2.2.

Theorem 2.16. *The minimization problem for $I^{(a)}$ on $(X(A,U), \rho)$ is generically strongly well posed with respect to $\mathcal{A}$.*

2.9 Variational Problems

We use the notations and definitions introduced in Sect. 2.1. Assume that $n = N$, $H(t,x,u) = u$, $(t,x,u) \in \Omega \times R^n \times R^n$ and B_1 and B_2 are singletons. Let $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, U : graph$(A) \to 2^{R^n} \setminus \{\emptyset\}$ and let graph(U) be a closed subset of the space $\Omega \times R^n \times R^n$ with the product topology. If $(x,u) \in X(A,U)$, then $u = \nabla x$ and (x,u) are identified with $x \in (W^{1,1}(\Omega))^n$. In this section we omit the notation u in describing the elements of $X(A,U)$. For the set $X(A,U)$ we consider the metric ρ introduced in Sect. 2.1 (see (2.5)) and the metric ρ_s defined by

$$\rho_s(x,y) = ||x-y||_{W^{1,1}(\Omega)} \text{ for all } x, y \in X(A,U).$$

Clearly $(X(A,U), \rho_s)$ is a complete metric space and its uniform structure is stronger than the uniformity which generates the metric ρ. Finally for the set $X(A,U)$ we consider the third uniformity which is determined by the following base:

$$\begin{aligned} E_{Xw}(\epsilon) = \{&(x_1,x_2) \in X(A,U) \times X(A,U) : \\ &\text{mes}\{t \in \Omega :\ |\nabla x_1(t) - \nabla x_2(t)| \geq \epsilon\} \leq \epsilon\}, \end{aligned} \tag{2.108}$$

where $\epsilon > 0$. (Note that if $x, y \in X(A,U)$ and $\nabla x = \nabla y$, then $x = y$ (Theorem 2.4.1 of [108].) It is easy to see that this uniform structure is metrizable (by a metric ρ_w) and weaker than the uniformity which generates the metric ρ.

For variational problems considered in this section we can obtain strong versions of Theorems 2.2 and 2.16. These strong versions establish generic strong well-posedness of the minimization problem on the space (X, ρ_s) while in Theorems 2.2 and 2.16 it is obtained on (X, ρ). They are derived from Theorems 2.2 and 2.16, Proposition 2.10, and the following proposition.

Proposition 2.17. *Let $f \in \mathcal{M}(A,U)$,*

$$c_0 > \inf\left\{\int_\Omega f(t, x(t), \nabla x(t))dt :\ x \in X(A,U)\right\} \tag{2.109}$$

and let

$$Y = \{x \in X(A, U) : \int_{\Omega} f(t, x(t), \nabla x(t))dt \leq c_0\}. \tag{2.110}$$

Then for each $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $x_1, x_2 \in Y$ *and* $(x_1, x_2) \in E_{Xw}(\delta)$*, then* $\rho_s(x_1, x_2) \leq \epsilon$*.*

Proof. Let $\epsilon > 0$. In the case $m > 1$ by Theorem 2.4.1 in [108] there exists a constant $c > 0$ such that $||h||_{L^1(\Omega)} \leq c||\nabla h||_{L^1(\Omega)}$ for all $h \in W_0^{1,1}(\Omega)$. In the case $m = 1$ set $c = 1$. Choose a positive number

$$\Delta < (32(c + 1)(\text{mes}(\Omega) + 1))^{-1}\epsilon.$$

By property (iii) (see the definition of $\mathcal{M}(A, U)$) and (2.110), the family of functions $\{|\nabla x(\cdot)| : \ x \in Y\}$ is uniformly integrable. Therefore there exists $\gamma \in (0, \Delta)$ such that for each $x \in Y$ and each measurable set $e \subset \Omega$ satisfying $\text{mes}(e) \leq \gamma$ the inequality $\int_e |\nabla x(t)|dt \leq \Delta$ holds. Choose a positive number $\delta < (8c + 8)^{-1}(\text{mes}(\Omega) + 1)^{-2}\gamma$.

Assume that $x_1, x_2 \in Y$ and $(x_1, x_2) \in E_{Xw}(\delta)$. There exists a measurable set $e \subset \Omega$ such that $\text{mes}(e) \leq \delta$ and $|\nabla x_1(t) - \nabla x_2(t)| \leq \delta$, $t \in \Omega \setminus e$. It follows from these inequalities and the definition of γ and δ that

$$\int_e |\nabla x_i(t)|dt \leq \Delta, \ i = 1, 2, \ \int_{\Omega} |\nabla x_1(t) - \nabla x_2(t)|dt, \tag{2.111}$$

$$\leq \int_e |\nabla x_1(t) - \nabla x_2(t)|dt + \int_{\Omega\setminus e} |\nabla x_1(t) - \nabla x_2(t)|dt \leq 2\Delta + \delta\text{mes}(\Omega).$$

In the case $m = 1$ we have

$$|x_1(t) - x_2(t)| \leq \int_{\Omega} |\nabla x_1(s) - \nabla x_2(s)|ds, \ t \in \Omega$$

and by (2.111) and the definition of δ and Δ

$$\begin{aligned}\rho_s(x_1, x_2) &= ||x_1 - x_2||_{W^{1,1}(\Omega)} \leq (\text{mes}(\Omega) + 1)||\nabla x_1 - \nabla x_2||_{L^1(\Omega)} \\ &\leq (\text{mes}(\Omega) + 1)(2\Delta + \delta\text{mes}(\Omega)) < \epsilon.\end{aligned}$$

In the case $m > 1$ it follows from (2.4), the definition of c, (2.111), and the definition of δ, Δ that

$$\begin{aligned}\rho_s(x_1, x_2) &= ||x_1 - x_2||_{L^1(\Omega)} + ||\nabla x_1 - \nabla x_2||_{L^1(\Omega)} \\ &\leq (c + 1)||\nabla x_1 - \nabla x_2||_{L^1(\Omega)} \leq (c + 1)(2\Delta + \delta\text{mes}(\Omega)) < \epsilon.\end{aligned}$$

This completes the proof of the proposition. □

Proposition 2.17 and the completeness of the space $(X(A,U),\rho_s)$ imply the following result.

Proposition 2.18. *Assume that $f \in \mathcal{M}(A,U)$, $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in the space $X(A,U)$ with the metric ρ_w and the sequence*

$$\left\{\int_{\Omega} f(t,x_i(t),\nabla x_i(t))dt\right\}_{i=1}^{\infty}$$

is bounded. Then there is $x_* \in X(A,U)$ such that $\rho_s(x_i,x_*) \to 0$ as $i \to \infty$ and moreover if $m=1$, then $x_i(t) \to x_*(t)$ as $i \to \infty$ uniformly on $[T_1,T_2]$.

From now on in this section we consider a fixed set-valued mapping $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ for which graph(A) is a closed subset of the space $\Omega \times R^n$ with the product topology and a set-valued mapping $\tilde{U}_A :$ graph$(A) \to 2^{R^n} \setminus \{\emptyset\}$ where $\tilde{U}_A(t,x) = R^n$, $(t,x) \in$ graph(A). For each $f \in \mathcal{M}(A,\tilde{U}_A)$ we define $I^{(f)} : X(A,\tilde{U}_A) \to R^1 \cup \{\infty\}$ by

$$I^{(f)}(x) = \int_{\Omega} f(t,x(t),\nabla x(t))dt,\ x \in X(A,\tilde{U}_A).$$

Consider the space of set-valued mappings $\mathcal{A}_2$ and the space of integrands $\mathcal{A}_{11}$ defined in Sect. 2.1. Denote by $\mathcal{A}_0$ the set of all functions $f \in \mathcal{A}_{11}$ which do not depend on x. Clearly $\mathcal{A}_0$ is a closed subset of $\mathcal{A}_{11}$ with the strong topology. We consider the topological subspace $\mathcal{A}_0 \subset \mathcal{A}_{11}$ with the relative weak and strong topologies.

Let a function F : graph$(A) \times R^n \to R^1 \cup \{\infty\}$ have the following properties:

F is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of Ω and Borel subsets of $R^n \times R^n$.

$F(t,\cdot,\cdot)$ is lower semicontinuous for almost every $t \in \Omega$.

There exists an integrable scalar function $\psi_F(t) \geq 0$, $t \in \Omega$, such that $F(t,x,u) \geq \psi_F(t)$ for all $(t,x,u) \in$ graph $(A) \times R^n$.

Clearly for each $g \in \mathcal{M}(A,\tilde{U}_A)$, $g + F \in \mathcal{M}(A,\tilde{U}_A)$.

For each $a = (a_1,a_2) \in \mathcal{A}_0 \times \mathcal{A}_2$ we define $J_a : X(A,\tilde{U}_A) \to R^1 \cup \{\infty\}$ by

$$J_a(x) = I^{(a_1+F)}(x),\ x \in S_{a_2},\ J_a(x) = \infty,\ x \in X(A,\tilde{U}_A) \setminus S_{a_2}.$$

Here $S_{a_2} = X(A,a_2)$ (see (2.19)). Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_0 \times \mathcal{A}_2 : \inf(J_a) < \infty\}$ in the space $\mathcal{A}_0 \times \mathcal{A}_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty.

Theorem 2.19. *The minimization problem for J_a on $(X(A,\tilde{U}_A),\rho_s)$ is generically strongly well posed with respect to $\mathcal{A}$.*

Proof. We will show that the following assertion holds:

The minimization problem for J_a on $(X(A,\tilde{U}_A),\rho_w)$ is generically strongly well posed with respect to $\mathcal{A}$.

This assertion is proved analogously to Theorem 2.2. Note that Propositions 2.6 and 2.18 imply the lower semicontinuity of J_a for all $a \in \mathcal{A}_0 \times \mathcal{A}_2$, (H2) follows from Proposition 2.18, and (A3) is derived from a modification of Lemma 2.13. In this modification the perturbation $g = g(t, x, u)$ does not depend on x and in the last line of the statement of Lemma 2.13 ρ is substituted by ρ_w. The proof of this modification is analogous to the proof of Lemma 2.13. In the relation (2.70) ρ is substituted by ρ_w and E_X is substituted by E_{Xw} and in (2.85) g is defined by

$$g(t, x, u) = \epsilon_0(\gamma)\bar{\phi}(|u - \bar{u}_s(t)|^2), \ (t, x, u) \in R^m \times R^n \times R^n.$$

Thus there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the assertions (1) and (2) of Theorem 2.3 hold with $(X, \rho) = (X(A, \tilde{U}_A), \rho_w)$ and $f_b = J_b, b \in \mathcal{A}$.

Let $a = (a_1, a_2) \in \mathcal{B}$. By the assertion (1) of Theorem 2.3 $\inf(J_a)$ is finite and attained at a unique element $\bar{x} \in X(A, \tilde{U}_A)$. In order to complete the proof of the theorem it is sufficient to show that the assertion (2) of Theorem 2.3 holds with $(X, \rho) = (X(A, \tilde{U}_A), \rho_s)$ and $f_b = J_b, b \in \mathcal{A}$.

By Proposition 2.10 there exists an open (in the weak topology) neighborhood $\mathcal{V}_1$ of a_1 in $\mathcal{A}_0$ such that for each $b \in \mathcal{V}_1$ and each $x \in X(A, \tilde{U}_A)$ satisfying $I^{(b+F)}(x) \le \inf(J_a) + 1$ the following relation holds:

$$I^{(a_1+F)}(x) \le I^{(b+F)}(x) + 1 \le \inf(J_a) + 2. \tag{2.112}$$

Let $\epsilon \in (0, 1)$. It follows from Proposition 2.17 that there exists $\epsilon_0 \in (0, \epsilon)$ such that for each $x_1, x_2 \in X(A, \tilde{U}_A)$ satisfying

$$I^{(a_1+F)}(x_i) \le \inf(J_a) + 2, \ i = 1, 2 \text{ and } \rho_w(x_1, x_2) \le \epsilon_0 \tag{2.113}$$

the relation $\rho_s(x_1, x_2) \le \epsilon$ holds. By the assertion (2) of Theorem 2.3 which holds for the space $(X(A, \tilde{U}_A), \rho_w)$ there are a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(J_b)$ is finite and if $z \in X(A, \tilde{U}_A)$ satisfies

$$J_b(z) \le \inf(J_b) + \delta, \tag{2.114}$$

then

$$\rho_w(\bar{x}, z) \le \epsilon_0 \text{ and } |J_b(z) - J_a(\bar{x})| \le \epsilon_0. \tag{2.115}$$

We may assume that

$$\mathcal{V} \subset \mathcal{V}_1 \times \mathcal{A}_2. \tag{2.116}$$

Now assume that $b = (b_1, b_2) \in \mathcal{V}$ and $z \in X(A, \tilde{U}_A)$ satisfies (2.114). Then (2.115) holds. By (2.115), (2.116), and the definition of $\mathcal{V}_1$ (see (2.112))

$$I^{(a_1+F)}(z) \le \inf(J_a) + 2.$$

It follows from this inequality, (2.115), and the definition of ϵ_0 (see (2.113)) that the relation $\rho_s(\bar{x}, z) \le \epsilon$ holds. Thus the assertion (2) of Theorem 2.3 holds with $(X, \rho) = (X(A, \tilde{U}_A), \rho_s)$ and $f_b = J_b$, $b \in \mathcal{A}$. This completes the proof of the theorem. □

Note that for the class of variational problems considered here we can also prove an analog of Theorem 2.16 in which only integrands are subject to variations.

2.10 Optimal Control Problems with Cinquini Growth Condition

In this chapter we also prove an extension of Theorem 2.2 to a class of optimal control problems satisfying the Cinquini growth condition obtained in [90]. We use the notations and definitions introduced in Sect. 2.1.

Let $n, N \ge 1$ be integers. We assume that $\Omega = (T_1, T_2)$, where T_1 and T_2 are fixed real numbers for which $T_1 < T_2$, $H(t, x, u)$ is a fixed continuous function defined on $\Omega \times R^n \times R^N$ with values in R^n such that $H(t, x, u) = (H_i)_{i=1}^n$ and B_1 and B_2 are fixed bounded nonempty closed subsets of R^n.

Suppose that $a(t)$, $t \in [T_1, T_2]$ is a given integrable scalar nonpositive function and $\phi, \psi : [0, \infty) \to [0, \infty)$ are nondecreasing functions such that ϕ is convex

$$\lim_{r \to \infty} \phi(r)/r = \infty \tag{2.117}$$

and

$$\lim_{r \to \infty} [\phi(2r(T_2 - T_1)^{-1}) - \psi(r + \sup\{|y| : \ y \in B_1 \cup B_2\})] = \infty.$$

Define set-valued mappings $\tilde{A} : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and $\tilde{U} : \Omega \times R^n \to 2^{R^N} \setminus \{\emptyset\}$ by

$$\tilde{A}(t) = R^n,\ t \in \Omega,\ \tilde{U}(t, x) = R^N,\ (t, x) \in \Omega \times R^n. \tag{2.118}$$

For each $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ and each U : graph$(A) \to 2^{R^N} \setminus \{\emptyset\}$ for which graph(U) is a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology, we denote by $X(A, U)$ the set of all pairs of functions (x, u), where $x = (x_1, \dots, x_n) \in (W^{1,1}(\Omega))^n$, $u = (u_1, \dots u_N) : \ \Omega \to R^N$ is measurable and the following relations hold:

$$x(t) \in A(t),\ t \in \Omega \text{ (a.e.)},\ u(t) \in U(t, x(t)),\ t \in \Omega \text{ (a.e.)}, \tag{2.119a}$$

$$x'(t) = H(t, x(t), u(t)),\ t \in \Omega \text{ (a.e.)}, \tag{2.119b}$$

$$x(T_i) \in B_i,\ i = 1, 2. \tag{2.119c}$$

Let $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, $U : \mathrm{graph}(A) \to 2^{R^N} \setminus \{\emptyset\}$ and let $\mathrm{graph}(U)$ be a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology.

For the set $X(A, U)$ defined above we consider the uniformity which is determined by the following base:

$$E_X(\epsilon) = \{((x_1, u_1), (x_2, u_2)) \in X(A, U) \times X(A, U) : \\ \mathrm{mes}\{t \in \Omega : \ |x_1(t) - x_2(t)| + |u_1(t) - u_2(t)| \geq \epsilon\} \leq \epsilon\}, \tag{2.120}$$

where $\epsilon > 0$. The uniform space $X(A, U)$ is metrizable (by a metric ρ). In the space $X(A, U)$ we consider the topology induced by the metric ρ.

Next we define spaces of integrands associated with the maps A and U. By $\mathfrak{M}(A, U)$ we denote the set of all functions $f : \mathrm{graph}(U) \to R^1 \cup \{\infty\}$ with the following properties:

(i) f is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of Ω and Borel subsets of $R^n \times R^N$.
(ii) $f(t, \cdot, \cdot)$ is lower semicontinuous for almost every $t \in \Omega$.
(iii) $f(t, x, u) \geq \phi(|H(t, x, u)|) - \psi(|x|) + a(t)$, $(t, x, u) \in \mathrm{graph}(U)$.

Due to property (i) for every $f \in \mathfrak{M}(A, U)$ and every $(x, u) \in X(A, U)$ the function $f(t, x(t), u(t))$, $t \in \Omega$ is measurable. The growth condition in (iii) was employed in [22, 41] to study a Bolza problem.

Denote by $\mathfrak{M}^l(A, U)$ (respectively, $\mathfrak{M}^c(A, U)$) the set of all lower semicontinuous (respectively, finite-valued continuous) functions $f : \mathrm{graph}(U) \to R^1 \cup \{\infty\}$ in $\mathfrak{M}(A, U)$. Now we equip the set $\mathfrak{M}(A, U)$ with the strong and weak topologies. For the space $\mathfrak{M}(A, U)$ we consider the uniformity determined by the following base:

$$E_{\mathfrak{M}s}(\epsilon) = \{(f, g) \in \mathfrak{M}(A, U) \times \mathfrak{M}(A, U) : \\ |f(t, x, u) - g(t, x, u)| \leq \epsilon, \ (t, x, u) \in \mathrm{graph}(U)\}, \tag{2.121}$$

where $\epsilon > 0$. It is easy to see that the uniform space $\mathfrak{M}(A, U)$ with this uniformity is metrizable (by a metric $d_{\mathfrak{M}}$) and complete. This uniformity generates in $\mathfrak{M}(A, U)$ the strong topology. Clearly $\mathfrak{M}^l(A, U)$ and $\mathfrak{M}^c(A, U)$ are closed subsets of $\mathfrak{M}(A, U)$ with this topology.

For each $\epsilon, r > 0$ we set

$$E_{\mathfrak{M}w}(\epsilon, r) = \{(f, g) \in \mathfrak{M}(A, U) \times \mathfrak{M}(A, U) : \ \text{there exists a nonnegative} \\ \phi \in L^1(\Omega) \text{ such that } \int_\Omega \phi(t) dt \leq 1, \text{ and for almost every } t \in \Omega, \\ |f(t, x, u) - g(t, x, u)| < \epsilon + \epsilon \max\{|f(t, x, u)|, |g(t, x, u)|\} + \epsilon\phi(t) \\ \text{for each } x \in A(t) \text{ satisfying } |x| \leq r \text{ and each } u \in U(t, x)\}. \tag{2.122}$$

Using Lemma 2.1 we can easily show that for the set $\mathfrak{M}(A, U)$ there exists the uniformity which is determined by the base $E_{\mathfrak{M}w}(\epsilon, r)$, $\epsilon, r > 0$. This uniformity induces in $\mathfrak{M}(A, U)$ the weak topology.

Denote by $C_l(B_1 \times B_2)$ the set of all lower semicontinuous functions $\xi : B_1 \times B_2 \to R^1 \cup \{\infty\}$ bounded from below. We also equip the set $C_l(B_1 \times B_2)$ with strong and weak topologies. For the set $C_l(B_1 \times B_2)$ we consider the uniformity determined by the following base:

$$E_c(\epsilon) = \{(\xi, h) \in C_l(B_1 \times B_2) \times C_l(B_1 \times B_2) : |\xi(z) - h(z)| \le \epsilon, \ z \in B_1 \times B_2\}, \tag{2.123}$$

where $\epsilon > 0$. It is easy to see that the uniform space $C_l(B_1 \times B_2)$ is metrizable (by a metric d_c) and complete. This metric induces in $C_l(B_1 \times B_2)$ the strong topology. We do not write down the explicit expressions for the metrics $d_{\mathfrak{M}}$ and d_c because we are not going to use them in the sequel.

For any $\epsilon > 0$ we set

$$\begin{aligned} E_{cw}(\epsilon) = \{(\xi, h) \in C_l(B_1 \times B_2) \times C_l(B_1 \times B_2) : |\xi(z) - h(z)| \\ < \epsilon + \epsilon \max\{|\xi(z)|, |h(z)|\}, \ z \in B_1 \times B_2\}, \end{aligned} \tag{2.124}$$

where $\epsilon > 0$. In view of Lemma 2.1, for the set $C_l(B_1 \times B_2)$ there exists a uniformity which is determined by the base $E_{cw}(\epsilon)$, $\epsilon > 0$. This uniformity induces in $C_l(B_1 \times B_2)$ the weak topology. Denote by $C(B_1 \times B_2)$ the set of all finite-valued continuous functions h in $C_l(B_1 \times B_2)$. Clearly it is a closed subset of $C_l(B_1 \times B_2)$ with the weak topology.

For each $f \in \mathfrak{M}(A, U)$ and each $\xi \in C_l(B_1 \times B_2)$ we define $I^{(f,\xi)} : X(A, U) \to R^1 \cup \{\infty\}$ by

$$I^{(f,\xi)}(x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt + \xi(x(T_1), x(T_2)), \ (x, u) \in X(A, U). \tag{2.125}$$

We will show that (2.125) defines lower semicontinuous functionals on $X(A, U)$.

From now on in this section we consider a fixed set-valued mapping $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$ for which graph(A) is a closed subset of the space $\Omega \times R^n$ with the product topology. Denote by $\tilde{U}_A$ the restriction of $\tilde{U}$ (see (2.118)) to the graph(A). Namely

$$\tilde{U}_A : \text{graph}(A) \to 2^{R^N}, \ \tilde{U}(t, x) = R^N, \ (t, x) \in \text{graph}(A). \tag{2.126}$$

We consider functionals $I^{(f,\xi)}$ with $(f, \xi) \in \mathfrak{M}(A, \tilde{U}_A) \times C_l(B_1 \times B_2)$ defined on the space $X(A, \tilde{U}_A)$ (see (2.119)). The main result of this section will be established for several classes of optimal control problems with different corresponding spaces of the integrands which are subsets of the space $\mathfrak{M}(A, \tilde{U}_A)$. The subspaces of lower semicontinuous and continuous integrands ($\mathfrak{M}^l(A, \tilde{U}_A)$ and $\mathfrak{M}^c(A, \tilde{U}_A)$) have already been defined. Now we define subspaces of $\mathfrak{M}(A, \tilde{U}_A)$ which consist of integrands differentiable with respect to the control variable u.

Let $k \geq 1$ be an integer. Denote by $\mathfrak{M}_k(A, \tilde{U}_A)$ the set of all finite-valued $f \in \mathfrak{M}(A, \tilde{U}_A)$ such that for each $(t, x) \in \text{graph}(A)$ the function $f(t, x, \cdot) \in C^k(R^N)$. We consider the topological subspace $\mathfrak{M}_k(A, \tilde{U}_A) \subset \mathfrak{M}(A, \tilde{U}_A)$ with the relative weak topology. The strong topology on $\mathfrak{M}_k(A, \tilde{U}_A)$ is induced by the uniformity which is determined by the following base:

$$E_{\mathfrak{M}k}(\epsilon) = \{(f, g) \in \mathfrak{M}_k(A, \tilde{U}_A) \times \mathfrak{M}_k(A, \tilde{U}_A) : \ |f(t, x, u) - g(t, x, u)| \leq \epsilon \tag{2.127}$$

$$\text{for all } (t, x, u) \in \ \text{graph}(A) \times R^N \text{ and}$$

$$||f(t, x, \cdot) - g(t, x, \cdot)||_{C^k(R^N)} \leq \epsilon \text{ for all } (t, x) \in \text{graph}(A)\},$$

where $\epsilon > 0$. It is easy to see that the space $\mathfrak{M}_k(A, \tilde{U}_A)$ with this uniformity is metrizable (by a metric $d_{\mathfrak{M},k}$) and complete. Define

$$\begin{aligned} \mathfrak{M}_k^l(A, \tilde{U}_A) &= \mathfrak{M}_k(A, \tilde{U}_A) \cap \mathfrak{M}^l(A, \tilde{U}_A), \\ \mathfrak{M}_k^c(A, \tilde{U}_A) &= \mathfrak{M}_k(A, \tilde{U}_A) \cap \mathfrak{M}^c(A, \tilde{U}_A). \end{aligned} \tag{2.128}$$

Clearly $\mathfrak{M}_k^l(A, \tilde{U}_A)$ and $\mathfrak{M}_k^c(A, \tilde{U}_A)$ are closed sets in $\mathfrak{M}_k(A, \tilde{U}_A)$ with the strong topology.

Finally we define subspaces of $\mathfrak{M}(\tilde{A}, \tilde{U})$ which consist of integrands differentiable with respect to the state variable x and the control variable u. Denote by $\mathfrak{M}_k^*(\tilde{A}, \tilde{U})$ the set of all $f : \Omega \times R^n \times R^N \to R^1$ in $\mathfrak{M}(\tilde{A}, \tilde{U})$ (see (2.118)) such that for each $t \in \Omega$ the function $f(t, \cdot, \cdot) \in C^k(R^n \times R^N)$. We consider the topological subspace $\mathfrak{M}_k^*(\tilde{A}, \tilde{U}) \subset \mathfrak{M}(\tilde{A}, \tilde{U})$ with the relative weak topology. The strong topology in $\mathfrak{M}_k^*(\tilde{A}, \tilde{U})$ is induced by the uniformity which is determined by the following base:

$$E_{\mathfrak{M}k}^*(\epsilon) = \{(f, g) \in \mathfrak{M}_k^*(\tilde{A}, \tilde{U}) \times \mathfrak{M}_k^*(\tilde{A}, \tilde{U}) : \tag{2.129}$$

$$|f(t, x, u) - g(t, x, u)| \leq \epsilon \text{ for all } (t, x, u) \in \Omega \times R^n \times R^N \text{ and}$$

$$||f(t, \cdot, \cdot) - g(t, \cdot, \cdot)||_{C^k(R^{n+N})} \leq \epsilon \text{ for all } t \in \Omega\},$$

where $\epsilon > 0$. It is easy to see that the space $\mathfrak{M}_k^*(\tilde{A}, \tilde{U})$ with this uniformity is metrizable (by a metric $d_{\mathfrak{M},k}^*$) and complete. Define

$$\mathfrak{M}_k^{*l}(\tilde{A}, \tilde{U}) = \mathfrak{M}_k^*(\tilde{A}, \tilde{U}) \cap \mathfrak{M}^l(\tilde{A}, \tilde{U}), \ \mathfrak{M}_k^{*c}(\tilde{A}, \tilde{U}) = \mathfrak{M}_k^*(\tilde{A}, \tilde{U}) \cap \mathfrak{M}^c(\tilde{A}, \tilde{U}). \tag{2.130}$$

Clearly $\mathfrak{M}_k^{*l}(\tilde{A}, \tilde{U})$ and $\mathfrak{M}_k^{*c}(\tilde{A}, \tilde{U})$ are closed sets in $\mathfrak{M}_k^*(\tilde{A}, \tilde{U})$ with the strong topology.

Thus we have defined all the spaces of integrands for which we will prove the main result of this section. Now we will define a space of constraint maps $\mathcal{P}_A$. Denote by $S(R^N)$ the set of all nonempty convex closed subsets of R^N. For each $x \in R^N$ and each $E \subset R^N$, set $d_H(x, E) = \inf_{y \in E} |x - y|$. For each pair of sets $C_1, C_2 \subset R^N$,

$$d_H(C_1, C_2) = \max\left\{ \sup_{y \in C_1} d_H(y, C_2),\ \sup_{x \in C_2} d_H(x, C_1) \right\}$$

is the Hausdorff distance between C_1 and C_2.

For the space $S(R^N)$ we consider the uniformity determined by the following base:

$$E_{R^N}(\epsilon) = \{(C_1, C_2) \in S(R^N) \times S(R^N) :\ d_H(C_1, C_2) \le \epsilon\}, \tag{2.131}$$

where $\epsilon > 0$. It is well known that the space $S(R^N)$ with this uniformity is metrizable and complete. Denote by $\mathcal{P}_A$ the set of all set-valued mappings U : graph$(A) \to S(R^N)$ such that graph(U) is a closed subset of the space graph$(A) \times R^N$ with the product topology. For the space $\mathcal{P}_A$ we consider the uniformity determined by the following base:

$$E_{\mathcal{P}_A}(\epsilon) = \{(U_1, U_2) \in \mathcal{P}_A \times \mathcal{P}_A :\ d_H(U_1(t, x), U_2(t, x)) \le \epsilon \text{ for all } (t, x) \in \text{graph}(A)\}, \tag{2.132}$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{P}_A$ with this uniformity is metrizable and complete. This uniformity generates in $\mathcal{P}_A$ the strong topology.

For the space $\mathcal{P}_A$ we also consider the uniformity determined by the following base:

$$E_{\mathcal{P}_A w}(\epsilon, r) = \{(U_1, U_2) \in \mathcal{P}_A \times \mathcal{P}_A :\ d_H(U_1(t, x), U_2(t, x)) \le \epsilon \text{ for all } (t, x) \in \text{graph}(A) \text{ satisfying } |x| \le r\}, \tag{2.133}$$

where $\epsilon, r > 0$. This uniformity generates in $\mathcal{P}_A$ the weak topology.

We consider the space $X(A, \tilde{U}_A)$ with the metric ρ (see 2.120)). For each $U \in \mathcal{P}_A$ define

$$S_U = X(A, U) = \{(x, u) \in X(A, \tilde{U}_A) :\ u(t) \in U(t, x(t)),\ t \in \Omega \text{ (a.e.)}\}. \tag{2.134}$$

For each $U \in \mathcal{P}_A$ and each $(f, \xi) \in \mathfrak{M}(A, \tilde{U}_A) \times C_l(B_1 \times B_2)$ we consider the optimal control problem

$$I^{(f,\xi)}(x, u) \to \min,\ (x, u) \in X(A, U).$$

We will state our generic well-posedness result, Theorem 2.20, in such a manner that it will be applicable for all the spaces of integrands defined above.

To meet this goal we set $\mathcal{A}_2 = \mathcal{P}_A$ and define a space $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1 = \mathcal{A}_{11} \times \mathcal{A}_{12},$$

where $\mathcal{A}_{12}$ is either $C_l(B_1 \times B_2)$ or $C(B_1 \times B_2)$ or a singleton $\{\xi\} \subset C_l(B_1 \times B_2)$, and $\mathcal{A}_{11}$ is one of the following spaces:

$$\mathfrak{M}(A, \tilde{U}_A);\ \mathfrak{M}^l(A, \tilde{U}_A);\ \mathfrak{M}^c(A, \tilde{U}_A);$$

$$\mathfrak{M}_k(A, \tilde{U}_A);\ \mathfrak{M}_k^l(A, \tilde{U}_A);\ \mathfrak{M}_k^c(A, \tilde{U}_A)\ \ \text{(here } k \geq 1 \text{ is an integer);}$$

$$\mathfrak{M}_k^*(\tilde{A}, \tilde{U});\ \mathfrak{M}_k^{*l}(\tilde{A}, \tilde{U});\ \mathfrak{M}_k^{*c}(\tilde{A}, \tilde{U})\ \text{(here } k \geq 1 \text{ is an integer and } A = \tilde{A}).$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define $J_a : X(A, \tilde{U}_A) \to R^1 \cup \{\infty\}$ by

$$J_a(x,u) = I^{(a_1)}(x,u),\ (x,u) \in S_{a_2},\ J_a(x,u) = \infty,\ (x,u) \in X(A, \tilde{U}_A) \setminus S_{a_2}.$$

We will show that J_a is lower semicontinuous for all $a \in \mathcal{A}_1 \times A_2$. Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 :\ \inf(J_a) < \infty\}$ in the space $\mathcal{A}_1 \times A_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty. We will prove the following result obtained in [90].

Theorem 2.20. *There exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the following assertions hold:*

(1) $\inf(J_a)$ *is finite and attained at a unique pair* $(\bar{x}, \bar{u}) \in X(A, \tilde{U}_A)$.
(2) For each $\epsilon > 0$ there are a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(J_b)$ *is finite and if* $(z,w) \in X(A, \tilde{U}_A)$ *satisfies* $J_b(z,w) \leq \inf(J_b) + \delta$, *then* $\rho((\bar{x}, \bar{u}), (z,w)) \leq \epsilon$ *and* $|J_b(z,w) - J_a(\bar{x}, \bar{u})| \leq \epsilon$.

Theorem 2.20 is an extension of Theorem 2.2 which was established for a class of optimal control problems with integrands satisfying the Cesari growth condition. If an integrand f satisfies the Cesari growth condition, then for each t the function $f(t,\cdot,\cdot)$ is bounded below while if f satisfies the Cinquini growth condition in (iii) and $H(t,x,u) = u$, then $f(t,x,u)$ can tend to $-\infty$ as $|x| \to \infty$ for fixed t, u. Note that using the Cinquini growth condition allows us to weaken the weak topology in the space $\mathcal{A}_1 \times \mathcal{A}_2$. Namely in Sect. 2.1 the space $\mathcal{P}_A$ was considered only with one uniformity determined by (2.132).

2.11 Variational Principles

We will obtain Theorem 2.20 as a realization of the variational principle which was introduced in Sect. 2.2.

We consider a metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology. (Note that these topologies can coincide.) We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$. In our study we use the hypotheses about the functions (H1) and (H2) introduced in Sect. 2.2 and Theorem 2.3.

The proof of Theorem 2.20 consists in verifying that the hypotheses (H1) and (H2) hold for the space of integrand–map pairs introduced in Sect. 2.10. In order to simplify the verification of (H1) we use auxiliary assumptions (A1)–(A4) which imply (H1).

Let (X, ρ) be a metric space with the topology generated by the metric ρ and let $(\mathcal{A}_1, d_1)$, $(\mathcal{A}_2, d_2)$ be metric spaces. For the space $\mathcal{A}_i$ $(i = 1, 2)$ we consider the topology generated by the metric d_i. This topology is called the strong topology. In addition to the strong topology we consider a weak topology on $\mathcal{A}_i$, $i = 1, 2$.

Assume that with every $a \in \mathcal{A}_1$ a lower semicontinuous function $\phi_a : X \to R^1 \cup \{\infty\}$ is associated and with every $a \in \mathcal{A}_2$ a set $S_a \subset X$ is associated. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \text{ for all } x \in S_{a_2}, \ \ f_a(x) = \infty \text{ for all } x \in X \setminus S_{a_2}.$$

Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 : \ \inf(f_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty.

We use the following hypotheses:

(A1) For each $a_1 \in \mathcal{A}_1$, $\inf(\phi_{a_1}) > -\infty$ and for each $a \in \mathcal{A}_1 \times \mathcal{A}_2$ the function f_a is lower semicontinuous.

(A2) For each $a \in \mathcal{A}_1$ and each $D, \epsilon > 0$ there is a neighborhood $\mathcal{U}$ of a in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{\phi_a(x), \phi_b(x)\} \le D$ the relation $|\phi_a(x) - \phi_b(x)| \le \epsilon$ holds.

(A3) For each $\gamma \in (0, 1)$ there exist positive numbers $\epsilon(\gamma)$ and $\delta(\gamma)$ such that $\epsilon(\gamma), \delta(\gamma) \to 0$ as $\gamma \to 0$ and the following property holds:

For each $\gamma \in (0, 1)$, each $a \in \mathcal{A}_1$, each nonempty set $Y \subset X$, and each $\bar{x} \in Y$ for which

$$\phi_a(\bar{x}) \le \inf\{\phi_a(z) : \ z \in Y\} + \delta(\gamma) < \infty$$

there is $\bar{a} \in \mathcal{A}_1$ such that the following conditions hold:

$$d_1(a, \bar{a}) \le \epsilon(\gamma), \ \phi_{\bar{a}}(z) \ge \phi_a(z), \ z \in X, \ \phi_{\bar{a}}(\bar{x}) \le \phi_a(\bar{x}) + \delta(\gamma);$$

for each $y \in Y$ satisfying

$$\phi_{\bar{a}}(y) \le \inf\{\phi_{\bar{a}}(z) : z \in Y\} + 2\delta(\gamma)$$

the inequality $\rho(y, \bar{x}) \le \gamma$ is valid.

(A4) For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying $\inf(f_a) < \infty$ and each $\epsilon, \delta > 0$ there exist $\bar{a}_2 \in \mathcal{A}_2$, $\bar{x} \in S_{\bar{a}_2}$, and an open set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology such that

$$d_2(a_2, \bar{a}_2) < \epsilon,\ \mathcal{U} \cap \{b \in \mathcal{A}_2 :\ d_2(b, a_2) < \epsilon\} \ne \emptyset,$$
$$\phi_{a_1}(\bar{x}) \le \inf\{\phi_{a_1}(z) :\ z \in S_{\bar{a}_2}\} + \delta < \infty,$$

and

$$\bar{x} \in S_b,\ \{y \in S_b :\ \phi_{a_1}(z) \le \phi_{a_1}(\bar{x}) + 1\} \subset S_{\bar{a}_2}$$

for all $b \in \mathcal{U}$.

The assumptions (A1)–(A3) were introduced in Sect. 2.3 while (A4) is a weakened version of the assumption (A4) in Sect. 2.3.

Analogously to Proposition 2.4 we can prove the following result.

Proposition 2.21. *Assume that (A1)–(A4) hold. Then (H1) holds for the space $\mathcal{A}$.*

2.12 Preliminary Results for Theorem 2.20

Assume that $A : \Omega \to 2^{R^n} \setminus \{\emptyset\}$, U : graph$(A) \to 2^{R^N} \setminus \{\emptyset\}$ and that graph(U) is a closed subset of the space $\Omega \times R^n \times R^N$ with the product topology. Consider the spaces $X(A, U)$, $\mathfrak{M}(A, U)$, and $C_l(B_1 \times B_2)$ introduced in Sect. 2.10.

Lemma 2.22. *Let N_0 be a positive number. Then there exists a number $N_1 > 0$ such that for each $f \in \mathfrak{M}(A, U)$ and each $(x, u) \in X(A, U)$ which satisfies $\int_{T_1}^{T_2} f(t, x(t), u(t))dt \le N_0$ the inequality $|x(t)| \le N_1$ holds for all $t \in [T_1, T_2]$.*

For the proof of this lemma see Lemma 2 and Theorem 2 in Sect. 9.1 of [41].

Proposition 2.23. *Let $f \in \mathfrak{M}(A, U)$, $(x, u) \in X(A, U)$, $\{(x_i, u_i)\}_{i=1}^{\infty} \subset X(A, U)$ and let $\rho((x_i, u_i), (x, u)) \to 0$ as $i \to \infty$. Then*

$$\int_{\Omega} f(t, x(t), u(t))dt \le \liminf_{i \to \infty} \int_{\Omega} f(t, x_i(t), u_i(t))dt.$$

Proof. We may assume that there is $\lim_{i\to\infty} \int_{\Omega} f(t, x_i(t), u_i(t))dt < \infty$. By Lemma 2.22

$$\sup\{|x_i(t)| :\ t \in [T_1, T_2],\ i = 1, 2, \dots\} < \infty. \tag{2.135}$$

It follows from (2.135) and property (iii) (see the definition of $\mathfrak{M}(A, U)$) that

$$\lim_{i\to\infty}\int_{T_1}^{T_2} f(t,x_i(t),u_i(t))dt$$

is finite. There is a subsequence $\{(x_{i_k},u_{i_k})\}_{k=1}^{\infty}$ such that

$$(x_{i_k}(t),u_{i_k}(t)) \to (x(t),u(t)) \text{ as } k\to\infty,\ t\in\Omega \text{ (a.e.)}$$

(see page 68 of [33]). By property (ii) (see the definition of $\mathfrak{M}(A,U)$) for almost every $t\in\Omega$

$$\liminf_{k\to\infty} f(t,x_{i_k}(t),u_{i_k}(t)) \ge f(t,x(t),u(t)).$$

The proposition now follows from Fatou's lemma. □

We recall that an integrand f : graph$(U)\to R^1\cup\{\infty\}$ satisfies the Cesari growth condition if for each $\epsilon>0$ there exists an integrable scalar function $\psi_\epsilon(t)\ge 0,\ t\in\Omega$, such that $|H(t,x,u)|\le\psi_\epsilon(t)+\epsilon f(t,x,u)$ for all $(t,x,u)\in$ graph (U).

The following proposition is an auxiliary result for the hypothesis (H2).

Proposition 2.24. *Assume that* $f\in\mathfrak{M}(A,U)$, $\{(x_i,u_i)\}_{i=1}^{\infty}\subset X(A,U)$ *is a Cauchy sequence and that the sequence* $\{\int_\Omega f(t,x_i(t),u_i(t))dt\}_{i=1}^{\infty}$ *is bounded. Then there is* $(x_*,u_*)\in X(A,U)$ *such that* (x_i,u_i) *converges to* (x_*,u_*) *as* $i\to\infty$ *in* $X(A,U)$ *and* $x_i(t)\to x_*(t)$ *as* $i\to\infty$ *uniformly on* $[T_1,T_2]$.

Proof. By Lemma 2.22 there exists $N_0>0$ such that

$$|x_i(t)|\le N_0,\ t\in[T_1,T_2],\ i=1,2,\dots \tag{2.136}$$

Define $A_1:\Omega\to 2^{R^n}\setminus\{\emptyset\}$ and U_1 : graph$(A_1)\to 2^{R^N}\setminus\{\emptyset\}$ by

$$A_1(t)=\{x\in A(t):\ |x|\le N_0\},\ t\in\Omega \text{ and}$$
$$U_1(t,x)=U(t,x),\ (t,x)\in \text{graph}(A_1).$$

It follows from (2.117), property (iii) (see the definition of $\mathfrak{M}(A,U)$), and (2.136) that the restriction of f to graph(U_1) satisfies the Cesari growth condition. Then by Proposition 2.7 there is $(x_*,u_*)\in X(A_1,U_1)\subset X(A,U)$ such that $\rho((x_i,u_i),(x_*,u_*))\to 0$ as $i\to\infty$ and $x_i(t)\to x_*(t)$ as $i\to\infty$ uniformly on $[T_1,T_2]$. The proposition is proved. □

The following proposition is an auxiliary result for the assumption (A2).

Proposition 2.25. *Let* $f\in\mathfrak{M}(A,U)$ *and* $\epsilon\in(0,1)$, $D>0$. *Then there exists a neighborhood* $\mathcal{W}$ *of* f *in* $\mathfrak{M}(A,U)$ *with the weak topology such that for each* $g\in\mathcal{W}$ *and each* $(x,u)\in X(A,U)$ *satisfying*

$$\min\left\{\int_\Omega f(t,x(t),u(t))dt,\ \int_\Omega g(t,x(t),u(t))dt\right\}\le D \tag{2.137}$$

the following relation holds:

$$\left| \int_{\Omega} f(t, x(t), u(t))dt - \int_{\Omega} g(t, x(t), u(t))dt \right| \leq \epsilon.$$

Proof. By Lemma 2.22 there exists a number $N_0 > 0$ such that for each $g \in \mathfrak{M}(A, U)$ and each $(x, u) \in X(A, U)$ which satisfies

$$\int_{T_1}^{T_2} g(t, x(t), u(t))dt \leq 4 + D \tag{2.138}$$

the following inequality holds:

$$|x(t)| \leq N_0, \ t \in [T_1, T_2]. \tag{2.139}$$

Choose $\epsilon_1 \in (0, 1)$ such that

$$\epsilon_1 D + \epsilon_1 (T_2 - T_1)(1 + \psi(N_0)) + \left(1 + \int_{\Omega} |a(t)|dt\right) \epsilon_1 < \epsilon/2 \tag{2.140}$$

and a positive number $\epsilon_0 < 1$ which satisfies

$$\epsilon_0 + \epsilon_0(1 - \epsilon_0)^{-1} < 4^{-1}\epsilon_1. \tag{2.141}$$

Define

$$\mathcal{W} = \{g \in \mathfrak{M}(A, U) : \ (g, f) \in E_{\mathfrak{M}w}(\epsilon_0, N_0)\} \tag{2.142}$$

(see (2.122)).

Assume that $g \in \mathcal{W}$, $(x, u) \in X(A, U)$, and (2.137) is valid. By the definition of N_0, (2.139) holds. By (2.142) and (2.122) there is a nonnegative function $\phi \in L^1(\Omega)$ such that $\int_{\Omega} \phi(t)dt \leq 1$ and for almost every $t \in \Omega$ the inequality

$$|f(t, y, v) - g(t, y, v)| < \epsilon_0 + \epsilon_0\phi(t) + \epsilon_0 \max\{|f(t, y, v)|, |g(t, y, v)|\} \tag{2.143}$$

is true for each $y \in A(t)$ satisfying $|y| \leq N_0$ and each $v \in U(t, y)$.

It follows from this inequality, Lemma 2.1, and (2.141) that for almost every $t \in \Omega$ the relation

$$\begin{aligned} |f(t, y, v) - g(t, y, v)| &< \epsilon_0 + \epsilon_0^2(1 - \epsilon_0)^{-1} + \phi(t)(\epsilon_0^2(1 - \epsilon_0)^{-1} + \epsilon_0) \\ &\quad + \epsilon_0(1 - \epsilon_0)^{-1} \min\{|f(t, y, v)|, |g(t, y, v)|\} \\ &< 4^{-1}\epsilon_1 + 4^{-1}\epsilon_1\phi(t) + 4^{-1}\epsilon_1 \min\{|f(t, y, v)|, |g(t, y, v)|\} \end{aligned} \tag{2.144}$$

is valid for each $y \in A(t)$ satisfying $|y| \leq N_0$ and each $v \in U(t, y)$.

Combined with property (iii) (2.144) implies that for almost every $t \in \Omega$ the inequality

$$g(t, y, v) \geq f(t, y, v) - 4^{-1}\epsilon_1 - 4^{-1}\epsilon_1\phi(t) - 4^{-1}\epsilon_1|f(t, y, v)| \qquad (2.145)$$
$$\geq -4^{-1}\epsilon_1\phi(t) - 2\psi(N_0) - 4^{-1}\epsilon_1 + 2a(t)$$

holds for each $y \in A(t)$ satisfying $|y| \leq N_0$ and each $v \in U(t, y)$. Set

$$\lambda(t) = \min\{f(t, x(t), u(t)), g(t, x(t), u(t))\}, \ t \in \Omega. \qquad (2.146)$$

It follows from (2.144), (2.139), (2.145), and (2.146) that for almost every $t \in \Omega$

$$|f(t, x(t), u(t)) - g(t, x(t), u(t))| < 4^{-1}\epsilon_1 + 4^{-1}\epsilon_1\phi(t)$$
$$+4^{-1}\epsilon_1 \min\{f(t, x(t), u(t)) + 2|a(t)| + 2\psi(N_0),$$
$$g(t, x(t), u(t)) + \phi(t) + 4\psi(N_0) + 4|a(t)|\} + 1$$
$$\leq 4^{-1}\epsilon_1 + 4^{-1}\epsilon_1\phi(t) + 4^{-1}\epsilon_1(4|a(t)| + 1 + \phi(t)) + 4^{-1}\epsilon_1\lambda(t) + \psi(N_0)\epsilon_1.$$

By this inequality, (2.146), and (2.137),

$$\int_\Omega |f(t, x(t), u(t)) - g(t, x(t), u(t))|dt \leq 4^{-1}\epsilon_1 \mathrm{mes}(\Omega) + \epsilon_1 \int_\Omega |a(t)|dt + 4^{-1}\epsilon_1$$
$$+ \epsilon_1\psi(N_0)(T_2 - T_1) + 4^{-1}\epsilon_1 \left(\int_\Omega \phi(t)dt + D + T_2 - T_1\right) < \epsilon.$$

This completes the proof of the proposition. □

Analogously to Corollary 2.9 we can prove the following result.

Proposition 2.26. *Let $f \in \mathfrak{M}(A, U)$ and $\epsilon > 0$. Then there exists a neighborhood $\mathcal{V}$ of f in $\mathfrak{M}(A, U)$ with the weak topology such that for each $g \in \mathcal{V}$*

$$\left|\inf\left\{\int_\Omega f(t, x(t), u(t))dt : \ (x, u) \in X(A, U)\right\} - \inf\left\{\int_\Omega g(t, x(t), u(t))dt :\right.\right.$$
$$\left.\left.(x, u) \in X(A, U)\right\}\right| < \epsilon.$$

The following proposition is an auxiliary result for the assumption (A2). We can prove it analogously to the proof of Proposition 2.12 by using Propositions 2.8, 2.25, and 2.26 and Corollary 2.9.

Proposition 2.27. *Let $f \in \mathfrak{M}(A, U)$, $h \in C_l(B_1 \times B_2)$, and $\epsilon \in (0, 1)$, $D > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}(A, U)$ with the weak topology and a neighborhood $\mathcal{V}$ of h in $C_l(B_1 \times B_2)$ with the weak topology such that for each $(\xi, g) \in \mathcal{V} \times \mathcal{U}$ and each $(x, u) \in X(A, U)$ which satisfies*

$$\min\{I^{(f,h)}(x, u), \ I^{(g,\xi)}(x, u)\} \leq D$$

the following relations are valid:

$$|h(x(T_1), x(T_2)) - \xi(x(T_1), x(T_2))| \leq \epsilon,$$

$$\left| \int_{T_1}^{T_2} [f(t, x(t), u(t)) - g(t, x(t), u(t))] dt \right| \leq \epsilon.$$

2.13 Proof of Theorem 2.20

By Propositions 2.23 and 2.24 and Lemma 2.22 (A1) holds and J_a is lower semicontinuous for all $a \in \mathcal{A}_1 \times \mathcal{A}_2$. By Theorem 2.3 we need to verify that (H1) and (H2) are valid. (H2) follows from Proposition 2.24. Therefore it is sufficient to show that (H1) holds. By Proposition 2.21 it is sufficient to show that (A2), (A3), and (A4) are valid. (A2) follows from Proposition 2.27. (A3) is proved analogously to Lemma 2.13. (A4) will follow from our next lemma.

Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_N = (0, \ldots, 1)$ be a standard basis in R^N.

Lemma 2.28. *Let $f \in \mathfrak{M}(A, \tilde{U}_A)$, $\xi \in C_l(B_1 \times B_2)$, $U \in \mathcal{P}_A$,*

$$\{(x, u) \in X(A, U) : \ I^{(f,\xi)}(x, u) < \infty\} \neq \emptyset, \tag{2.147}$$

and let $\epsilon, \delta > 0$. Then there are $U_* \in \mathcal{P}_A$, $(\bar{x}, \bar{u}) \in X(A, U_*)$, and an open set $\mathcal{W}$ in $\mathcal{P}_A$ with the weak topology such that

$$(U_*, U) \in E_{\mathcal{P}_A}(\epsilon), \ \mathcal{W} \cap \{V \in \mathcal{P}_A : \ (U, V) \in E_{\mathcal{P}_A}(\epsilon)\} \neq \emptyset, \tag{2.148}$$

$$I^{(f,\xi)}(\bar{x}, \bar{u}) \leq \inf\{I^{(f,\xi)}(x, u) : \ (x, u) \in X(A, U_*)\} + \delta < \infty \tag{2.149}$$

and for all $V \in \mathcal{W}$

$$(\bar{x}, \bar{u}) \in X(A, V), \tag{2.150}$$

$$\{(x, u) \in X(A, U) : \ I^{(f,\xi)}(x, u) \leq I^{(\bar{x},\bar{u})}(\bar{x}, \bar{u}) + 1\} \subset X(A, U_*).$$

Proof. For each $r \in [0, 1]$ define $U_r \in \mathcal{P}_A$ by

$$U_r(t, x) = \{u \in R^N : \ d_H(u, U(t, x)) \leq r\}, \ (t, x) \in \text{graph}(A) \tag{2.151}$$

and define

$$\mu(r) = \inf\{I^{(f,\xi)}(x, u) : \ (x, u) \in X(A, U_r)\}. \tag{2.152}$$

Clearly $\mu(r)$ is finite for all $r \in [0, 1]$ and the function μ is monotone decreasing. There is $r_0 \in (0, 8^{-1}\epsilon)$ such that μ is continuous at r_0. Choose $r_1 \in (0, r_0)$ such that

$$|\mu(r_1) - \mu(r_0)| < 16^{-1}\delta. \tag{2.153}$$

There is

$$(\bar{x}, \bar{u}) \in X(A, U_{r_1}) \tag{2.154}$$

such that

$$I^{(f,\xi)}(\bar{x}, \bar{u}) \le \mu(r_1) + 16^{-1}\delta. \tag{2.155}$$

Equations (2.153) and (2.155) imply that

$$I^{(f,\xi)}(\bar{x}, \bar{u}) \le \mu(r_0) + 8^{-1}\delta. \tag{2.156}$$

Set

$$r_2 = 2^{-1}(r_0 + r_1). \tag{2.157}$$

Clearly

$$(U_{r_i}, U) \in E_{\mathcal{P}_A}(\epsilon),\ i = 0, 1, 2. \tag{2.158}$$

By Lemma 2.22 there exists a natural number N_0 such that

$$|x(t)| \le N_0,\ t \in [T_1, T_2] \tag{2.159}$$

for each $(x, u) \in X(A, \tilde{U}_A)$ satisfying

$$\int_{T_1}^{T_2} f(t, x(t), u(t))dt \le |I^{(f,\xi)}(\bar{x}, \bar{u})| + 2 + |\inf\{\xi(z) :\ z \in B_1 \times B_2\}|. \tag{2.160}$$

Choose a positive number γ for which

$$\gamma < \min\{4^{-1}\delta,\ (16NN_0)^{-1}(r_0 - r_1)\} \tag{2.161}$$

and define

$$\mathcal{W} = \{V \in \mathcal{P}_A :\ (U_{r_2}, V) \in E_{\mathcal{P}_A w}(\gamma, N_0)\},\ U_* = U_{r_0}. \tag{2.162}$$

It follows from (2.158) and (2.156) that (2.148) and (2.149) are true.

Assume that $V \in \mathcal{W}$. We will show that $(\bar{x}, \bar{u}) \in X(A, V)$. By the definition of N_0 (see (2.159) and (2.160)),

$$|x(t)| \le N_0,\ t \in [T_1, T_2]. \tag{2.163}$$

It is sufficient to show that

$$\bar{u}(t) \in V(t, \bar{x}(t)) \text{ for almost every } t \in \Omega. \tag{2.164}$$

By (2.154) for almost every $t \in \Omega$

$$\bar{u}(t) \in U_{r_1}(t, \bar{x}(t)). \tag{2.165}$$

Assume that $t \in \Omega$ and (2.165) is true. By (2.165), (2.157), and (2.151) for $i = 1, \ldots, N$

$$\bar{u}(t) + 2^{-1}(r_0 - r_1)e_i, \ \bar{u}(t) - 2^{-1}(r_0 - r_1)e_i \in U_{r_2}(t, \bar{x}(t))$$

and by (2.162) and (2.163) there are $z_{i1}, z_{i2} \in R^N$ such that

$$\bar{u}(t) + z_{i1}, \ \bar{u}(t) + z_{i2} \in V(t, \bar{x}(t)), \ |z_{i1} - 2^{-1}(r_0 - r_1)e_i|, \ |z_{i2} + 2^{-1}(r_0 - r_1)e_i| \le \gamma. \tag{2.166}$$

Since the set $V(t, \bar{x}(t))$ is convex it follows from (2.161) and Proposition 2.14 that

$$0 \in \mathrm{conv}\{z_{ij} : i = 1, \ldots N, \ j = 1, 2\}, \ \bar{u}(t) \in V(t, \bar{x}(t)).$$

Thus (2.164) holds for a.e. $t \in \Omega$.

We will show that

$$\{(x, u) \in X(A, U) : I^{(f,\xi)}(x, u) \le I^{(f,\xi)}(\bar{x}, \bar{u}) + 1\} \subset X(A, U_*) = X(A, U_{r_0}).$$

Assume that

$$(x, u) \in X(A, U) \text{ and } I^{(f,\xi)}(x, u) \le I^{(f,\xi)}(\bar{x}, \bar{u}) + 1.$$

By the definition of N_0 (see (2.159) and (2.160)),

$$|x(t)| \le N_0, \ t \in [T_1, T_2].$$

It follows from this relation and (2.162) that for almost every $t \in \Omega$

$$\begin{aligned} u(t) \in V(t, x(t)) &\subset \{z \in R^N : d_H(z, U_{r_2}(t, x(t)) \le \gamma\} \\ &\subset \{z \in R^N : d_H(z, U(t, x(t)) \le r_0\} = U_{r_0}(t, x(t)). \end{aligned}$$

Thus $(x, u) \in X(A, U_{r_0})$. The lemma is proved. □

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