

Chapter 2

Kinematics

Abstract Kinematics is focused on the mathematical description of motion and does not concern its dynamical causes. We shall consider the motion of particles in different coordinate systems, including Cartesian, polar (in two-dimensional $2D$ case), cylindrical, spherical and also in the co-moving basis which is useful in reducing $3D$ motion to $2D$. Also we are going to deal with the kinematics of rigid bodies, which can be decomposed into a translational motion and a rotation around an instantaneous axis. Finally, we will consider the motion in an accelerated reference frame and introduce the notion of covariant derivative which is helpful, in particular, for the kinematics in rotating coordinates.

2.1 Kinematics of a Particle

To parameterize the motion of a material point, we use clocks and a space coordinate system. We assume that all clocks (idealized devices that measure time), regardless of their positions, velocities or accelerations, indicate the same measure of time. The coordinate system (also called reference system or reference frame) is used to parameterize the three-dimensional space (for the sake of brevity, we use the notation $3D$, or $2D$ for the two-dimensional plane, or $1D$ for one-dimensional space). As a first step, we define the Cartesian coordinates. This coordinate system includes the point of origin O , and the orthonormal basis $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$. These three vectors must have right-handed orientation (see Fig. 1.1), i.e., they must satisfy the relations (1.1).

For any point M we can define the position vector (also called radius-vector) $\mathbf{OM} = \mathbf{r} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y + \hat{\mathbf{k}}z$. The coefficients (x, y, z) are called coordinates of the point in space. In the case of a laboratory reference frame, we assume that the vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are constant and static (Fig. 2.1).

For a material point (also called a particle or point-like body) in motion, the coordinates depend on time:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}. \quad (2.1)$$

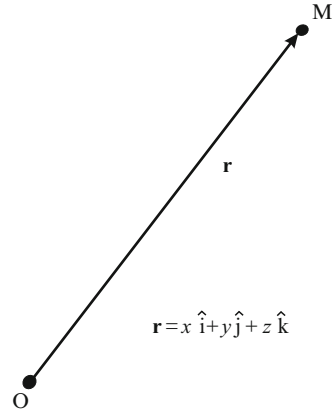


Fig. 2.1 Position vector (or radius-vector) in the Cartesian basis

The relation above describes the particle motion, i.e., it shows how the position vector depends on time.

In some cases we are not interested in this dependence and just want to know the curve of spatial motion of the particle, called a trajectory. In fact, the relationship (2.1) is a particular case of the trajectory equation, where time t serves as a parameter. When the time dependence does not matter, we can choose another parameter. In this case, there are two special options, namely:

1) We can use one of the coordinates as a parameter. For example, consider the following motion:

$$x = vt \cos \alpha, \quad y = 0, \quad z = vt \sin \alpha - \frac{gt^2}{2}, \quad (2.2)$$

where v , α , g are constants. To obtain the trajectory parameterized by x , we need to solve the first equation in (2.2) with respect to t and replace this solution into the third equation. Thus, we obtain $t = x/(v \cos \alpha)$ and the equation of the trajectory is cast into the form

$$y = 0, \quad z = x \tan \alpha - \frac{gx^2}{2v^2 \cos^2 \alpha}. \quad (2.3)$$

The system of Eqs. (2.3) has one less equation compared to (2.2). This reflects the fact that some information has been lost on the way, namely the Eqs. (2.3) do not contain information about the time dependence. They only show the spatial line, in other words they contain geometrical information about the movement, but not the dynamical content.

2) We can use the natural parameter, l , along the trajectory. If we choose some point on the curve (trajectory) and define it as a starting point, the natural parameter represents the distance along the curve between the starting point and the point of our interest. To set this parameter, we consider an infinitesimal displacement along the curve. We have

$$dl^2 = dx^2 + dy^2 + dz^2,$$

where (x, y, z) satisfy the equation of the trajectory. Correspondingly, if the starting point is A and the point of interest is M , the value of the parameter l is given by the line integral of first kind (see Mathematical Appendix for more details)

$$l = \int_{(AM)} dl. \quad (2.4)$$

We assume that initially the path is parameterized by some parameter λ such that $\mathbf{r} = \mathbf{r}(\lambda)$. In this case, the integral (2.4) can be written as a Riemann integral (definite integral)

$$l = \int_{\lambda_A}^{\lambda_M} \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2} d\lambda,$$

where λ_A and λ_M are the values that correspond to the points A and M , correspondingly. For example, when $\lambda = x$, the last equation is written as

$$l = \int_{x_A}^{x_M} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx.$$

Another interesting case, when $\lambda = t$ (time) will be discussed later on.

A major advantage in using the natural parameter along the curve (also called the canonical parameter) is that a change in the position of the particle in the trajectory always corresponds to a change of the natural parameter. None of the other parameters considered above (e.g., coordinate x or time t) have this property. At the same time, our main objective is the study of the dynamics of particles, so the dependence on time plays a key role in our considerations.

The velocity (instantaneous velocity) of the particle is defined as the time derivative of the position vector,

$$\mathbf{v}(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t},$$

or

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \hat{\mathbf{i}} \frac{dx}{dt} + \hat{\mathbf{j}} \frac{dy}{dt} + \hat{\mathbf{k}} \frac{dz}{dt} = \dot{x}\hat{\mathbf{i}} + \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}. \quad (2.5)$$

Henceforth, we will adopt the dot over the quantity to denote its time derivative. To determine the magnitude of velocity, we can use the theorem of Pythagoras, as

$$v = |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}.$$

It is easy to see that this is nothing else but the time derivative of the natural parameter along the trajectory,

$$v = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = \frac{dl}{dt}. \quad (2.6)$$

Dividing the velocity vector by its magnitude, we find the unit vector

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{v}$$

which is directed parallel to the tangent line of the trajectory in a given point and points to the side of growth of the parameter l .

We can invert the relation (2.5). Integrating $\mathbf{v}(t)$ and taking into account the initial condition $\mathbf{r}(t_0) = \mathbf{r}_0$, we obtain

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^t \mathbf{v}(\tau) d\tau.$$

For example, if $\mathbf{v}(t) = \mathbf{v}_0 = \text{const}$, we have

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0).$$

In order to simplify the formulas, we can assume that $t_0 = 0$. In this case, for example, $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 \cdot t$.

Now we define the instantaneous acceleration, $\mathbf{a}$, as the time derivative of the velocity vector,

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = \dot{v}_x \hat{\mathbf{i}} + \dot{v}_y \hat{\mathbf{j}} + \dot{v}_z \hat{\mathbf{k}}, \quad (2.7)$$

or, with respect to the position vector,

$$\mathbf{a} = \ddot{\mathbf{r}} = \frac{d^2 \mathbf{r}}{dt^2} = \ddot{x} \hat{\mathbf{i}} + \ddot{y} \hat{\mathbf{j}} + \ddot{z} \hat{\mathbf{k}}.$$

Obviously,

$$a_x = \dot{v}_x = \ddot{x}, \quad a_y = \dot{v}_y = \ddot{y}, \quad a_z = \dot{v}_z = \ddot{z}.$$

In the general case, the components of acceleration depend on time. We can integrate the relation (2.7) using the initial condition $\mathbf{v}(t = 0) = \mathbf{v}_0$,

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \mathbf{a}(\tau) d\tau.$$

In particular, for $\mathbf{a} \equiv \mathbf{a}_0 = \text{const}$, we have

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}_0 \int_0^t d\tau = \mathbf{v}_0 + \mathbf{a}_0 t. \quad (2.8)$$

In this case, for the radius-vector we find

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{\mathbf{a}_0 t^2}{2}.$$

In the general case of an arbitrary time-dependent acceleration, one has a more general expression,

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + \int_0^t d\tau \mathbf{v}(\tau) = \mathbf{r}_0 + \int_0^t d\tau \left\{ \mathbf{v}_0 + \int_0^\tau \mathbf{a}(\tau') d\tau' \right\} \\ &= \mathbf{r}_0 + \mathbf{v}_0 t + \int_0^t d\tau \int_0^\tau d\tau' \mathbf{a}(\tau'). \end{aligned} \quad (2.9)$$

As an example, consider the motion of a point-like body in a homogeneous and uniform gravitational field. In this case, the acceleration is constant, $\mathbf{a} \equiv \mathbf{g} = -g\hat{\mathbf{k}}$. We can always choose the reference frame such that the vector basis satisfies $\hat{\mathbf{j}} \perp \mathbf{v}_0$. In this case, we can write

$$\mathbf{v}_0 = v_{ox}\hat{\mathbf{i}} + v_{oz}\hat{\mathbf{k}} = v_0 (\hat{\mathbf{i}} \cos \alpha + \hat{\mathbf{k}} \sin \alpha),$$

where α is the angle between $\mathbf{v}_0$ and $\hat{\mathbf{i}}$. Using the previous results (2.8) and (2.9), we can write the solution

$$\begin{aligned} \mathbf{v}(t) &= v_{ox}\hat{\mathbf{i}} + (v_{oz} - gt)\hat{\mathbf{k}}, \\ \mathbf{r} &= \mathbf{r}_0 + v_{ox}t\hat{\mathbf{i}} + \left(v_{oz}t - \frac{gt^2}{2} \right) \hat{\mathbf{k}}, \end{aligned}$$

i.e., in components,

$$\begin{aligned} x &= x_0 + v_{ox}t = x_0 + v_0 t \cos \alpha, & y &= y_0, \\ z &= z_0 + v_{oz}t \sin \alpha - \frac{gt^2}{2}. \end{aligned}$$

The last formula describes the motion of a particle in a homogeneous and uniform gravitational field. Eliminating the time t , we obtain the equation of the trajectory (2.3) for the case $x_0 = y_0 = z_0 = 0$.

As far as the description of the motion of a material point involves the radius-vector, velocity or acceleration, a relation between them represents, typically, a differential equation. The solution of this equation provides the kinematical description by a vector function $\mathbf{r}(t)$.

Let us consider a simple example of this sort. Suppose we know that for some particle $\mathbf{a} = -k\mathbf{v}$, with $k = \text{const}$ and $k > 0$. The dynamics of the point is completely defined by this equation and by the necessary initial data (in mathematical terminology, this combination is called the Cauchy problem),

$$\frac{d\mathbf{v}}{dt} = -k\mathbf{v} \quad \text{and} \quad \mathbf{v}(t=0) = \mathbf{v}_0. \quad (2.10)$$

The first formula represents a first order differential equation with respect to the velocity $\mathbf{v}$, and the second one is an initial condition. In fact, it is a system of three equations for v_x, v_y, v_z . One can solve this problem, e.g., in arbitrary Cartesian coordinates. At the same time, the problem can be essentially simplified if we choose a special basis. Let us introduce the orthonormal basis consisting of the three vectors $\hat{\mathbf{v}}, \hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, where $\hat{\mathbf{v}} = \mathbf{v}/v$, also $\hat{\mathbf{n}}_{1,2} \perp \hat{\mathbf{v}}$ and $\hat{\mathbf{n}}_1 \perp \hat{\mathbf{n}}_2$. It is easy to see that the direction of velocity $\hat{\mathbf{v}}$ remains constant, and so we can write $\mathbf{v}(t) = \hat{\mathbf{v}} \cdot v(t)$. The first equation (2.10) becomes

$$\frac{d\mathbf{v}}{dt} = \hat{\mathbf{v}} \frac{dv}{dt} = -k\hat{\mathbf{v}}v.$$

Finally, we have the only one differential equation, $\dot{v}(t) = -kv(t)$. To solve it, we will use the method of separation of variables. Grouping all terms involving v on the left side and the terms involving t on the right side, we write

$$\frac{dv}{v} = -kdt.$$

Assuming that $v(t)$ is a function that satisfies the last relation and integrating both sides, we arrive at the formula

$$\ln \left| \frac{v}{C} \right| = -kt, \quad (2.11)$$

where C is a constant of integration. To define this constant, one has to use the initial condition. Substituting $t = 0$ and $v = v_0$ in Eq. (2.11), we find $\ln |v_0/C| = 0$, i.e., $C = v_0$. Finally, the solution for the velocity is

$$\mathbf{v} = \hat{\mathbf{v}} v_0 e^{-kt} = \mathbf{v}_0 e^{-kt}.$$

It is easy to see that the last formula satisfies both relations (2.10). In the limit $t \rightarrow \infty$ we have $\mathbf{v} \rightarrow 0$. To find the kinematical equations, it is necessary to introduce a new initial condition $\mathbf{r}(0) = \mathbf{r}_0$, and use the general prescription (2.9),

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 + \hat{\mathbf{v}} v_0 \int_0^t e^{-k\tau} d\tau \\ &= \mathbf{r}_0 + \mathbf{v}_0 \cdot \left(-\frac{1}{k} \right) e^{-k\tau} \Big|_0^t = \mathbf{r}_0 + \frac{\mathbf{v}_0}{k} (1 - e^{-kt}). \end{aligned}$$

In the limit $t \rightarrow \infty$, we find

$$\mathbf{r} \rightarrow \mathbf{r}_0 + \frac{\mathbf{v}_0}{k}.$$

The last example shows the great importance of choosing a useful reference frame. Here we use a basis related to the motion of the particle itself and got a real simplification in the solution. We will develop the same idea in a more general form in the next section.

Exercise 1. A particle moves in space with coordinates x, y, z , according to the equations of motion

$$x = 3vt, \quad y = 3L \cos \omega t, \quad z = -5L \sin \omega t.$$

Write the same equations in vector form and find the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ as functions of time.

Answers:

$$\begin{aligned} \mathbf{r} &= 3v\hat{\mathbf{i}} + 3\hat{\mathbf{j}}L \cos \omega t - 5\hat{\mathbf{k}}L \sin \omega t; \\ \mathbf{v} &= 3v\hat{\mathbf{i}} - 3\omega\hat{\mathbf{j}}L \sin \omega t - 5\omega\hat{\mathbf{k}}L \cos \omega t; \\ \mathbf{a} &= -3\omega^2\hat{\mathbf{j}}L \cos \omega t + 5\omega^2\hat{\mathbf{k}}L \sin \omega t. \end{aligned}$$

2.2 Motion of a Particle in the Co-moving Basis

Let us introduce the so-called co-moving basis, i.e., the one which is attached to the particle. The advantage of this basis compared to some static basis, $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, is related to the fact that, in many situations, only two vectors of the co-moving basis are relevant. In many cases the laws of mechanics involve the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ vectors of the particle. Then we can always choose a single plane in which the motion occurs.¹ This plane should include both vectors, $\mathbf{v}$ and $\mathbf{a}$. Now we can build the corresponding orthonormal basis. The first vector of this basis is $\hat{\mathbf{v}} = \mathbf{v}/v$, this means it has the same direction as velocity. To choose the second vector, we assume that the vectors $\mathbf{v}$ and $\mathbf{a}$ are not parallel. In this case one can choose the unit vector $\hat{\mathbf{n}}$, which is normal to the trajectory, such that

- (i) $\hat{\mathbf{v}} \perp \hat{\mathbf{n}}$;
- (ii) $\hat{\mathbf{n}}$ belongs to the same plane as the vectors $\mathbf{v}$ and $\mathbf{a}$;
- (iii) The plane is divided into two semi-planes by the line that contains the vector $\mathbf{v}$. The vector $\hat{\mathbf{n}}$ lies in the same half-plane that the vector $\mathbf{a}$.

Figure 2.2 shows the positions of the vectors defined above.

It is important to remember that the vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{n}}$, in general, depend on time, such that $\hat{\mathbf{v}} = \hat{\mathbf{v}}(t)$ and $\hat{\mathbf{n}} = \hat{\mathbf{n}}(t)$. It is clear that this dependence does not concern their absolute values, but only the directions of the two vectors.

Let us consider the velocity and acceleration vectors on the co-moving basis. The velocity can be written as

$$\mathbf{v}(t) = v(t) \cdot \hat{\mathbf{v}}(t). \quad (2.12)$$

For the sake of convenience, in what follows we shall omit the time argument. For acceleration, one can write

¹ The particular case where $\mathbf{v} \parallel \mathbf{a}$ does not require the basis that we are discussing now and can be treated in the same manner as we did in the example discussed above, at the end of the previous section.

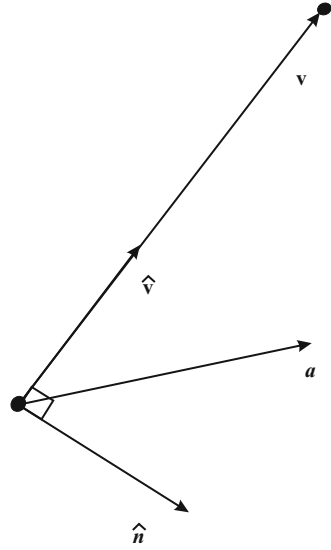


Fig. 2.2 Construction of the co-moving basis. All the vectors belong to the same plane

$$\mathbf{a} = a_t \cdot \hat{\mathbf{v}} + a_n \cdot \hat{\mathbf{n}}. \quad (2.13)$$

The components of acceleration on the co-moving base have the following nomenclature: a_t is the tangential acceleration and a_n is the normal acceleration. We can establish some useful relations between the terms that appear in the formula (2.13). First, $a^2 = a_t^2 + a_n^2$.

Taking the time derivative of Eq. (2.12), we obtain

$$\mathbf{a} = \frac{dv}{dt} \cdot \hat{\mathbf{v}} + v \cdot \frac{d\hat{\mathbf{v}}}{dt}. \quad (2.14)$$

Comparing (2.13) and (2.14), we can see that the tangential component of acceleration, a_t , is the time derivative of the absolute value of the velocity vector,

$$a_t = \frac{dv}{dt} = \frac{d^2l}{dt^2}. \quad (2.15)$$

We know that (see Fig. 2.3)

$$\frac{d\hat{\mathbf{v}}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \hat{\mathbf{v}}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\hat{\mathbf{v}}(t + \Delta t) - \hat{\mathbf{v}}(t)}{\Delta t}$$

and, at the same time, the magnitude of the vector $\hat{\mathbf{v}}$ remains constant, $|\hat{\mathbf{v}}| \equiv 1$. Obviously, this is only possible in the case

$$\frac{d\hat{\mathbf{v}}}{dt} \perp \hat{\mathbf{v}}, \quad \text{that is,} \quad \frac{d\hat{\mathbf{v}}}{dt} \parallel \hat{\mathbf{n}}.$$

Finally we can write Eq. (2.14) in the following way:

$$v \frac{d\hat{\mathbf{v}}}{dt} = \mathbf{a} - \dot{v}\hat{\mathbf{v}}, \quad \text{that is,} \quad a_n \hat{\mathbf{n}} = v \frac{d\hat{\mathbf{v}}}{dt}.$$

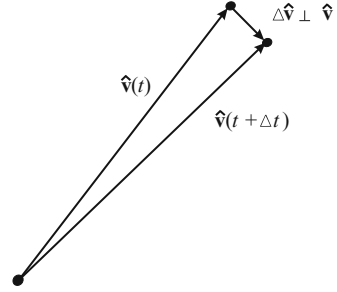


Fig. 2.3 Vector representation of the vectors $\Delta \mathbf{v}$ and $\mathbf{v}$

At this stage we can use the natural parameter, l , to define one more important characteristic of the trajectory, called the radius of curvature, R . As a next step, we will find a relationship between this quantity and the normal component of acceleration. The value of R is defined by the relation

$$\frac{d\hat{\mathbf{v}}}{dl} = \frac{\hat{\mathbf{n}}}{R}. \quad (2.16)$$

It is clear that R does not necessarily have the same value at different points of the trajectory. At the same time, the radius of curvature is a geometric feature, i.e., it depends only on the trajectory and not on the given particle's motion along it. This means that it does not matter if the particle passes through a given point of the trajectory fast or slow, R is the same. At the same time, one can establish an interesting relation between the normal component of acceleration, the magnitude of the velocity and the radius of curvature,

$$a_n = \frac{v^2}{R}. \quad (2.17)$$

To demonstrate this relation, we write again

$$\mathbf{v} = \hat{\mathbf{v}} \cdot v, \quad v = \frac{dl}{dt}, \quad a_t = \frac{dv}{dt}$$

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \hat{\mathbf{v}} \cdot \frac{dv}{dt} + a_n \cdot \hat{\mathbf{n}}. \quad (2.18)$$

Let us consider, on the other hand,

$$\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\hat{\mathbf{v}} \cdot v) = \frac{d\hat{\mathbf{v}}}{dt} v + \hat{\mathbf{v}} \frac{dv}{dt} = \frac{d\hat{\mathbf{v}}}{dt} v + \hat{\mathbf{v}} \cdot a_t. \quad (2.19)$$

Comparing (2.18) with (2.19), we have

$$\hat{\mathbf{v}}a_t + v \frac{d\hat{\mathbf{v}}}{dt} = \hat{\mathbf{v}}a_t + a_n \cdot \hat{\mathbf{n}}, \quad \text{so that} \quad \frac{d\hat{\mathbf{v}}}{dt} = \frac{a_n \hat{\mathbf{n}}}{v}. \quad (2.20)$$

Finally, we have

$$\frac{d\hat{\mathbf{v}}}{dt} = \frac{d\hat{\mathbf{v}}}{dl} \cdot \frac{dl}{dt} = v \frac{d\hat{\mathbf{v}}}{dl}. \quad (2.21)$$

Taking into account the definition (2.16), the Eqs. (2.21) and (2.20) give the Eq. (2.17).

Example 1. To illustrate the physical meaning of R , consider a particular case of circular motion of a particle along a circumference of radius R with constant angular velocity ω (See Fig. 2.4). We choose the initial instant $t = 0$ in which the particle has coordinates $x(0) = R$ and $y(0) = 0$.

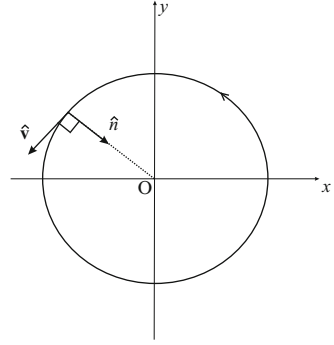


Fig. 2.4 Uniform circular motion

In this case, we have

$$x = R \cos \omega t, \quad y = R \sin \omega t,$$

or, in vector form,

$$\mathbf{r} = R (\hat{\mathbf{i}} \cos \omega t + \hat{\mathbf{j}} \sin \omega t).$$

For the velocity vector, we obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = R\omega (-\hat{\mathbf{i}} \sin \omega t + \hat{\mathbf{j}} \cos \omega t),$$

It is easy to see that $r = |\mathbf{r}| = R = \text{const}$ and also $v = |\mathbf{v}| = \omega R = \text{const}$. For acceleration, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -R\omega^2 (\hat{\mathbf{i}} \cos \omega t + \hat{\mathbf{j}} \sin \omega t) = -\hat{\mathbf{n}} R\omega^2 = \frac{v^2}{R} \hat{\mathbf{n}}.$$

One can see that in this case $a_t = 0$ and $a_n = v^2/R$. The acceleration vector is directed to the center. The curvature radius is exactly the circumference radius. In the case of circular motion, the normal component of acceleration is called centripetal acceleration. The normal component is not always the only component of acceleration. The tangential component of the acceleration is related to the angular acceleration, β , as follows:

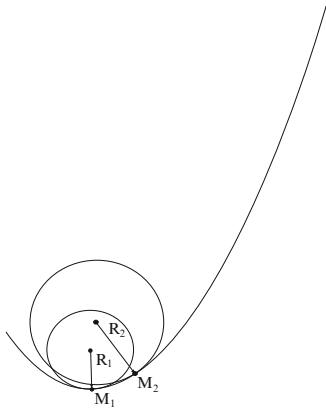
$$a_t = \beta R, \quad \text{where} \quad \beta = \frac{d\omega}{dt}. \quad (2.22)$$

In the general case, when the movement is not necessarily circular, for each point of the trajectory we can imagine a circumference with radius R such that a_n is the centripetal acceleration corresponding to this circle (see illustration in Fig. 2.5). For any motion, the magnitude of the acceleration is

$$|\mathbf{a}| = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{R}\right)^2}$$

and the direction of $\mathbf{a}$ depends on the relation between a_t and a_n . For example, if $a_t = 0$, the acceleration has only a normal component and the absolute value of the velocity is constant.

Fig. 2.5 R_1 and R_2 are the radii of curvature at the local respective points M_1 and M_2 of the trajectory



Example 2. A particle is in circular motion such that its angular acceleration is positive and proportional to its angular velocity. Its initial angular velocity is ω_0 . Find the direction of acceleration as a function of time.

Solution. We consider the angle of rotation as a function of time. We have

$$\omega(t) = \frac{d\varphi(t)}{dt}, \quad \beta(t) = \frac{d\omega(t)}{dt} = k\omega(t),$$

where k is some coefficient. We choose the initial condition $\varphi(0) = \varphi_0$. As the first step, we solve the differential equation for $\omega(t)$, using the method of separation of variables. Thus,

$$\frac{d\omega}{\omega} = k dt \quad \Rightarrow \quad \ln \left| \frac{\omega}{\omega_0} \right| = kt,$$

where we already used the initial condition $\omega(0) = \omega_0$. So, $\omega(t) = \omega_0 e^{kt}$. This means that

$$v(t) = R\omega(t) = R\omega_0 e^{kt}$$

and also

$$a_t = \frac{dv}{dt} = R\omega_0 k e^{kt} = R\beta(t).$$

For the normal component of acceleration, we find

$$a_n = \frac{v^2}{R} = R\omega_0^2 e^{2kt}.$$

To find the direction of the acceleration, we remember that

$$\hat{\mathbf{n}} = -\frac{\mathbf{r}}{R} = -(\hat{\mathbf{i}} \cos \varphi + \hat{\mathbf{j}} \sin \varphi) \quad \text{and} \quad \hat{\mathbf{v}} = -\hat{\mathbf{i}} \sin \varphi + \hat{\mathbf{j}} \cos \varphi.$$

To obtain $\varphi(t)$, one needs to integrate $\omega(t)$:

$$\varphi(t) = \varphi_0 + \int_0^t \omega(t) dt = \varphi_0 + \frac{\omega_0}{k} (e^{kt} - 1). \quad (2.23)$$

Finally,

$$\begin{aligned} \mathbf{a} &= a_t \cdot \hat{\mathbf{v}} + a_n \hat{\mathbf{n}} \\ &= R\omega_0 k e^{kt} \cdot (-\hat{\mathbf{i}} \sin \varphi + \hat{\mathbf{j}} \cos \varphi) - R\omega_0^2 e^{2kt} \cdot (\hat{\mathbf{i}} \cos \varphi + \hat{\mathbf{j}} \sin \varphi) \\ &= R\omega_0 k e^{kt} \left[-\hat{\mathbf{i}} \left(\sin \varphi + \frac{\omega_0 e^{kt}}{k} \cos \varphi \right) + \hat{\mathbf{j}} \left(\cos \varphi - \frac{\omega_0 e^{kt}}{k} \sin \varphi \right) \right], \end{aligned}$$

where φ is given by the relation (2.23). This formula defines the magnitude and direction of acceleration $\mathbf{a}$. To find the unit direction of $\mathbf{a}$, we normalize it,

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{a} = \frac{-\hat{\mathbf{i}} (\sin \varphi + \omega_0 k e^{kt} \cos \varphi) + \hat{\mathbf{j}} (\cos \varphi - \omega_0 k e^{kt} \sin \varphi)}{\sqrt{1 + \frac{\omega_0^2 e^{2kt}}{k^2}}}.$$

Exercises

1. A projectile is launched from the origin O with the initial velocity which has magnitude v_0 and makes angle α with the horizon. Consider a straight line which passes through the origin and makes an angle β with the horizon, such that $\beta < \alpha$. After some time, the projectile crosses the line at the point P . What is the choice of α such that the distance between P and the origin be a maximal one?

Solution. The equation for the trajectory and the equation of the straight line are respectively

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha} \quad \text{and} \quad y = x \tan \beta. \quad (2.24)$$

By replacing the second equation in (2.24) into the first one, we obtain the coordinate x of the point P :

$$x = (\tan \alpha - \tan \beta) \frac{2v_0^2 \cos^2 \alpha}{g}.$$

As far as x corresponds to the distance between P and the origin, multiplied by $\cos \beta$, the maximum value of x will result in a maximum for that distance. So, one only needs to find out what is the value of α that maximizes x . Taking the derivative $dx/d\alpha = 0$ (or, simpler, with respect to $t = \tan \alpha$), after a small algebra we arrive at the solution for the angle,

$$\alpha = \arctan \left(\frac{\sin \beta + 1}{\cos \beta} \right) = \frac{\pi}{4} + \frac{\beta}{2}.$$

2. A small rubber ball is dropped from rest at a height h over the ground and falls vertically. It bounces from the ground and returns to the height $h/2$. Every time ball hits the ground, it returns to half the previous height.

(a) Under these conditions, will it be possible to observe a finite duration of time τ , in which the ball stops on the floor? If so, calculate τ .

(b) Considering the interval of time between the instant in which the ball is released and the instant when it stops on the floor, calculate the magnitude of its average velocity and the average of absolute value of velocity (note that there is a difference between these two quantities).

Answers: $\tau = \sqrt{\frac{2h}{g}}(2\sqrt{2} + 3), \quad |\langle \mathbf{v} \rangle| = \frac{h}{\tau}, \quad \langle v \rangle = \frac{3\sqrt{gh}}{2}(3\sqrt{2} - 4).$

3. For the motion of particle on the $2D$ plane in the co-moving reference frame, prove the relation which is dual to (2.16),

$$\frac{d\hat{\mathbf{n}}}{dl} = -\frac{\hat{\mathbf{v}}}{R}. \quad (2.25)$$

Discuss whether this formula holds for the general $3D$ motion.

2.3 Polar Coordinates in the Plane, Cylindrical and Spherical in the Space

Depending on the problem, the use of Cartesian coordinates is not always the most useful. Sometimes it is better to work in some other reference frame, for example, in the co-moving coordinates discussed above. Almost always we want to use coordinates that correspond to an orthogonal and normalized (orthonormal) basis. In this section we review some standard and most frequently used choices. As a first example, let us consider polar coordinates on the plane.

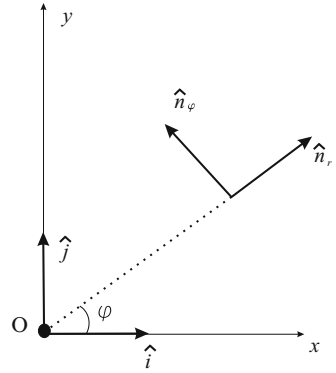


Fig. 2.6 Cartesian and polar bases

The polar coordinates (see Fig. 2.6) r and φ can be defined by the relations

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi. \end{cases}$$

To use a vector formalism, we define unit basis vectors

$$\begin{aligned} \hat{\mathbf{n}}_r &= \frac{\mathbf{r}}{r} = \hat{\mathbf{i}} \cos \varphi + \hat{\mathbf{j}} \sin \varphi, \\ \hat{\mathbf{n}}_\varphi &= -\hat{\mathbf{i}} \sin \varphi + \hat{\mathbf{j}} \cos \varphi. \end{aligned} \quad (2.26)$$

Exercise 2. Verify that the vectors (2.26) are orthonormal, that means the relations

$$\hat{\mathbf{n}}_r \cdot \hat{\mathbf{n}}_r = \hat{\mathbf{n}}_\varphi \cdot \hat{\mathbf{n}}_\varphi = 1 \quad \text{and} \quad \hat{\mathbf{n}}_r \cdot \hat{\mathbf{n}}_\varphi = 0,$$

where the dot indicates the scalar product.

Now we can imagine a particle with polar coordinates r, φ such that $r = r(t)$ and $\varphi = \varphi(t)$. Our next goal is to calculate the velocity and acceleration of the material point in polar coordinates. As the first step, we compute the time derivatives of the basis vectors (2.26),

$$\frac{d\hat{\mathbf{n}}_r}{dt} \quad \text{and} \quad \frac{d\hat{\mathbf{n}}_\varphi}{dt}.$$

It is easy to obtain, using Eqs. (2.26),

$$\begin{aligned}\frac{d\hat{\mathbf{n}}_r}{dt} &= \dot{\varphi}(-\hat{\mathbf{i}}\sin\varphi + \hat{\mathbf{j}}\cos\varphi) = \dot{\varphi}\hat{\mathbf{n}}_\varphi, \\ \frac{d\hat{\mathbf{n}}_\varphi}{dt} &= \dot{\varphi}(-\hat{\mathbf{i}}\cos\varphi - \hat{\mathbf{j}}\sin\varphi) = -\dot{\varphi}\hat{\mathbf{n}}_r.\end{aligned}\quad (2.27)$$

Now we are at the point of writing the velocity and acceleration in polar coordinates. From Eqs. (2.27) we obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\hat{\mathbf{n}}_r) = \dot{r}\hat{\mathbf{n}}_r + r\dot{\varphi}\hat{\mathbf{n}}_\varphi.$$

The physical interpretation of this formula is obvious. The first term corresponds to the radial velocity of the particle, and the second one to its angular motion. One can see the illustration for this formula in Fig. 2.7. For the square of the velocity, we have $v^2 = \dot{r}^2 + r^2\dot{\varphi}^2$.

The next step is to calculate the acceleration. By using the relations (2.27), one can write

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{\mathbf{n}}_r + r\dot{\varphi}\hat{\mathbf{n}}_\varphi) = \ddot{r}\hat{\mathbf{n}}_r + \dot{r}\dot{\varphi}\hat{\mathbf{n}}_\varphi + \dot{r}\dot{\varphi}\hat{\mathbf{n}}_\varphi + r\ddot{\varphi}\hat{\mathbf{n}}_\varphi - r\dot{\varphi}^2\hat{\mathbf{n}}_r \\ &= \hat{\mathbf{n}}_r(\ddot{r} - r\dot{\varphi}^2) + \hat{\mathbf{n}}_\varphi(2\dot{r}\dot{\varphi} + r\ddot{\varphi}).\end{aligned}\quad (2.28)$$

In order to interpret different terms in this expression, we consider different types of movements. The radial part represents the sum of the term containing $\ddot{r}$, which does not depend on the angular motion, and the term that can be identified as centripetal acceleration (pointing to the origin). For the circular motion, this term coincides with the normal acceleration. The second term of the angular part is related to the angular acceleration and it is the same as the one in the formula (2.22). The term of less immediate interpretation is the first one in the angular part. To interpret this term, it is not enough to consider the movement which is purely circular or purely radial, because in both cases this term vanishes. The origin of this term is that the motion of the vector basis is (generally) accelerated. In what follows we will see that this term has connection to the so-called Coriolis force.

Exercise 3. Consider cylindrical coordinates in three-dimensional space,

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z.$$

Check that, in this case, the formulas for $\hat{\mathbf{n}}_r$, $\hat{\mathbf{n}}_\varphi$ and their derivatives are the same as for the polar coordinates. On the top of that there is a constant basis vector $\hat{\mathbf{n}}_z = \hat{\mathbf{k}}$. Write the formulas to $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$ in cylindrical coordinates.

$$\begin{aligned}\text{Answer: } \mathbf{r} &= r\hat{\mathbf{n}}_r + \hat{\mathbf{k}}z; \\ \mathbf{v} &= \dot{r}\hat{\mathbf{n}}_r + r\dot{\varphi}\hat{\mathbf{n}}_\varphi + \hat{\mathbf{k}}\dot{z}; \\ \mathbf{a} &= \hat{\mathbf{n}}_r(\ddot{r} - r\dot{\varphi}^2) + \hat{\mathbf{n}}_\varphi(2\dot{r}\dot{\varphi} + r\ddot{\varphi}) + \hat{\mathbf{k}}\ddot{z}.\end{aligned}$$

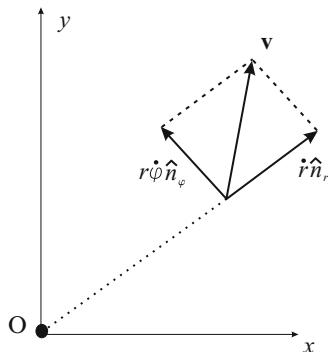


Fig. 2.7 Velocity components in polar coordinates

As a final example, consider spherical coordinates (r, θ, φ) , defined according to Fig. 2.8 and related to the Cartesian coordinates by the equations

$$\begin{cases} x = r \cos \varphi \sin \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \theta. \end{cases}$$

Let $\hat{\mathbf{n}}_r$, $\hat{\mathbf{n}}_\varphi$ and $\hat{\mathbf{n}}_\theta$ be the basis vectors indicated in the figure above. The vectors $\hat{\mathbf{n}}_r$ and $\hat{\mathbf{n}}_\theta$ belong to the same vertical plane containing the axis z . Obviously,

$$\begin{aligned} \hat{\mathbf{n}}_r &= \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r} \\ &= \sin \theta (\hat{\mathbf{i}} \cos \varphi + \hat{\mathbf{j}} \sin \varphi) + \hat{\mathbf{k}} \cos \theta \end{aligned} \quad (2.29)$$

and also

$$\hat{\mathbf{n}}_\varphi = -\hat{\mathbf{i}} \sin \varphi + \hat{\mathbf{j}} \cos \varphi,$$

as in the case of polar coordinates in the plane.

Exercise 4. Verify that

$$\hat{\mathbf{n}}_r \cdot \hat{\mathbf{n}}_r = \hat{\mathbf{n}}_\varphi \cdot \hat{\mathbf{n}}_\varphi = 1 \quad \text{and} \quad \hat{\mathbf{n}}_r \cdot \hat{\mathbf{n}}_\varphi = 0.$$

To define the last component of the basis, the vector $\hat{\mathbf{n}}_\theta$, we remember that this should be a unit vector and orthogonal to the other two. So the easiest way is to define this vector as

$$\hat{\mathbf{n}}_\theta = \hat{\mathbf{n}}_\varphi \times \hat{\mathbf{n}}_r. \quad (2.30)$$

Using the last formula, we have

$$\hat{\mathbf{n}}_\theta = (\hat{\mathbf{i}} \cos \varphi + \hat{\mathbf{j}} \sin \varphi) \cos \theta - \hat{\mathbf{k}} \sin \theta. \quad (2.31)$$

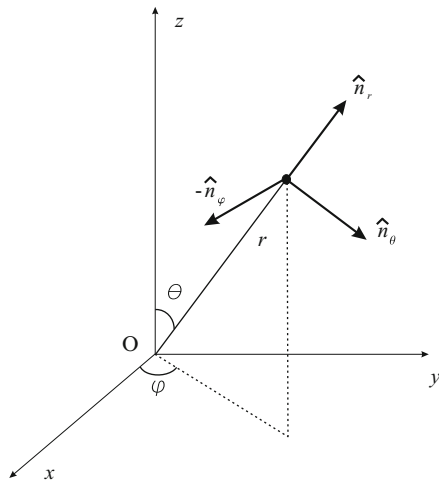


Fig. 2.8 Spherical coordinates

Looking at Fig. 2.8, we can note that the projections of $\hat{\mathbf{n}}_\theta$ and $\hat{\mathbf{n}}_r$ in the plane xy are parallel,² and also that the angle between these vectors is $\pi/2$, which can be verified by inspection of (2.29) and (2.31).

Exercise 5. (a) Verify that the solution (2.31) satisfies relation (2.30) and $|\hat{\mathbf{n}}_\theta| = 1$. (b) The property (2.30) means that the set of vectors $(\hat{\mathbf{n}}_r, \hat{\mathbf{n}}_\theta, \hat{\mathbf{n}}_\phi)$ has dextrorotatory direction, just like the set of vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$. Check that the cyclic property is also valid in this case, i.e.,

$$\hat{\mathbf{n}}_r = \hat{\mathbf{n}}_\theta \times \hat{\mathbf{n}}_\phi, \quad \hat{\mathbf{n}}_\phi = \hat{\mathbf{n}}_r \times \hat{\mathbf{n}}_\theta.$$

(c) Looking at Fig. 2.8, verify that all the three basis vectors are pointing in the growing direction of the corresponding parameter.

Exercise 6. Check if there are positions of $\mathbf{r}$ for which $\hat{\mathbf{n}}_r = \hat{\mathbf{i}}$ and the other two vectors $\hat{\mathbf{n}}_\theta$ and $\hat{\mathbf{n}}_\phi$ coincide respectively with (a) $-\hat{\mathbf{k}}$ and $+\hat{\mathbf{j}}$; or with (b) $-\hat{\mathbf{k}}$ and $-\hat{\mathbf{j}}$. Explain why the solution exists in only one of these two cases.

Answer: In the case (b) no solution is possible because the set of vectors $(\hat{\mathbf{i}}, -\hat{\mathbf{k}}, -\hat{\mathbf{j}})$ has different orientation from that of the set of vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$.

In order to find the velocity and acceleration in spherical coordinates, we will first calculate

$$\frac{d\hat{\mathbf{n}}_r}{dt}, \quad \frac{d\hat{\mathbf{n}}_\phi}{dt}, \quad \frac{d\hat{\mathbf{n}}_\theta}{dt}.$$

To simplify the calculation, we introduce the new notations,

² Without considering the case $\hat{\mathbf{r}} = \pm \hat{\mathbf{k}}$.

$$\begin{cases} \hat{\mathbf{m}} = \hat{\mathbf{i}} \cos \varphi + \hat{\mathbf{j}} \sin \varphi \\ \hat{\mathbf{l}} = -\hat{\mathbf{i}} \sin \varphi + \hat{\mathbf{j}} \cos \varphi. \end{cases}$$

Note that these vectors are the same that we have already found for the polar coordinates in the plane, so they satisfy the relations (2.27),

$$\frac{d\hat{\mathbf{m}}}{dt} = \dot{\varphi} \hat{\mathbf{l}}, \quad \frac{d\hat{\mathbf{l}}}{dt} = -\dot{\varphi} \hat{\mathbf{m}}. \quad (2.32)$$

We can see that $\hat{\mathbf{n}}_\varphi$ is exactly the vector $\hat{\mathbf{l}}$. As far as the basis is orthonormal, we get by rotating the basis $\{\hat{\mathbf{m}}, \hat{\mathbf{k}}\}$, the following result:

$$\begin{aligned} \hat{\mathbf{n}}_r &= \hat{\mathbf{m}} \sin \theta + \hat{\mathbf{k}} \cos \theta, \\ \hat{\mathbf{n}}_\varphi &= \hat{\mathbf{l}}, \\ \hat{\mathbf{n}}_\theta &= \hat{\mathbf{m}} \cos \theta - \hat{\mathbf{k}} \sin \theta. \end{aligned} \quad (2.33)$$

Using formulas (2.33) and (2.32), one can find

$$\begin{aligned} \frac{d\hat{\mathbf{n}}_\varphi}{dt} &= -\dot{\varphi} \hat{\mathbf{m}}, \\ \frac{d\hat{\mathbf{n}}_\theta}{dt} &= \dot{\varphi} \cos \theta + \dot{\theta} (-\hat{\mathbf{m}} \sin \theta - \hat{\mathbf{k}} \cos \theta), \\ \frac{d\hat{\mathbf{n}}_r}{dt} &= \dot{\varphi} \sin \theta + \dot{\theta} (\hat{\mathbf{m}} \cos \theta - \hat{\mathbf{k}} \sin \theta). \end{aligned}$$

At this point, we must invert the relations for $\hat{\mathbf{n}}_\theta$ and $\hat{\mathbf{n}}_r$.

Exercise 7. Check that the relationship between different basis vectors can be written in a matrix form,

$$\begin{pmatrix} \hat{\mathbf{n}}_\theta \\ \hat{\mathbf{n}}_r \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{m}} \\ \hat{\mathbf{k}} \end{pmatrix}.$$

This is a rotation matrix (See Fig. 2.9) and its inverse matrix has the form

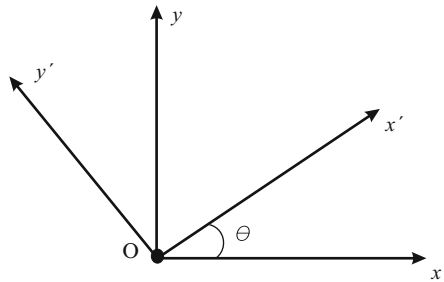


Fig. 2.9 Rotation of a basis

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Then, the inverse relationship for the basis vectors is given by

$$\begin{pmatrix} \hat{\mathbf{m}} \\ \hat{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{\mathbf{n}}_\theta \\ \hat{\mathbf{n}}_r \end{pmatrix},$$

or

$$\hat{\mathbf{m}} = \hat{\mathbf{n}}_\theta \cos \theta + \hat{\mathbf{n}}_r \sin \theta, \quad \hat{\mathbf{k}} = -\hat{\mathbf{n}}_\theta \sin \theta + \hat{\mathbf{n}}_r \cos \theta.$$

Hence,

$$\begin{aligned} \frac{d\hat{\mathbf{n}}_\varphi}{dt} &= -\dot{\varphi} (\hat{\mathbf{n}}_\theta \cos \theta + \hat{\mathbf{n}}_r \sin \theta); \\ \frac{d\hat{\mathbf{n}}_\theta}{dt} &= \dot{\varphi} \hat{\mathbf{n}}_\varphi \cos \theta - \dot{\theta} \hat{\mathbf{n}}_r; \\ \frac{d\hat{\mathbf{n}}_r}{dt} &= \dot{\varphi} \hat{\mathbf{n}}_\varphi \sin \theta + \dot{\theta} \hat{\mathbf{n}}_\theta. \end{aligned} \quad (2.34)$$

Using the last formulas, we calculate the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ in spherical coordinates. The radius-vector of the particle can be written as

$$\mathbf{r} = r\hat{\mathbf{n}}_r.$$

Taking the time derivative and using (2.34), we have

$$\mathbf{v} = \dot{r}\hat{\mathbf{n}}_r + r\dot{\varphi}\hat{\mathbf{n}}_\varphi \sin \theta + r\dot{\theta}\hat{\mathbf{n}}_\theta. \quad (2.35)$$

An important result is the square of the absolute value of the velocity,

$$\mathbf{v} \cdot \mathbf{v} = v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 \sin^2 \theta + r^2\dot{\theta}^2. \quad (2.36)$$

To obtain the acceleration, one needs to do a bit more involved calculation. The result is given by the expression

$$\begin{aligned} \mathbf{a} &= \hat{\mathbf{n}}_r(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2 \theta) \\ &\quad + \hat{\mathbf{n}}_\varphi(2r\dot{\varphi}\dot{\theta} \cos \theta + 2\dot{\varphi}\dot{r} \sin \theta + r\ddot{\varphi} \sin \theta) \\ &\quad + \hat{\mathbf{n}}_\theta(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2 \sin \theta \cos \theta). \end{aligned} \quad (2.37)$$

Exercises

1. (a) Verify the formulas (2.35), (2.36) and (2.37);
 (b) Consider the plane $\theta = \frac{\pi}{2}$ and check the coincidence of the above results with the ones for the polar system in the plane;

(c) Consider $\theta = \theta_0$ and $\varphi = \varphi_0$ (θ_0 and φ_0 constants). Calculate the velocity and acceleration. Give a physical interpretation for the result. What kind of movement is that?

Answer: It is a rectilinear motion, defined by the angles θ_0 and φ_0 .

2. Consider the motion of a point mass along the circumference of the radius R , centered at the origin O in the vertical plane $y = 3x$. The particle has a constant angular velocity, ω , and at the initial moment, $t = 0$, has $z = 0$, $\dot{z} > 0$ and $x > 0$. Write components of $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ in (a) Cartesian coordinates, (b) cylindrical coordinates, (c) spherical coordinates.

Answers:

$$(a) \quad x = \frac{R \cos \omega t}{\sqrt{10}}, \quad y = \frac{3R \cos \omega t}{\sqrt{10}}, \quad z = R \sin \omega t.$$

$$(b) \quad r = \sqrt{x^2 + y^2} = R |\cos \omega t|, \quad z = R \sin \omega t.$$

$$(c) \quad r \equiv R, \quad \theta = \frac{\pi}{2} - \omega t.$$

In both cases (b) and (c) we have, for the angular coordinate φ ,

$$\begin{aligned} \varphi &= \arctan 3 \quad \text{for} \quad \frac{(2n-1)\pi}{2\omega} \leq t \leq \frac{(2n+1)\pi}{2\omega}, \\ \varphi &= \arctan 3 + \pi \quad \text{for} \quad \frac{(2n+1)\pi}{2\omega} \leq t \leq \frac{(2n+3)\pi}{2\omega} \\ &\text{where} \quad n = \pm 1, \pm 2, \pm 3, \dots \end{aligned}$$

3. Derive the Eq. (2.27) for circular motion using Eq. (2.37).

4. A body is moving along the curve $y = Bx^2$, with $B = \text{const}$ and $\ddot{x} = a_0 = \text{const}$. Determine the absolute value of the acceleration as a function of x , if $\dot{x}(t=0) = 0$ and also $x(t=0) = 0$. Calculate the curvature radius of the trajectory as a function of x .

Answers: $a = a_0 \sqrt{1 + 36B^2x^2}$ and $R = \frac{1}{2B} (1 + 4B^2x^2)^{3/2}$.

5. A material point is moving in a spiral trajectory, according to the equations

$$x = b \cos \omega t, \quad y = b \sin \omega t, \quad z = v_0 t,$$

where b , ω and v_0 are constants. Calculate the radius of curvature of the trajectory at any point and the value of the tangential and normal components of acceleration.

Answers: $a_t = 0$, $a_n = \omega^2 b$ and $R = b + v_0^2/a_n$.

6. A train is moving in a circle of radius $R = 400$ m and its tangential acceleration is 0.2 m/s^2 . Determine the magnitude of its acceleration as well as its normal acceleration at the time instant when its velocity reaches a value of 10 m/s .

Answers: $a_n = 0.25 \text{ m/s}^2$ and $a = 0.32 \text{ m/s}^2$.

7. Knowing that the velocity of a particle is given by

$$\mathbf{v} = 4\omega R \hat{\mathbf{i}} \cos \omega t + 4\omega R \hat{\mathbf{j}} \sin \omega t + 3\omega R \hat{\mathbf{k}}$$

and the coordinates at the initial moment are $x(0) = R$, $y(0) = -4R$ and $z(0) = 0$, calculate the position vector, $\mathbf{r}(t)$, and the acceleration, $\mathbf{a}(t)$. Get the formulas for the normal and tangential components of acceleration. Compare the results with the one in Exercise 5.

8. Calculate the normal and tangential components of acceleration of a particle that is moving in a homogeneous gravitational field, $\mathbf{g}$. The initial velocity, $\mathbf{v}_0$, makes an angle α with the vector $\mathbf{g}$. Determine the radius of curvature of this trajectory as a function of time.

$$\text{Answer: } R = \frac{(v_0^2 + g^2 t^2 + 2v_0 g t \cos \alpha)^{3/2}}{v_0 g \sin \alpha}.$$

9. A particle moves so that its polar coordinates are given by $\rho(t) = v_0 t$ and $\phi(t) = \omega_0 t$, where v_0 and ω_0 are positive constants. Sketch the path of the particle. Write $\mathbf{v}$ and $\mathbf{a}$ in the polar basis with its constituents as a function of time. Calculate the instantaneous radius of curvature of the trajectory, R in terms of ϕ . After a large number of turns, N , calculate R .

Solution. We can write $\mathbf{v} = \dot{\rho} \hat{\rho} + \rho \dot{\phi} \hat{\phi} = v_0 \hat{\rho} + v_0 \omega_0 t \hat{\phi}$. Taking the derivative, we obtain $\mathbf{a} = 2\omega_0 v_0 \hat{\phi} - v_0 \omega_0^2 t \hat{\rho}$. To calculate R , just take the modulus of the equation

$$2\omega_0 v_0 \hat{\phi} - v_0 \omega_0^2 t \hat{\rho} = \frac{dv}{dt} \hat{v} + \frac{v^2}{R} \hat{n}.$$

Using $v = v_0 \sqrt{1 + \phi^2}$, then we have

$$R(\phi) = \frac{v_0 (\phi^2 + 1)^{3/2}}{\omega_0 \phi^2 + 2}.$$

For $\phi = 2\pi N \gg 1$, we have $R \rightarrow v_0 \phi / \omega_0 = 2\pi N v_0 / \omega_0$.

10. The problem of the fox. A fox is running along a straight line with a constant velocity v , trying to escape from the dog that always runs in the direction of the fox with constant velocity u . At the initial time instant $t = 0$ the dog is at the distance L from the fox and the line connecting both of them is orthogonal to the path of the fox.

- Calculate the relationship between the distance that separates the fox from the dog, l , and the angle φ between their velocity vectors;
- Determine the acceleration of the dog as a function of the same angle, φ .
- For the case $v > u$, calculate the time required for the dog to reach the fox.

Solution. The angle between the vector with components l_x, l_y joining the positions of the animals and the line of motion of the fox is φ . We have $\varphi = \varphi(t)$ and $l = l(t)$ and always

$$l_x = l \cos \varphi, \quad l_y = l \sin \varphi.$$

For an infinitesimal interval of time t , we obviously have

$$dl_x = (u - v \cos \varphi) dt, \quad dl_y = -v \sin \varphi dt,$$

so that, using $l dl = l_x dl_x + l_y dl_y$, we have

$$dl = (u \cos \varphi - v) dt. \quad (2.38)$$

On the other hand, considering only the terms of first order in the differentials, we obtain

$$l_x + dl_x = (l + dl) \cos(\varphi + d\varphi) = l_x + \cos \varphi dl - l \sin \varphi d\varphi,$$

$$l_y + dl_y = (l + dl) \sin(\varphi + d\varphi) = l_y + \sin \varphi dl + l \cos \varphi d\varphi.$$

Using these relations, we can write

$$dl_x = \cos \varphi dl - l \sin \varphi d\varphi, \quad dl_y = \sin \varphi dl + l \cos \varphi d\varphi.$$

Inverting the above relations, we obtain

$$dl = \cos \varphi dl_x + \sin \varphi dl_y, \quad l d\varphi = -\sin \varphi dl_x + \cos \varphi dl_y.$$

As a result, we conclude that

$$l d\varphi = -u \sin \varphi dt. \quad (2.39)$$

Using the relations (2.38) and (2.39), we can eliminate time and arrive at the differential equation

$$\frac{l d\varphi}{dl} = \frac{u \sin \varphi}{v - u \cos \varphi}.$$

This equation is separable and its solution can be written as

$$\frac{l(\varphi)}{C} = (1 - \cos \varphi)^{\frac{v-u}{2u}} (1 + \cos \varphi)^{-\frac{v+u}{2u}}, \quad (2.40)$$

where C is a constant of integration. Using the initial condition $l(\varphi = \pi/2) = L$, the constant is defined by $C = L$. It is easy to see that the dog reaches the fox in the case $v > u$, while in the case $v \leq u$ the fox will escape from the dog.

To find the acceleration of the dog, first we note that its tangential component is zero, because the module of its velocity is constant. To determine the normal component of acceleration, one can use (2.39) and obtain

$$a_n = \dot{\phi} v = -\frac{u \sin \phi}{l(\phi)},$$

where the function $l(\phi)$ is given by Eq. (2.40) with $C = L$.

Finally, to calculate the time of the race, T in the case $u > v$ (otherwise, T is infinite) we note that, according to the result (2.38),

$$L = \int_0^T (u \cos \phi - v) dt. \quad (2.41)$$

On the other hand, by the time T , the projections of the positions of the two animals on the line that defines the trajectory of the fox suffers the same increment, i.e.,

$$\int_0^T v \cos \phi dt = uT. \quad (2.42)$$

Comparing the Eqs. (2.41) and (2.42), we obtain the result

$$T = \frac{vL}{u^2 - v^2}.$$

2.4 Kinematics of a Rigid Body

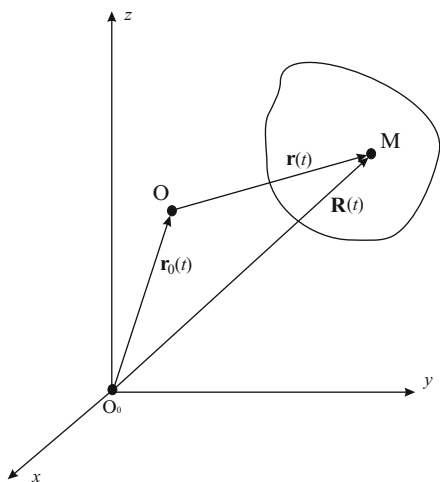
In this section, we consider the kinematics of the rigid body. This part of mechanics has a special importance because of its applications, apart from the fact that in many cases it is useful to associate a reference frame with a rigid body. Sometimes it is useful, for example, to work in a rotating reference frame, and find its relation to the inertial (laboratory) reference frame.

To study the motion of a rigid body, it proves useful to introduce a static reference frame with the origin O_o (we will identify it as the laboratory reference frame), and also another, non-inertial reference system attached to the rigid body, with a mobile origin, O , as shown in Fig. 2.10. It is not necessary that the point O belongs to the body, but the distances between this point and all the points of the body must be fixed.

Any movement of the rigid body can be considered as the sum of two independent motions. The first one is the movement of the point O in relation to the reference frame of the laboratory, point O_o . This movement is nothing else, but the motion of a point which can be parameterized by the radius-vector $\mathbf{r}_0(t)$. The second movement is the one of the rigid body with a static (fixed) point O . For a point M of the body, this movement can be parameterized by the radius-vector $\mathbf{r}(t)$. Then we have $\mathbf{R}(t) = \mathbf{r}_0(t) + \mathbf{r}(t)$.

It is clear that the point O can perform complicated motions with respect to O_o but anyway, it is a motion of a single point. For its description we can use the formalism developed in the previous chapter. So, in order to get qualitatively new information,

Fig. 2.10 Rigid body in the co-moving reference system and at the laboratory reference frame



it is sufficient to consider the reference frame where O is a static point. In this case, the movement of the rigid body is reduced to a sequence of rotations around an instantaneous axis.

A complete description of the motion of a rigid body, with the variable directions of the axis can only be performed in the context of analytical mechanics. In this book we are going to consider the motion around a instantaneous axis in a simplified form. We will mainly consider the motion of bodies with fixed axis or with a relatively simple geometry and mass distribution.

The first step is to study the rotation of a rigid body around a fixed axis passing through the point O (see Fig. 2.11).

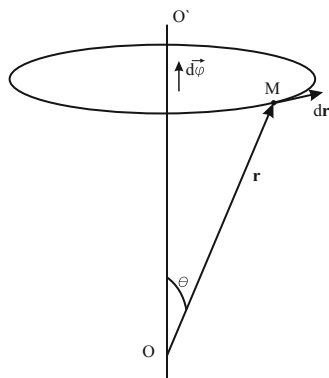


Fig. 2.11 Rotation of a point of the rigid body, M , around the axis OO'

When the rotation axis is fixed, we can focus the attention on the angle of rotation, φ . We adopt the *radian* as the unit of angular measure. For an infinitesimal rotation, we can associate an angle of rotation with the vector $d\vec{\varphi}$, such that

$d\vec{\phi} \parallel OO'$ and it has the direction defined by the right-hand rule (or rule of the corkscrew). For a point M , with its position specified by the radius-vector $\mathbf{r}$, its displacement due to $d\vec{\phi}$ is given by

$$d\mathbf{r} = d\vec{\phi} \times \mathbf{r} = [d\vec{\phi}, \mathbf{r}], \quad (2.43)$$

where we have used two different notations for the vector product (see Mathematical Appendix).

An important observation is related to the possibility of reversing the coordinate axes (this transformation is called space inversion, or parity inversion),

$$x \rightarrow -x, \quad y \rightarrow -y, \quad z \rightarrow -z. \quad (2.44)$$

What is the behavior of the components of the vectors $\mathbf{r}$, $d\mathbf{r}$, and $d\vec{\phi}$ in this case? Obviously, for any radius-vector, including $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$, the components change their signs. In particular, besides (2.44), we have $x + dx \rightarrow -(x + dx)$ etc. We can describe this situation as the transformation $\mathbf{r} \rightarrow -\mathbf{r}$, in a sense that all components of the vector change their signs. At the same time $\mathbf{r} + d\mathbf{r} \rightarrow -(\mathbf{r} + d\mathbf{r})$, that means $d\mathbf{r} \rightarrow -d\mathbf{r}$. In this case, in order to maintain the orientation of the vectors $\mathbf{r}$, $d\vec{\phi}$, $d\mathbf{r}$ and the relation (2.43), we need to define the rule of transformation of the vector $d\vec{\phi}$ to be different, namely we need it to transform without change of sign,

$$d\vec{\phi} \rightarrow d\vec{\phi}. \quad (2.45)$$

In other words, in order to keep the consistency of the formulas, the components of the vector $d\vec{\phi}$ should not suffer a change when the basis vectors turn to the opposite sides and the components of the radius-vector change their signs. We conclude that there are two different kinds of vectors:

1) Simple vectors, such as $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{a}$. Under the inversion of axes their components change their signs. In this sense we can say that the vectors change their directions to the opposite ones, e.g., $\mathbf{r} \rightarrow -\mathbf{r}$, $\mathbf{v} \rightarrow -\mathbf{v}$, etc.

2) Axial vectors, such as $d\vec{\phi}$. They do not change the signs of their components (one can say that they do not change their directions, in this sense) under the inversion of the axes.

Both types of vectors have important applications in physics. Typically, axial vectors appear as a consequence of the cross product. For example, the position, velocity and acceleration of a particle are vectors (simple vectors). But if we recall the formula for the Lorentz force exerted by the magnetic field, $\mathbf{H}$, on a charged particle moving with velocity $\mathbf{v}$,

$$\mathbf{F} = m\mathbf{a} = e\mathbf{v} \times \mathbf{H},$$

it becomes clear that the magnetic field must be an axial vector.

Dividing $d\vec{\phi}$ by dt , we find the angular velocity $\vec{\omega} = d\vec{\phi}/dt$. In the general case, the direction of $\vec{\omega}$ depends on time. This means that the axis of rotation may depend on time as well as the magnitude of $\vec{\omega}$. The derivative of $\vec{\omega}$ is the angular

acceleration vector, $\vec{\beta} = d\vec{\omega}/dt$. The units for $\vec{\omega}$ and $\vec{\beta}$ are rad/s and rad/s² respectively.

Now we can find a relationship between linear and angular quantities. Dividing (2.43) by dt , we obtain

$$\frac{d\mathbf{r}}{dt} = \frac{d\vec{\phi}}{dt} \times \mathbf{r}, \quad \text{i.e.,} \quad \mathbf{v} = \vec{\omega} \times \mathbf{r}. \quad (2.46)$$

The last formula is valid for the rotations of a rigid body relative to the axis OO' that passes through a static point, O . In order to find the analogous relation for the acceleration, one has to take the derivative of the last relationship (see Mathematical Appendix for the derivative of the vector product):

$$\frac{d\mathbf{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \mathbf{r} + \vec{\omega} \times \frac{d\mathbf{r}}{dt} = \vec{\beta} \times \mathbf{r} + \vec{\omega} \times \mathbf{v}.$$

Using (2.46), we obtain

$$\mathbf{a} = [\vec{\beta}, \mathbf{r}] + [\vec{\omega}, [\vec{\omega}, \mathbf{r}]].$$

The last term can be rewritten using the formula for the double vector product,

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b}), \quad (2.47)$$

and so we arrive at the general expression for the acceleration,

$$\mathbf{a} = \vec{\beta} \times \mathbf{r} + \vec{\omega} (\vec{\omega} \cdot \mathbf{r}) - \mathbf{r} \omega^2. \quad (2.48)$$

The last formula becomes simpler when we consider a particle moving on the plane which includes the origin, point O , and is perpendicular to the rotation axis. In this case, $\mathbf{r} \perp \vec{\omega}$ and the expression (2.48) reduces to

$$\mathbf{a} = \vec{\beta} \times \mathbf{r} - \mathbf{r} \omega^2. \quad (2.49)$$

In the particular case when the axis of rotation does not depend on time, the direction of the angular velocity is constant, i.e., $\vec{\beta} \parallel \vec{\omega}$ and we can see that the two terms in (2.49) are vectors with orthogonal directions. The first term, $\vec{\beta} \times \mathbf{r}$, always has the direction perpendicular to the radius-vector, and the second term, $-\mathbf{r} \omega^2$, is the usual expression for the centripetal acceleration.

Finally, we write the formulas for the motion of the point M of the rigid body (see Fig. 2.11) in the reference of the laboratory, i.e., referred to the point O_o (starting point of the static coordinate system). As we know, the radius-vector in the laboratory reference system is given by

$$\mathbf{R}(t) = \mathbf{r}_0(t) + \mathbf{r}(t).$$

Taking the derivative of both sides, we find

$$\mathbf{V}(t) = \frac{d\mathbf{R}}{dt} = \mathbf{v}_0(t) + \mathbf{v}(t),$$

where $\mathbf{v}_0(t)$ is the velocity of the point O related to the point O_ρ . Using the previous results, we obtain

$$\mathbf{V}(t) = \mathbf{v}_0(t) + \vec{\omega} \times \mathbf{r} = \mathbf{v}_0(t) + \vec{\omega} \times (\mathbf{R} - \mathbf{r}_0).$$

The last form represents a relationship between the radius-vector $\mathbf{R}$ and its derivative $\mathbf{V}$.

For the acceleration we can use the relation (2.48) and write a similar formula

$$\mathbf{a} = \mathbf{a}_0 + \vec{\beta} \times \mathbf{r} + \vec{\omega} (\vec{\omega} \cdot \mathbf{r}) - \mathbf{r} \omega^2,$$

where

$$\mathbf{a}_0 = \frac{d\mathbf{v}_0}{dt} = \frac{d^2\mathbf{r}_0}{dt^2} \quad \text{and} \quad \mathbf{r} = \mathbf{R} - \mathbf{r}_0.$$

Exercise 8. (a) Verify the formula (2.47), using the definition of vector product

$$\mathbf{b} \times \mathbf{c} = [\mathbf{b}, \mathbf{c}] = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

(b) Consider the case where $\vec{r} \parallel \vec{\omega}$ and interpret the result for $\mathbf{v}$ and $\mathbf{a}$.

Exercise 9. (a) Consider the vector quantities listed below and identify which ones are vectors (simple vectors) and which are axial vectors.

$$\begin{array}{lll} (i) \quad \mathbf{r}, & (ii) \quad \mathbf{v}, & (iii) \quad \mathbf{a}, \\ (iv) \quad \mathbf{l}_1 = \mathbf{r} \times \mathbf{v}, & (v) \quad \mathbf{l}_2 = \mathbf{r} \times \mathbf{a}, & (vi) \quad \mathbf{l}_3 = \mathbf{v} \times \mathbf{a}, \\ (vii) \quad \mathbf{l}_4 = \mathbf{r} \times \mathbf{l}_1, & (viii) \quad \mathbf{l}_5 = \mathbf{r} \times \mathbf{l}_3. & \end{array}$$

(b) Try to get the same answers using different approaches.

(c) Try to elaborate a general criteria for identifying the nature of the vectors with respect to inversion of the coordinates, namely whether a given vector is a simple one or an axial vector.

(d) Check the result for the vector $\mathbf{l}_4$ using the Eq. (2.47).

Exercise 10. Consider the motion of a rigid body with constant rotation axis which coincides with the axis OX . Compare the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the points of the radius-vectors

$$\mathbf{r}_1 = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} \quad \text{and} \quad \mathbf{r}_2 = 5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}.$$

- (a) What is the ratio between the magnitudes of these velocities?
- (b) Why does one of the components of each $\mathbf{r}$ have no importance for the answer?
- (c) Find a rotation axis that passes through the point O , for which the magnitudes of the velocities are equal. Is this the unique solution for this problem?

Exercise 11. Considering the previous problem, assume that the angular velocity is given by $\vec{\omega} = \omega \hat{\mathbf{i}}$, with $\omega = \text{const}$, and calculate the instantaneous accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ for the points of the rigid body with radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$.

2.5 Transformation Between Reference Systems

In this section, we consider the transformation of coordinates, velocities and accelerations between different reference frames. Remember that the reference frame is a spatial coordinate system defined by an origin O , and by the linearly independent vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, and $\hat{\mathbf{n}}_3$. Any position vector which starts at the origin O and ends at the given point, can be written as

$$\mathbf{r} = x^1 \hat{\mathbf{n}}_1 + x^2 \hat{\mathbf{n}}_2 + x^3 \hat{\mathbf{n}}_3,$$

where x^1 , x^2 and x^3 are coordinates in the given reference frame. Our goal is to accomplish the transformation of the components of the vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ from one reference frame to another. In this book, we consider only the cases when the basis is orthonormal, i.e., $\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j = \delta_{ij}$. Here we introduced a useful notion of the Kronecker delta δ_{ij} (or Kronecker symbol) which is equal to 1 when $i = j$ and to 0 when $i \neq j$.

First we will study the cases when, for both reference frames, the basis vectors are the same, namely: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$; and only the initial points O and O' are different. Under these conditions, the relationship between the reference points can be described as a transformation called *translation*, or *displacement*. Suppose that $\vec{OO'} = \mathbf{r}_0$. For a spatial point M , we have in this case (see Fig. 2.12),

$$\mathbf{r} = \mathbf{r}' + \mathbf{r}_0.$$

At this point we can introduce the notion of inertial reference frame, which is very important in mechanics. For example, Newton's first law establishes the existence of an inertial reference frame. When the coordinate system K is inertial, any free particle (i.e., free of interactions with other objects) remains in motion with constant velocity, $\mathbf{v} = \text{const}$. In practice, it is impossible to find a physical body that has no interaction with any other object. We can isolate an object, for example, taking him into space, far from the Earth. But he will still be under the action of the gravitational field of the Sun. Taking the same object away from the Solar system, there is still a galaxy's gravitational field and so on. A free body is an idealization.

A possible option to achieve inertial reference frame in reality is to let the body to move freely in a weak gravitational field which should be approximately

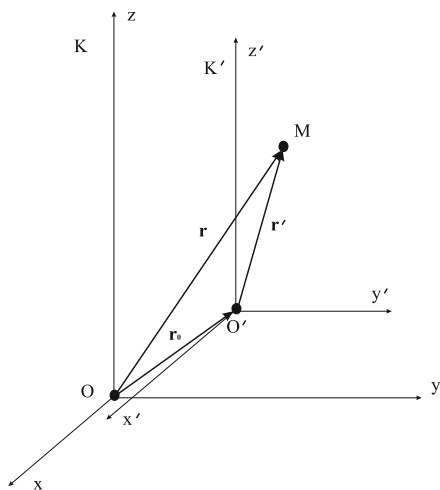


Fig. 2.12 Two reference frames related by a translation

homogeneous. In this case, the gravitational force effectively disappears for this body, and we can consider the reference frame attached to it as being inertial. We will discuss this issue later in the chapters on the laws of dynamics, when considering gravity.

In Mechanics, we can make a description of movements both in an inertial reference frame, and in a non-inertial one. In the present chapter, dedicated to kinematics, we will start by considering the coordinate transformations between inertial reference frames, and after that we will consider transformations to non-inertial reference frames.

Suppose K is an inertial reference frame, which can be considered static, for simplicity. Another reference frame, K' , is moving with respect to K with constant velocity, $\mathbf{v}_0$. The relationship between the radius-vectors $\mathbf{r}$ and $\mathbf{r}'$ of the same point M , in both reference frames is

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}_0 \cdot t. \quad (2.50)$$

The last formula is called Galileo's transformation. If the relative velocity (velocity of K' with respect to K) is directed along the OX axis to the positive side, i.e., $\mathbf{v}_0 = v_0 \hat{\mathbf{i}}$, we will have the following relations between the two sets of coordinates:

$$\begin{aligned} x &= x' + v_0 t, \\ y &= y', \\ z &= z', \\ t &= t'. \end{aligned} \quad (2.51)$$

In order to obtain the velocities of the particles in the two reference frames, we take the time derivative of (2.50) and obtain

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_0.$$

Now it is easy to see that if K is an inertial reference frame in the way defined above, the same is true for K' . The reason is that if $\mathbf{v} = \text{const}$, then $\mathbf{v}' = \mathbf{v} - \mathbf{v}_0 = \text{const}$ too. Thus, the Galileo transformation (2.51) shows the relationship between coordinates (or radius-vectors) in the two inertial frames. Since $\mathbf{v}_0 = \text{const}$, the acceleration of the test particle is always the same in two reference frames, $\mathbf{a}' = \mathbf{a}$.

The second step is to consider a reference frame K' , which is uniformly accelerated with respect to the inertial frame K . The acceleration of K' with respect to K is represented by the vector $\mathbf{a}_0 = d\mathbf{v}_0/dt = \text{const}$. In this case, we can easily find the relation between the velocities in the two frames,

$$\mathbf{v}' = \mathbf{v} - \mathbf{a}_0 t \quad (2.52)$$

and the following relation between the accelerations in the two reference frames:

$$\mathbf{a}' = \mathbf{a} - \mathbf{a}_0. \quad (2.53)$$

For example, if the material point has constant velocity, $\mathbf{v} = \text{const}$, in the frame K , the same point will move with acceleration $\mathbf{a}' = -\mathbf{a}_0$ with respect to K' . According to the formula (2.52), K' is not an inertial frame, because even if no force acts on this body, its velocity $\mathbf{v}'$ is not a constant.

The third step is to consider the case where K is an inertial reference frame and K' is spinning around a fixed axis. To simplify the consideration, let us choose the axis OZ to be the axis of rotation, and assume that the origins of the two coordinate systems coincide, $O = O'$ (see Fig. 2.13). In this case, we have $\vec{\omega} = -\omega \hat{\mathbf{k}}$ and can use the results concerning rotation of the rigid body from the previous chapter. The radius-vector of the point M in both frames is the same at every instant, i.e.,

$$\mathbf{r}' = \overrightarrow{O'M} = \overrightarrow{OM} = \mathbf{r}. \quad (2.54)$$

At the same time, the velocities and accelerations in the two reference frames differ from each other, as we shall see.

Let us suppose that the point M is moving with the speed $\mathbf{u}'$ with respect to the reference frame K' , at the moment when its radius-vector is given by $\mathbf{r} = \mathbf{r}'$. Consider a very small (infinitesimal) time interval Δt . The displacement of the point M with respect to the rotating reference frame K' is $\Delta \mathbf{r}' = \mathbf{u}' \Delta t$. At the same time, the displacement of a point which is fixed in the reference frame K' at the same point M , with respect to K is $\mathbf{v} \Delta t = \vec{\omega} \times \mathbf{r} \Delta t$. Thus, the displacement of the point M with respect to static reference frame K will be given by the sum of the two terms,

$$\Delta \mathbf{r} = (\mathbf{u}' + \vec{\omega} \times \mathbf{r}) \Delta t.$$

Dividing both sides by Δt and taking the limit $\Delta t \rightarrow 0$, we observe that the velocity of this point in K is

$$\mathbf{u} = \mathbf{u}' + \vec{\omega} \times \mathbf{r}. \quad (2.55)$$

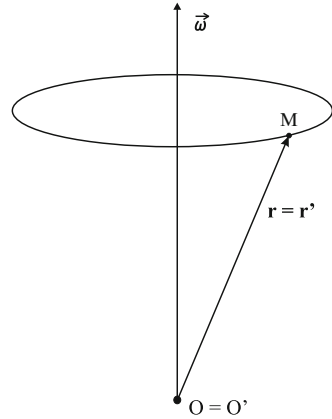


Fig. 2.13 The non-inertial reference frame K' is rotated relative to the inertial one K . The starting points are the same, so $\mathbf{r} = \mathbf{r}'$

The result (2.55) is very interesting because it can be seen in other way. Let us remember, once more, that the position of the vectors M in both frames, K' and K , coincide according to (2.54). Of course, time runs identically in both frames, as always in Classical Mechanics. Apparently, this yields to a contradiction to the relation (2.55) for velocities.

To explain what is the origin for the relation (2.55), one has to understand the main difference between the reference frames K and K' . The frame K is inertial and for the time interval Δt , the displacement of the particle M is given by $\Delta \mathbf{r}$, whereas in respect to K' , it is given by $\Delta \mathbf{r}'$. So, despite the coincidence of the radius vectors, their variations are different, because the reference system K' is rotating, i.e., performs its own motion, additional to the one of the motion of the particle in the rotating frame. This additional movement of K' produces an extra term in (2.55). The procedure that generates this term can be seen as a more sophisticated rule to take the time derivative of any vector at the reference frame K' , because all vectors which are static in K' are rotating with respect to K with the same angular speed. We can define a “generalized” derivative (called *covariant derivative* by mathematicians) of a vector $\mathbf{b}$, taking into account the rotation of this vector,

$$\frac{D\mathbf{b}}{dt} = \frac{d\mathbf{b}}{dt} - \vec{\omega} \times \mathbf{b}. \quad (2.56)$$

In this case, we rewrite (2.55) as follows:

$$\frac{D\mathbf{r}'}{dt} = \mathbf{u}' = \frac{d\mathbf{r}'}{dt} - \vec{\omega} \times \mathbf{r}' = \frac{d\mathbf{r}}{dt} - \vec{\omega} \times \mathbf{r}, \quad (2.57)$$

where we used, once more, the relationship (2.54). For the radius-vector, the “true” time derivative in the reference frame K' is given by D/dt from the Eq. (2.57), and not by the simple derivative d/dt . The additional term appears because of the rotation, which produces an extra displacement of the radius-vector.

The fact that we know how to take the time derivative of an arbitrary vector in a non-inertial rotating frame K' turns out to be very helpful in calculating the acceleration in this frame. An important observation is that (as we have already mentioned before) all vectors which are static in K' are rotating with respect to K with the same angular speed. As a consequence, the rule for taking a derivative in the rotating frame should be the same for all kind of vectors. By applying the covariant derivative in the reference system K' according to (2.56), to the velocity of the particle, $\mathbf{u}' = \mathbf{u} - \vec{\omega} \times \mathbf{r}'$, and remembering that the angular velocity, $\vec{\omega}$, is constant, we obtain

$$\begin{aligned} \mathbf{a}' &= \frac{D\mathbf{u}'}{dt} = \frac{d\mathbf{u}'}{dt} - \vec{\omega} \times \mathbf{u}' \\ &= \frac{d}{dt} (\mathbf{u} - \vec{\omega} \times \mathbf{r}) - \vec{\omega} \times (\mathbf{u} - \vec{\omega} \times \mathbf{r}) \\ &= \mathbf{a} - \vec{\omega} \times \mathbf{u} - \vec{\omega} \times \mathbf{u} + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ &= \mathbf{a} - 2\vec{\omega} \times \mathbf{u} + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}). \end{aligned}$$

In order to obtain the inverse relation between accelerations, one can simply use the formula $\mathbf{r} = \mathbf{r}'$. As a result, we arrive at the expression for acceleration in the frame K ,

$$\begin{aligned} \mathbf{a} &= \mathbf{a}' + 2\vec{\omega} \times \mathbf{u} - \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) \\ &= \mathbf{a}' + 2\vec{\omega} \times \mathbf{u}' + 2\vec{\omega} \times (\vec{\omega} \times \mathbf{r}') - \vec{\omega} \times (\vec{\omega} \times \mathbf{r}') \\ &= \mathbf{a}' + 2\vec{\omega} \times \mathbf{u}' + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}'). \end{aligned} \tag{2.58}$$

Exercises

1. Write the equations similar to Eq. (2.51) for the cases when the relative velocity between the two frames is directed along the OY and OZ axis and for the general case, when the orientation is arbitrary, e.g.,

$$\mathbf{v}_0 = v_0 (\hat{\mathbf{i}} \cos \alpha + \hat{\mathbf{j}} \cos \beta + \hat{\mathbf{k}} \cos \gamma),$$

where α , β , and γ are the angles between $\mathbf{v}_0$ and the vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, respectively.

Answer: In the general case

$$x = x' + v_0 t \cos \alpha, \quad y = y' + v_0 t \cos \beta, \quad z = z' + v_0 t \cos \gamma, \quad t = t'.$$

2. A train began to accelerate in the straight part of its path with acceleration a . A small lead block, inside the train, begin to slide horizontally without friction. Determine the acceleration of this block with respect to (a) the Earth; (b) the train.

Write the expression for $v(t)$ of the block in relation to the train.

Answers: (a) zero. (b) $-a$, $v = -at$.

3. Consider a particular case of the Eq. (2.57),

$$\mathbf{u}' = \frac{D\mathbf{r}'}{dt} = \mathbf{u} - \vec{\omega} \times \mathbf{r},$$

in the case where the velocity $\mathbf{u}'$ is zero. (a) What is the physical interpretation of $\mathbf{u}$ in this case? What is the origin of $\mathbf{u}$? (b) What is the physical result of the condition $\mathbf{u} = 0$? Suggest an example of a system where such a movement occurs.

Answers: (a) The condition $\mathbf{u}' = 0$ means that the point is rotating in relation to the reference K and is at rest relative to the reference K' . The velocity $\mathbf{u}$ is only caused by rotation.

(b) The condition $\mathbf{u} = 0$ means that the point is at rest with respect to the inertial reference frame K . Such a point is not rotating together with K' .

Lecture Notes on Newtonian Mechanics

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