

Chapter 2

The Bergman Metric

2.1 Smoothness to the Boundary of Biholomorphic Mappings

Poincaré's theorem (see [KRA1, KRA9] for discussion) that the ball and polydisc are biholomorphically inequivalent shows that there is no Riemann mapping theorem (at least in the traditional sense) in several complex variables. More recent results of Burns, Shnider, and Wells [BSW] and of Greene and Krantz [GRK1, GRK2] confirm how truly dismal the situation is. First, we need a definition.

Definition 2.1.1. Let ρ_0 be a C^k defining function for a bounded domain $\Omega_0 \subseteq \mathbb{R}^N$, $k \geq 2$. We define a neighborhood basis for Ω_0 in the C^k topology as follows: Let $\epsilon > 0$ be so small that if $\|\rho - \rho_0\|_{C^k} < \epsilon$, then ρ has nonvanishing gradient on $\{x : \rho(x) = 0\}$. For any such ρ , let $\Omega_\rho = \{x \in \mathbb{R}^N : \rho(x) < 0\}$. Define

$$\mathcal{U}_\epsilon^k(\Omega_0) = \{\Omega_\rho \subseteq \mathbb{R}^N : \|\rho - \rho_0\|_{C^k} < \epsilon\}.$$

Observe that $\mathcal{U}_\epsilon^k(\Omega_0)$ is a *collection of domains*. Then the sets $\mathcal{U}_\epsilon \equiv \mathcal{U}_\epsilon^k(\Omega_0)$ are called *neighborhoods of Ω_0 in the C^k topology*. Of course neighborhoods in the C^∞ topology are defined similarly.

If Ω_1, Ω_2 are domains in $\mathbb{C}^n$, then we say that $\Omega_1 \sim \Omega_2$ if Ω_1 is biholomorphic to Ω_2 .

Theorem 2.1.2 (Burns–Shnider–Wells). *Let $k \in \mathbb{N}$. Let $\epsilon > 0$ be small. Let $\mathcal{U}_\epsilon^k \equiv \mathcal{U}_\epsilon^k(B)$ be any neighborhood of the ball $B \subseteq \mathbb{C}^n$ in the C^k topology as defined above. If $n \geq 2$, then $\mathcal{U}_\epsilon^k / \sim$ is uncountable, no matter how small $\epsilon > 0$ is or how large k is (even $k = \infty$ or $k = \omega$). By contrast, if $n = 1$ and $\epsilon < 1/5$, then $\mathcal{U}_\epsilon^k / \sim$ has just one element.*

The last statement of the theorem perhaps merits some explanation. If $n = 1$ and $\epsilon < 1/5$, then perforce any equivalence class in $\mathcal{U}_\epsilon^k / \sim$ will contain only bounded

domains that are simply connected. Thus any such domain will, by the Riemann mapping theorem, be conformally equivalent to the disc.

Greene and Krantz [GRK1, GRK2] have refined the theorem to show that, when $n \geq 2$, in fact each of the equivalence classes is closed and nowhere dense.

We now give a brief accounting of some of the differences between $n = 1$ and $n > 1$. A more detailed discussion appears in Greene and Krantz [GRK9], in [GKK] and also in the original work of Poincaré. This subject begins with the following breakthrough of Fefferman [FEF1, Part I]:

Theorem 2.1.3. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be strictly pseudoconvex domains with C^∞ boundary. If $\phi : \Omega_1 \rightarrow \Omega_2$ is biholomorphic, then ϕ extends to a C^∞ diffeomorphism of $\overline{\Omega}_1$ onto $\overline{\Omega}_2$.*

The Fefferman's theorem enables one to see that, if Ω_1 and Ω_2 are biholomorphic under ϕ , then there are certain differential invariants of $\partial\Omega_1, \partial\Omega_2$ that must be preserved under ϕ . More precisely, if k is large, then the k^{th} -order Taylor expansion of the defining function ρ_1 for Ω_1 (resp. of the defining function ρ_2 for Ω_2) has more coefficients than the k^{th} -order Taylor expansion for ϕ (the disparity in the number grows rapidly with k). Since ρ_1 is mapped to ρ_2 under composition with ϕ^{-1} , it follows that some of these coefficients, or combinations thereof, must be invariant under biholomorphic mappings. Tanaka [TAN] and Chern–Moser [CHM] have made these remarks precise and have shown how to calculate these invariants. A more leisurely discussion of these matters appears in Greene and Krantz [GRK9].

Now it is easy to see intuitively that two domains Ω and Ω_0 can be close in the C^k topology, any k , and have entirely different Chern–Moser–Tanaka invariants. This notion is made precise, for instance, in Burns, Shnider, and Wells [BSW], by using a transversality argument. [Note that everything we are saying is vacuous in $\mathbb{C}^1$ because the invariants must live in the complex tangent space to the boundary—which is empty in dimension one. See [KRA1].] It is essentially a foregone conclusion that things will go badly in higher dimensions.

If one seeks positive results in the spirit of the Riemann mapping theorem in dimension $n \geq 2$, then one must find statements of a different nature. Fridman [FRI] has constructed a “universal domain” Ω^* which can be used to exhaust any other. He has obtained a number of variants of this idea, using elementary but clever arguments. Semmes [SEM] has yet another approach to the Riemann mapping theorem that is more in the spirit of the work of Lempert [LEM1]. We next present, mainly for background, a substitute for the Riemann mapping theorem whose statement and proof is more in the spirit of the Fefferman's theorem. We continue to use the notation $\Omega_1 \sim \Omega_2$ to mean that Ω_1 is biholomorphic to Ω_2 .

In what follows, we let $\text{Aut}(\Omega)$ denote the group (under composition of mappings) of biholomorphic self-maps of the domain Ω .

Theorem 2.1.4 (Greene–Krantz [GRK2]). *Let $B \subseteq \mathbb{C}^n$ be the unit ball. Let $\rho_0(z) = |z|^2 - 1$ be the usual defining function for B . If $\epsilon > 0$ is sufficiently small, $k = k(n)$ is sufficiently large, and $\Omega \in \mathcal{U}_\epsilon^k(B)$ then either*

$$\Omega \sim B \tag{2.1.4.1}$$

or

$$\Omega \text{ is not biholomorphic to the ball and} \quad (2.1.4.2)$$

- (a) $\text{Aut}(\Omega)$ is compact.
 (b) $\text{Aut}(\Omega)$ has a fixed point. Moreover,

If $K \subset\subset B$, $\epsilon > 0$ is sufficiently small (depending on K), and $\Omega \in \mathcal{U}_\epsilon^k(B)$ has the property that its fixed point set lies in K , then there is a biholomorphic mapping $\Phi : \Omega \rightarrow \Phi(\Omega) \equiv \Omega' \subseteq \mathbb{C}^n$ such that $\text{Aut}(\Omega')$ is the restriction to Ω' of a subgroup of the group of unitary matrices.

The collection of domains to which (2.1.4.2) applies is both dense and open.

Theorem 2.1.4 shows, in a weak sense, that domains near the ball that have any automorphisms other than the identity are (biholomorphic to) domains with only Euclidean automorphisms. It should be noted that (2.1.4.2a) is already contained in the theorem of Bun Wong and Rosay [WON, ROS] and that the denseness of the domains to which (2.1.4.2) applies is contained in the work of Burns–Shnider–Wells. The proof of Theorem 2.1.4 involves a detailed analysis of Fefferman’s asymptotic expansion for the Bergman kernel and of the $\bar{\partial}$ -Neumann problem and would double the length of this book if we were to treat it in any detail.

The purpose of this lengthy introduction has been to establish the importance of Theorem 2.1.4 and to set the stage for what follows. It may be noted that the proof of the result analogous to Fefferman’s in $\mathbb{C}^1$, that a biholomorphic mappings of smooth domains extends smoothly to the boundary, was proved in the nineteenth century by Painlevé [PAI]. The result in one complex dimension has been highly refined, beginning with work of Kellogg [KEL] and more recently by Warschawski [WAR1, WAR2, WAR3], Rodin and Warschawski [ROW], and others. This classical work uses harmonic estimation, potential theory, and the Jordan curve theorem, devices which have no direct analogue in higher dimensions. A short, self-contained, proof of the one-variable result—using ideas closely related to those presented here—appears in [BEK].

We conclude this section by presenting a short and elegant proof of the Fefferman’s Theorem 2.1.3. The techniques are due to Bell [BEL1] and Bell and Ligocka [BELL]. The proof uses an important and nontrivial fact (known as “Condition R ” of Bell and Ligocka) about the $\bar{\partial}$ -Neumann problem. We will actually prove Condition R for a strictly pseudoconvex domain in Theorem 4.4.5. (Condition R , and more generally the solution of the $\bar{\partial}$ -Neumann problem, is considered in detail in the book Krantz [KRA4].)

Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^∞ boundary. We define

Condition R (Bell [BEL1]) Define an operator on $L^2(\Omega)$ by

$$Pf(z) = \int_{\Omega} K(z, \zeta) f(\zeta) dV(\zeta),$$

where $K(z, \zeta)$ is the Bergman kernel for Ω . This is the Bergman projection. Then, for each $j > 0$, there is an $m = m(j) > 0$ such that P satisfies the estimates

$$\|Pf\|_{W^j(\Omega)} \leq C_j \|f\|_{W^m(\Omega)}$$

for all testing functions f .

Using a little Sobolev theory (see [KRA4]), one can easily see that this formulation of Condition R is equivalent to the condition that the Bergman kernel map $C^\infty(\overline{\Omega})$ to $C^\infty(\overline{\Omega})$.

The deep fact, which we shall prove in Sect. 4.4, is that *Condition R holds on any strictly pseudoconvex domain*.

In fact we can and should say what is the key idea in establishing this last assertion. Let $P : L^2(\Omega) \rightarrow L^2(\Omega)$. The operator

$$\bar{\partial} : \bigwedge^{0,j} \rightarrow \bigwedge^{0,j+1} \quad (2.1.5)$$

is the usual exterior differential operator of complex analysis. One may show that the second-order, elliptic partial differential operator $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ has a canonical right inverse called N . This is the $\bar{\partial}$ -Neumann operator. These operators are treated in detail in [FOK] and [KRA4]. Then it is a straightforward exercise in Hilbert space theory to verify that

$$P = I - \bar{\partial}^* N \bar{\partial},$$

where P is the Bergman projection. Now the references [FOK] and [KRA4] prove in detail that N maps W^s (the Sobolev space of order s) to W^{s+1} for every s . It follows from this and formula (2.1.5) that P maps W^s to W^{s-1} . That is enough to verify Condition R .

We remark in passing that, in general, it does not matter whether $m(j)$ is much larger than j or whether the $m(j)$ in Condition R depends polynomially on j or exponentially on j . It so happens that, for a strictly pseudoconvex domain, we may take $m(j) = j$. This assertion is proved in [KRA4] in detail. On the other hand, Barrett [BAR1] has shown that, on the Diederich–Fornaess worm domain [DIF1], we must take $m(j) > j$. Later on, Christ [CHR1] showed that Condition R fails altogether on the worm.

Now we build a sequence of lemmas leading to the Fefferman's theorem. First we record some notation.

We let $W^j(\Omega)$ be the usual Sobolev space. See [KRA4] for this idea.

If $\Omega \subset \subset \mathbb{C}^n$ is any smoothly bounded domain and if $j \in \mathbb{N}$, we let

$$WH^j(\Omega) = W^j(\Omega) \cap \{\text{holomorphic functions on } \Omega\},$$

$$WH^\infty(\Omega) = \bigcap_{j=1}^{\infty} WH^j(\Omega) = C^\infty(\overline{\Omega}) \cap \{\text{holomorphic functions on } \Omega\}.$$

Here W^j is the standard Sobolev space on a domain (for which see [KRA4, ADA]). Let $W_0^j(\Omega)$ be the W^j closure of $C_c^\infty(\Omega)$. [Exercise: if j is sufficiently large, then the Sobolev embedding theorem implies trivially that $W_0^j(\Omega)$ is a proper subset of $W^j(\Omega)$.¹]

Let us say that $u, v \in C^\infty(\overline{\Omega})$ agree up to order k on $\partial\Omega$ if

$$\left(\frac{\partial}{\partial z} \right)^\alpha \left(\frac{\partial}{\partial \bar{z}} \right)^\beta (u - v) \Big|_{\partial\Omega} = 0 \quad \forall \alpha, \beta \quad \text{with} \quad |\alpha| + |\beta| \leq k.$$

Lemma 2.1.6. *Let $\Omega \subset \subset \mathbb{C}^n$ be smoothly bounded and strictly pseudoconvex. Let $w \in \Omega$ be fixed. Let K denote the Bergman kernel. There is a constant $C_w > 0$ such that*

$$\|K(w, \cdot)\|_{\sup} \leq C_w.$$

Proof. The function $K(z, \cdot)$ is harmonic. Let $\phi : \Omega \rightarrow \mathbb{R}$ be a radial, C_c^∞ function centered at w . Assume that $\phi \geq 0$ and $\int \phi(\zeta) dV(\zeta) = 1$. Then the mean value property implies that

$$K(z, w) = \int_{\Omega} K(z, \zeta) \phi(\zeta) dV(\zeta).$$

But the last expression equals $P\phi(z)$. Therefore

$$\begin{aligned} \|K(w, \cdot)\|_{\sup} &= \sup_{z \in \Omega} |K(w, z)| \\ &= \sup_{z \in \Omega} |K(z, w)| \\ &= \sup_{z \in \Omega} |P\phi(z)|. \end{aligned}$$

By Sobolev's Theorem, this is

$$\leq C(\Omega) \cdot \|P\phi\|_{WH^{2n+1}}.$$

By Condition R , this is

$$\leq C(\Omega) \cdot \|\phi\|_{W^{m(2n+1)}} \equiv C_w. \quad \square$$

¹For the readers's convenience, we recall here that the Sobolev embedding theorem says that if a function on $\mathbb{R}^N$ has more than $N/2$ derivatives in L^2 , then in fact it has a continuous derivative. See [STE1], for instance, for the details.

Lemma 2.1.7. *Let $u \in C^\infty(\overline{\Omega})$ be arbitrary. Let $s \in \{0, 1, 2, \dots\}$. Then there is a $v \in C^\infty(\overline{\Omega})$ such that $Pv = 0$ and the functions u and v agree to order s on $\partial\Omega$.*

Proof. After a partition of unity, it suffices to prove the assertion in a small neighborhood U of $z_0 \in \partial\Omega$. After a rotation, we may suppose that $\partial\rho/\partial z_1 \neq 0$ on $U \cap \overline{\Omega}$, where ρ is a defining function for Ω . Define the differential operator

$$v = \frac{\operatorname{Re} \left\{ \sum_{j=1}^n \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} \right\}}{\sum_{j=1}^n \left| \frac{\partial \rho}{\partial z_j} \right|^2}.$$

Notice that $v\rho = 1$. Now we define v by induction on s .

For the case $s = 0$, let

$$w_1 = \frac{\rho u}{\partial\rho/\partial\zeta_1}.$$

Define

$$\begin{aligned} v_1 &= \frac{\partial}{\partial\zeta_1} w_1 \\ &= u + O(\rho). \end{aligned}$$

Then u and v_1 agree to order 0 on $\partial\Omega$. Also

$$Pv_1(z) = \int K(z, \zeta) \frac{\partial}{\partial\zeta_1} w_1(\zeta) dV(\zeta).$$

This equals, by integration by parts,

$$- \int \frac{\partial}{\partial\zeta_1} K(z, \zeta) w_1(\zeta) dV(\zeta).$$

Notice that the integration by parts is valid by Lemma 2.1.6 and because $w_1|_{\partial\Omega} = 0$. Also, the integrand in this last line is zero because $K(z, \cdot)$ is conjugate holomorphic.

Suppose inductively that $w_{s-1} = w_{s-2} + \theta_{s-1}\rho^s$ and $v_{s-1} = (\partial/\partial z_1)(w_{s-1})$ have been constructed. We show that there is a w_s of the form

$$w_s = w_{s-1} + \theta_s \cdot \rho^{s+1}$$

such that $v_s = (\partial/\partial z_1)(w_s)$ agrees to order s with u on $\partial\Omega$. By the inductive hypothesis,

$$\begin{aligned}
v_s &= \frac{\partial}{\partial z_1} w_s \\
&= \frac{\partial w_{s-1}}{\partial z_1} + \frac{\partial}{\partial z_1} [\theta_s \cdot \rho^{s+1}] \\
&= v_{s-1} + \rho^s \left[(s+1) \theta_s \frac{\partial \rho}{\partial z_1} + \rho \cdot \frac{\partial \theta_s}{\partial z_1} \right]
\end{aligned}$$

agrees to order $s-1$ with u on $\partial\Omega$ so long as θ_s is smooth. So we need to examine $D(u-v_s)$, where D is an s -order differential operator. But if D involves a tangential derivative D_0 , then write $D = D_0 \cdot D_1$. It follows that $D(u-v_s) = D_0(\alpha)$, where α vanishes on $\partial\Omega$ so that $D_0\alpha = 0$ on $\partial\Omega$. So we need only check $D = v^s$.

We have seen that θ_s must be chosen so that

$$v^s(u-v_s) = 0 \quad \text{on} \quad \partial\Omega.$$

Equivalently,

$$v^s(u-v_{s-1}) - v^s \left(\frac{\partial}{\partial z_1} \right) (\theta_s \rho^{s+1}) = 0 \quad \text{on} \quad \partial\Omega$$

or

$$v^s(u-v_{s-1}) - \theta_s \left(v^s \frac{\partial}{\partial z_1} \rho^{s+1} \right) = 0 \quad \text{on} \quad \partial\Omega$$

or

$$v^s(u-v_{s-1}) - \theta_s \cdot (s+1)! \frac{\partial \rho}{\partial z_1} = 0 \quad \text{on} \quad \partial\Omega.$$

It follows that we must choose

$$\theta_s = \frac{v^s(u-v_{s-1})}{(s+1)! \frac{\partial \rho}{\partial z_1}},$$

which is indeed smooth on U . As in the case $s=0$, it holds that $Pv_s = 0$. This completes the induction and the proof. $\square$

Remark 2.1.8. A retrospection of the proof reveals that we have constructed v by subtracting from u a Taylor-type expansion in powers of ρ . $\square$

Lemma 2.1.9. For each $s \in \mathbb{N}$ we have $WH^\infty(\Omega) \subseteq P(W_0^s(\Omega))$.

Proof. Let $u \in C^\infty(\overline{\Omega})$. Choose v according to Lemma 2.1.7. Then $u-v \in W_0^s$ and $Pu = P(u-v)$. Therefore

$$P(W_0^s) \supseteq P(C^\infty(\overline{\Omega})) \supseteq P(WH^\infty(\Omega)) = WH^\infty(\Omega). \quad \square$$

Henceforth, let Ω_1, Ω_2 be fixed C^∞ strictly pseudoconvex domains in $\mathbb{C}^n$, with K_1, K_2 their Bergman kernels and P_1, P_2 the corresponding Bergman projections. Let $\phi : \Omega_1 \rightarrow \Omega_2$ be a biholomorphic mapping, and let $u = \det \text{Jac}_{\mathbb{C}} \phi$. For $j = 1, 2$, let $\delta_j(z) = \delta_{\Omega_j}(z) = \text{dist}(z, {}^c\Omega_j)$.

Lemma 2.1.10. *For any $g \in L^2(\Omega_2)$, we have*

$$P_1(u \cdot (g \circ \phi)) = u \cdot ((P_2(g)) \circ \phi).$$

Proof. Notice that $u \cdot (g \circ \phi) \in L^2(\Omega_1)$ by change of variables. Therefore

$$\begin{aligned} P_1(u \cdot (g \circ \phi))(z) &= \int_{\Omega_1} K_1(z, \zeta) u(\zeta) g(\phi(\zeta)) dV(\zeta) \\ &= \int_{\Omega_1} u(z) K_2(\phi(z), \phi(\zeta)) \overline{u(\zeta)} u(\zeta) g(\phi(\zeta)) dV(\zeta) \end{aligned}$$

by Proposition 1.1.14. Change of variable now yields

$$\begin{aligned} P_1(u \cdot (g \circ \phi))(z) &= u(z) \int_{\Omega_2} K_2(\phi(z), \xi) g(\xi) dV(\xi) \\ &= u(z) \cdot [(P_2(g)) \circ \phi](z). \end{aligned}$$

□

Lemma 2.1.11. *Let $\psi : \Omega_1 \rightarrow \Omega_2$ be a C^j diffeomorphism that satisfies*

$$\left| \frac{\partial^\alpha \psi}{\partial z^\alpha}(z) \right| \leq C \cdot (\delta_1(z))^{-|\alpha|}, \quad (2.1.11.1)$$

for all multi-indices α with $|\alpha| \leq j \in \mathbb{N}$ and

$$|\nabla \psi^{-1}(w)| \leq C(\delta_2(w))^{-1}. \quad (2.1.11.2)$$

Suppose also that

$$\delta_2(\psi(z)) \leq C \delta_1(z). \quad (2.1.11.3)$$

Then there is a number $J = J(j)$ such that whenever $g \in W_0^{j+J}(\Omega_2)$, then $g \circ \psi \in W_0^j(\Omega_1)$.

Proof. The subscript 0 causes no trouble by the definition of W_0^j . Therefore it suffices to prove an estimate of the form

$$\|g \circ \psi\|_{W_0^j} \leq C \|g\|_{W_0^{j+J}}, \quad \text{all } g \in C_c^\infty(\Omega).$$

By the chain rule and Leibniz's rule, if α is a multi-index of modulus not exceeding j , then

$$\left(\frac{\partial}{\partial z}\right)^\alpha (g \circ \psi) = \sum [(D^\beta g) \circ \psi] \cdot D^{\gamma_1} \psi \cdots D^{\gamma_\ell} \psi,$$

where $|\beta| \leq |\alpha|$, $\sum |\gamma_i| \leq |\alpha|$, and the number of terms in the sum depends only on α (a classical formula of Faà de Bruno—see [ROM]—actually gives this sum quite explicitly, but we do not require such detail). Note here that $D^{\gamma_i} \psi$ is used to denote a derivative of *some component* of ψ . By hypothesis, it follows that

$$\left| \left(\frac{\partial}{\partial z}\right)^\alpha (g \circ \psi) \right| \leq C \sum |(D^\beta g) \circ \psi| \cdot (\delta_1(z))^{-j}.$$

Therefore

$$\begin{aligned} \int_{\Omega_1} \left| \left(\frac{\partial}{\partial z}\right)^\alpha (g \circ \psi) \right|^2 dV &\leq \\ &\leq C \sum \int_{\Omega_1} |(D^\beta g) \circ \psi|^2 (\delta_1(z))^{-2j} dV(z) \\ &= C \sum \int_{\Omega_2} |D^\beta g(w)|^2 \delta_1(\psi^{-1}(w))^{-2j} \\ &\quad \times |\det J_{\mathbb{C}} \psi^{-1}|^2 dV(w). \end{aligned}$$

But (2.1.11.2) and (2.1.11.3) imply that the last line is majorized by

$$C \sum \int_{\Omega_2} |D^\beta g(w)|^2 \delta_2(w)^{-2j} \delta_2(w)^{-2n} dV(w). \quad (2.1.11.4)$$

Now if J is large enough, depending on the Sobolev embedding Theorem, then

$$|D^\beta g(w)| \leq C \|g\|_{W_0^{j+J}} \cdot \delta_2(w)^{2n+2j}.$$

(Remember that g is compactly supported in Ω_2 .) Hence, (2.1.11.4) is majorized by $C \|g\|_{W_0^{j+J}}$. $\square$

Lemma 2.1.12. *For each $j \in \mathbb{N}$, there is an integer J so large that if $g \in W_0^{j+J}(\Omega_2)$, then $g \circ \phi \in W_0^j(\Omega_1)$.*

Proof. The Cauchy estimates give (since ϕ is bounded) that

$$\left| \frac{\partial^\alpha \phi_\ell}{\partial z^\alpha}(z) \right| \leq C \cdot (\delta_1(z))^{-|\alpha|}, \quad \ell = 1, \dots, n \quad (2.1.12.1)$$

and

$$|\nabla \phi^{-1}(w)| \leq C(\delta_2(w))^{-1}, \quad (2.1.12.2)$$

where $\phi = (\phi_1, \dots, \phi_n)$. We will prove that

$$C \cdot \delta_1(z) \geq \delta_2(\phi(z)). \quad (2.1.12.3)$$

Then Lemma 2.1.9 gives the result.

To prove (2.1.12.3), let ρ be a smooth strictly plurisubharmonic defining function for Ω_1 . Then $\rho \circ \phi^{-1}$ is a smooth plurisubharmonic function on Ω_2 . Since ρ vanishes on $\partial\Omega_1$ and since ϕ^{-1} is proper, we conclude that $\rho \circ \phi^{-1}$ extends continuously to $\overline{\Omega}_2$. If $P \in \partial\Omega_2$ and ν_P is the unit outward normal to $\partial\Omega_2$ at P , then Hopf's lemma implies that the (lower) one-sided derivative $(\partial/\partial\nu_P)(\rho \circ \phi^{-1})$ satisfies

$$\frac{\partial}{\partial\nu_P}(\rho \circ \phi^{-1}(P)) \geq C.$$

So, for $w = P - \epsilon\nu_P$, ϵ small, it holds that

$$-\rho \circ \phi^{-1}(w) \geq C \cdot \delta_2(w).$$

These estimates are uniform in $P \in \partial\Omega_2$. Using the comparability of $|\rho|$ and δ_1 yields

$$C\delta_1(\phi^{-1}(w)) \geq \delta_2(w).$$

Setting $z = \phi^{-1}(w)$ now gives

$$C'\delta_1(z) \geq \delta_2(\phi(z)),$$

which is (2.1.12.3). □

Exercise for the Reader: Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain. Let $j \in \mathbb{N}$. There is an $N = N(j)$ so large that $g \in W_0^N$ implies that g vanishes to order j on $\partial\Omega$.

Lemma 2.1.13. *The function u is in $C^\infty(\overline{\Omega}_1)$.*

Proof. It suffices to show that $u \in W^j(\Omega_1)$, every j . So fix j . Let $m = m(j)$ as in Condition R. According to (2.1.12.1), $|u(z)| \leq C\delta_1(z)^{-2n}$. Then, by Lemma 2.1.12 and the Exercise for the Reader following it, there is a J so large

that $g \in W_0^{m+J}(\Omega_2)$ implies $u \cdot (g \circ \phi) \in W^m(\Omega_1)$. Choose, by Lemma 2.1.7, a $g \in W_0^{m+J}(\Omega_2)$ such that $P_2g \equiv 1$. Then Lemma 2.1.10 yields

$$P_1(u \cdot (g \circ \phi)) = u.$$

By Condition R , it follows that $u \in W^j(\Omega_1)$. □

Lemma 2.1.14. *The function u is bounded from 0 on $\overline{\Omega}_1$.*

Proof. By symmetry, we may apply Lemma 2.1.13 to ϕ^{-1} and $\det J_{\mathbb{C}}(\phi^{-1}) = 1/u$. We conclude that $1/u \in C^\infty(\overline{\Omega}_2)$. Thus u is nonvanishing on $\overline{\Omega}$. □

Proof of the Fefferman's Theorem (Theorem 2.1.3): Use the notation of the proof of Lemma 2.1.12. Choose $g_1, \dots, g_n \in W_0^{m+J}(\Omega_2)$ such that $P_2g_i(w) = w_i$ (here w_i is the i^{th} coordinate function). Then Lemma 2.1.10 yields that $u \cdot \phi_i \in W^j(\Omega_1)$, $i = 1, \dots, n$. By Lemma 2.1.12, $\phi_i \in W^j(\Omega_1)$, $i = 1, \dots, n$. By symmetry, $\phi^{-1} \in W^j(\Omega_2)$. Since j is arbitrary, the Sobolev embedding theorem finishes the proof. □

It is important to understand the central role of Condition R in this proof. With some emendations, the proof we have presented shows that, if $\Omega_1, \Omega_2 \subset \subset \mathbb{C}^n$ are smoothly bounded, pseudoconvex, and both satisfy Condition R , then a biholomorphic mapping from Ω_1 to Ω_2 extends smoothly to the boundary (in fact Bell [BEL1] has shown that it suffices for just one of the domains to satisfy Condition R). Condition R is known to hold on domains that have real analytic boundaries (see Diederich and Fornæss [DIF2]), and more generally on domains of finite type (see [CAT1, CAT2]). There are a number of interesting examples of non-pseudoconvex domains on which Condition R fails (see Barrett [BAR1] and Kiselman [KIS]). It had been conjectured that Condition R holds on all smoothly bounded pseudoconvex domains. But Christ [CHR1] showed that in fact Condition R fails on the Diederich–Fornæss worm domain (which is smoothly bounded and pseudoconvex).

Lempert [LEM1] has derived a sharp boundary regularity result for biholomorphic mappings of strictly pseudoconvex domains with C^k boundary. Pinchuk and Tsyganov [PIT] have an analogous result. The correct conclusion turns out to be that there is a loss of smoothness in some directions. So the sharp regularity result is formulated in terms of nonisotropic spaces. It is too technical to describe here.

2.2 Boundary Behavior of the Bergman Metric

The Bergman metric on the disc D is given by

$$g_{jk} = \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log K(z, \bar{z}) = \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} [-\log \pi - 2 \log(1 - |z|^2)] = \frac{2}{(1 - |z|^2)^2}.$$

Thus the length of a curve $\gamma : [0, 1] \rightarrow D$ is given by

$$\ell_B(\gamma) = \int_0^1 \frac{2\|\gamma'(t)\|}{(1 - |\gamma(t)|^2)} dt .$$

It is natural to wonder what one can say about the Bergman metric on a more general class of domains. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with C^4 boundary, and suppose for the moment that Ω is simply connected. Then the Riemann mapping theorem tells us that there is a conformal mapping $\Phi : \Omega \rightarrow D$. Of course then we know that

$$K_\Omega(z, z) = |\Phi'(z)|^2 K_D(\Phi(z), \Phi(z)) = \frac{|\Phi'(z)|^2}{\pi \cdot (1 - |\Phi(z)|^2)^2} .$$

Then the Bergman metric on Ω is given by

$$\begin{aligned} g_{jk} &= \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log \left[\frac{|\Phi'(z)|^2}{\pi \cdot (1 - |\Phi(z)|^2)^2} \right] \\ &= \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \left[\log |\Phi'(z)|^2 - \log \pi - 2 \log(1 - |\Phi(z)|^2) \right] . \end{aligned}$$

Of course $|\Phi'(z)|$ is bounded and bounded from 0 (see [BEK]). Also the derivatives of Φ , up to order 3, are bounded. So the second derivative of $\log |\Phi'(z)|^2$ is a bounded term. The second derivative of $\log \pi$ is of course 0.

The second derivative of the remaining (and most interesting) term may be calculated to be

$$\begin{aligned} \frac{\partial^2}{\partial z \partial \bar{z}} \log K_\Omega(z, z) &= \frac{\partial^2}{\partial \bar{z} \partial z} \left[-2 \log(1 - |\Phi(z)|^2) \right] \\ &= \frac{\partial}{\partial \bar{z}} \left(\frac{-2}{1 - |\Phi(z)|^2} \cdot [-\Phi'(z) \bar{\Phi}(z)] \right) \\ &= \frac{2}{(1 - |\Phi(z)|^2)^2} \cdot [-\bar{\Phi}'(z) \Phi(z)] \cdot [-\Phi'(z) \bar{\Phi}(z)] \\ &\quad - \frac{2}{1 - |\Phi(z)|^2} \cdot [-\Phi'(z) \bar{\Phi}'(z)] . \end{aligned}$$

This in turn, after some simplification, equals

$$\frac{2|\Phi'(z)|^2}{(1 - |\Phi(z)|^2)^2} . \tag{2.2.1}$$

As previously noted, the numerator is bounded and bounded from 0. Hopf's lemma (see [KRA1]) tells us that $(1 - |\Phi(z)|^2) \approx \text{dist}_{\partial\Omega}(z)$. So that the Bergman metric on Ω blows up like the reciprocal of the square of the distance to the boundary—just as on the disc.

In the case that Ω has C^2 boundary and is finitely connected—not necessarily simply connected—then one may use the Ahlfors map (see [KRA5]) instead of the Riemann mapping and obtain a result similar to that in (2.2.1). We omit the details.

Bremermann [BRE] showed that any domain in $\mathbb{C}^n$ with complete Bergman metric is a domain of holomorphy. This is in fact not difficult, as one can use the hypothesis of the completeness of the metric to confirm the *Kontinuitätssatz* (see [KRA1]), hence derive pseudoconvexity. Bremermann also gave an example to demonstrate that the converse is not true. However, Ohsawa [OHS] has shown that any pseudoconvex domain with C^1 boundary has complete Bergman metric. This result is important both conceptually and practically. There is no analogous result for either the Carathéodory or Kobayashi metrics.

In the paper [KOB1, Theorem 9.2], Kobayashi shows that any bounded analytic polyhedron has complete Bergman metric.

2.3 The Biholomorphic Inequivalence of the Ball and the Polydisc

In this section we give a Bergman-geometric proof of the following classical result of Poincaré (Poincaré’s original proof was more group theoretic).

Theorem 2.3.1. *There is no biholomorphic map of the bidisc $D^2(0, 1)$ to the ball $B(0, 1) \subseteq \mathbb{C}^2$.*

Proof. Suppose, seeking a contradiction, that there is such a map. Since Möbius transformations act transitively on the disc, pairs of them act transitively on the bidisc. Therefore we may compose ϕ with a self-map of the bidisc and assume that ϕ maps 0 to 0.

If $Y \in \partial B$, then the disc $d_Y = \{z \in B : z = \zeta Y, \zeta \in \mathbb{C}, |\zeta| < 1\}$ is a totally geodesic submanifold of B (informally, this means that if P, Q are points of d_Y , then the geodesic connecting them in the Riemannian manifold d_Y is the same as the geodesic connecting them in the Riemannian manifold B —see Kobayashi and Nomizu [KON]).

By our discussion in the calculation of the Poincaré metric, we may conclude that the geodesics, or paths of least length, emanating from the origin in the ball are the rays $\tau_Y : t \mapsto tY$. (This assertion may also be derived from symmetry considerations.)

Likewise, if $\alpha, \beta \in \mathbb{C}, |\alpha| = 1, |\beta| = 1$, then the disc $e_0 = \{(\zeta\alpha, \zeta\beta) : \zeta \in D\} \subseteq D^2(0, 1)$ is a totally geodesic submanifold of $D^2(0, 1)$. Again we may apply our discussion of the Poincaré metric on the disc to conclude that the geodesic curve emanating from the origin in the bidisc in the direction $X = (\alpha, \beta)$ is $\psi_{\alpha\beta} : t \mapsto tX$. A similar argument shows that the curve $t \mapsto (t, 0)$ is a geodesic in the bidisc.

Now if $t \mapsto tX$ is one of the above-mentioned geodesics on the bidisc, then it will be mapped under ϕ to a geodesic $t \mapsto tY$ in the ball. If $0 < t_1 < t_2 < 1$, then the points $t_1X, t_2X \in D^2$ will be mapped to points $t'_1Y, t'_2Y \in B$ and it must be that

$0 < t'_1 < t'_2 < 1$ since ϕ is an isometry hence must map the point t_2X to a point further from the origin than it maps t_1X (because t_2X is further from the origin than t_1X). It follows that the limit

$$\lim_{t \rightarrow 1^-} \phi(tX)$$

exists for every choice of X and the limit lies in ∂B . After composing ϕ with a rotation, we may suppose that $\{\phi(t(1, 0))\}$ terminates at $(1, 0)$.

Now consider the function $f(z_1, z_2) = (z_1 + 1)/2$ on B . This function has the property that $f(1, 0) = 1$, f is holomorphic on a neighborhood of $\overline{B}$, and $|f(z)| < 1$ for $z \in \overline{B} \setminus \{(1, 0)\}$. For $0 < r < 1$ we invoke the mean-value property for a harmonic function to write

$$\frac{1}{2\pi} \int_0^{2\pi} f \circ \phi(r, re^{i\theta}) d\theta = f \circ \phi(r, 0). \quad (2.3.1.1)$$

As $r \rightarrow 1^-$ the right-hand side tends to $\lim_{t \rightarrow 1^-} f(t, 0) = 1$. However, each of the paths $r \rightarrow (r, re^{i\theta})$ is a geodesic in the bidisc, as discussed above, and for different $\theta \in [0, 2\pi)$ they are distinct. Thus the curves $r \rightarrow \phi(r, re^{i\theta})$ have distinct limits in ∂B , and these limits will be different from the point $(1, 0) \in \partial B$. In particular, $\lim_{r \rightarrow 1^-} f \circ \phi(r, re^{i\theta})$ exists for each $\theta \in [0, 2\pi)$ and assumes a value of modulus strictly less than 1.

By the Lebesgue dominated convergence theorem, we may pass to the limit as $r \rightarrow 1^-$ in the left side of (2.3.1.1) to obtain a limit that must be strictly less than one in absolute value. That is the required contradiction. $\square$

Exercises

1. Let us construct an invariant metric on the disc D in the complex plane by hand. Take it that the length of the vector $\langle 1, 0 \rangle$ at the base point 0 is 1. Now use the conformal invariance of the metric to calculate the length of any other vector at any other base point. Of course the metric that you obtain in this way should be (a constant multiple of) the Poincaré metric.
2. Imitate Exercise 1 for the unit ball in $\mathbb{C}^n$.
3. Consider the domain $\Omega = \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta < 0 \text{ or } \operatorname{Im} \zeta < 0\}$. This domain is conformally equivalent to the upper half plane, which is in turn conformally equivalent to the unit disc. Therefore one can calculate the Bergman kernel for Ω using its mapping invariance property. What can you say about the boundary behavior of this kernel at the origin? What can you say about the boundary behavior of the Bergman metric at the origin?
4. Calculate the Bergman kernel and metric for the punctured disc $D' \equiv \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$. How do they differ from the Bergman kernel and metric for the usual disc D ?

Calculate the Bergman kernel and metric for the punctured ball $B' \equiv \{z \in \mathbb{C}^n : 0 < |z| < 1\}$. How do they differ from the Bergman kernel and metric for the usual ball B ?

5. Use the results from the text about the Bergman kernel for the annulus to calculate the Bergman metric on the annulus.
6. Let $\Omega = B(0, 1) \setminus B(1, 1/2)$, where $B(P, r)$ denotes the open ball with center P and radius r . Estimate the distance from the origin to the point $(1/2, 0)$ in the Bergman metric of Ω .
7. The infinitesimal Kobayashi–Royden metric on a domain $\Omega \subseteq \mathbb{C}^n$ is defined by $F_K : \Omega \times \mathbb{C}^n \rightarrow \mathbb{R}$, where

$$\begin{aligned}
 F_K(z, \xi) &\equiv \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(B) \\
 &\quad \text{with } f(0) = z, (f'(0))(e_1) = \xi/\alpha\} \\
 &= \inf\left\{ \frac{|\xi|}{|(f'(0))(e_1)|} : f \in \Omega(B), (f'(0))(e_1) \text{ is a} \right. \\
 &\quad \left. \text{constant multiple of } \xi\right\} \\
 &= \frac{|\xi|}{\sup\{|(f'(0))(e_1)| : f \in \Omega(B), (f'(0))(e_1) \text{ is a constant multiple of } \xi\}}.
 \end{aligned}$$

Use the Schwarz lemma to calculate the Kobayashi–Royden metric on the disc. How does it compare with the Poincaré metric? Now calculate the Kobayashi–Royden metric on the ball.

8. The infinitesimal Carathéodory metric on a domain $\Omega \subseteq \mathbb{C}^n$ is defined by $F_C : \Omega \times \mathbb{C}^n \rightarrow \mathbb{R}$, where

$$F_C(z, \xi) = \sup_{\substack{f \in B(\Omega) \\ f(z)=0}} |f_*(z)\xi| \equiv \sup_{\substack{f \in B(\Omega) \\ f(z)=0}} \left| \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) \cdot \xi_j \right|.$$

Use the Schwarz lemma to calculate the Carathéodory metric on the disc. How does it compare with the Poincaré metric? Now calculate the Carathéodory metric on the ball.

9. Refer to Exercises 7 and 8 for terminology. Prove that the Carathéodory metric is always less than or equal to the Kobayashi metric.
10. Refer to Exercises 7 and 8 for terminology. Prove that, if $\Phi : \Omega_1 \rightarrow \Omega_2$ is a holomorphic mapping of complex domains, then its distance is non-increasing in the Carathéodory metric. Prove an analogous statement for the Kobayashi–Royden metric.

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