

Preface

The Bergman kernel and metric have been a seminal part of geometric analysis and partial differential equations since their invention by Stefan Bergman in 1922. Applications to holomorphic mappings, to function theory, to partial differential equations, and to differential geometry have kept the techniques plugged into the mainstream of mathematics for 90 years.

The Bergman kernel is based on a very simple idea: that the square-integrable holomorphic functions on a bounded domain in that complex space form a Hilbert space. Moreover, a simple formal argument shows that Hilbert space possesses a so-called reproducing kernel. This is an integration kernel which reproduces each element of the space. The kernel has wonderful invariance properties, leading to the Bergman metric. The Bergman kernel and metric have developed into powerful tools for function theory, analysis, differential geometry, and partial differential equations. The purpose of this book is to exposit this theory (particularly in the context of several complex variables), examine its key features, and bring the reader up to speed with some of the latest developments.

Bergman wrote several books about his kernel and contributed mightily to its development. The idea caught on widely, and the kernel became a standard device in the field. The Bergman metric was the first-ever Kähler metric, and that in turn spawned the vital subject of complex differential geometry. Ahlfors (in one variable), Chern (in several variables), Greene–Wu, and many others played a decisive role in this development.

Bergman’s ideas received a major boost in the 1970s when Charles Fefferman did his Fields Medal-winning work on the boundary behavior of biholomorphic mappings. The key device in his analysis was the Bergman kernel and metric. Since then, a myriad of workers, from Bell and Ligocka to Webster to Greene–Krantz, Krantz–Li, Kim–Krantz, Isaev–Krantz, and many others, have developed and extended Bergman’s theory. It is now part of the lingua franca of complex analysis, and the technique of reproducing kernels which it has spawned is part of every analyst’s toolkit.

Fefferman’s work inspired many others to examine the utility of the Bergman theory in the study of biholomorphic mappings. Bell’s condition R , formulated

in terms of regularity properties of the Bergman projection, has proved to be an influential and powerful weapon in the subject. In turn, condition R can be formulated in terms of the regularity theory of subelliptic partial differential equations, and this connection has had a key influence on the directions of research.

The Bergman metric was the first “universal” (in the sense that it can be constructed on virtually any domain) example of a metric that is invariant under biholomorphic mappings. [The Poincaré metric was of course the primordial example on the disc.] Today there are the Kobayashi–Royden metric, the Sibony metric, the Carathéodory metric, and many others. This is a useful tool in geometric analysis and function theory.

The connections of the Bergman kernel with partial differential equations, especially the extremal properties of the kernel and metric, are profound. Bergman himself explored applications of his theory to elliptic partial differential equations. Today we see the Bergman kernel as inextricably linked with the $\bar{\partial}$ -Neumann problem. This link played a vital role in Fefferman’s work and later proved crucial to Greene–Krantz and many of the other workers in the subject. Certainly Donald Spencer and J. J. Kohn were the pioneers of this symbiosis. Today the interaction is prospering. Another development is that the Bergman theory enjoys connections with the Monge–Ampère equation. That nonlinear partial differential equation contains important information about biholomorphic mappings and about the construction of geometries.

Connections with harmonic analysis are another exciting, and relatively new, aspect of the Bergman paradigm. Coifman–Rochberg–Weiss used the Bergman kernel in their proof of the H^1/BMO duality theorem on the ball, and Krantz–Li exploited it further in their study on strictly pseudoconvex domains. Many of the natural artifacts of harmonic analysis—including approach regions for Fatou theorems—are most propitiously formulated in terms of Bergman geometry or the boundary asymptotics of the kernel. The Bergman kernel is now a standard artifact of the harmonic analysis of several complex variables.

This text will in fact be a thoroughgoing treatment of all the basic analytic and geometric aspects of Bergman’s theory. This will include

- Definitions of the Bergman kernel
- Definition and basic properties of the Bergman metric
- Calculation of the Bergman kernel and metric
- Invariance properties of the kernel and metric
- Boundary asymptotics of the kernel and metric
- Asymptotic expansions for the Bergman kernel
- Applications to function theory
- Applications to geometry
- Applications to partial differential equations
- Interpretations in terms of functional analysis
- The geometry of the Bergman metric
- Curvature of the Bergman metric
- The Bergman kernel and metric on manifolds

There are a few recent treatises on the Bergman kernel, notably those by Hedenmalm–Korenblum–Zhu and Duren–Schuster. But these books concentrate on the one-variable theory; they are also oriented towards the functional analysis aspects of the Bergman kernel. Our focus instead is the geometry of several complex variables and the contexts of real analysis, complex analysis, harmonic analysis, and differential geometry. This puts Bergman’s ideas into a much broader arena and provides many more opportunities for applications and illustrations. We will certainly touch on the functional analysis properties of the Bergman projection, but these will not be our main focus. We shall also cover selected topics of the one complex variable theory. There is little overlap between this book and the two books cited above.

We would also be remiss not to mention the book of Ma and Marinescu on the Kähler geometry aspects of the Bergman theory. Certainly the Bergman metric was the very first Kähler metric, and this in turn has spawned the active and fruitful area of multivariable complex differential geometry. Various parts of the present book touch on this Kähler theory.

Lurking in the background behind Fefferman’s biholomorphic mapping theorem were Bergman representative coordinates—yet another outgrowth of the Bergman kernel and metric. This is a much-underappreciated aspect of the theory and one that we shall treat in detail in the text. In fact there are many aspects of the Bergman theory that tend only to be known to experts and are not readily accessible in the literature. We intend to treat many of those. Several of the topics in this text appear here for the first time in book form.

We intend this to be a book for students as well as seasoned researchers. All needed background will be provided. The reader is only assumed to have had a solid course in complex variables and some basic background in real and functional analysis. A little exposure to geometry will be helpful, but is not a requirement. There are many illustrative examples and some useful figures. The book abounds with useful and instructive calculations, many of which cannot be found elsewhere. Every chapter ends with a selection of exercises, which should serve to help the reader get more directly involved in the subject matter. It will cause him/her to consult the literature, to calculate, and to learn by doing.

The book will help the novice reader to see how analysis is used in practice and how it can be evolved into a seminal tool for research. It is important for the student to see fundamental mathematics used *in vitro* in order to understand how research develops and grows.

It is a pleasure to thank E. M. Stein for introducing me to the Bergman kernel and Robert E. Greene for teaching me the geometric aspects. I have had many collaborators in my study of the kernel, and I offer them all my gratitude. I thank my editor, Elizabeth Loew, for her constant enthusiasm and support. And I thank the several referees for this book, who contributed a wealth of ideas and information.

Geometric Analysis of the Bergman Kernel and Metric

Krantz, S.G.

2013, XIII, 292 p. 7 illus., Hardcover

ISBN: 978-1-4614-7923-9