Chapter 2
Nonlinear Monotone Stochastic Partial Differential Equations

2.1 Solutions of Monotone Stochastic Equations

Let $\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^*$ be a Gelfand triple, i.e., $(\mathbb{H}, \langle\cdot,\cdot\rangle, |\cdot|)$ is a separable Hilbert space, $\mathbb{V}$ is a reflexive Banach space continuously and densely embedded into $\mathbb{H}$, and $\mathbb{V}^*$ is the duality of $\mathbb{V}$ with respect to $\mathbb{H}$. Let ${}_{\mathbb{V}^*}\langle\cdot,\cdot\rangle_{\mathbb{V}}$ be the dualization between $\mathbb{V}$ and $\mathbb{V}^*$. We have ${}_{\mathbb{V}^*}\langle u,v\rangle_{\mathbb{V}} = \langle u,v\rangle$ for $u \in \mathbb{H}$ and $v \in \mathbb{V}$. Let $\mathscr{L}(\mathbb{H})$ (respectively $\mathscr{L}_b(\mathbb{H})$, $\mathscr{L}_{HS}(\mathbb{H})$) be the set of all densely defined (respectively bounded, Hilbert–Schmidt) linear operators on $\mathbb{H}$. Let $\|\cdot\|$ and $\|\cdot\|_{HS}$ denote the operator norm and the Hilbert–Schmidt norm respectively.

Let $W = (W(t))_{t\ge 0}$ be the cylindrical Brownian motion on $\mathbb{H}$ (see the beginning of Sect. 1.2) with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$. Consider the following stochastic equation:

$$\mathrm{d}X(t) = b(t,X(t))\mathrm{d}t + \sigma(t,X(t))\mathrm{d}W(t), \tag{2.1}$$

where

$$b : [0,\infty) \times \mathbb{V} \times \Omega \to \mathbb{V}^*, \quad \sigma : [0,\infty) \times \mathbb{V} \times \Omega \to \mathscr{L}_{HS}(\mathbb{H})$$

are progressively measurable and satisfy the following assumptions for some constant $\alpha > 0$, adapted $\phi \in L^1_{loc}([0,\infty) \to L^1(\mathbb{P}); \mathrm{d}t)$, $K \in C([0,\infty))$, and strictly positive $\psi \in C([0,\infty))$:

(A2.1) Hemicontinuity. For every $t \ge 0$ and $v_1, v_2, v \in \mathbb{V}$,

$$\mathbb{R} \ni s \mapsto {}_{\mathbb{V}^*}\langle b(t, v_1 + s v_2), v\rangle_{\mathbb{V}}$$

is continuous.

(A2.2) Monotonicity. For every $v_1, v_2 \in \mathbb{V}, t \ge 0$,

$$2\,{}_{\mathbb{V}^*}\langle b(t,v_1) - b(t,v_2), v_1 - v_2\rangle_{\mathbb{V}} + \|\sigma(t,v_1) - \sigma(t,v_2)\|_{HS}^2 \le K(t)|v_1 - v_2|^2.$$

F.-Y. Wang, *Harnack Inequalities for Stochastic Partial Differential Equations*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4614-7934-5_2, © Feng-Yu Wang 2013

(A2.3) Coercivity. For every $t \geq 0, v \in \mathbb{V}$,

$$2_{\mathbb{V}^*}\langle b(t,v),v\rangle_{\mathbb{V}} + \|\sigma(t,v)\|_{HS}^2 \leq \phi(t) + K(t)|v|^2 - \psi(t)\|v\|_{\mathbb{V}}^{\alpha+1}.$$

(A2.4) Growth. For every $u, v \in \mathbb{V}, t \geq 0$,

$$|_{\mathbb{V}^*}\langle b(t,v),u\rangle_{\mathbb{V}}| \leq \phi(t) + K(t)\{\|v\|_{\mathbb{V}}^{\alpha} + \|u\|_{\mathbb{V}}^{\alpha+1} + |u|^2 + |v|^2\}.$$

Definition 2.1.1 *A continuous $\mathbb{H}$-valued adapted process X is called a (strong or variational) solution to (2.1) if*

$$\mathbb{E}\int_0^T \big(\|X(t)\|_{\mathbb{V}}^{\alpha+1} + |X(t)|^2\big)\mathrm{d}t < \infty, \ \ T > 0,$$

and $\mathbb{P}$-a.s.

$$X(t) = X(0) + \int_0^t b(s,X(s))\mathrm{d}s + \int_0^t \sigma(s,X(s))\mathrm{d}W(s), \ \ t \geq 0.$$

According to [24, Theorems II.2.1, II.2.2], for every $\mathscr{F}_0$-measurable $X(0) \in L^2(\Omega \to \mathbb{H};\mathbb{P})$, (2.1) has a unique solution $X = (X(t))_{t\geq 0}$; see also [40, Theorem 2.1] for

$$\mathbf{K} := L^{1+\alpha}([0,T]\times\Omega \to \mathbb{V};\mathrm{d}t\times\mathbb{P}) \cap L^2([0,T]\times\Omega \to \mathbb{H};\mathrm{d}t\times\mathbb{P}).$$

Moreover, $|X|^2$ satisfies the Itô formula

$$\mathrm{d}|X(t)|^2 = 2_{\mathbb{V}^*}\langle b(t,X(t)),X(t)\rangle_{\mathbb{V}}\mathrm{d}t + \|\sigma(t,X(t))\|_{HS}^2\mathrm{d}t + 2\langle X(t),\sigma(t,X(t))\mathrm{d}W(t)\rangle,$$

and hence

$$\mathbb{E}\sup_{t\in[0,T]}|X(t)|^2 + \mathbb{E}\int_0^T \|X(t)\|_{\mathbb{V}}^{\alpha+1}\mathrm{d}t < \infty, \ \ T > 0.$$

The study of (2.1) with the above assumptions goes back to [37, 38], and such equations are called (nonlinear) monotone SPDEs. A typical example is that of the following stochastic generalized porous media and fast-diffusion equations. Generalizations with local conditions or for nonmonotone equations can be found in [29, 30, 42].

Example 2.1.1 (Stochastic generalized porous media/fast-diffusion equations) *Let $(\Delta, \mathscr{D}(\Delta))$ be the Dirichlet Laplacian on a bounded open domain $D \subset \mathbb{R}^d$. Let $\alpha > 0$ be a constant. Consider the following PDE (partial differential equation):*

$$\partial_t u = \Delta u^{\alpha}, \ \ t \geq 0, u(t,\cdot)|_{\partial D} = 0,$$

where $u^{\alpha} := |u|^{\alpha}\mathrm{sign}(u)$. This equation is called the (Dirichlet) heat equation if $\alpha = 1$, the fast-diffusion equation if $\alpha \in (0,1)$, and the porous medium equation if $\alpha > 1$. Since $\Delta \leq -\lambda_1$ for some $\lambda_1 > 0$, $\mathscr{D}(\sqrt{-\Delta})$ under the inner product

$$\langle u,v\rangle_{\mathscr{D}} := \int_D \big(\sqrt{-\Delta}u\big)\big(\sqrt{-\Delta}v\big)\mathrm{d}\mathbf{m}$$

is a separable Hilbert space, known as the (first-order) Sobolev space associated to Δ, where $\mathbf{m}$ is the normalized volume measure on D. Let $\mathbb{H}$ be the dual space of $\mathscr{D}(\sqrt{-\Delta})$ with respect to $L^2(\mathbf{m})$, and let $(W(t))_{\geq 0}$ be a cylindrical Brownian motion on $\mathbb{H}$. Consider the SPDE

$$\mathrm{d}X(t) = \Delta X(t)^\alpha \mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

where $\sigma : [0,\infty) \to \mathscr{L}_{HS}(\mathbb{H})$ is measurable such that $\|\sigma(\cdot)\|_{HS}^2 \in L^1_{loc}([0,\infty);\mathrm{d}t)$. Let $\mathbb{V} = \mathbb{H}\cap L^{\alpha+1}(\mathbf{m})$ be equipped with $\|\cdot\|_{\mathbb{V}} := |\cdot| + \|\cdot\|_{L^{\alpha+1}(\mathbf{m})}$. By Sobolev's inequality, if $\alpha \geq \frac{d-2}{d+2}$, then $\mathbb{V} = L^{\alpha+1}(\mathbf{m})$. It is easy to see that **(A2.1)–(A2.4)** *hold for $K = \psi = 1$ and $\phi = \|\sigma(\cdot)\|_{HS}^2$. Therefore, for every $X(0) \in L^2(\Omega \to \mathbb{H};\mathbb{P})$, this equation has a unique solution.*

More generally, let $(E,\mathscr{B},\mathbf{m})$ be a probability measure space, and $(L,\mathscr{D}(L))$ a self-adjoint operator on $L^2(\mathbf{m})$ with $L \leq -\lambda_1$ for some constant $\lambda_1 > 0$. Let $\mathbb{H}$ be the dual space of $\mathscr{D}(\sqrt{-L})$ with respect to $L^2(\mathbf{m})$. Let $\mathbb{V} = L^{\alpha+1}(\mathbf{m})\cap\mathbb{H}$, and let $W(t)$ be a cylindrical Brownian motion on $\mathbb{H}$. Consider the following SPDE on $\mathbb{H}$:

$$\mathrm{d}X(t) = \big\{L\Psi(t,X(t)) + \Phi(t,X(t))\big\}\mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

where $\sigma : [0,\infty) \to \mathscr{L}_{HS}(\mathbb{H})$ is measurable with $\|\sigma(\cdot)\|_{HS}^2 \in L^1_{loc}([0,\infty);\mathrm{d}t)$, and $\Psi,\Phi : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ are measurable and continuous in the second variable, satisfying the condition that for some functions $\delta,\zeta,\gamma,h \in C([0,\infty))$ with $\delta > 0$,

$$|\Psi(t,s)| + |\Phi(t,s) - h(t)s| \leq \zeta(t)(1+|s|^\alpha), \quad s\in\mathbb{R}, t\geq 0, \tag{2.2}$$

$$\begin{aligned} &2\langle\Psi(t,x)-\Psi(t,y),y-x\rangle_2 + 2\langle\Phi(t,x)-\Phi(t,y),(-L)^{-1}(x-y)\rangle_2 \\ &\quad \leq -\delta(t)\|x-y\|_{L^{\alpha+1}(\mathbf{m})}^{\alpha+1} + \gamma(t)|x-y|^2, \quad x,y\in L^{\alpha+1}(\mathbf{m}),\ t\geq 0, \end{aligned} \tag{2.3}$$

where here and in the sequel, $\langle\cdot,\cdot\rangle_2$ denotes the inner product in $L^2(\mathbf{m})$. A very simple example of Ψ,Φ satisfying these conditions is $\Psi(t,s) := |s|^{\alpha-1}s$ and $\Phi(t,s) = \gamma(t)s$. If $\Phi(t,s)$ is nonlinear in s, we also assume that L^{-1} is bounded in $L^{\alpha+1}(\mathbf{m})$, which is the case if L is a Dirichlet operator. Under these conditions, **(A2.1)–(A2.4)** hold for $b(t,v) := L\Psi(t,v) + \Phi(t,v)$ and some ϕ,ψ,K. Thus, for every $X(0)\in L^2(\Omega\to\mathbb{H};\mathbb{P})$, the equation has a unique solution. A simple example for this condition to hold is that $\Psi(\cdot,s) = \eta(\cdot)s^\alpha$ for some strictly positive $\eta\in C([0,\infty))$. See Sect. 3 in [40, 55] for more general results where $L^{\alpha+1}(\mathbf{m})$ is replaced by an Orlicz space.

The next example can be found in [27, 39].

Example 2.1.2 (Stochastic p-Laplace equation) *Let $D\subset\mathbb{R}^d$ be an open domain, let $\mathbf{m}$ be the normalized volume measure on D, and let $p\geq 2$ be a constant. Let $\mathbb{H}_0^{1,p}(D)$ be the closure of $C_0^\infty(D)$ with respect to the norm*

$$\|f\|_{1,p} := \|f\|_{L^p(\mathbf{m})} + \|\nabla f\|_{L^p(\mathbf{m})}.$$

Let $\mathbb{H} = L^2(\mathbf{m})$ *and* $\mathbb{V} = \mathbb{H}_0^{1,p}(D)$. *By the* L^p*-Poincaré inequality, there exists a constant* $C > 0$ *such that* $\|f\|_{1,p} \le C\|\nabla f\|_{L^p(\mathbf{m})}$. *Consider the SPDE*

$$\mathrm{d}X(t) = \mathrm{div}\big(|\nabla X(t)|^{p-2}\nabla X(t)\big)\mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

where $W(t)$ *is a cylindrical Brownian motion on* $\mathbb{H}$, *and* $\sigma : [0,\infty) \to \mathscr{L}_{HS}(\mathbb{H})$ *is measurable with* $\|\sigma(\cdot)\|_{HS}^2 \in L^1_{loc}([0,\infty);\mathrm{d}t)$. *Then* **(A2.1)–(A2.4)** *hold for* $b(t,v) := \mathrm{div}\big(|\nabla v|^{p-2}\nabla v\big)$ *and some* ϕ, ψ, K. *Thus, for every* $\mathscr{F}_0$*-measurable* $X(0) \in L^2(\Omega \to \mathbb{H};\mathbb{P})$, *the equation has a unique solution.*

2.2 Harnack Inequalities for $\alpha \ge 1$

In this section, we consider monotone stochastic equations with deterministic coefficients and additive noise, i.e., equations of the form

$$\mathrm{d}X(t) = b(t,X(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \tag{2.4}$$

where $W(t)$ is a cylindrical Brownian motion on $\mathbb{H}$, $\sigma : [0,\infty) \to \mathscr{L}_{HS}(\mathbb{H})$ is measurable with $\|\sigma\|_{HS} \in L^2_{loc}([0,\infty);\mathrm{d}t)$, and $b : [0,\infty) \times \mathbb{V} \to \mathbb{V}^*$ is measurable such that **(A2.1)–(A2.4)** hold for $\sigma(t,v) := \sigma(t)$ and some $\alpha \ge 1$. For every $x \in \mathbb{H}$, let $X^x(t)$ be the solution with $X(0) = x$. We aim to establish Harnack inequalities for the associated Markov operators $P_t, t > 0$:

$$P_t f(x) := \mathbb{E} f(X^x(t)), \ \ f \in \mathscr{B}_b(\mathbb{H}), x \in \mathbb{H},$$

using Theorem 1.1.1. In this case, the idea for the construction of coupling is due to [54], where stochastic generalized porous media equations were considered, and it was extended in [27, 28, 31] to more general equations. The construction of Harnack inequalities for nonlinear monotone SPDEs with multiplicative noise is still open at the moment. Although coupling by change of measure has been constructed in [57] to establish Harnack inequalities for SDEs with multiplicative noise (see Sect. 3.4 below), it is very hard to apply the construction to nonlinear SPDEs with multiplicative noise.

Throughout this section, we assume that the noise is nondegenerate, i.e., $\sigma(t)v=0$ for some $v \in \mathbb{H}$ implies that $v = 0$. For every $v \in \mathbb{H}$, we define the intrinsic norm induced by $\sigma(t)$ by

$$\|v\|_{\sigma(t)} = |\sigma(t)^{-1}v|,$$

where $|\sigma(t)^{-1}v| := |y|$ if $y \in \mathbb{H}$ such that $\sigma(t)y = v$; $|\sigma(t)^{-1}v| = \infty$ if $v \notin \sigma(t)\mathbb{H}$.

The main result of this section is the following.

Theorem 2.2.1 *If there exist a constant* $\theta \in [2,\infty) \cap (\alpha - 1,\infty)$ *and* $\eta, \gamma \in C([0,\infty))$ *with* $\eta > 0$ *such that*

$$2_{\mathbb{V}^*}\langle b(t,u) - b(t,v), u - v\rangle_{\mathbb{V}} \le -\eta(t)\|u - v\|_{\sigma(t)}^{\theta}|u - v|^{\alpha+1-\theta} + \gamma(t)|u - v|^2 \tag{2.5}$$

holds for $t \geq 0$ and $u, v \in \mathbb{V}$, then for every $T > 0, x, y \in \mathbb{H}$, and strictly positive $f \in \mathscr{B}_b(\mathbb{H})$,

$$(P_T f(y))^p \leq (P_T f^p(x)) \exp\left[\frac{p\left(\frac{\theta+2}{\theta+1-\alpha}\right)^{\frac{2(\theta+1)}{\theta}} |x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}}{2(p-1)\left(\int_0^T \eta(t)^{\frac{2}{\theta+2}} \mathrm{e}^{-\frac{\theta+1-\alpha}{\theta+2}\int_0^t \gamma(s)\mathrm{d}s} \mathrm{d}t\right)^{\frac{\theta+2}{\theta}}}\right],$$

$$P_T \log f(y) \leq \log P_T f(x) + \frac{\left(\frac{\theta+2}{\theta+1-\alpha}\right)^{\frac{2(\theta+1)}{\theta}} |x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}}{2\left(\int_0^T \eta(t)^{\frac{2}{\theta+2}} \exp\left[-\frac{\theta+1-\alpha}{\theta+2}\int_0^t \gamma(s)\mathrm{d}s\right] \mathrm{d}t\right)^{\frac{\theta+2}{\theta}}}.$$

We now explain the idea of the proof using coupling by change of measure. Let $T > 0$ and $x, y \in \mathbb{H}$ be fixed, and let $X(t) = X^x(t)$ be the solution to (2.4) with $X(0) = x$. We intend to construct $Y(t)$ with $Y(0) = y$ such that $Y(T) = X(T)$ and the law of $Y(T)$ under a weighted probability $\mathrm{d}\mathbb{Q} := R\mathrm{d}\mathbb{P}$ coincides with that of $X^y(T)$. To this end, let $Y(t)$ solve the equation

$$\mathrm{d}Y(t) = \left\{b(t, Y(t)) + \frac{\xi(t)(X(t) - Y(t))}{|X(t) - Y(t)|^{\varepsilon}}\right\}\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \ \ Y(0) = y, \tag{2.6}$$

where $\varepsilon \in (0,1)$ and $\xi \in C([0,\infty))$ are to be determined such that $Y(T) = X(T)$, and if $u = v$, we set $\frac{u-v}{|u-v|^{\varepsilon}} = 0$. As shown in the proof of [54, Theorem A.2], we have that **(A2.1)–(A2.4)** hold with $\sigma(t,v) = \sigma(t)$ and $b(t,v)$ replaced by

$$\tilde{b}(t,v) := b(t,v) + \frac{\xi(t)(X(t) - v)}{|X(t) - v|^{\varepsilon}}, \ \ t \geq 0, v \in \mathbb{V}.$$

Therefore, (2.6) has a unique solution.

Now we aim to choose ξ and ε such that

$$\tau := \inf\{t \geq 0 : X(t) = Y(t)\} \leq T,$$

which implies that $X(T) = Y(T)$ by the uniqueness of solutions to (2.4) for $t \geq \tau$.

Lemma 2.2.2 *Let $\xi \in C([0,\infty))$ and $\varepsilon \in (0,1)$ be such that*

$$\int_0^T \xi(t)\mathrm{e}^{-\frac{\varepsilon}{2}\int_0^t \gamma(s)\mathrm{d}s}\mathrm{d}t \geq \frac{1}{\varepsilon}|x-y|^{\varepsilon}. \tag{2.7}$$

Then $X(T) = Y(T)$.

Proof. By (2.5) and Itô's formula, we have

$$\mathrm{d}|X(t) - Y(t)|^2 \leq \left(\gamma(t)|X(t) - Y(t)|^2 - 2\xi(t)|X(t) - Y(t)|^{2-\varepsilon}\right)\mathrm{d}t, \ \ t < \tau.$$

Then

$$
\begin{aligned}
&\mathrm{d}\Big\{|X(t)-Y(t)|^2\mathrm{e}^{-\int_0^t\gamma(s)\mathrm{d}s}\Big\}^{\frac{\varepsilon}{2}}\\
&=\frac{\varepsilon}{2}\Big\{|X(t)-Y(t)|^2\mathrm{e}^{-\int_0^t\gamma(s)\mathrm{d}s}\Big\}^{\frac{\varepsilon-2}{2}}\mathrm{d}\Big\{|X(t)-Y(t)|^2\mathrm{e}^{-\int_0^t\gamma(s)\mathrm{d}s}\Big\}\\
&\le -\varepsilon\xi(t)\mathrm{e}^{-\frac{\varepsilon}{2}\int_0^t\gamma(s)\mathrm{d}s}\mathrm{d}t,\ \ t<\tau.
\end{aligned}
$$

This implies

$$
|X(t)-Y(t)|^{\varepsilon}\mathrm{e}^{-\frac{\varepsilon}{2}\int_0^t\gamma(s)\mathrm{d}s}\le|x-y|^{\varepsilon}-\varepsilon\int_0^t\xi(s)\mathrm{e}^{-\frac{\varepsilon}{2}\int_0^s\gamma(r)\mathrm{d}r}\mathrm{d}s,\ \ t<\tau. \tag{2.8}
$$

If $\tau>T$, then $|X(T)-Y(T)|>0$, but by (2.7) and (2.8), we have $|X(T)-Y(T)|^{\varepsilon}\le 0$, which is a contradiction. □

Proof of Theorem 2.2.1.

(a) Take

$$
\begin{aligned}
&\varepsilon=\frac{\theta+1-\alpha}{\theta+2},\\
&\xi(t)=\frac{(\theta+2)\eta(t)^{\frac{2}{\theta+2}}\exp\big[-\frac{\theta+1-\alpha}{2(\theta+2)}\int_0^t\gamma(s)\mathrm{d}s\big]}{(\theta+1-\alpha)\int_0^T\eta(t)^{\frac{2}{\theta+2}}\exp\big[-\frac{\theta+1-\alpha}{\theta+2}\int_0^t\gamma(s)\mathrm{d}s\big]\mathrm{d}t}\big|x-y\big|^{\frac{\theta+1-\alpha}{\theta+2}},\ \ t\ge 0.
\end{aligned}
$$

Then

$$
\int_0^T\xi(t)\mathrm{e}^{-\frac{\varepsilon}{2}\int_0^t\gamma(s)\mathrm{d}s}\mathrm{d}t=\frac{\theta+2}{\theta+1-\alpha}\big|x-y\big|^{\frac{\theta+1-\alpha}{\theta+2}}=\frac{|x-y|^{\varepsilon}}{\varepsilon}.
$$

By Lemma 2.2.2, we have $X(T)=Y(T)$.

(b) To formulate the changed probability $\mathrm{d}\mathbb{Q}=R\mathrm{d}\mathbb{P}$, we rewrite (2.6) by

$$
\mathrm{d}Y(t)=b(t,Y(t))\mathrm{d}t+\sigma(t)\mathrm{d}\tilde{W}(t),\ \ Y(0)=y,
$$

where

$$
\tilde{W}(t):=W(t)+\int_0^{\tau\wedge t}\frac{\xi(s)\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^{\varepsilon}}\mathrm{d}s.
$$

We shall prove

$$
\begin{aligned}
&\int_0^{\tau}\bigg|\xi(s)\frac{\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^{\varepsilon}}\bigg|^2\mathrm{d}s\\
&\le\frac{\big(\frac{\theta+2}{\theta+1-\alpha}\big)^{\frac{2(\theta+1)}{\theta}}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}}{\big(\int_0^T\eta(t)^{\frac{2}{\theta+2}}\exp\big[-\frac{\theta+1-\alpha}{\theta+2}\int_0^t\gamma(s)\mathrm{d}s\big]\mathrm{d}t\big)^{\frac{\theta+2}{\theta}}},
\end{aligned} \tag{2.9}
$$

which ensures by Girsanov's theorem that $\tilde{W}(t)$ is a cylindrical Brownian motion on $\mathbb{H}$ under $\mathrm{d}\mathbb{Q}:=R\mathrm{d}\mathbb{P}$, where

$$R := \exp\Big[-\int_0^\tau \frac{\xi(s)}{|X(s)-Y(s)|^\varepsilon}\big\langle \sigma(s)^{-1}(X(s)-Y(s)), \mathrm{d}W(s)\big\rangle \\ -\frac{1}{2}\int_0^\tau \Big|\xi(s)\frac{\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^\varepsilon}\Big|^2 \mathrm{d}s\Big]. \tag{2.10}$$

By (2.5) and Itô's formula, we have

$$\mathrm{d}|X(t)-Y(t)|^2 \leq \Big\{-\eta(t)\|X(t)-Y(t)\|_{\sigma(t)}^\theta |X(t)-Y(t)|^{\alpha+1-\theta} + \gamma(t)|X(t)-Y(t)|^2\Big\}\mathrm{d}t$$

for $t < \tau$. Then,

$$\begin{aligned}
&\mathrm{d}\Big\{|X(t)-Y(t)|^2 \mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\Big\}^\varepsilon \\
&\leq -\varepsilon\Big\{|X(t)-Y(t)|^2 \mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\Big\}^{\varepsilon-1} \eta(t)\|X(t)-Y(t)\|_{\sigma(t)}^\theta \\
&\quad \times |X(t)-Y(t)|^{\alpha+1-\theta}\mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\mathrm{d}t \\
&= -\varepsilon\eta(t)\mathrm{e}^{-\varepsilon\int_0^t \gamma(s)\mathrm{d}s}\frac{\|X(t)-Y(t)\|_{\sigma(t)}^\theta}{|X(t)-Y(t)|^{\theta\varepsilon}}\mathrm{d}t, \quad t < \tau,
\end{aligned}$$

where in the last step we have used $\varepsilon = \frac{\theta+1-\alpha}{\theta+2}$. Thus, again using $\varepsilon = \frac{\theta+1-\alpha}{\theta+2}$, we obtain

$$\int_0^\tau \eta(t)\mathrm{e}^{-\varepsilon\int_0^t \gamma(s)\mathrm{d}s}\frac{\|X(t)-Y(t)\|_{\sigma(t)}^\theta}{|X(t)-Y(t)|^{\theta\varepsilon}}\mathrm{d}t \leq \frac{\theta+2}{\theta+1-\alpha}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta+2}}.$$

Since $\theta \geq 2$ and $\tau \leq T$, by Hölder's inequality, we obtain from this that

$$\begin{aligned}
&\int_0^\tau \Big|\xi(s)\frac{\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^\varepsilon}\Big|^2 \mathrm{d}s = \int_0^\tau \xi(s)^2\frac{\|X(s)-Y(s)\|_{\sigma(s)}^2}{|X(s)-Y(s)|^{2\varepsilon}}\mathrm{d}s \\
&\leq \Big(\int_0^\tau \eta(t)\mathrm{e}^{-\varepsilon\int_0^t \gamma(s)\mathrm{d}s}\frac{\|X(t)-Y(t)\|_{\sigma(t)}^\theta}{|X(t)-Y(t)|^{\theta\varepsilon}}\mathrm{d}t\Big)^{\frac{2}{\theta}} \Big(\int_0^T \mathrm{e}^{\frac{2\varepsilon}{\theta-2}\int_0^t \gamma(s)\mathrm{d}s}\frac{\xi(t)^{\frac{2\theta}{\theta-2}}}{\eta(t)^{\frac{2}{\theta-2}}}\mathrm{d}t\Big)^{\frac{\theta-2}{\theta}} \\
&\leq \Big(\frac{2+\theta}{\theta+1-\alpha}\Big)^{\frac{2}{\theta}}|x-y|^{\frac{4(\theta+1-\alpha)}{\theta(\theta+2)}}\Big(\int_0^T \mathrm{e}^{\frac{2(\theta+1-\alpha)}{\theta^2-4}\int_0^t \gamma(s)\mathrm{d}s}\frac{\xi(t)^{\frac{2\theta}{\theta-2}}}{\eta(t)^{\frac{2}{\theta-2}}}\mathrm{d}t\Big)^{\frac{\theta-2}{\theta}}.
\end{aligned} \tag{2.11}$$

By the definition of $\xi(t)$, we have

$$\begin{aligned}
&\int_0^T \mathrm{e}^{\frac{2(\theta+1-\alpha)}{\theta^2-4}\int_0^t \gamma(s)\mathrm{d}s}\frac{\xi(t)^{\frac{2\theta}{\theta-2}}}{\eta(t)^{\frac{2}{\theta-2}}}\mathrm{d}t \\
&= \frac{\big(\frac{\theta+2}{\theta+1-\alpha}|x-y|^{\frac{\theta+1-\alpha}{\theta+2}}\big)^{\frac{2\theta}{\theta-2}}}{\big(\int_0^T \eta(t)^{\frac{2}{\theta+2}}\exp\big[-\frac{\theta+1-\alpha}{\theta+2}\int_0^t \gamma(s)\mathrm{d}s\big]\mathrm{d}t\big)^{\frac{\theta+2}{\theta-2}}}.
\end{aligned}$$

Combining this with (2.11), we prove (2.9).

(c) By Theorem 1.1.1, the proof is finished, since according to (2.10) and (2.9), we have

$$\Big(\mathbb{E}R^{\frac{p}{p-1}}\Big)^{p-1} \le \operatorname*{ess\,sup}_{\Omega} \exp\left[\frac{p}{2(p-1)}\int_0^\tau \left|\xi(s)\frac{\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^\varepsilon}\right|^2 \mathrm{d}s\right]$$
$$\le \exp\left[\frac{p\big(\frac{\theta+2}{\theta+1-\alpha}\big)^{\frac{2(\theta+1)}{\theta}}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}}{2(p-1)\big(\int_0^T \eta(t)^{\frac{2}{\theta+2}}\exp\big[-\frac{\theta+1-\alpha}{\theta+2}\int_0^t \gamma(s)\mathrm{d}s\big]\mathrm{d}t\big)^{\frac{\theta+2}{\theta}}}\right]$$

and

$$\mathbb{R}[R\log R] = \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\int_0^\tau \left|\xi(s)\frac{\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^\varepsilon}\right|^2 \mathrm{d}s$$
$$\le \frac{\big(\frac{\theta+2}{\theta+1-\alpha}\big)^{\frac{2(\theta+1)}{\theta}}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}}{2\big(\int_0^T \eta(t)^{\frac{2}{\theta+2}}\exp\big[-\frac{\theta+1-\alpha}{\theta+2}\int_0^t \gamma(s)\mathrm{d}s\big]\mathrm{d}t\big)^{\frac{\theta+2}{\theta}}}.$$

□

As applications of the Harnack inequalities, we consider estimates of P_t in terms of its invariant probability measure. To this end, we assume that the equation is time-homogeneous, i.e., $b(t,\cdot)$ and $\sigma(t)$ are independent of t. In this case, P_t is a Markov semigroup.

Proposition 2.2.3 *Assume that $b(t,\cdot)$ and $\sigma(t)$ are independent of t such that* **(A2.1)–(A2.4)** *hold with constants ϕ, K, Ψ such that $\Psi > 0$. If the embedding $\mathbb{V}\subset\mathbb{H}$ is compact and*

$$K|v|^2 - \Psi\|v\|_{\mathbb{V}}^{\alpha+1} \le C - \delta\|v\|_{\mathbb{V}}^{\alpha+1},\ \ v\in\mathbb{V} \tag{2.12}$$

holds for some constants $C,\delta>0$, then P_t has an invariant probability measure μ such that

$$\int_{\mathbb{H}}\Big(\|\cdot\|_{\mathbb{V}}^{\alpha+1} + \mathrm{e}^{c|\cdot|^{\alpha+1}}\Big)\mathrm{d}\mu < \infty \tag{2.13}$$

holds for some constant $c>0$, where $\|v\|_{\mathbb{V}} := \infty$ for $v\in\mathbb{H}\setminus\mathbb{V}$.

Proof. By **(A2.3)**, (2.12), and Itô's formula, we obtain

$$\mathrm{d}|X(t)|^2 \le \big\{C_1 - \delta\|X(t)\|_{\mathbb{V}}^{\alpha+1}\big\}\mathrm{d}t + 2\langle X(t), \sigma\mathrm{d}W(t)\rangle \tag{2.14}$$

for some constants $C_1,\delta>0$. This implies that

$$\frac{1}{T}\mathbb{E}\int_0^T \|X^0(t)\|_{\mathbb{V}}^{\alpha+1}\mathrm{d}t \le \frac{C_1}{\delta},\ \ T>0.$$

Since the embedding $\mathbb{V}\subset\mathbb{H}$ is compact, the sequence $\{\frac{1}{n}\int_0^n \delta_0 P_t \mathrm{d}t\}_{n\ge1}$ is tight, where δ_0 is the Dirac measure at the point $0\in\mathbb{H}$. By a standard argument, a weak limit μ of a subsequence is an invariant probability measure and $\mu(\|\cdot\|_{\mathbb{V}}^{\alpha+1}) \leq \frac{C_1}{\delta}$. It remains to find a constant $c > 0$ such that $\mu(\mathrm{e}^{c|\cdot|^{\alpha+1}}) < \infty$. For every $\varepsilon > 0$, it follows from (2.14) that

$$\begin{aligned} \mathrm{d}\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}} \leq\ & \mathrm{d}M_t + \frac{(\alpha+1)\varepsilon}{2}\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}}|X(t)|^{\alpha-1} \\ & \times\big(C_2 + (\alpha+1)\varepsilon|X(t)|^{\alpha+1} - \delta\|X(t)\|_{\mathbb{V}}^{\alpha+1}\big)\mathrm{d}t \end{aligned}$$

holds for some local martingale M_t and some constant $C_2 > 0$. Since $|\cdot| \leq c_0\|\cdot\|_{\mathbb{V}}$ holds for some constant $c_0 > 0$, we see that for small enough $\varepsilon > 0$,

$$\begin{aligned} & \mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}}|X(t)|^{\alpha-1}\big(C_2 + (\alpha+1)\varepsilon|X(t)|^{\alpha+1} - \delta\|X(t)\|_{\mathbb{V}}^{\alpha+1}\big) \\ & \leq \tilde{C} - \tilde{\delta}|X(t)|^{2\alpha}\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}} \leq C' - \delta'\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}} \end{aligned}$$

holds for some constants $\tilde{C}, \tilde{\delta}, C', \delta' > 0$. Therefore, for small $\varepsilon > 0$,

$$\mathrm{d}\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}} - \mathrm{d}M_t \leq \big\{\tilde{C} - \tilde{\delta}|X(t)|^{2\alpha}\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}}\big\}\mathrm{d}t \leq \big\{C' - \delta'\mathrm{e}^{\varepsilon|X(t)|^{\alpha+1}}\big\}\mathrm{d}t. \quad (2.15)$$

In particular, there exists a constant $C'' > 0$ such that for small enough $\varepsilon > 0$, we have

$$\mathbb{E}\mathrm{e}^{\varepsilon|X^0(t)|^{\alpha+1}} \leq C'', \ \ t \geq 0.$$

This implies that $\mu(\mathrm{e}^{\varepsilon|\cdot|^{\alpha+1}}) < \infty$. $\square$

Corollary 2.2.4 *In the situation of Proposition 2.2.3 and under the assumption that (2.5) holds for some constant functions $\eta > 0$ and γ, we have that P_t is strong Feller and has a unique invariant probability measure μ, and μ has full support on $\mathbb{H}$. Moreover:*

(1) *If $\alpha = 1$ and $\gamma \leq 0$, then P_t is hyperbounded, i.e., $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$ for some $t > 0$.*
(2) *If $\alpha > 1$, then P_t is ultracontractive with*

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leq \exp\Big[C + Ct^{-\frac{\alpha+1}{\alpha-1}}\Big], \ \ t > 0,$$

holding for some constant $C > 0$.
(3) *Let P_t^* be the adjoint operator of P_t in $L^2(\mu)$. Let W_1^ρ denote the L^1-transportation cost induced by the cost function $\rho(x,y) := |x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}$; see Theorem 1.4.2. Then*

$$\mu\big((P_T^*f)\log P_T^*f\big) \leq \frac{\theta+2}{2\eta^{\frac{2}{\theta}}(\theta+1-\alpha)}\left(\frac{\gamma}{1-\exp\big[-\frac{\gamma(\theta+1-\alpha)}{\theta+2}T\big]}\right)^{\frac{\theta+2}{\theta}} W_1^\rho(f\mu,\mu)$$

holds for all $T > 0$ and $f \geq 0$ with $\mu(f) = 1$.

Proof. (a) By Theorem 2.2.1 with constants η, γ, and $p=2$, (1.2) holds for $P=P_T$, $\Phi(r)=r^2$, and

$$\Psi(x,y)=c_1\Big(\frac{\gamma}{1-\mathrm{e}^{-c_2\gamma T}}\Big)^{\frac{\theta+2}{\theta}}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}$$

for some constants $c_1,c_2>0$, i.e.,

$$(P_T f(x))^2 \le (P_T f^2(y))\exp\bigg[c_1\Big(\frac{\gamma}{1-\mathrm{e}^{-c_2\gamma T}}\Big)^{\frac{\theta+2}{\theta}}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}\bigg] \tag{2.16}$$

holds for $T>0$, $x,y\in\mathbb{H}$, and $f\in\mathscr{B}_b^+(\mathbb{H})$. Then by Theorem 1.4.1(3) and Proposition 2.2.3, P_t has a unique invariant measure μ. To see that μ has full support on $\mathbb{H}$, it suffices to show that $\mu(B(x,\varepsilon))>0$ for every $x\in\mathbb{H}$ and $\varepsilon>0$. Applying (2.16) to $f=1_{B(x,\varepsilon)}$, we obtain

$$(P_T 1_{B(x,\varepsilon)}(x))^2\exp\bigg[-c_1\Big(\frac{\gamma}{1-\mathrm{e}^{-c_2\gamma T}}\Big)^{\frac{\theta+2}{\theta}}|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}\bigg]\le P_T 1_{B(x,\varepsilon)}(y).$$

If $\mu(B(x,\varepsilon))=0$, then taking the integral of both sides with respect to $\mu(\mathrm{d}y)$ yields

$$\mathbb{P}(|X^x(T)-x|\ge\varepsilon)=1-P_T 1_{B(x,\varepsilon)}(x)=1,\ \ T>0.$$

This is impossible, since the solution is continuous, so that $X^x(T)\to x$ as $T\to 0$.

(b) Let $\alpha=1$ and $\gamma\le 0$. Then (2.16) implies

$$(P_T f(x))^2\le (P_T f^2(y))\exp\Big[\frac{c|x-y|^2}{T^{\frac{\theta+2}{\theta}}}\Big]$$

for some constant $c>0$. So by Theorem 1.4.1(6), we obtain

$$\begin{aligned}\sup_{\mu(f^2)\le 1}|P_T f(x)| &\le \frac{1}{\int_{B(0,|x|+1)}\exp[-\frac{c(1+2|x|)^2}{T^{\frac{\theta+2}{\theta}}}]\mu(\mathrm{d}y)}\\ &\le \frac{1}{\mu(B(0,1))}\exp\Big[\frac{c(1+2|x|)^2}{T^{\frac{\theta+2}{\theta}}}\Big].\end{aligned}$$

Combining with (2.13), we see that for sufficiently large $T>0$, $\|P_T\|_{L^2(\mu)\to L^4(\mu)}<\infty$.

(c) Let $\alpha>1$. Then (2.16) implies

$$(P_T f(x))^2\le (P_T f^2(y))\exp\Big[\frac{c|x-y|^{\frac{2(\theta+1-\alpha)}{\theta}}}{(1\wedge T)^{\frac{\theta+2}{\theta}}}\Big]$$

for some constant $c>0$. This implies

$$\sup_{\mu(f^2)\le 1} |P_T f(x)| \le C_1 \exp\left[\frac{C_2|x|^{\frac{2(\theta+1-\alpha)}{\theta}}}{(1\wedge T)^{\frac{\theta+2}{\theta}}}\right]$$

for some constants $C_1, C_2 > 0$. So

$$\sup_{\mu(f^2)\le 1} |P_T f(x)| \le \sup_{\mu(f^2)\le 1} \left|P_{\frac{T}{2}}(P_{\frac{T}{2}} f)(x)\right| \le C_1 \mathbb{E}\exp\left[\frac{C_2|X^x(\frac{T}{2})|^{\frac{2(\theta+1-\alpha)}{\theta}}}{(1\wedge T)^{\frac{\theta+2}{\theta}}}\right]. \tag{2.17}$$

By (2.15) and Jensen's inequality, we see that $h(t) := \mathbb{E}\mathrm{e}^{\varepsilon|X^x(t)|^{\alpha+1}}$ satisfies

$$h'(t) \le \tilde{C} - \tilde{\delta}\mathbb{E}\left\{|X^x(t)|^{2\alpha}\mathrm{e}^{\varepsilon|X^x(t)|^{\alpha+1}}\right\} \le \tilde{C} - \tilde{\delta}\varepsilon^{-\frac{2\alpha}{1+\alpha}} h(t)\{\log h(t)\}^{\frac{2\alpha}{\alpha+1}}.$$

Since $\frac{2\alpha}{\alpha+1} > 1$, this implies

$$\mathbb{E}\mathrm{e}^{\varepsilon|X^x(t)|^{\alpha+1}} \le \exp\left[c + ct^{-\frac{\alpha+1}{\alpha-1}}\right], \quad t>0, x\in\mathbb{H}$$

for some constant $c > 0$. Noting that there exists a constant $c' > 0$ such that

$$\frac{C_2|X^x(t)|^{\frac{2(\theta+1-\alpha)}{\theta}}}{(1\wedge t)^{\frac{\theta+2}{\theta}}} \le \varepsilon|X^x(t)|^{\alpha+1} + \frac{c'}{(1\wedge t)^{\frac{\alpha+1}{\alpha-1}}}$$

holds for some constant $c' > 0$ and all $t > 0$, we arrive at

$$\mathbb{E}\exp\left[\frac{C_2|X^x(\frac{T}{2})|^{\frac{2(\theta+1-\alpha)}{\theta}}}{(1\wedge T)^{\frac{\theta+2}{\theta}}}\right] \le \exp\left[c'' + c''T^{-\frac{\alpha+1}{\alpha-1}}\right]$$

for some constant $c'' > 0$ and all $x \in \mathbb{H}$, $T > 0$. Combining this with (2.17), we prove (2).

(d) The entropy–cost inequality in (3) follows from Theorem 1.4.2(3) and the log-Harnack inequality in Theorem 2.2.1 with constants η and γ. $\square$

2.3 Harnack Inequalities for $\alpha \in (0,1)$

When $\alpha \in (0,1)$, for the typical model that $b(t,u) = \Delta u^\alpha$ as in Example 2.1.1, the upper bound of ${}_{\mathbb{V}^*}\langle b(t,u) - b(t,v), u-v\rangle_{\mathbb{V}}$ behaves like $-\mathbf{m}\big(|u-v|^2(|u|\vee|v|)^{\alpha-1}\big)$, so that the condition (2.5) does not hold. In this section, we investigate Harnack inequalities for (2.4) under assumptions **(A2.1)–(A2.4)** with $\alpha \in (0,1)$ by introducing the following conditions (2.18) and (2.19) to replace (2.5): for some measurable function $h : \mathbb{V} \to (0,\infty)$, some constant $\theta \ge \frac{4}{\alpha+1}$, some $\gamma \in C([0,\infty))$, and strictly positive $q, \delta, \eta \in C([0,\infty))$,

$$
\begin{aligned}
&2_{\mathbb{V}^*}\langle b(t,u),u\rangle_{\mathbb{V}} + \|\sigma(t)\|_{HS}^2 \le q(t) - \delta(t)h(u)^{\alpha+1} + \gamma(t)|u|^2, &(2.18)\\
&2_{\mathbb{V}^*}\langle b(t,u)-b(t,v),u-v\rangle_{\mathbb{V}}\\
&\le -\frac{\eta(t)\|u-v\|_{\sigma(t)}^{\theta}}{|u-v|^{\theta-2}(h(u)\vee h(v))^{1-\alpha}} + \gamma(t)|u-v|^2 &(2.19)
\end{aligned}
$$

hold for $t \ge 0$ and $u,v \in \mathbb{V}$.

For any constants $\theta \ge \frac{4}{\alpha+1}$, $p>1$, $T>0$, and $x,y \in \mathbb{H}$, let

$$
\begin{aligned}
&\tilde{q}(t) = q(t)\mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}, \quad \tilde{\delta}(t) = \delta(t)\mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}, \quad t \in [0,T].\\
&c_1(\theta,T) = \int_0^T \eta(t)^{\frac{2(\alpha+1)}{4+\theta+\theta\alpha}} \mathrm{e}^{-\frac{\theta(\alpha+1)}{\theta\alpha+\theta+4}\int_0^t \gamma(s)\mathrm{d}s}\mathrm{d}t,\\
&c_2(\theta,T,x,y) = \frac{1}{\min_{[0,T]}\tilde{\delta}}\bigg\{|x|^2+|y|^2+2\int_0^T \tilde{q}(t)\mathrm{d}t\\
&\qquad + \frac{(\theta+2)^2}{\theta}\Big(\frac{2}{\theta+4}\Big)^{\frac{\theta+4}{\theta+2}}|x-y|^{\frac{\theta}{\theta+2}}\Big(|y|^2+\int_0^T \tilde{q}(t)\mathrm{d}t\Big)^{\frac{\theta+4}{2(\theta+2)}}\bigg\}.
\end{aligned}
$$

Theorem 2.3.1 *Let $\alpha \in (0,1)$ and assume that (2.18) and (2.19) hold for some constant $\theta \ge \frac{4}{\alpha+1}$, measurable $h : \mathbb{V} \to (0,\infty)$, $\gamma \in C([0,\infty))$, and strictly positive $q,\delta,\eta \in C([0,\infty))$.*

(1) *For every $T>0, x,y \in \mathbb{H}$ and strictly positive $f \in \mathscr{B}_b(\mathbb{H})$,*

$$
P_T \log f(y) - \log P_T f(x) \le \frac{|x-y|^2}{2}\Big(\frac{\theta+2}{\theta}\Big)^{\frac{2(\theta+1)}{\theta}} \frac{c_2(\theta,T,x,y)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}}{c_1(\theta,T)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}.
$$

Consequently, there exists a constant $C>0$ such that

$$
P_T \log f(y) - \log P_T f(x) \le \frac{C|x-y|^2(|x|^2+|y|^2+1\wedge T)^{\frac{2(1-\alpha)}{\theta(1+\alpha)}}}{(T\wedge 1)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}.
$$

(2) *There exists a constant $C>0$ such that for every $T>0$, $p>1$, $x,y \in \mathbb{H}$, and $f \in \mathscr{B}_b^+(\mathbb{H})$,*

$$
\begin{aligned}
&(P_T f(y))^p \le (P_T f^p(x))\\
&\times \exp\left[\frac{Cp}{p-1}\left(\frac{|x-y|^2(1+|x|^2+|y|^2)}{(T\wedge 1)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + \Big(\frac{p}{p-1}\Big)^{\frac{4(1-\alpha)}{\alpha(\theta+2)+\theta-2}} \frac{|x-y|^{\frac{2\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}}{(1\wedge T)^{\frac{\theta(\alpha+1)+4}{\alpha(\theta+2)+\theta-2}}}\right)\right].
\end{aligned}
$$

To prove this theorem using coupling by change of measure, let $X(t) = X^x(t)$ solve (2.4) with $X(0)=x$, and let $Y(t)$ solve (2.6) for some $\varepsilon \in (0,1)$ and

$$
\xi(t) := \frac{|x-y|^{\varepsilon}\eta(t)^{\alpha_2}}{\varepsilon c_1(\theta,T)}\mathrm{e}^{-\alpha_1\int_0^t \gamma(s)\mathrm{d}s}, \quad t \ge 0,
$$

where

$$\varepsilon = \frac{\theta}{\theta+2}, \ \ \alpha_1 = \frac{\theta(\theta\alpha+\theta+4\alpha)}{2(\theta+2)(\theta\alpha+\theta+4)}, \ \ \alpha_2 = \frac{2(\alpha+1)}{4+\theta+\theta\alpha}.$$

Then it is easy to see that

$$\int_0^T \xi(t)\mathrm{e}^{-\frac{\varepsilon}{2}\int_0^t \gamma(s)\mathrm{d}s}\mathrm{d}t = \frac{|x-y|^\varepsilon}{\varepsilon}, \tag{2.20}$$

so that $X(T) = Y(T)$ (i.e., $\tau \le T$) according to Lemma 2.2.2. Moreover,

$$c_1(\theta,T) = \int_0^T \eta(t)^{\alpha_2}\mathrm{e}^{-(\alpha_1+\frac{\theta}{2(\theta+2)})\int_0^t \gamma(s)\mathrm{d}s}\mathrm{d}t. \tag{2.21}$$

Next, it is easy to see from Itô's formula and (2.18) that

$$\mathbb{E}\int_0^t \{h(X(s))\vee h(Y(s))\}^{\alpha+1}\mathrm{d}s < \infty, \ \ t > 0.$$

Since $\alpha+1 \ge \frac{2(1-\alpha)}{\theta-2}$, we have $S(t) := \int_0^t \{h(X(s))\vee h(Y(s))\}^{\frac{2(1-\alpha)}{\theta-2}}\mathrm{d}s < \infty$ for all $t > 0$. Let

$$\tau_n = \tau \wedge \inf\{t \ge 0 : S(t) \ge n\}, \ \ n \ge 1.$$

We have $\tau_n \uparrow \tau$ as $n \uparrow \infty$.

Lemma 2.3.2 *Assume* (2.18) *and* (2.19). *For every* $n \ge 1$,

$$R_n := \exp\bigg[-\int_0^{\tau_n}\frac{\xi(s)}{|X(s)-Y(s)|^\varepsilon}\big\langle\sigma(s)^{-1}(X(s)-Y(s)),\mathrm{d}W(s)\big\rangle \\ -\frac{1}{2}\int_0^{\tau_n}\Big|\xi(s)\frac{\sigma(s)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|^\varepsilon}\Big|^2\mathrm{d}s\bigg]$$

is a well-defined probability density such that

$$\mathbb{E}\big\{R_n\log R_n\big\} \\ \le \frac{|x-y|^2}{2}\Big(\frac{\theta+2}{\theta}\Big)^{\frac{2(\theta+1)}{\theta}}\frac{\big(\int_0^T\mathbb{E}_{\mathbb{Q}_n}\big(h(X(t\wedge\tau_n))\vee h(Y(t\wedge\tau_n))\big)^{\alpha+1}\mathrm{d}t\big)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}}{c_1(\theta,T)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}},$$

and for every $p > 1$,

$$\big(\mathbb{E}R_n^{\frac{p}{p-1}}\big)^{p-1} \\ \le \bigg(\mathbb{E}_{\mathbb{Q}_n}\exp\bigg[\frac{c(\theta,p)|x-y|^2}{c_1(\theta,T)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\bigg(\int_0^{\tau_n}\big(h(X(t))\vee h(Y(t))\big)^{\alpha+1}\mathrm{d}t\bigg)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\bigg]\bigg)^{\frac{p-1}{2}},$$

where

$$\mathrm{d}\mathbb{Q}_n = R_n \mathrm{d}\mathbb{P}, \;\; c(\theta,p) := \frac{p+1}{(p-1)^2}\Big(\frac{\theta+2}{\theta}\Big)^{\frac{2(\theta+1)}{\theta}}.$$

Proof. By (2.19) and Itô's formula, we have

$$\mathrm{d}|X(t)-Y(t)|^2 \le -\eta(t)\frac{\|X(t)-Y(t)\|_{\sigma(t)}^{\theta}|X(t)-Y(t)|^{2-\theta}}{(h(X(t))\vee h(Y(t)))^{1-\alpha}}\,\mathrm{d}t + \gamma(t)|X(t)-Y(t)|^2\mathrm{d}t$$

for $t<\tau$. Since $\theta-2+2(1-\varepsilon)=\varepsilon\theta$, this implies

$$\mathrm{d}\Big\{|X(t)-Y(t)|^2\mathrm{e}^{-\int_0^t\gamma(s)\mathrm{d}s}\Big\}^{\varepsilon} \le -\frac{\varepsilon\eta(t)\|X(t)-Y(t)\|_{\sigma(t)}^{\theta}\mathrm{e}^{-\varepsilon\int_0^t\gamma(s)\mathrm{d}s}}{|X(t)-Y(t)|^{\varepsilon\theta}(h(X(t))\vee h(Y(t)))^{1-\alpha}}\,\mathrm{d}t$$

for $t<\tau$. So,

$$\int_0^{\tau}\frac{\eta(t)\|X(t)-Y(t)\|_{\sigma(t)}^{\theta}\mathrm{e}^{-\varepsilon\int_0^t\gamma(s)\mathrm{d}s}}{|X(t)-Y(t)|^{\varepsilon\theta}(h(X(t))\vee h(Y(t)))^{1-\alpha}}\,\mathrm{d}t \le \frac{|x-y|^{2\varepsilon}}{\varepsilon}. \tag{2.22}$$

Let

$$M(t) = -\int_0^t\frac{\xi(s)}{|X(s)-Y(s)|^{\varepsilon}}\big\langle\sigma(s)^{-1}(X(s)-Y(s)),\mathrm{d}W(s)\big\rangle.$$

By (2.22), Hölder's inequality, and $\tau_n\le\tau\le T$, we see that

$$\begin{aligned}\langle M\rangle(\tau_n\rangle &= \int_0^{\tau_n}\xi(t)^2\frac{\|X(t)-Y(t)\|_{\sigma(t)}^2}{|X(t)-Y(t)|^{2\varepsilon}}\,\mathrm{d}t\\ &\le\Big(\int_0^{\tau_n}\frac{\eta(t)\|X(t)-Y(t)\|_{\sigma(t)}^{\theta}\mathrm{e}^{-\varepsilon\int_0^t\gamma(s)\mathrm{d}s}}{|X(t)-Y(t)|^{\varepsilon\theta}(h(X(t))\vee h(Y(t)))^{1-\alpha}}\,\mathrm{d}t\Big)^{\frac{2}{\theta}}\\ &\quad\times\Big(\int_0^{\tau_n}\frac{\xi(t)^{\frac{2\theta}{\theta-2}}(h(X(t))\vee h(Y(t)))^{\frac{2(1-\alpha)}{\theta-2}}\mathrm{e}^{\frac{2\varepsilon}{\theta-2}\int_0^t\gamma(s)\mathrm{d}s}}{\eta(t)^{\frac{2}{\theta-2}}}\,\mathrm{d}t\Big)^{\frac{\theta-2}{\theta}}\\ &\le\frac{|x-y|^{\frac{4\varepsilon}{\theta}}}{\varepsilon^{\frac{2}{\theta}}}\Big(\int_0^{\tau_n}\frac{\xi(t)^{\frac{2\theta}{\theta-2}}(h(X(t))\vee h(Y(t)))^{\frac{2(1-\alpha)}{\theta-2}}\mathrm{e}^{\frac{2\varepsilon}{\theta-2}\int_0^t\gamma(s)\mathrm{d}s}}{\eta(t)^{\frac{2}{\theta-2}}}\,\mathrm{d}t\Big)^{\frac{\theta-2}{\theta}}\end{aligned} \tag{2.23}$$

is bounded. Then, by Girsanov's theorem, $\mathrm{d}\mathbb{Q}_n := R_n\mathrm{d}\mathbb{P}$ is a probability measure under which

$$\tilde{W}_n(t) := W(t) + \int_0^{t\wedge\tau_n}\frac{\xi(s)}{|X(s)-Y(s)|^{\varepsilon}}\sigma(s)^{-1}(X(s)-Y(s))\mathrm{d}s$$

is a cylindrical Brownian motion.

By $\varepsilon=\frac{\theta}{\theta+2}$ and $\frac{2(1-\alpha)}{\theta-2}\le\alpha+1$, (2.23) and Hölder's inequality yield

$$\langle M\rangle(\tau_n) \le \Big(\frac{\theta+2}{\theta}\Big)^{\frac{2}{\theta}} |x-y|^{\frac{4}{\theta+2}} \tilde C(T) \bigg(\int_0^{\tau_n} \big(h(X(t))\vee h(Y(t))\big)^{\alpha+1} \mathrm{d}t\bigg)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}},$$

where due to the definition of ξ and (2.21),

$$\begin{aligned} \tilde C(T) \ &:= \bigg(\int_0^T \frac{\xi(t)^{\frac{2\theta(\alpha+1)}{\theta(\alpha+1)-4}} \mathrm{e}^{\frac{2\varepsilon(\alpha+1)}{\theta(\alpha+1)-4}\int_0^t \gamma(s)\mathrm{d}s}}{\eta(t)^{\frac{2(\alpha+1)}{\theta(\alpha+1)-4}}} \mathrm{d}t\bigg)^{\frac{\theta(\alpha+1)-4}{\theta(\alpha+1)}} \\ &= \Big(\frac{\theta+2}{\theta}\Big)^2 c_1(\theta,T)^{-\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}} |x-y|^{\frac{2\theta}{\theta+2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle M\rangle(\tau_n) \le \ & \Big(\frac{\theta+2}{\theta}\Big)^{\frac{2(\theta+1)}{\theta}} c_1(\theta,T)^{-\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}} |x-y|^2 \\ & \times \bigg(\int_0^{\tau_n} \big(h(X(t))\vee h(Y(t))\big)^{\alpha+1} \mathrm{d}t\bigg)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}. \end{aligned} \tag{2.24}$$

Finally, we have

$$\mathbb{E}\big\{R_n \log R_n\big\} = \mathbb{E}_{\mathbb{Q}_n} \log R_n = \frac{1}{2}\mathbb{E}_{\mathbb{Q}} \langle M\rangle(\tau_n),$$

and for $p>1$ and $\tilde M_n$ defined as M using the $\mathbb{Q}_n$-cylindrical Brownian motion $\tilde W_n$ to replace $W(t)$ (note that $\langle M\rangle = \langle \tilde M_n\rangle$), we have

$$\begin{aligned} \mathbb{E}R_n^{\frac{p}{p-1}} &= \mathbb{E}_{\mathbb{Q}_n} R_n^{\frac{1}{p-1}} = \mathbb{E}_{\mathbb{Q}} \mathrm{e}^{\frac{1}{p-1}\tilde M(\tau_n) + \frac{1}{2(p-1)}\langle M\rangle(\tau_n)} \\ &\le \Big(\mathbb{E}_{\mathbb{Q}_n} \mathrm{e}^{\frac{2}{p-1}\tilde M(\tau_n) - \frac{2}{(p-1)^2}\langle M\rangle(\tau_n)}\Big)^{\frac12} \Big(\mathbb{E}_{\mathbb{Q}_n} \mathrm{e}^{\frac{p+1}{(p-1)^2}\langle M\rangle(\tau_n)}\Big)^{\frac12} \\ &= \Big(\mathbb{E}_{\mathbb{Q}_n} \mathrm{e}^{\frac{p+1}{(p-1)^2}\langle M\rangle(\tau_n)}\Big)^{\frac12}. \end{aligned}$$

Combining these with (2.24) and using Jensen's inequality, we finish the proof. □

Lemma 2.3.3 *Assume* (2.18) *and* (2.19). *Let*

$$\lambda(T) = \frac{1}{8\max_{t\in[0,T]} \|\sigma(t)\|_{HS}^2 \mathrm{e}^{-\int_0^t (\gamma(s)+1)\mathrm{d}s}}, \quad \tilde\lambda(T) = \mathrm{e}^{-T}\lambda(T) \min_{t\in[0,T]} \tilde\delta(t).$$

Then

$$\mathbb{E}\exp\bigg[\tilde\lambda(T)\int_0^T h(X(t))^{\alpha+1}\mathrm{d}t\bigg] \le \exp\bigg[2\lambda(T)|x|^2 + 2\lambda(T)\int_0^T \tilde q(t)\mathrm{d}t\bigg].$$

Proof. By (2.18) and Itô's formula,

$$\begin{aligned} \mathrm{d}|X(t)|^2 \le \ & \big\{q(t) - \delta(t) h(X(t))^{\alpha+1} + \gamma(t)|X(t)|^2\big\}\mathrm{d}t \\ & + 2\langle X(t), \sigma(t)\mathrm{d}W(t)\rangle. \end{aligned} \tag{2.25}$$

So

$$\begin{aligned}\mathrm{d}\big\{|X(t)|^2\mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}\big\} \le\ & \mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}\big\{q(t)-|X(t)|^2\big\}\mathrm{d}t\\ &+2\mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}\langle X(t),\sigma(t)\mathrm{d}W(t)\rangle.\end{aligned}$$

This and the fact that $\mathbb{E}\mathrm{e}^{M(t)} \le (\mathbb{E}\mathrm{e}^{2\langle M\rangle(t)})^{\frac{1}{2}}$ for a continuous martingale $M(t)$ imply that

$$\begin{aligned}&\mathbb{E}\exp\left[\lambda(T)\int_0^T \mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}|X(t)|^2\mathrm{d}t-\lambda(T)|x|^2-\lambda(T)\int_0^T\tilde{q}(t)\mathrm{e}^{-t}\mathrm{d}t\right]\\ &\le\left(\mathbb{E}\exp\left[8\lambda(T)^2\int_0^T\mathrm{e}^{-2\int_0^t(\gamma(s)+1)\mathrm{d}s}\|\sigma(t)\|_{HS}^2|X(t)|^2\mathrm{d}t\right]\right)^{\frac{1}{2}}\\ &\le\left(\mathbb{E}\exp\left[\lambda(T)\int_0^T\mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}|X(t)|^2\mathrm{d}t\right]\right)^{\frac{1}{2}}.\end{aligned}$$

By an approximation argument, we may assume that the expectations in this display are finite, so that

$$\begin{aligned}&\mathbb{E}\exp\left[\lambda(T)\int_0^T\mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}|X(t)|^2\mathrm{d}t\right]\\ &\le\exp\left[2\lambda(T)|x|^2+2\lambda(T)\int_0^T\tilde{q}(t)\mathrm{e}^{-t}\mathrm{d}t\right].\end{aligned}\tag{2.26}$$

Combining (2.26) with (2.25), we obtain

$$\begin{aligned}&\mathbb{E}\exp\left[\tilde{\lambda}(T)\int_0^T h(X(t))^{\alpha+1}\mathrm{d}t\right]\le\mathbb{E}\exp\left[\lambda(T)\mathrm{e}^{-T}\int_0^T\tilde{\delta}(t)h(X(t))^{\alpha+1}\mathrm{d}t\right]\\ &\le\exp\left[\lambda(T)\mathrm{e}^{-T}|x|^2+\lambda(T)\mathrm{e}^{-T}\int_0^T\tilde{q}(t)\mathrm{d}t\right]\\ &\qquad\times\mathbb{E}\exp\left[2\lambda(T)\mathrm{e}^{-T}\int_0^T\mathrm{e}^{-\int_0^t\gamma(s)\mathrm{d}s}\langle X(t),\sigma(t)\mathrm{d}W(t)\rangle\right]\\ &\le\exp\left[\lambda(T)|x|^2+\lambda(T)\int_0^T\tilde{q}(t)\mathrm{e}^{-t}\mathrm{d}t\right]\\ &\qquad\times\left(\mathbb{E}\exp\left[\lambda(T)\int_0^T\mathrm{e}^{-\int_0^t(\gamma(s)+1)\mathrm{d}s}|X(t)|^2\mathrm{d}t\right]\right)^{\frac{1}{2}}\\ &\le\exp\left[2\lambda(T)|x|^2+2\lambda(T)\int_0^T\tilde{q}(t)\mathrm{e}^{-t}\mathrm{d}t\right].\end{aligned}$$

□

Proof of Theorem 2.3.1. According to Theorem 1.1.1 and Lemma 2.3.2, we have only to estimate the expectation and the exponential moment of $\int_0^T(h(X(t))\vee h(Y(t)))^{\alpha+1}\mathrm{d}t$ with respect to $\mathrm{d}\mathbb{Q}:=R\mathrm{d}\mathbb{P}$.

(1) By (2.18) and Itô's formula, we have

$$\mathrm{d}|Y(t)|^2 \le \{q(t) - \delta(t)h(Y(t))^{\alpha+1} + \gamma(t)|Y(t)|^2\}\mathrm{d}t + 2\langle Y(t), \sigma(t)\mathrm{d}\tilde{W}_n(t)\rangle, \ t \le \tau_n.$$

Then

$$\mathbb{E}_{\mathbb{Q}_n}\int_0^{\tau_n} h(Y(t))^{\alpha+1}\mathrm{d}t \le \frac{|y|^2 + \int_0^T \tilde{q}(t)\mathrm{d}t}{\min_{t\in[0,T]}\tilde{\delta}(t)}, \tag{2.27}$$

$$\mathbb{E}_{\mathbb{Q}_n}|Y(t\wedge\tau_n)|^2 \le \left(|y|^2 + \int_0^t \tilde{q}(s)\mathrm{d}s\right)\mathrm{e}^{\int_0^t \gamma(s)\mathrm{d}s}, \ t \in [0,T]. \tag{2.28}$$

Next, by (2.18) and Itô's formula, we have

$$\begin{aligned}
\mathrm{d}|X(t)|^2 &\le \{q(t) - \delta(t)h(X(t))^{\alpha+1} + \gamma(t)|X(t)|^2\}\mathrm{d}t + 2\langle X(t), \sigma(t)\mathrm{d}W(t)\rangle \\
&\le \Big\{q(t) + \gamma(t)|X(t)|^2 - \delta(t)h(X(t))^{\alpha+1} - \frac{2\xi(t)\langle X(t), X(t)-Y(t)\rangle}{|X(t)-Y(t)|^\varepsilon}\Big\}\mathrm{d}t \\
&\quad + 2\langle X(t), \sigma(t)\mathrm{d}\tilde{W}_n(t)\rangle, \quad t \le \tau_n.
\end{aligned} \tag{2.29}$$

Noting that

$$-\frac{\langle u, u-v\rangle}{|u-v|^\varepsilon} \le |v|\cdot|u-v|^{1-\varepsilon} - |u-v|^{2-\varepsilon} \le \frac{|v|^{2-\varepsilon}}{1-\varepsilon}\Big(\frac{1-\varepsilon}{2-\varepsilon}\Big)^{2-\varepsilon},$$

we obtain

$$\begin{aligned}
\mathrm{d}\big\{|X(t)|^2\mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\big\} &\le 2\mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\langle X(t), \sigma(t)\mathrm{d}\tilde{W}_n(t)\rangle \\
&+ \mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\Big\{q(t) - \delta(t)h(X(t))^{\alpha+1} + \frac{2\xi(t)|Y(t)|^{2-\varepsilon}}{1-\varepsilon}\Big(\frac{1-\varepsilon}{2-\varepsilon}\Big)^{2-\varepsilon}\Big\}\mathrm{d}t, \ t \le \tau_n.
\end{aligned}$$

Combining this with (2.28) and (2.20), we arrive at

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_n}\int_0^{\tau_n} h(X(t))^{\alpha+1}\mathrm{d}t \\
&\le \frac{|x|^2 + \int_0^T \mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}\{q(t) + \frac{2\xi(t)}{1-\varepsilon}(\frac{1-\varepsilon}{2-\varepsilon})^{2-\varepsilon}(\mathbb{E}_{\mathbb{Q}_n}|Y(t\wedge\tau_n)|^2)^{\frac{2-\varepsilon}{2}}\}\mathrm{d}t}{\min_{t\in[0,T]}\delta(t)\mathrm{e}^{-\int_0^t \gamma(s)\mathrm{d}s}} \\
&\le \frac{|x|^2 + \int_0^T \tilde{q}(t)\mathrm{d}t + \frac{2|x-y|^\varepsilon}{\varepsilon(1-\varepsilon)}(\frac{1-\varepsilon}{2-\varepsilon})^{2-\varepsilon}(|y|^2 + \int_0^T \tilde{q}(t)\mathrm{d}t)^{\frac{2-\varepsilon}{2}}}{\min_{[0,T]}\tilde{\delta}}.
\end{aligned}$$

Combining this with (2.27), we conclude that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_n}\int_0^{\tau_n}\big(h(X(t)) \vee h(Y(t))\big)^{\alpha+1}\mathrm{d}t \\
&\le \frac{|x|^2 + |y|^2 + 2\int_0^T \tilde{q}(t)\mathrm{d}t + \frac{2|x-y|^\varepsilon}{\varepsilon(1-\varepsilon)}(\frac{1-\varepsilon}{2-\varepsilon})^{2-\varepsilon}(|y|^2 + \int_0^T \tilde{q}(t)\mathrm{d}t)^{\frac{2-\varepsilon}{2}}}{\min_{t\in[0,T]}\tilde{\delta}(t)} \\
&= c_2(\theta, T, x, y).
\end{aligned}$$

Thus, by Lemma 2.3.2, we obtain

$$\mathbb{E}\{R_n \log R_n\} \le \frac{|x-y|^2}{2}\Big(\frac{\theta+2}{\theta}\Big)^{\frac{2(\theta+1)}{\theta}} \frac{c_2(\theta,T,x,y)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}}{c_1(\theta,T)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}.$$

Letting $n \uparrow \infty$, we conclude that R given in (2.10) is a probability density with

$$\mathbb{E}\{R \log R\} \le \frac{|x-y|^2}{2}\Big(\frac{\theta+2}{\theta}\Big)^{\frac{2(\theta+1)}{\theta}} \frac{c_2(\theta,T,x,y)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}}{c_1(\theta,T)^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}.$$

Let $\mathrm{d}\mathbb{Q} = R\mathrm{d}\mathbb{P}$. Then the first log-Harnack inequality in (1) follows from Theorem 1.1.1. As for the second inequality, by the Markov property and Jensen's inequality, we need consider only $T \in (0,1]$. In this case, the second inequality follows from the first, since for some constants $C_1, C_2, C_3 > 0$, we have $c_1(\theta,T) \ge C_1 T$ and

$$c_2(\theta,T,x,y) \le C_2\Big\{|x|^2+|y|^2+T+|x-y|^{\frac{\theta}{\theta+2}}\big(|y|^2+T\big)^{\frac{\theta+4}{2(\theta+2)}}\Big\} \le C_3\Big\{|x|^2+|y|^2+T\Big\}.$$

(2) Again we need to prove the result only for $T \in (0,1]$. According to Theorem 1.1.1, it suffices to find a constant $C>0$ such that for every $T \in (0,1]$, $p>1$, and $x,y \in \mathbb{H}$,

$$\begin{aligned}&\big(\mathbb{E}R^{\frac{p}{p-1}}\big)^{p-1} \\ &\le \exp\left[C\left(\frac{p|x-y|^2(1+|x|^2+|y|^2)}{(p-1)T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + \frac{\big(\frac{p}{p-1}\big)^{\frac{\alpha\theta+\theta+2-2\alpha}{\alpha(\theta+2)+\theta-2}}|x-y|^{\frac{2\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}}{T^{\frac{\theta(\alpha+1)+4}{\alpha(\theta+2)+\theta-2}}}\right)\right].\end{aligned} \tag{2.30}$$

By Lemma 2.3.3 with $n \uparrow \infty$ and noting that the distribution of $Y(t)$ under $\mathbb{Q}$ coincides with that of $X^y(t)$ under $\mathbb{P}$, there exists a constant $C_1 > 0$ such that for every $r>0$ and $s,T \in (0,1]$,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}} \exp\left[r\left(\int_0^T h(Y(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right] \\ &\le \mathbb{E}_{\mathbb{Q}} \exp\left[s\tilde{\lambda}(T)\int_0^T h(Y(t))^{\alpha+1}\mathrm{d}t + C_1 r^{\frac{\theta(1+\alpha)}{\alpha(\theta+2)+\theta-2}}\big(\tilde{\lambda}(T)s\big)^{-\frac{2(1-\alpha)}{\alpha(\theta+2)+\theta-2}}\right] \\ &\le \exp\left[C_1 s(|y|^2+1) + C_1 r^{\frac{\theta(1+\alpha)}{\alpha(\theta+2)+\theta-2}}\big(\tilde{\lambda}(T)s\big)^{-\frac{2(1-\alpha)}{\alpha(\theta+2)+\theta-2}}\right].\end{aligned}$$

Taking $s = 1 \wedge r$, we see that $\inf_{T\in(0,1]}\tilde{\lambda}(T) > 0$ implies

$$r^{\frac{\theta(1+\alpha)}{\alpha(\theta+2)+\theta-2}}\big(\tilde{\lambda}(T)s\big)^{-\frac{2(1-\alpha)}{\alpha(\theta+2)+\theta-2}} \le C'\Big(r + r^{\frac{\theta(1+\alpha)}{\alpha(\theta+2)+\theta-2}}\Big)$$

for some constant $C' > 0$ and all $r > 0$, $T \in (0,1]$. Hence,

$$\mathbb{E}_{\mathbb{Q}} \exp\left[r\left(\int_0^T h(Y(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right] \le \exp\left[C_2\left(r + r|y|^2 + r^{\frac{\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}\right)\right]$$

holds for some constant $C_2 > 0$ and all $r > 0$, $T \in (0,1]$. Consequently, for every constant $r > 0$ (recall that the distribution of Y under $\mathbb{Q}$ coincides with that of X^y under $\mathbb{P}$),

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{r|x-y|^2}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\left(\int_0^T h(Y(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right] \\
&\le \exp\left[\frac{C_2 r|x-y|^2(|y|^2+1)}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + C_2\left(\frac{r|x-y|^2}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\right)^{\frac{\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}\right],
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
&\mathbb{E} \exp\left[\frac{r|x-y|^2}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\left(\int_0^T h(X(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right] \\
&\le \exp\left[\frac{C_2 r|x-y|^2(|x|^2+1)}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + C_2\left(\frac{r|x-y|^2}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\right)^{\frac{\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}\right].
\end{aligned} \tag{2.32}$$

Combining these with Lemma 2.3.2 and noting that $\frac{p+1}{p} \le 2$, we obtain for some constants $c, C_3 > 0$ that

$$\begin{aligned}
&\left(\mathbb{E}R^{\frac{p}{p-1}}\right)^4 \le \left(\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{cp|x-y|^2}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\left(\int_0^T \{h(X(t)) \vee h(Y(t))\}^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right]\right)^2 \\
&\le \left(\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{2cp|x-y|^2}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\left(\int_0^T h(X(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right]\right) \\
&\qquad \times \left(\mathbb{E}_{\mathbb{Q}} \exp\left[\frac{2cp|x-y|^2}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\left(\int_0^T h(Y(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right]\right) \\
&\le \left(\mathbb{E}R^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\mathbb{E} \exp\left[\frac{2cp^2|x-y|^2}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\left(\int_0^T h(X(t))^{\alpha+1}\mathrm{d}t\right)^{\frac{2(1-\alpha)}{\theta(\alpha+1)}}\right]\right)^{\frac{1}{p}} \\
&\qquad \times \exp\left[\frac{2C_2 cp|x-y|^2(|y|^2+1)}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + C_2\left(\frac{2cp|x-y|^2}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}\right)^{\frac{\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}\right] \\
&\le \left(\mathbb{E}R^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&\times \exp\left[C_3\left(\frac{p|x-y|^2(1+|x|^2+|y|^2)}{(p-1)^2 T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}} + \frac{\left(\frac{p}{p-1}\right)^{\frac{\alpha\theta+\theta+2-2\alpha}{\alpha(\theta+2)+\theta-2}}|x-y|^{\frac{2\theta(\alpha+1)}{\alpha(\theta+2)+\theta-2}}}{(p-1)T^{\frac{\theta(\alpha+1)+4}{\alpha(\theta+2)+\theta-2}}}\right)\right].
\end{aligned}$$

This implies (2.30) for some constant $C > 0$. $\square$

2.4 Applications to Specific Models

In this section we apply Theorems 2.2.1, 2.3.1 and Corollary 2.2.4 to specific models presented in Sect. 2.1.

2.4.1 Stochastic Generalized Porous Media Equations

Let $(E,\mathscr{B},\mathbf{m})$ be a separable probability space and $(L,\mathscr{D}(L))$ a negative definite self-adjoint linear operator on $L^2(\mathbf{m})$ having discrete spectrum. Let

$$(0<)\lambda_1 \le \lambda_2 \le \cdots$$

be all the eigenvalues of $-L$ including multiplicities with unit eigenfunctions $\{e_i\}_{i\ge 1}$.

Let $\mathbb{H}$ be the dual space of $\mathscr{D}((-L)^{\frac{1}{2}})$ with respect to $L^2(\mathbf{m})$; i.e., $\mathbb{H}$ is the completion of $L^2(\mathbf{m})$ under the inner product

$$\langle x,y\rangle := \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\langle x,e_i\rangle_2 \langle y,e_i\rangle_2.$$

Recall that $\langle\ ,\ \rangle_2$ is the inner product in $L^2(\mathbf{m})$. Let $W(t)$ be the cylindrical Brownian motion on $\mathbb{H}$ with respect to a complete filtered probability space $(\Omega,\{\mathscr{F}_t\}_{t\ge 0},\mathbb{P})$. Let

$$\Psi,\Phi : [0,\infty)\times\mathbb{R}\to\mathbb{R}$$

be measurable and continuous in the second variable, and let

$$\sigma : [0,\infty)\to\mathscr{L}_{HS}(\mathbb{H})$$

be measurable such that $\|\sigma\|_{HS}\in L^2_{loc}([0,\infty);\mathrm{d}t)$. We consider the equation

$$\mathrm{d}X(t) = \big\{L\Psi(t,X(t))+\Phi(t,X(t))\big\}\mathrm{d}t+\sigma(t)\mathrm{d}W(t). \tag{2.33}$$

Let $\alpha\ge 1$ be a fixed number and assume that L^{-1} is bounded in $L^{\alpha+1}(\mathbf{m})$ if $\Phi\ne 0$. It is easy to see that this assumption holds if L is a Dirichlet operator. To ensure the existence and uniqueness of the solution to (2.33), we assume (2.2) and (2.3) as in Example 2.1.1.

Theorem 2.4.1 *Assume $\|\sigma(\cdot)\|_{HS}\in L^2_{loc}([0,\infty);\mathrm{d}t)$, (2.2), and (2.3). If there exists a nonnegative constant $\theta\in[2,\infty)\cap(\alpha-1,\infty)$ such that*

$$\|x\|^{\alpha+1}_{L^{\alpha+1}(\mathbf{m})} \ge \xi(t)\|x\|^{\theta}_{\sigma(t)}|x|^{\alpha+1-\theta},\quad x\in L^{\alpha+1}(\mathbf{m}),\ t\ge 0, \tag{2.34}$$

for some strictly positive function $\xi\in C([0,\infty))$, then:

(1) *The Harnack inequalities in Theorem 2.2.1 hold for* $\eta(t) := \delta(t)\xi(t)$.
(2) *When the equation is time-homogeneous, i.e.,* $\sigma(t)$, $\Psi(t,\cdot)$, *and* $\Phi(t,\cdot)$ *are independent of* t *such that* δ, γ, *and* ξ *are constants, and either* $\alpha > 1$ *or* $\alpha = 1$ *but* $\delta\lambda_1 > \gamma$, *then* P_t *has a unique invariant probability measure* μ *that has full support such that (2.13) and the assertions in Corollary 2.2.4 hold.*

Proof. Simply note that (2.3) and (2.34) imply (2.5) for $\eta(t) = \delta(t)\xi(t)$, and since $\|\cdot\|^2_{L^2(\mathbf{m})} \geq \lambda_1|\cdot|^2$, $\alpha > 1$ or $\alpha = 1$ but $\delta\lambda_1 > \gamma$ implies (2.12) for some $C \geq 0$ and $\delta > 0$. $\square$

It is easy to construct examples of Φ and Ψ such that condition (2.3) holds. Below we present a simple example to illustrate condition (2.34) and that $\sigma \in \mathscr{L}_{HS}(\mathbb{H})$, so that Theorem 2.4.1 applies.

Example 2.4.1 Let $\sigma e_i = \sigma_i e_i$, $i \geq 1$, for a sequence $\{\sigma_i\}_{i\geq 1} \subset \mathbb{R}$. If

$$\sum_{i=1}^{\infty} \sigma_i^2 < \infty, \quad \inf_{i\geq 1}\{\lambda_i\sigma_i^2\} > 0, \tag{2.35}$$

then $\sigma \in \mathscr{L}_{HS}(\mathbb{H})$ and (2.34) holds for some constant $\xi > 0$. Indeed, the first inequality implies that $\sigma \in \mathscr{L}_{HS}(\mathbb{H})$, while the second inequality implies (2.34) for some $\xi > 0$, since

$$\|\cdot\|^2_{L^{\alpha+1}(\mathbf{m})} \geq \|\cdot\|^2_{L^2(\mathbf{m})} \geq \inf_{i\geq 1}(\sigma_i^2\lambda_i)\|\cdot\|^2_{\sigma}.$$

In particular, (2.35) holds if $\sum_{i=1}^{\infty}\frac{1}{\lambda_i} < \infty$ and

$$\frac{1}{C\lambda_i} \leq \sigma_i^2 \leq \frac{C}{\lambda_i}, \quad i \geq 1,$$

holds for some constant $C \geq 1$. When $\sum_{i\geq 1}\frac{1}{\lambda_i} < \infty$, one may easily choose $(\sigma_i)_{i\geq 1}$ satisfying (2.35). That is the case for $L = \Delta$ the Dirichlet Laplacian on a bounded interval.

2.4.2 Stochastic p-Laplacian Equations

We simply consider the equation in Example 2.1.2 for $p \geq 2$, $d = 1$, and $\mathbf{m}(\mathrm{d}x) = \mathrm{d}x$ on $(0,1)$. In this case, we have $\mathbb{H} = L^2(\mathbf{m})$ and $\mathbb{V} = \mathbb{H}_0^{1,p}$. Let Δ be the Dirichlet Laplacian on $(0,1)$. Then $\{(\pi i)^2\}_{i\geq 1}$ are all eigenvalues of $-\Delta$ with unit eigenfunctions $e_i(x) := \sqrt{2}\sin(i\pi x)$. Assume that there exist a sequence of constants $(\sigma_i)_{i\geq 1}$ and a constant $c > 0$ such that

$$\sigma e_i = \sigma_i e_i, \quad \sigma_i^2 \geq \frac{c}{i^4}, \quad \sum_{i\geq 1}\sigma_i^2 < \infty. \tag{2.36}$$

Then $b(t,u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. According to Example 3.3 in [27], we have

$$2_{\mathbb{V}^*}\langle b(t,u) - b(t,v), u - v\rangle_{\mathbb{V}} \le -2^{p-1}\mathbf{m}(|\nabla(u-v)|^2)^{\frac{p}{2}}.$$

Since by (2.36),

$$\mathbf{m}(|\nabla(u-v)|^2) = \sum_{i\ge 1}\pi^{2i}\mathbf{m}((u-v)e_i)^2 \ge c\|u-v\|_\sigma^2,$$

we obtain

$$2_{\mathbb{V}^*}\langle b(t,u) - b(t,v), u - v\rangle_{\mathbb{V}} \le -2^{p-1}c^{\frac{p}{2}}\|u-v\|_\sigma^p.$$

So the assertion in Theorem 2.2.1 holds for $\theta = p$, $\alpha = p-1$, $\eta(t) = 2^{p-1}c^{\frac{p}{2}}$, and $\gamma(t) = 0$. Therefore, there exists a constant $C > 0$ such that for every positive $f \in \mathscr{B}_b(\mathbb{H})$, $x, y \in \mathbb{H}$, and $T > 0$, we have

$$P_T \log f(y) \le \log P_T f(x) + \frac{C|x-y|^{\frac{4}{p}}}{T^{\frac{p+2}{p}}},$$

and for every $p' > 1$,

$$(P_T f(y))^{p'} \le (P_T f^{p'}(x)) \exp\left[\frac{p'C|x-y|^{\frac{4}{p}}}{(p'-1)T^{\frac{p+2}{p}}}\right].$$

2.4.3 Stochastic Generalized Fast-Diffusion Equations

Let $(E,\mathscr{B},\mathbf{m}), (L,\mathscr{D}(L))$, $\mathbb{H}$, σ, and $W(t)$ be as in Sect. 2.4.1. We consider the equation

$$\mathrm{d}X(t) = \Big\{L\Psi(t,X(t)) + \beta(t)X(t)\Big\}\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \tag{2.37}$$

where $\beta \in C([0,\infty))$, $\Psi : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is measurable, continuous in the second variable, such that for some constant $\alpha \in (0,1)$ and strictly positive $\zeta, \delta \in C([0,\infty))$,

$$2\big(\Psi(t,s_1) - \Psi(t,s_2)\big)(s_1 - s_2) \ge \zeta(t)|s_1 - s_2|^2(|s_1| \vee |s_2|)^{\alpha-1}, \quad s_1, s_2 \in \mathbb{R}, t \ge 0, \tag{2.38}$$

$$s\Psi(t,s) \ge \delta(t)|s|^{\alpha+1}, \quad \sup_{t\in[0,T], s\ge 0}\frac{|\Psi(t,s)|}{1+|s|^\alpha} < \infty, \quad s \in \mathbb{R}, t \ge 0. \tag{2.39}$$

By the mean value theorem and $\alpha < 1$, one has $(s_1 - s_2)(s_1^\alpha - s_2^\alpha) \ge \alpha|s_1 - s_2|^2(|s_1| \vee |s_2|)^{\alpha-1}$. So a simple example of Ψ for these conditions to hold is $\Psi(t,s) = s^\alpha := |s|^\alpha \mathrm{sgn}(s)$, for which (2.37) is known as the stochastic fast-diffusion equation with linear drift.

Let $\mathbb{V} = L^{\alpha+1}(\mathbf{m}) \cap \mathbb{H}$. It is easy to see that **(A2.1)–(A2.4)** hold (see [40, Theorem 3.9] for a more general result). Let $X^x(t)$ be the unique solution for $X(0) = x \in \mathbb{H}$, and let P_t be the associated Markov operator.

Theorem 2.4.2 *Assume that (2.38) and (2.39) hold for some strictly positive functions $\zeta, \delta \in C([0,\infty))$. If there exist a constant $\theta \geq \frac{4}{\alpha+1}$ and a strictly positive function $\xi \in C([0,\infty))$ such that*

$$\|u\|^2_{L^{\alpha+1}(\mathbf{m})} \cdot |u|^{\theta-2} \geq \xi(t)\|u\|^{\theta}_{\sigma(t)}, \quad u \in L^{\alpha+1}(\mathbf{m}),\ t \geq 0, \tag{2.40}$$

then the assertions in Theorem 2.3.1 hold for

$$\gamma(t) := 2\beta(t),\ \ q(t) := \|\sigma(t)\|^2_{HS},\ \ \eta(t) := 2^{\frac{\alpha-1}{1+\alpha}}\zeta(t)\xi(t).$$

Proof. Let $h(u) := \|u\|_{L^{\alpha+1}(\mathbf{m})}$. Obviously, (2.37) and the first inequality in (2.39) imply (2.18) for $q(t) = \|\sigma(t)\|^2_{HS}$ and $\gamma(t) = 2\beta(t)$. It remains to verify (2.19) for the above η and h. By Hölder's inequality, we have

$$\begin{aligned}\|u-v\|^{\alpha+1}_{L^{\alpha+1}(\mathbf{m})} &= \mathbf{m}(|u-v|^{\alpha+1}) \leq \mathbf{m}\big(|u-v|^2(|u|\vee|v|)^{\alpha-1}\big)^{\frac{\alpha+1}{2}}\mathbf{m}\big((|u|\vee|v|)^{\alpha+1}\big)^{\frac{1-\alpha}{2}}\\ &\leq 2^{\frac{1-\alpha}{2}}\mathbf{m}\big(|u-v|^2(|u|\vee|v|)^{\alpha-1}\big)^{\frac{\alpha+1}{2}}(h(u)\vee h(v))^{\frac{1-\alpha^2}{2}}.\end{aligned}$$

Combining this with (2.38) and (2.40), we prove (2.19) for $\eta(t) = 2^{\frac{\alpha-1}{1+\alpha}}\zeta(t)\xi(t)$. $\square$

To conclude this subsection, we consider the stochastic fast-diffusion equation for which $\Psi(t,s) = s^{\alpha}$ for some $\alpha \in (0,1)$.

Corollary 2.4.3 *Consider (2.37) for $\Psi(t,s) = s^{\alpha} := |s|^{\alpha}\mathrm{sgn}(s)$. Let $(-L, \mathscr{D}(L))$ be a nonnegative definite self-adjoint operator on $L^2(\mathbf{m})$ with discrete spectrum $(0 <)\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \uparrow \infty$, counting multiplicities. Let $\{e_n\}_{n\geq 1}$ be the corresponding eigenvectors, which form an orthonormal basis of $L^2(\mathbf{m})$. Let $\sigma(t) = \sigma$ be such that*

$$\sigma e_i = \sigma_i e_i,\ \ i \geq 1,$$

for some constants $\{\sigma_i\}_{i\geq 1}$ satisfying

$$\|\sigma\|^2_{HS} = \sum_{i=1}^{\infty}\sigma_i^2 < \infty. \tag{2.41}$$

If there exist constants $\varepsilon \in (0,1), \theta \geq \frac{4}{\alpha+1}$ and $C_1, C_2 > 0$ such that

$$|\sigma_i| \geq C_1\lambda_i^{-\frac{1-\varepsilon}{\theta}},\ \ i \geq 1, \tag{2.42}$$

and the Nash inequality

$$\|f\|^{2+\frac{4}{d}}_{L^2(\mathbf{m})} \leq -C_2\mathbf{m}(fLf),\ \ f \in \mathscr{D}(L),\ \mathbf{m}(|f|) = 1, \tag{2.43}$$

holds for some $d \in (0, \frac{2\varepsilon(\alpha+1)}{1-\alpha})$*, then the assertions in Theorem 2.3.1 hold for*

$$\gamma(t) = 2\beta(t), \ \ q(t) = \|\sigma\|_{HS}^2, \ \ \delta(t) = 1, \ \text{ and } \eta(t) = c,$$

for some constant $c > 0$*. In particular, if* $\beta \le 0$*, we may take* $\gamma = 0$ *such that the log-Harnack inequality*

$$P_T \log f(y) - \log P_T f(x) \le \frac{C|x-y|^2(|x|^2 + |y|^2 + T)^{\frac{2(1-\alpha)}{\theta(1+\alpha)}}}{T^{\frac{\theta(\alpha+1)+4}{\theta(\alpha+1)}}}$$

holds for some constant $C > 0$ *and all* $T > 0$*,* $x, y \in \mathbb{H}$*, and positive* $f \in \mathscr{B}_b(\mathbb{H})$*.*

Proof. By Jensen's inequality, it suffices to prove the result for $T \in (0,1]$. Obviously, (2.38) and (2.39) hold for $\zeta(t) = 2\alpha$ and $\delta(t) = 1$. To apply Theorem 2.4.2, it remains to verify (2.40). By (2.42), we have

$$\begin{aligned} \|x\|_\sigma^\theta &= \Big(\sum_{i\ge 1} \frac{\mathbf{m}(xe_i)^2}{\sigma_i^2 \lambda_i}\Big)^{\frac{\theta}{2}} \le \Big(\sum_{i\ge 1} \frac{\mathbf{m}(xe_i)^2}{|\sigma_i|^\theta \lambda_i}\Big)\Big(\sum_{i\ge 1} \frac{\mathbf{m}(xe_i)^2}{\lambda_i}\Big)^{\frac{\theta-2}{2}} \\ &\le C_1^{-\theta} |x|^{\theta-2} \Big(\sum_{i\ge 1} \frac{\mathbf{m}(xe_i)^2}{\lambda_i^\varepsilon}\Big). \end{aligned}$$

According to the proof of Corollary 3.2 in [31], (2.43) for some $d \in (0, \frac{2\varepsilon(\alpha+1)}{1-\alpha})$ implies that

$$\|x\|_{\alpha+1}^2 \ge c \sum_{i\ge 1} \frac{\mathbf{m}(xe_i)^2}{\lambda_i^\varepsilon}$$

holds for some constant $c > 0$. Therefore, (2.40) holds for $\xi(t) = c'$ for some constant $c' > 0$. Combining these with (2.41) and using Theorem 2.4.2, we may apply Theorem 2.3.1 to

$$\gamma(t) = 2\beta(t), \ \ q(t) = \|\sigma\|_{HS}^2, \ \ \delta(t) = 1, \ \text{ and } \eta(t) = c,$$

for some constant $c > 0$. Finally, if $\gamma = 0$, then it is easy to see that for some constants $C_1', C_2' > 0$,

$$c_1(\theta, T) \ge C_1' T, \ \ c_2(\theta, T, x, y) \le C_2'(|x|^2 + |y|^2 + T), \quad T > 0, x, y \in \mathbb{H},$$

which implies the desired log-Harnack inequality according to Theorem 2.3.1(1). □

Example 2.4.2 *Let* $\Psi(t,s) = s^\alpha$ *for some* $\alpha \in (\frac{1}{3}, 1)$*, and let* $L = \Delta$ *be the Dirichlet Laplacian on the open interval* $(0, \pi)$*. Then* $\lambda_i = i^2$ *and the Nash inequality (2.43) holds for* $d = 1$*. For every* $\theta \in (\frac{4}{\alpha+1}, \frac{6\alpha+2}{\alpha+1})$ *and* $\varepsilon \in (\frac{1-\alpha}{2(\alpha+1)}, 1 - \frac{\theta}{4})$*, we have* $d = 1 \in (0, \frac{2\varepsilon(\alpha+1)}{1-\alpha})$*, as required. So taking* $\sigma_i = i^{-\frac{2(1-\varepsilon)}{\theta}}$*, we see that (2.41) and (2.42) are satisfied. Therefore, the assertions in Corollary 2.4.3 hold.*

http://www.springer.com/978-1-4614-7933-8

Harnack Inequalities for Stochastic Partial Differential Equations
Wang, F.-Y.
2013, X, 125 p., Softcover
ISBN: 978-1-4614-7933-8