Chapter 2
A Compact Symmetric Space: The Sphere

It is important that you establish a steady, rhythmic tempo while doing the exercises and that you rest as little as possible between them. Also remember not to strain yourself.

—From *Jane Fonda's Year of Fitness and Health/1984, Desk Diary*, Simon and Schuster, NY, 1983. Reprinted by permission.

2.1 Fourier Analysis on the Sphere

2.1.1 The Sphere as a Symmetric Space

Whenever there is a large earthquake the Earth vibrates for days afterwards. The vibrations consist of the superposition of the elastic–gravitational normal modes of the Earth that are excited by the earthquake.

—From F. Gilbert [212, p. 107].

A (surface or Laplace) **spherical harmonic** is an eigenfunction of the Laplacian on the sphere. These are the analogues of exponentials for Fourier analysis on the sphere. Laplace and Legendre introduced these functions in order to study gravitational theory in the 1780s. Spherical harmonics are necessary for the analysis of any phenomena with spherical symmetry; e.g., earthquakes, the hydrogen atom, and the solar corona. Some of these topics will be discussed later in this section.

Since our treatment of harmonic analysis on the sphere is rather condensed, the reader may want to consult some of the following references for more information: Coifman and Weiss [98], Courant and Hilbert [111], Dym and McKean [147], Erdélyi et al. [164], Lebedev [401], Müller [481], Sugiura [648], Talman [655], Wawrzyńczyk [722], and Vilenkin [704]. For the history of the subject, see Wangerin [716].

Before discussing spherical harmonics, we need to understand something about the geometry of the sphere. This symmetric space is closely related to the **orthogonal group** $O(n)$ of real $n \times n$ matrices U such that ${}^tUU = I$, where ${}^tU$ denotes the transpose of U and I denotes the $n \times n$ identity matrix. The **special orthogonal**

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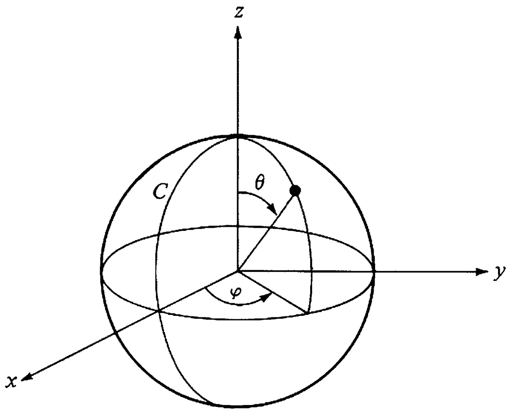

Fig. 2.1 Angular coordinates on the unit sphere

group, $SO(n)$, is the subgroup of matrices U in $O(n)$ such that the determinant of U is one. You can regard U in $SO(n-1)$ as an element of $SO(n)$ by forming

$$\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \text{ in } SO(n).$$

The group $O(n)$ is a compact group and the sphere is a compact symmetric space, making some of the analysis on it somewhat easier.

Exercise 2.1.1 (The Sphere as a Quotient or Homogeneous Space). Consider the sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Show that S^{n-1} can be identified with the quotient space $SO(n)/SO(n-1)$, using the preceding identification of $SO(n-1)$ as a subgroup of $SO(n)$.

Hint. *Map the coset $gSO(n-1)$ to the vector ge_n for g in $SO(n-1)$ and $e_n = {}^t(0,\ldots,0,1)$.*

You may also want to read the discussion of the topology of spheres and orthogonal groups in Chevalley [85, pp. 52–67]. In particular, the fundamental (or Poincaré) group of $SO(2)$ is isomorphic to $\mathbb{Z}$, while that of $SO(n)$, $n \geq 3$, has order 2.

From now on we shall consider only the sphere S^2. See the references for the general case. Now the sphere S^2 is a **differentiable manifold.** This means that locally it looks like two-dimensional Euclidean space. To make this precise, we use the usual angular coordinates (θ, φ), $0 < \varphi < 2\pi$, $0 < \theta < \pi$, pictured in Fig. 2.1, to parameterize S^2 except for the semi-circle C through the poles and $(1,0,0)$. A similar coordinate patch can be constructed to cover the rest of the sphere. The equations for the rectangular coordinates (x,y,z) of a point on S^2 in terms of the angular coordinates are

$$\left.\begin{aligned} x &= \sin\theta\cos\varphi \\ y &= \sin\theta\sin\varphi \\ z &= \cos\theta \end{aligned}\right\} \tag{2.1}$$

More information on differentiable manif olds can be found in Yvonne Choquet-Bruhat et al. [87], Helgason [276–278], Loomis and Sternberg [428], Singer and Thorpe [604], or Spivak [614], for example.

Before proceeding further with our discussion of spherical geometry, let us review how to change variables in the Laplacian, using the method of Courant and Hilbert [111, pp. 224–225]. Assume that the substitution mapping (x,y,z) to (u_1,u_2,u_3) is differentiable with differentiable inverse. Let the **Jacobian matrix** of the change of variables be

$$A = \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}. \tag{2.2}$$

Then the **volume element** is

$$dx\,dy\,dz = \|A\|\,du_1\,du_2\,du_3, \quad \|A\| = \text{absolute value of determinant of } A. \tag{2.3}$$

The **Euclidean arc length element** is

$$ds^2 = (dx\,dy\,dz)\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = (du_1\,du_2\,du_3)\ {}^tAA \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}.$$

Thus we obtain

$$ds^2 = \sum_{i,j=1}^{3} g_{ij}du_idu_j \quad \text{where} \quad G = {}^tAA = (g_{ij})_{1\le i,j\le 3}. \tag{2.4}$$

Similarly, one uses the fact that

$$\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial u_1},\frac{\partial f}{\partial u_2},\frac{\partial f}{\partial u_3}\right)A^{-1},$$

to see that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \sum_{i,j=1}^{3} g^{ij}\frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j} \text{ where } G^{-1} = (g^{ij})_{1\le i,j\le 3}. \tag{2.5}$$

To see what happens to the Laplacian in the new coordinate system, one can use the calculus of variations (see Courant and Hilbert [111, Chap. 4]). Suppose that f minimizes the integral

$$J(f) = \int (f_x^2 + f_y^2 + f_z^2)\,dx\,dy\,dz$$

subject to the constraint

$$K(f) = \int f^2 \, dx \, dy \, dz = \text{constant}.$$

Then f must satisfy the **Euler–Lagrange equation**

$$f_{xx} + f_{yy} + f_{zz} = \Delta f = \lambda f$$

(see Courant and Hilbert [111, p. 192]). That is, f must be an eigenfunction of the Euclidean Laplacian. Suppose now that we change variables in the integrals J and K. The new constrained minimization problem should lead to a differential equation involving the transformed version of the Laplacian. And this is indeed the case. From (2.3) and (2.5), one obtains

$$J(f) = \int \sum_{i,j=1}^{3} g^{i,j} f_{u_i u_j} \sqrt{|G|} \, du_1 \, du_2 \, du_3, \quad \text{with} \quad |G| = \det G,$$

and

$$K(f) = \int f^2 \sqrt{|G|} \, du_1 \, du_2 \, du_3.$$

The Euler–Lagrange equation for the transformed problem is

$$\sum_i \frac{\partial}{\partial u_i} \sqrt{|G|} \sum_k g^{i,k} \frac{\partial f}{\partial u_k} = \lambda f \sqrt{|G|}.$$

Thus the **Laplacian** in the new coordinate system is

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = |G|^{-\frac{1}{2}} \sum_{i=1}^{3} \frac{\partial}{\partial u_i} |G|^{\frac{1}{2}} \sum_{k=1}^{3} g^{i,k} \frac{\partial f}{\partial u_k}. \tag{2.6}$$

There are other ways to obtain these formulas. See, for example, the following references: Arfken [13, Chap. 2], Yvonne Choquet-Bruhat et al. [87, p. 307], Churchill and Brown [89, pp. 16–19], or Helgason [277, pp. 386–387]).

Now we can show that the sphere is a **Riemannian manifold**, meaning that there is a notion of arc length defined from an inner product on the tangent space to the surface at each point. In order to obtain the arc length element on S^2, we shall use the preceding discussion to write the Euclidean arc length element in spherical polar coordinates. This is done in the next exercise.

Exercise 2.1.2 (Spherical Polar Coordinates). Spherical polar coordinates for a point (x,y,z) in $\mathbb{R}^3$ are defined by

$$x = r \sin\theta \cos\varphi, \;\; y = r \sin\theta \sin\varphi, \;\; z = r\cos\theta,$$

where $0 \leq \varphi \leq 2\pi,\ 0 \leq \theta \leq \pi,\ 0 \leq r$. Use formulas (2.3), (2.4), and (2.6) above to transform the Euclidean volume, arc length, and Laplacian to spherical polar coordinates.

Answer.

$$\begin{aligned}
&d\mu = dx\,dy\,dz = r^2 \sin\theta\; dr\, d\theta\; d\varphi,\\
&ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\; d\varphi^2,\\
&\Delta f = \tfrac{1}{r^2 \sin\theta}\left(\tfrac{\partial}{\partial r}(r^2 \sin\theta \tfrac{\partial f}{\partial r}) + \tfrac{\partial}{\partial \theta}\left(\sin\theta \tfrac{\partial f}{\partial \theta}\right) + \tfrac{\partial}{\partial \varphi}\left(\tfrac{1}{\sin\theta}\tfrac{\partial f}{\partial \varphi}\right)\right).
\end{aligned}$$

It follows from Exercise 2.1.2 that the **element of arc length on the unit sphere** S^2 is

$$ds^2 = d\theta^2 + \sin^2\theta\; d\varphi^2. \tag{2.7}$$

This element of arc length is invariant under $O(3)$ as is the corresponding **volume or area element**:

$$d\mu = \sin\theta\; d\theta\; d\varphi. \tag{2.8}$$

Finally the **Laplacian** on S^2 is

$$\Delta^* = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{2.9}$$

Exercise 2.1.3. Show that the Euclidean Laplacian $\Delta f = f_{xx} + f_{yy} + f_{zz}$ is invariant under $O(3)$ (i.e., that Δ commutes with rotation). Deduce that the spherical Laplacian (2.9) is also invariant under $O(3)$.

Hint. *Dym and McKean [147, p. 243], have a proof using the Fourier transform.*

Because the sphere is a Riemannian manifold, we can consider geodesics on the sphere. The **geodesic** through two points on the sphere is the curve through these two points that minimizes distance. Airplane pilots know that geodesics in S^2 are great circles; i.e., the intersection of S^2 with a plane through the origin and the two given points on S^2. To see this, suppose that you are given points p and q on S^2. You can rotate S^2 so that both points p and q have φ-coordinate in (2.1) equal to zero. See Exercise 2.1.11 if you do not believe this. Then the distance between p and q on S^2 is

$$\int_p^q \sqrt{d\theta^2 + \sin^2\theta\; d\varphi^2} \geq \int_p^q d\theta = \theta(q) - \theta(p).$$

Thus the great circle route minimizes distance on the sphere (assuming that you go in the proper direction around the circle).

Exercise 2.1.4 (The Sphere Is a Symmetric Space). Show that S^2 is a symmetric space. To do this, it only remains to show that for each point p in S^2 there is a diffeomorphism $f_p : S^2 \to S^2$ that preserves arc length (i.e., f_p is an **isometry** of the Riemannian manifold) and reverses geodesics.

Hint. *At* $p = (0,0,1)$, *use* $(\theta, \varphi) \mapsto (-\theta, -\varphi)$.

Spherical geometry is non-Euclidean because the geodesics cannot be extended indefinitely, which violates Euclid's second postulate. In fact, Euclid's fifth postulate fails as well. And Klein noticed that if you identify antipodal points on the sphere, then any two geodesic lines have a unique point in common. The geometry obtained by identifying antipodal points on the sphere is called **elliptic geometry** and the resulting compact surface is also called the **projective plane**. The term *elliptic* is due to Klein (from a Greek work *elleipein*, to fall short). It is motivated by the following picture. Suppose that two geodesic rays, R_1 and R_2, emanate from the ends of a geodesic segment perpendicular to them both. The distance between the rays R_1 and R_2 will decrease or fall short.

In elliptic or spherical geometry, when A, B, C are the three angles of a geodesic triangle, then the area of the triangle is $A+B+C-\pi$, measuring the angles in radians, of course. This result goes back to Albert Girard (1595–1632) for the sphere. References for non-Euclidean geometry are Coxeter [113] and Hilbert and Cohn-Vossen [297].

Exercise 2.1.5. Prove Girard's formula for the area of a spherical triangle, which was mentioned above.

Note. This could be proved using the Gauss–Bonnet formula (see Singer and Thorpe [604]) and, in fact, it was the first known case of the Gauss–Bonnet theorem.

2.1.2 Spherical Harmonics

It is now possible to make sense of the definition of a surface spherical harmonic, which was given at the beginning of this section. A **surface spherical harmonic** Y is a function $Y : S^2 \to \mathbb{C}$ such that

$$\Delta^* Y = \frac{1}{\sin\theta}\left((\sin\theta\, Y_\theta)_\theta + \frac{1}{\sin\theta} Y_{\varphi\varphi} \right) = \lambda Y \tag{2.10}$$

for some eigenvalue $\lambda \in \mathbb{C}$. The theorem that follows characterizes these spherical harmonics very explicitly in terms of (associated) Legendre polynomials. Legendre had discussed these polynomials before Laplace's work, but Legendre did not manage to publish his results until after Laplace did (see Legendre [404, 405] and Laplace [396]). Spherical harmonics also played a role in Laplace's long treatise on celestial mechanics, as well as in Gauss's work on terrestrial magnetism.

The following exercises give some of the properties of **Legendre polynomials** $P_n(x)$ defined by

$$2^n n! P_n(x) = \frac{d^n}{dx^n}\left((x^2-1)^n\right), \quad \text{for } n = 0, 1, 2, \ldots, \tag{2.11}$$

and **associated Legendre functions** $P_n^m(x)$ defined by

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}. \tag{2.12}$$

Note that some authors replace $(1-x^2)$ by (x^2-1) in formula (2.12). We follow Courant and Hilbert [111] and Arfken [13] rather than Lebedev [401] in our choice of notation.

Exercise 2.1.6 (Legendre Polynomials and Associated Legendre Functions). (a) Show that the associated Legendre functions are solutions of the Sturm–Liouville equation:

$$(1-x^2)u'' - 2xu' + \left(n(n+1) - \frac{m^2}{1-x^2}\right)u = 0.$$

(b) If $n = 0, 1, 2, \ldots$, show that $P_n^m(x) = 0$ for all x, unless $m = 0, \pm 1, \pm 2, \ldots, \pm n$. Note that we can allow m to be negative by writing $(d/dx)^{-1}(d/dx) =$ identity. Show also that

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x).$$

(c) Show that for fixed m in $\{0, \pm 1, \pm 2, \ldots\}$, the set

$$\{\, P_n^m \mid n = |m|,\ |m|+1,\ |m|+2, \ldots \}$$

is a complete orthonormal set for $L^2[-1,\ +1]$.

Hint. *The orthogonality is easy. For completeness when $m = 0$, see Courant and Hilbert [111, pp. 82–83]. In the general case, write $P_{m+k}^m(x) = (1-x^2)^{m/2} f_k^m(x)$. Then $\{f_k^m,\ k = 0, 1, 2, \ldots\}$ is a complete set of orthogonal polynomials for $L^2([-1,+1],(1-x^2)^m)$. This is the weighted L^2-space with inner product given by*

$$(f,g) = \int_{-1}^{+1} f(x)\overline{g(x)}\left(1-x^2\right)^m dx.$$

This means that you must show that f_k^m is orthogonal to x^r when $r = 0, 1, \ldots, k-1$. Integration by parts, using the case $m = 0$, will do the trick.

Exercise 2.1.7 (Integral Formula for Associated Legendre Functions). Show that the associated Legendre P_n^m is represented by the following integral:

$$P_n^m(x) = i^m \frac{\Gamma(n+m+1)}{\pi\Gamma(n+1)} \int_0^\pi \left(x + \sqrt{x^2-1}\cos\varphi\right)^n \cos(m\varphi)\, d\varphi$$

for $\operatorname{Re} x > 0,\ m = 0,1,2,\ \ldots$.

Hint. *See Courant and Hilbert [111, p. 505].*

Exercise 2.1.7 has been generalized to symmetric spaces by Harish-Chandra. This will be discussed in later chapters.

Theorem 2.1.1 (Spherical Harmonics).

(a) *The only eigenvalues λ of the spherical Laplacian Δ^* defined in (2.10) are $\lambda = -n(n+1),\ n = 0,1,2,\ldots$. The vector space of eigenfunctions of Δ^* corresponding to the eigenvalue $\lambda = -n(n+1)$ has dimension $2n+1$. A complete orthogonal set of eigenfunctions of Δ^* is*

$$Y(\theta,\varphi) = \exp(im\varphi)P_n^m(\cos\theta) \quad \text{for} \quad n = 0,1,2,\ldots, \quad |m| \le n. \tag{2.13}$$

Here P_n^m is the associated Legendre function (2.12) and the coordinates (θ,φ) are defined by (2.1). We call cY, with Y given by formula (2.13), a (surface) ***spherical harmonic of degree n and order m****, where c is a normalizing constant.*

(b) *Let $f(x,y,z) = r^n Y(\theta,\varphi)$, using spherical polar coordinates (r,θ,φ) corresponding to the rectangular coordinates (x,y,z) in Exercise 2.1.2. Then Y is a surface spherical harmonic satisfying*

$$\Delta^* Y = -n(n+1)Y$$

if and only $f(x,y,z)$ is a homogeneous harmonic polynomial of degree n. When we say that f is ***harmonic****, we mean that f satisfies Laplace's equation*

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = 0.$$

Exercise 2.1.8. Prove part (a) of Theorem 2.1.1, using separation of variables on $\Delta^* Y = \lambda Y$ and Exercise 2.1.6.

Note. There are other proofs that $\{\lambda = -n(n+1),\ n = 0,1,2,\ \ldots\}$ are the only possible eigenvalues of Δ^* on S^2. For example, see Dym and McKean [147, pp. 252–253], Kirillov [353, pp. 271–274], or Van der Waerden [691, p. 21].

Exercise 2.1.9. Use separation of variables on $\Delta f = 0$ to prove part (b) of Theorem 2.1.1. Note that Exercise 2.1.2 says that if $f(x,y,z) = R(r)Y(\theta,\varphi)$, then

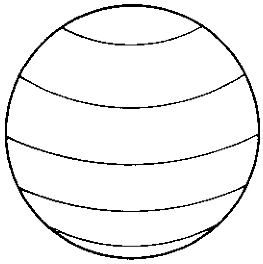

Fig. 2.2 Zero set of $P_n(\cos\theta)$

$$\Delta f = \frac{1}{r^2}(r^2 R'(r))' Y + \frac{R}{r^2}\Delta^* Y \qquad \text{where} \qquad ' = \frac{d}{dr}.$$

If $\Delta f = 0$, then $\Delta^* Y = \lambda Y$ and $(r^2 R'(r))' = -\lambda R$, for some constant λ.

You should not be surprised that the eigenvalues of the Laplacian are real since the operator is self-adjoint. The next exercise explains why they are negative.

Exercise 2.1.10 (Why the Eigenvalues of the Laplacian on the Sphere Are Negative). Suppose that f and $\Delta^* f$ are in $L^2(S^2)$. Show that

$$(f, \Delta^* f) \leq 0.$$

Use Green's theorem. Then show that the eigenvalues of Δ^* are all negative or zero.

According to part (a) of Theorem 2.1.1, separation of variables in $\Delta^* Y = \lambda Y$ leads from functions $Y(\theta, \varphi)$ of the two angle variables in (2.1) to functions $\exp(im\varphi)P_n^m(\cos\theta)$. If we assume that Y is constant with respect to the angle φ, then $Y = Y(\theta) = P_n(\cos\theta)$. Such a spherical function $Y = Y(\theta)$ is called a **zonal spherical function**. The name results from the fact that the zeros of Y cut the sphere up into zones, as in Fig. 2.2. In general, $P_n(\cos\theta)$ has n distinct zeros in $0 \leq \theta \leq \pi$ which are positioned symmetrically about $\theta = \pi/2$ (see Fig. 2.2). There is a more group theoretical definition of zonal spherical function, which is developed in the following exercise.

Exercise 2.1.11 (Zonal Spherical Functions on $K\backslash G/K$, where $G = SO(3)$, $K = SO(2)$). Let $G = SO(3)$ and $K = SO(2)$ be considered as a subgroup of G as in Exercise 2.1.1. We know from Exercise 2.1.2 that the sphere can be identified with G/K. Show that the space $K\backslash G/K$ of double cosets KgK, $g \in G$, can be represented by the cosets

$$Kg_\theta K, \quad \text{with} \quad g_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad 0 \leq \theta < \pi.$$

Table 2.1 Surface spherical harmonics

	$\boldsymbol{Y_n^m(\theta,\varphi)} = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} P_n^m(\cos\theta)\exp(im\varphi)$
s	$Y_0^0 = \frac{1}{\sqrt{4\pi}}$
p	$\begin{cases} Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\ \exp(\pm i\varphi) \\ Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \end{cases}$
d	$\begin{cases} Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\ \exp(\pm 2i\varphi) \\ Y_2^{\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\ \cos\theta\ \exp(\pm i\varphi) \\ Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \end{cases}$
f	$\begin{cases} Y_3^{\pm 3} = \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta\ \exp(\pm 3i\varphi) \\ Y_3^{\pm 2} = \sqrt{\frac{105}{32\pi}}\sin^2\theta\ \cos\theta\ \exp(\pm 2i\varphi) \\ Y_3^{\pm 1} = \mp\sqrt{\frac{21}{64\pi}}\sin\theta\ \left(5\cos^2\theta - 1\right)\exp(\pm i\varphi) \\ Y_3^0 = \sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \end{cases}$

When do two of these cosets coincide? The corresponding points to gK on the sphere are ge_3, where $e_3 = {}^t(0,0,1)$. So if

$$k_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find that $k_{-\varphi}g_\theta e_3 = {}^t(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. Thus a function on $K\backslash G/K$ is a function only of the angle θ.

Let $A = \{g_\theta, |0 \le \theta \le \pi\}$, with g_θ from Exercise 2.1.11. Then the decomposition $SO(3) = KAK$, which follows from the preceding exercise, is the **Euler angle decomposition** of $SO(3)$. See Volume II [667] for a generalization of this decomposition and its application to harmonic analysis on general symmetric spaces.

Exercise 2.1.12. Check Table 2.1 which lists the first few surface spherical harmonics. The table uses the Condon–Shortley [101] convention.

We can use Mathematica to visualize spherical harmonics using density plots on the sphere. See Fig. 2.3 for two such pictures. On the left is that of $\mathrm{Re}\,Y_{14}^7(\theta,\varphi)$ obtained using the Mathematica command:

```
ParametricPlot3D[{Cos[φ]*Sin[θ], Sin[φ]*Sin[θ],Cos[θ]},{φ,0,2π},{θ,0,π},
PlotPoints ->300,Mesh->None,ColorFunction->Function[{x,y,z,φ,θ},
Hue[Re[SphericalHarmonicY[14,7,θ,φ]]]],ColorFunctionScaling->
False].
```

On the right is that of $\mathrm{Re}\,Y_{14}^7(\theta,\varphi)$+2$\mathrm{Re}\,Y_7^3(\theta,\varphi)$ obtained using a similar command.

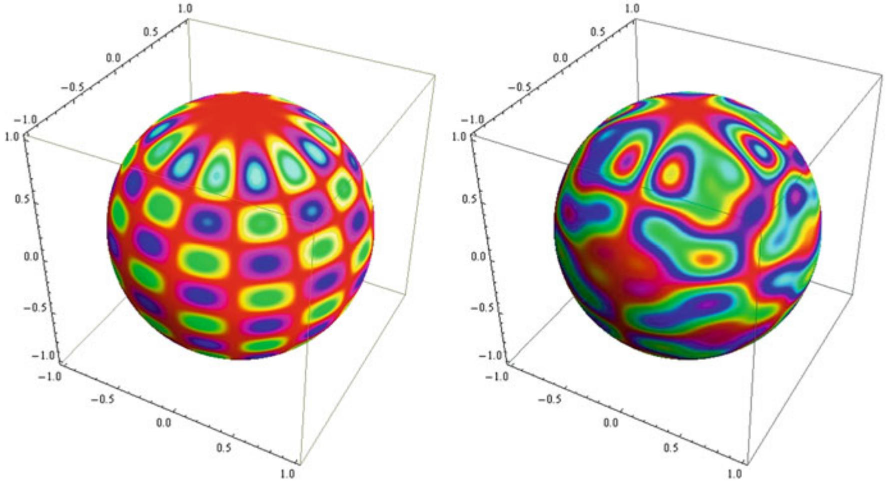

Fig. 2.3 Mathematica density plot for $Y_{14}^{7}(\theta, \varphi)$ on the *left* and for $\mathrm{Re}\,Y_{14}^{7}(\theta, \varphi) + 2\mathrm{Re}\,Y_{7}^{3}(\theta, \varphi)$ on the *right*

These should be compared with analogous plots for the analogous functions on a unit square in the plane in Fig. 1.4, which we considered in Sect. 1.3.

Exercise 2.1.13. (a) Make some similar plots to those in Fig. 2.3 using the Mathematica command

$$\mathrm{Re}[\mathrm{SphericalHarmonic}Y[m, n, \theta, \varphi]].$$

(b) Solve the wave equation on the sphere with initial condition modelling what would happen if the sphere were struck with a hammer at a point. Then make a Mathematica movie of your solution.

(c) Consider the effect of forcing a sphere to vibrate at some frequency near an eigenfrequency of Δ.

Corollary 2.1.1 (Harmonic Analysis on the Sphere). *Every function $f : S^2 \to \mathbb{C}$ with continuous second-order derivatives can be expanded in an absolutely and uniformly convergent series of spherical harmonics. If we take Y_n^m, $|m| \leq n$, to be an orthonormal basis for $L^2(S^2, d\mu)$, with $d\mu$ as in Exercise 2.1.2, then*

$$f(\theta, \varphi) = \sum_{n \geq 0} \sum_{|m| \leq n} \hat{f}(n, m) Y_n^m(\theta, \varphi),$$

where

$$\hat{f}(n, m) = \int_{S^2} f \, \overline{Y_n^m} \, d\mu.$$

Exercise 2.1.14. Prove the preceding corollary of Theorem 2.1.1.

Hint. *The corollary can be proved from the existence of the Green's function for the problem (see Courant and Hilbert [111, p. 369]) or you might try to relate this problem to Theorem 1.3.1 of Sect. 1.3.*

Exercise 2.1.15 (The Gibbs Phenomenon for the Sphere). Find an analogue of Exercises 1.3.1 and 1.3.2 of Sect. 1.3 for the sphere.

Hint. *There is a discussion of the Gibbs phenomenon as well as summability procedures in Weyl [730, Vol. I, pp. 305–353, 376–389 and Vol. IV, pp. 432–456].*

There are many facts about Fourier series that would be worthwhile to extend to Laplace series of spherical harmonics. For example, it is possible to obtain the analogue of the results of Slepian and Pollak (see Grünbaum et al. [234]). And it is possible to obtain a central limit theorem for the sphere or even $SO(3)$ (see Clerc and Roynette [92]).

Expansions in spherical harmonics are useful in many areas. For example, Backus and Gilbert [19], among others, have used such expansions relative to the vibration of the earth due to large earthquakes to study the structure of the interior of the earth. Some of the flavor of their work can be derived from the following quote.

> The Chilean earthquake of May 22, 1960 excited oscillations of the earth with periods from six to one hundred minutes which were observed continuously on strain gauges for 265 hours after the main shock and at one-minute intervals on a gravimeter for 110 hr. In the Fourier spectra of these records, peaks were observed which corresponded well with the characteristic seismic oscillations computed by Pekeris and Gilbert and MacDonald for certain earth models. Some of the observed peaks, however, differed in period from the theoretical spectral lines, and some of the observed peaks appeared to be double or triple, the components being separated by as much as two minutes and the theoretical line falling approximately at their mean position.
>
> The displacements of observed peaks relative to theoretical frequencies presumably reflect slight errors in the earth model used in the theory and will not be discussed further here. The line splitting cannot be explained thus; we propose to explain it quantitatively as an effect of the diurnal rotation of the earth.

2.1.3 Quantum Mechanics: The Hydrogen Atom

According to the most simplistic version of quantum mechanics (i.e., neglecting spin and relativistic effects), the wave function ψ for the hydrogen atom satisfies the **Schrödinger equation**

$$-\left(\frac{\hbar^2}{2m}\Delta+\frac{e^2}{r}\right)\psi=E\psi,\qquad r=(x^2+y^2+z^2)^{1/2}. \tag{2.14}$$

Here $-e$ is the charge of the electron, $\hbar=$ Planck's constant divided by 2π, Δ is the Laplacian $(\Delta f=f_{xx}+f_{yy}+f_{zz})$, $m=m_em_p/(m_e+m_p)$ is the reduced mass of the system if m_e is the mass of the electron and m_p the mass of the proton. The eigenvalue E is the **energy level** of the system. Some references for these matters are Biedenharn and Louck [39], Castellan [80], Condon and Odabaşi [100], Courant and Hilbert [111, pp. 341–343], Eyring, Walter, and Kimball [170], Mackey [442, pp. 159–271], Messiah [465, especially pp. 362 and 412], Sommerfeld [610, pp. 200–206], B. Thaller [676], Van der Waerden [691, Chap. 1], Dorothy Wallace and J. BelBruno [713], Weyl [731, pp. 41–70], and Wigner [738].

The eigenvalues E that lie in the discrete spectrum of the Schrödinger operator in equation (2.14) will be the only ones of interest for our discussion. Such eigenvalues are negative (in fact, there is also a continuous spectrum of positive real numbers). Physicists interpret the values of E that lie in the discrete spectrum as **energy levels of bound states of hydrogen**. Spectroscopists going back to Balmer in the 1880s have found various series of lines in the spectrum of hydrogen corresponding to these energy levels. We can obtain some understanding of these spectral lines by finding the discrete spectrum of the Schrödinger operator very explicitly.

We shall use spherical harmonics to separate variables via $\psi(x,y,z) = w(r)Y(\theta,\varphi)$ in equation (2.14) for the purpose of obtaining the discrete spectrum eigenvalues E. Then Exercise 2.1.2 shows that (2.14) becomes

$$\frac{(r^2w')' + (2mr^2/\hbar^2)(E+e^2/r)w}{w} = -\frac{\Delta^* Y}{Y}, \quad \text{where} \quad ' = \frac{d}{dr}.$$

Both sides of this equation must be constant. Thus, by Theorem 2.1.1,

$$\Delta^* Y = -n(n+1)Y, \quad n = 0,1,2,\ldots,$$
$$(r^2w')' + \frac{2mr^2}{\hbar^2}\left(E + \frac{e^2}{r}\right)w = n(n+1)w.$$

Using Exercise 2.1.16 on Laguerre polynomials we find that

$$w(r) = r^n \exp(-cr/(2k))L_{k+n}^{(2n+1)}(cr/k), \quad k \in \mathbb{Z}, k > n,\ c = 2me^2/(\hbar^2).$$

Furthermore, the corresponding **eigenvalues** are

$$E_k = -\frac{me^4}{2\hbar^2 k^2}. \tag{2.15}$$

Physicists call k the **principal quantum number**, n the **azimuthal quantum number**. If $Y = Y_n^p$, $|p| \le n$, from Theorem 2.1.1, then p is called the **magnetic quantum number**. For each k, there are

$$k^2 = \sum_{n=0}^{k-1}(2n+1) \tag{2.16}$$

linearly independent eigenfunctions of (2.14) with eigenvalue E_k. Physicists call this $\boldsymbol{k^2}$**-fold degeneracy**. For example, spectroscopists use the letters $(s, p, d, f, q, \ldots)$ to indicate the value of n corresponding to the state of hydrogen. They would say the ground state corresponds to $k = 1$ and is a $1s$ state. The first excited state corresponds to $k = 2$. It is said to be fourfold degenerate because it contains one $2s$ state and three $2p$ states.

Thus for each value $k,\ n$ of the principal and azimuthal quantum numbers there are $(2n+1)$ experimentally indistinguishable states of the hydrogen atom. This mathematically explains the **Zeeman effect** in which spectral lines of hydrogen split into an odd number of lines when a magnetic field is switched on. The effect was first observed by Zeeman in 1896.

Note that E_k given by formula (2.15) must equal $h\nu$, where ν is the frequency of a spectral line. And we obtain

$$\nu = Rk^{-2},$$

where R is the Rydberg constant, $R = 2\pi^2 me^4 h^{-3}$. When an atom moves from initial state 1 with principal quantum number k_1 to final state 2 with principal quantum number k_2, the frequency of the associated spectral line is

$$\nu = R(k_2^{-2} - k_1^{-2}).$$

The series of spectral lines of hydrogen observed by Balmer in 1885 has $k_2 = 2$. Thus the Balmer series lines have frequencies

$$\nu = R(2^{-2} - k^{-2}), \quad k = 3, 4, 5, \ldots.$$

These lines are in the visible range. Lyman found a series of spectral lines in the ultraviolet range in 1909 with $k_2 = 1$. The series with $k_2 = 3, 4$, are in the infrared range. The Balmer series is not obtained in the laboratory because it requires the temperature of stellar atmospheres.

The theory just described does not account for the fine structure of the spectrum. Relativistic effects and spin must be considered. Even then, agreement between theory and experiment is not perfect. See Messiah [465, Vol. II, pp. 930–933], for a discussion of the Lamb shift.

There is an additional degeneracy of $(n+1)^2$ rather than $2n+1$, which was explained by Fock, who showed that if one formulates the Schrödinger equation for the hydrogen atom as an integral equation, then it is also invariant under $SO(4)$. See Louck and Metropolis [430] for a discussion of a related problem which was motivated by Fock's result.

Spectroscopists daily analyze all sorts of materials—atoms, molecules, crystals, solids, gases. But, of course, the theory becomes much more complicated for the many-body problem. Group theory helps, as many of the references mentioned at the beginning of this discussion show.

Exercise 2.1.16 (Laguerre Polynomials). Set

$$L_p = e^z \frac{d^p}{dz^p}(e^{-z}z^p) = the\ pth\ Laguerre\ polynomial.$$

Show that $w(r) = r^n \exp(-cr/(2k))L_{k+n}^{(2n+1)}(cr/k)$ satisfies

$$(r^2w')' + \frac{2mr^2}{\hbar^2}\left(E + \frac{e^2}{r}\right) w = n(n+1)w,$$

for E given by equation (2.15), $k \in \mathbb{Z},\ k > n,\ c = 2me^2/(\hbar^2)$. Then show that we have found the only continuous bounded eigenfunctions $w(r)$ for the differential operator involved here.

Hint. *See Arfken [13, Chap. 13], or Courant and Hilbert [111, p. 330].*

We have found a wave function ψ solving equation (2.14) of the form

$$\psi(r,\theta,\varphi) = Y_n^m(\theta,\varphi)r^n \exp(-cr/2k)L_{k+n}^{(2n+1)}(cr/k),$$

where Y_n^m denotes a spherical harmonic as in Theorem 2.1.1 and $L_a^{(b)}$ is the b$^{\text{th}}$ derivative of the Laguerre polynomial L_a. Then $|\psi|^2$ can be interpreted as the probability density of the electron for the various states of the hydrogen atom. Fig. 2.4 from White [733, p. 71], illustrates the first few states.

2.1.4 The Sun's Magnetic Field

Using the measured magnetic field of the sun's surface, the problem is to determine the magnetic field of the corona, assuming that the latter is current-free. The corona itself was not recognized until the last half of the nineteenth century. A new green spectral line was found to appear only in the corona and just above the thin layer next to the solar surface which emits the bright red Balmer line of hydrogen. By 1930, eighteen spectral lines had been discovered in the solar corona. And by the early 1940s it was realized that the green line comes from forbidden transitions in highly ionized metal atoms. The lack of spectral lines of hydrogen and helium in the corona implies that these elements are completely ionized. The corona is mostly hydrogen plasma at one to two million degrees K. There are also cooler regions and warmer regions. Since the corona is too hot to be in thermal equilibrium, there is a solar wind. And observed coronal holes are sources of streams of plasma, which ultimately give rise to terrestrial phenomena such as aurorae and disturbances in radio transmission. A more complete discussion of these basic facts can be found in Altschuler's article in Herman [290, pp. 105–145]. A fascinating account of solar theory can be found in Zirin [755].

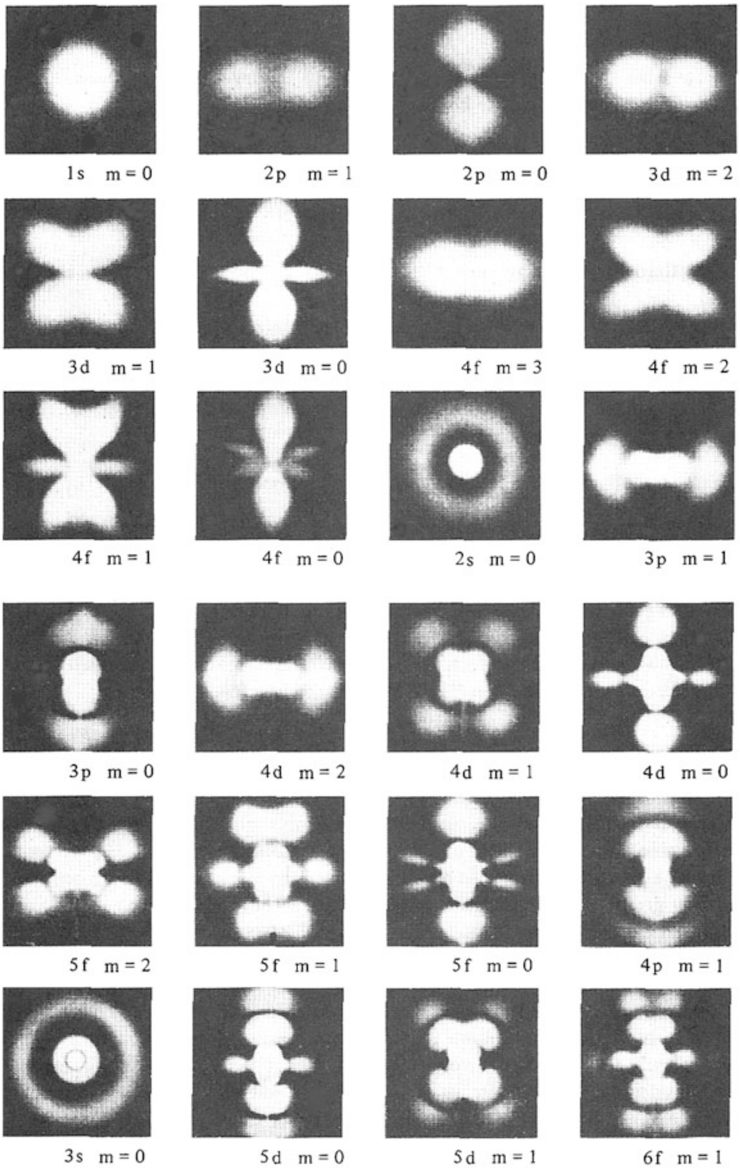

Fig. 2.4 Photographs of the electron cloud for various states of the hydrogen atom as made from a spinning mechanical model. The probability-density distribution $\psi\psi^*$ is symmetrical about the φ-axis, which is vertical and in the plane of paper. (From H.E. White, [733, p. 71]. Reprinted by permission of McGraw-Hill)

The idealized mathematical problem involved in determining the magnetic field of the solar corona using measurements from the solar surface can be posed as follows: find ψ, where

$$\begin{aligned} &\Delta\psi = 0, && R < \|x\| < R_w, \\ &\psi(x) = f(x), && \text{if } \|x\| = R, \text{ the sun's surface}, \\ &\psi(x) = 0, && \text{if } \|x\| = R_w, \text{ sufficiently far from the sun.} \end{aligned}$$

The function $f(x)$ is obtained from the measured magnetic field of the sun's surface. The relation $B = -\text{grad } \psi$ gives the magnetic field itself. The assumption is that there are no magnetic monopoles and thus ψ must be harmonic. So we can use Exercise 2.1.9 and Corollary 2.1.1 to Theorem 2.1.1 to obtain

$$\psi(r,\theta,\varphi) = R \sum_{\substack{n\geq 0 \\ |m|\leq n}} \frac{1}{1-a^{2n+1}} \left(\left(\frac{R}{r}\right)^{n+1} - a^{2n+1} \left(\frac{r}{R}\right)^{n} \right) \hat{f}(n,m) Y_n^m(\theta,\varphi),$$

where $a = R/R_w < 1$ and

$$\hat{f}(n,m) = \int_{S^2} f \, \overline{Y_n^m} \, d\mu.$$

This formula is satisfactory as a solution in a textbook, but the numerical problem remains to be solved. See Altschuler's article mentioned above for a description of the numerical work that led to the illustrations in Fig. 2.5.

Another interesting application of spherical harmonics can be found in Chandrasekhar [83, Chap. VI]. The same sort of picture that gave us an idea of the shape of the probability distribution of the electron in a hydrogen atom should give an idea of the distribution of heat in a uniform sphere. Several applications of spherical harmonics to geophysics can be found in Gilbert [212], Kato [341], Stacey [616], and Trefil [683, pp. 106–111].

It is possible to generalize Theorem 2.1.1 to the unit sphere S^n in $\mathbb{R}^{n+1}$. In fact, spherical harmonics for such higher-dimensional spheres have been of use in quantum mechanics, as have the representations of many higher-dimensional Lie groups. In the Lecture Note-Reprint collection Dyson [148] gives an interesting glimpse into the arguments concerning the choice of symmetry groups for nuclear and particle physics. The compact groups $SU(n)$, $n = 3, 6, 12$, seem to have attracted the most attention. Here $SU(n)$ is the **special unitary group** of $n \times n$ complex matrices U such that ${}^t\overline{U}U = I$ and $|U| = 1$. Some of the papers in the Dyson collection are devoted to showing that the representations of $SU(6)$ are somehow incompatible with those of the Lorentz group $O(3,1)$ which leaves the Maxwell equations invariant. See R. Hermann [291, pp. 144–149] and Talman [655, pp. 186–187] for discussions of Fock's work on applications of four-dimensional spherical harmonics to quantum mechanics. The standard model in particle physics has as its local gauge group $SU(3) \times SU(2) \times SU(1)$. Representations of the group

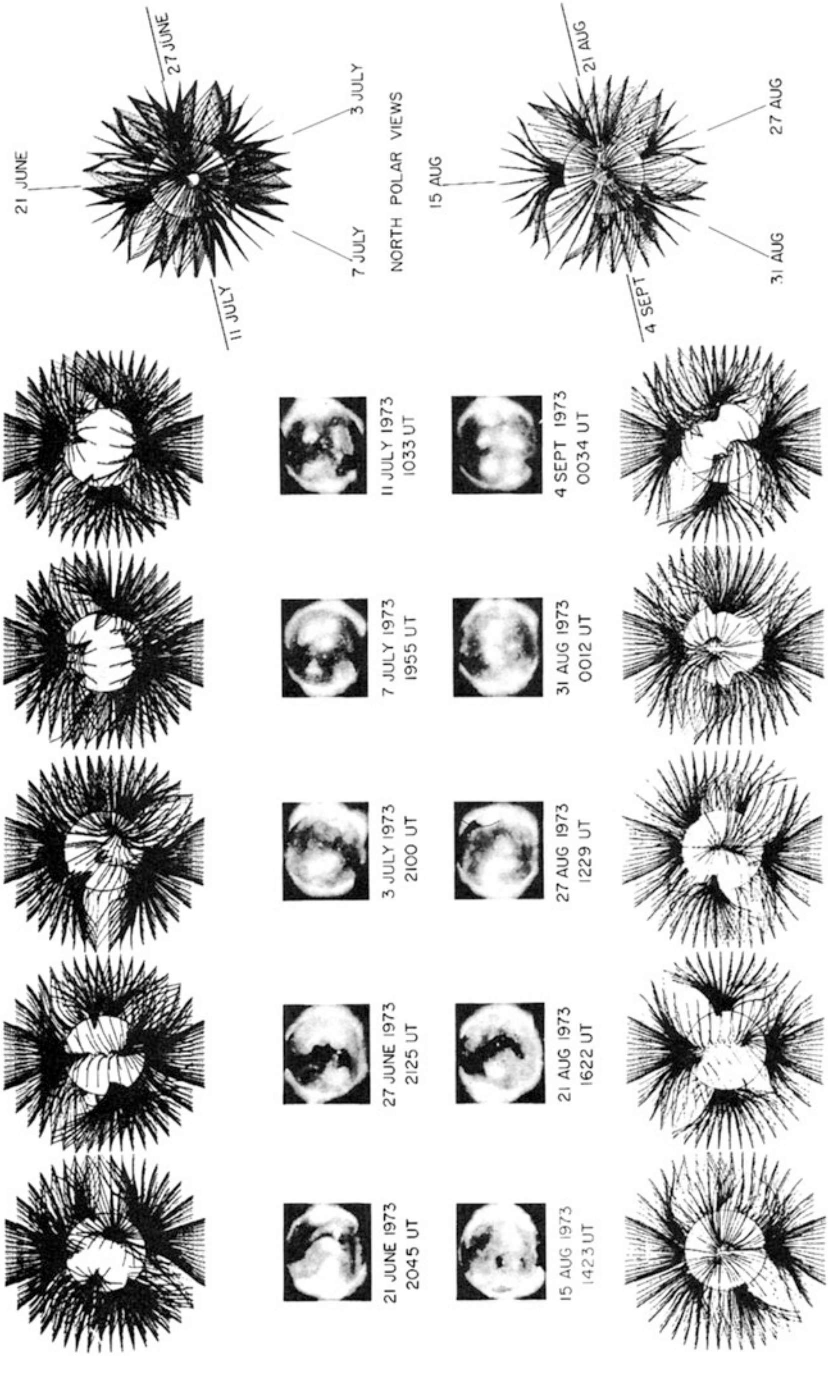

Fig. 2.5 Open magnetic structures on the sun. (From Altschuler's article in Herman [290, p. 125]. Reprinted by permission of Springer-Verlag)

correspond to the fields. There are many books on the subject. We leave it to you to Google them, or see Wikipedia. There is also the Grand Unified Theory, or GUT. This involves $SU(5), SO(10)$ and even the exceptional Lie group E_8. Again there is much on the Internet including some beautiful movies related to E_8.

Hecke and others have made use of these higher-dimensional spherical harmonics in number theory and the kinetic theory of gases (see Hecke [258, pp. 361–373, 849–854]) and Ogg [505, Chap. VI, p. 6]. We shall see in the next section that spherical harmonics also give a new description of harmonic analysis on $\mathbb{R}^n$.

2.1.5 Connections with Group Representations

It is not difficult to connect spherical harmonics with representations of $SO(3)$. Before doing so, let us sketch some of the **basic facts about group representations**. We will be sketchy because we do not intend to emphasize the representation-theoretic point of view. It would be useful for the reader to study these things however, and the following references will provide plenty of food for thought: Barut and Raçzka [28], Boerner [45], Gelfand, Minlos, and Shapiro [205], Gurarie [238], Hamermesh [246], Knapp [356], Mackey [442–444], Maurin [458], Sugiura [648], Talman [655], Van der Waerden [691], Varadarajan [692], Michelle Vergne [697], Vilenkin [704], Wallach [714,715], Warner [717], Wawrzyńczyk [722], Weyl [730–732], Wigner [738], and Williams [741]. Mackey's introduction to Biedenharn and Louck [39] provides a translation of physicists' terminology for mathematicians (or vice versa).

Suppose that H is a **separable Hilbert space** and let $GL(H)$ be the group of invertible continuous linear maps from H to H. We want to consider representations of a topological group G by elements of $GL(H)$. By a **topological group** G we mean a group with a topology such that multiplication $(x,y) \to xy$ and inversion $x \mapsto x^{-1}$ are continuous maps. A topological group is **locally Euclidean** if there is a neighborhood of the identity e in G which is homeomorphic to an open subset of $\mathbb{R}^n$. A **Lie group** is a locally Euclidean topological group with group operations that are infinitely differentiable maps. Most, if not all, of the Lie groups considered here are groups of matrices like $O(n)$.

A **representation** of a topological group G is a pair (T,H) where H is a separable Hilbert space and $T : G \to GL(H)$ is a continuous group homomorphism, using the strong operator topology on $GL(H)$. Actually a stronger definition is made when G is not locally compact (see Kirillov [353, p. 111]). You might wonder why we want to represent groups of matrices by their infinite-dimensional analogues. The answer is that this cannot be avoided if you wish to do Fourier analysis on a noncompact group.

Let (v,w), v, $w \in H$, denote the Hilbert space inner product on H. We say that the representation (T,H) is **unitary** if $(T(g)v, T(g)w) = (v,w)$ for all v,w in H and g in G. The representation (T,H) is said to be **finite-dimensional** if H is finite-dimensional and then $\dim H =$ **degree** of the representation.

Suppose that v, w are elements of H. The function $g \mapsto (T(g)v,w) = m_{w,v}(g)$, for $g \in G$, is called a **matrix entry** of the representation (T,H). Many of the special functions which are useful in applied mathematics are either matrix entries of representations of well-known Lie groups or are part of an orthogonal basis for the Hilbert space of such a representation.

The theory of group representations is closely connected with the search for eigenfunctions of differential operators. For if a group G leaves a differential operator L invariant, then G takes eigenfunctions of L into eigenfunctions of L with the same eigenvalue. So G acts on the vector space H of eigenfunctions of L for a fixed eigenvalue λ to give a representation of G.

We say that a representation (T,H) of G is **irreducible** if there is no closed subspace W of H with $\{0\} \subsetneqq W \subsetneqq H$ such that $T(g)W \subset W$ for all $g \in G$.

Two representations (T_1,H_1) and (T_2,H_2) of a Lie group G are said to be **equivalent** if there is a linear bicontinuous bijection $A : H_1 \to H_2$ such that $T_2(g)A = AT_1(g)$ for all g in G. Then A is called an **intertwining operator**. Or even more generally, if A is only continuous linear, it is still called an intertwining operator.

If the group is $\mathbb{R}^m$ under addition, the irreducible unitary representations are one-dimensional and a complete list of inequivalent irreducible unitary representations of $\mathbb{R}^m$ under addition consists of the exponentials

$$\widehat{\mathbb{R}^m} = \left\{ e_a(x) = \exp\left({}^t ax\right), \text{for} \quad x \in \mathbb{R}^m \mid a \in \mathbb{R}^m \right\} \tag{2.17}$$

Exercise 2.1.17 (Schur's Lemma).

(a) Suppose that (T_1,H_1) and (T_2,H_2) are finite-dimensional irreducible representations of G that are inequivalent. If $A : H_1 \to H_2$ is a linear map such that $AT_1(g) = T_2(g)A$, for all g in G, then $A = 0$. Prove this.

(b) Suppose that (T,H) is an irreducible representation of G on a finite-dimensional complex vector space H. If $A : H \to H$ is a linear map such that $AT(g) = T(g)A$, for all g in G, show that there is a complex number c such that $A = cI$, where I is the identity operator; i.e., $Iv = v$, for all v in H.

Note. The spectral theorem for unitary operators allows you to extend the result of the preceding exercise to unitary representations on infinite-dimensional Hilbert spaces.

We say that a finite-dimensional representation (T,H) of G is **completely reducible** if $H = H_1 \oplus \cdots \oplus H_n$, with $T(g)H_i \subset H_i$, for all $g \in G$, provided that the representations (T_i,H_i) of G obtained by restricting $T(g)$ to H_i are irreducible for all $i = 1, \ldots, n$. We say $T = T_1 \oplus \cdots \oplus T_n =$ the **direct sum** of the T_i.

Exercise 2.1.18. Given a basis $e_1, \ldots, e_m$ of the representation space H of a finite-dimensional representation (T,H) of G, you can form the matrix corresponding to $T(g)$. If $e_1, \ldots, e_m$ is an orthonormal basis of H, then this matrix has (i,j) entry $(T(g)e_i,e_j) = m_{j,i}(g)$, for $g \in G$.

(a) Show that two finite-dimensional representations (T_1, H) and (T_2, H) are equivalent if and only if the matrix of T_1 is obtained from the matrix of T_2 by changing the basis of H that is used.
(b) Show that a finite-dimensional representation (T, H) of G is completely reducible if and only if you can find a basis of H which puts the matrix of T in block diagonal form:

$$\begin{pmatrix} T_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_n \end{pmatrix}$$

so that the representation determined by the matrix T_i is irreducible for each $i = 1, \ldots, n$.

We will often need to be able to integrate functions on topological groups. The integral involved is an analogue of the Lebesgue integral on $\mathbb{R}^n$ and is supposed to come from a countably additive positive measure on the Borel sets in the group. Such an integral on a topological group G is **right invariant** if

$$\int_G f(x)\, dx = \int_G f(xa)\, dx \quad \text{for all } a \text{ in } G.$$

The invariant integral on a compact Lie group was used by Hurwitz, Schur, and Weyl beginning in the 1890s. Haar proved the existence of a invariant integral on any locally compact topological group in 1933 and thus the integral is called the **Haar integral**. If $G = \mathbb{R}^m$, under addition, then the Haar integral is the standard Lebesgue integral. Proofs of the existence of the Haar integral may be found in Helgason [277, p. 365], Lang [387], Pontryagin [518], or Weil [725], for example.

The invariant integral can be used to show that any irreducible unitary representation of a compact group must be finite-dimensional (see Kirillov [353, p. 135]), for example. And you will need the invariant integral to do Exercise 2.1.20. In fact, integration on topological groups will be a necessary tool for the rest of this book. Of course, our groups will be matrix groups and thus the Haar integral for these groups is not so mysterious.

The right-invariant Haar measure dx is unique up to a positive constant multiple. This means that for each g in G, there is a positive constant $\delta(g)$ (not to be confused with the like-named distribution) defined by

$$\int f(gx)\, dx = \delta(g) \int f(x)\, dx.$$

Then $\delta : G \to \mathbb{R}^+$ is a continuous homomorphism, which is called the **modular function** of G (not to be confused with the modular functions that appear in Chap. 3). Clearly this function relates the right- and left-invariant Haar integrals. In particular, one has

$$\int f(x)\ \delta(x)\ dx = \int f(x^{-1})\ dx.$$

The group G is said to be **unimodular** when $\delta(g) = 1$ for all g in G. This means that right-invariant Haar integrals are also left invariant. Many groups that we shall consider are unimodular. In particular, all compact or abelian groups are unimodular. Most of the Lie groups that we shall consider in this book are what is called "semisimple"; for example, the special linear group of all $n \times n$ real matrices of determinant one. It can be shown that all semisimple Lie groups are unimodular (see Helgason [277, p. 366]). However, not all groups are unimodular. For example, the group of upper triangular matrices with positive diagonal entries is not unimodular if $n \geq 3$ (Exercise).

Exercise 2.1.19. Show that any finite-dimensional unitary representation is completely reducible and that its decomposition into irreducible representations is unique up to equivalence.

Exercise 2.1.20 (The Haar Integral on a Compact Group).

(a) Show that any compact group is unimodular; i.e., right Haar measure = left Haar measure.
(b) Use Haar measure to show that any representation of a compact group is equivalent to a unitary representation.

Suppose that (T_1, H_1) and (T_2, H_2) are representations of G. Define the **tensor product representation** $(T_1 \otimes T_2, H_1 \otimes H_2)$ by

$$(T_1 \otimes T_2)(g)(v_1 \otimes v_2) = (T_1(g)v_1) \otimes (T_2(g)v_2) \quad \text{for}\, g \in G,\ v_i \in H_i,\ i = 1,2.$$

See the references mentioned above for more information on tensor products. The decomposition of tensor products of representations into direct sums of irreducible representations has much importance for quantum mechanics. For example, if one ignores the interaction between two electrons in the field of a positive nucleus, the Schrödinger operator of the system has eigenfunctions which are products $\psi_1\ \psi_2$ of eigenfunctions ψ_i corresponding to representations T_i of $O(3)$, $i = 1,2$. And the product $\psi_1 \psi_2$ corresponds to the representation $T_1 \otimes T_2$. The Clebsch–Gordon series breaks $T_1 \otimes T_2$ up into its irreducible components. Thus one can conclude what sort of spectral lines should occur for such a situation. See the references mentioned in Sect. 2.1.3 on quantum mechanics for more details.

There are many other useful topics in representation theory such as induced representations, Frobenius reciprocity, Cartan's theorem on the highest weight, and Weyl's character formula. See the references on group representations for a discussion of these matters.

The systematic study of group representations of finite groups began in the 1890s with work of Frobenius, Schur, and others. In the 1920s, Weyl obtained the irreducible representations of the compact simple Lie groups such as $G = SO(3)$ (see Weyl [730, Vol. II, pp. 543–647]). To do this, Weyl used his formula for the character of a representation T, which is the trace of $T(g), g \in G$. The representations of compact Lie groups G are studied by restricting them to a maximal abelian subgroup A or **torus** in G. Any representation of A decomposes into a direct sum of one-dimensional representations called **weights**. Among the weights there is a highest one and this characterizes the original representation of G, up to equivalence. We have actually seen an example of the theorem on the highest weight in our study of spherical harmonics, but this will not be developed here. Another result called the Borel–Weil–Bott theorem realizes the representations of compact Lie groups on sheaf cohomology groups (see Warner [717]).

The **character** χ_T of a finite-dimensional representation T is defined by $\chi_T(g) = \text{Trace}(T(g))$ for $g \in G$. Define $\hat{G} =$ the **dual** of G to be the set of equivalence classes of irreducible unitary representations of G. For example, since the complete list of irreducible unitary representations is given by formula (2.17), $\widehat{\mathbb{R}^m}$ can be identified with $\mathbb{R}^m$. It is said that $\mathbb{R}^m$ is **self-dual**. If G is compact, describing $\hat{G}$ is equivalent to describing the characters of G.

When infinite-dimensional representations are required, the character is defined as a distribution when possible. If $f \in L^1(G)$, define

$$T(f) = \int_G f(g)T(g)\,dg,$$

which means that for $x, y \in H$, we have

$$(T(f)x, y) = \int_G f(g)(T(g)x, y)\,dg,$$

with $(,) =$ the Hilbert space inner product on H. If H has an orthonormal basis e_i, then we can often define the character of T via

$$\text{Tr}\,T(f) = \sum_i (T(f)e_i, e_i) \quad \text{where Tr} = \text{Trace},$$

considering the map $f \mapsto \text{Tr}\,T(f)$ as a distribution. When H is finite-dimensional,

$$\text{Tr}\,T(f) = \int_G (\text{Tr}\,T(g))f(g)\,dg.$$

When G is a “tame” unimodular Lie group the **abstract Plancherel theorem** (or **Fourier inversion theorem**) says there is a measure $d\mu$ on $\hat{G}$ called the **Plancherel measure** such that

$$f(e) = \int_{\hat{G}} \text{Tr}\,T(f)\,d\mu(T),$$

for infinitely differentiable functions f on G with compact support. See Michelle Vergne [697] for a interesting discussion of this subject with many examples. Harish-Chandra obtained the Plancherel measure for real semisimple Lie groups, for which the Plancherel inversion formula does not always involve an integral over all of the dual $\hat{G}$. In Chap. 1 we saw the result for $\mathbb{R}^m$ and $\mathbb{R}^m/\mathbb{Z}^m$. In this chapter we have considered the result for functions on $SO(3)/SO(2)$.

One of the main goals of these volumes is the explicit description of the Plancherel measure—not for functions on the group $GL(n,\mathbb{R})$ but instead for functions on the symmetric space $GL(n,\mathbb{R})/O(n)$. See Volume II (i.e., [667]) for this result and its history.

Formula (2.18) below gives the Plancherel theorem for the group $SO(3)$.

Exercise 2.1.21 (The Irreducible Unitary Representations of $SO(3)$). Let $\{Y_n^m,\ |m| \leq n,\ n = 0,1,2,\ \ldots\}$ denote a complete orthonormal set of spherical harmonics as in Theorem 2.1.1. Define a $(2n+1)\times(2n+1)$ matrix $A_n(g)$ for g in $SO(3)$ by

$$Y_n^m(gx) = \sum_{|k|\leq n} (A_n)_{m,k} Y_n^k(x).$$

(a) Justify this formula for $A_n(g)$ and then show that $A_n(g)$ defines a representation of $SO(3)$.
(b) Show that $A_n(g)$ is a unitary representation using

$$\int_{S^2} Y_n^m(gx)\overline{Y_n^k(gx)}\,dx = \int_{S^2} Y_n^m(x)\overline{Y_n^k(x)}\,dx.$$

(c) Show that the representation $A_n(g)$ is irreducible.

In fact, the representations $A_n(g)$ form a complete set of inequivalent irreducible unitary representations of $SO(3)$. This means that any function f in $L^2(SO(3))$ with respect to Haar measure has a Fourier series expression, converging in the L^2 norm

$$\left.\begin{aligned} &f(g) = \textstyle\sum_{n=0}^{\infty}(2n+1)\ \mathrm{Trace}\,[\hat{f}(n)A_n(g)], \quad \text{where} \\ &\hat{f}(n) = \textstyle\int_{SO(3)} f(g)\ \overline{{}^tA_n(g)}\,dg, \quad dg = \text{Haar measure on } G. \end{aligned}\right\} \tag{2.18}$$

This is proved, for example, in Dym and McKean [147, pp. 256–261]. Vilenkin [704, pp. 440–457], examines the representations of $SO(n)$ defined in an analogous way to that used for $n = 3$ in Exercise 2.1.21 and shows that if $n > 3$, then these representations do not exhaust all of the irreducible unitary representations of $SO(n)$. Only the class-one representations are obtained in this way. A **class-one representation** (A,H) of G has a vector v in H such that $A(k)v = v$ for all k in K. Here $G = SO(n)$ and $K = SO(n-1)$ (embedded in G as in Exercise 2.1.1).

Formula (2.18) giving Fourier analysis on the group $SO(3)$ is a special case of the **Peter–Weyl theorem** for any compact group (see Weyl [730, Vol. III, pp. 58–75], Pontryagin [518], or Weil [725]). The theorem says that if $\left\{T^\alpha = (m_{ij}^\alpha),\ \alpha \in A\right\}$ is a complete system of irreducible unitary representations of a compact topological group G, then $\sqrt{d_\alpha}m_{ij}^\alpha$ forms a complete orthonormal set in $L^2(G)$, where $d_\alpha =$ degree of T^α. The proof is not hard using the spectral theorem for compact self-adjoint operators (Theorem 1.3.7 of Sect. 1.3). The compact self-adjoint operator used in the proof of the Peter–Weyl theorem is the convolution operator $Tf = g*f$, where g is a fixed nonzero continuous function on G such that $g(x) = g(x^{-1})$ for all x in G. The convolution is taken with respect to Haar measure on G. The eigenfunctions for a fixed eigenvalue of T form a finite-dimensional vector space on which G operates by sending $f(x)$ to $f(xa)$ for $a \in G$. This gives a representation of G. The eigenfunctions for T form a complete orthonormal set in $L^2(G)$ and thus so do the matrix entries of the unitary irreducible representations of G. It follows that the analogue of formula (2.18) replaces A_n by the irreducible unitary representations of G and $(2n+1)$ by the degree of the irreducible representation.

The fact that special functions come from representations often leads to an easier understanding of the myriads of formulas listed in books such as Erdélyi et al. [164, 165].

Exercise 2.1.22 (Addition Formulas for Matrix Entries). Suppose that (T,H) is a finite-dimensional representation of G and that $e_1, \ldots, e_n$ is an orthonormal basis of H. Let the i,j th matrix entry of T be $m_{ij}(g) = (T(g)e_i, e_j)$ for g in G. Here $(,)$ denotes the inner product on H. Show that $T(gh) = T(g)T(h)$ implies the following addition formula for the matrix entries:

$$m_{ij}(gh) = \sum_{k=1}^{n} m_{ik}(h)m_{kj}(g).$$

Exercise 2.1.23 (Addition Formula for Spherical Harmonics). For x, y in S^2, prove that

$$\sum_{|m|\le n} Y_n^m(x)\overline{Y_n^m(y)} = \frac{2n+1}{4\pi} P_n({}^t xy).$$

Note that both sides are unchanged if you replace x by gx and y by gy for g in $SO(3)$. So the right and left sides of the equation can only differ by a constant. Find the constant by setting $x = y$ and integrating over S^2.

2.1.6 Integral Equations for Spherical Harmonics

It is possible to characterize spherical harmonics by integral equations rather than differential equations. This method often simplifies the theory, for the same reason that Green's functions simplify the theory of differential operators (see Sect. 1.3 and the discussion surrounding equation (1.16)). Examples of the method appear in Weyl's work on spherical harmonics (see Weyl [730, Vol. III, pp. 386–399]), Selberg's work on the trace formula (see Selberg [569]), and Harish-Chandra's work on group representations (see Harish-Chandra [253, 254]). The method of integral equations also makes it easier to extend results to finite, p-adic and adelic groups (see Gelbart [200], Godement [213], Macdonald [440], Tamagawa [656]). This should not, however, cause us to forget the differential equations and the contact with applied mathematics.

The following theorem was proved in 1916 by Funk [187] and in 1918 by Hecke [258, pp. 208–214].

Theorem 2.1.2 (The Funk–Hecke Theorem). *Suppose that Y_n is a spherical harmonic of degree n. Let $f : [-1,+1] \to \mathbb{C}$ be continuous. Then*

$$\int_{S^2} f({}^t ux) Y_n(u)\, du = 2\pi Y_n(x) \int_{-1}^{+1} f(t) P_n(t)\, dt.$$

Proof. Since every continuous function on $[-1,+1]$ can be uniformly approximated by zonal spherical functions $P_n(x)$, it suffices to do the following exercises. □

Exercise 2.1.24 (Properties of Spherical Harmonics).

(a) Show that

$$\int_{S^2} Y_m(x) P_n({}^t xy)\, dx = \frac{4\pi}{2n+1} Y_n(y) \delta_{mn}.$$

Here $\delta_{mn} = 0$ if $m \neq n$ and $\delta_{mn} = 1$ if $m = n$.

Hint. *Use the addition formula of Exercise 2.1.23.*

(b) Prove that

$$\int_{-1}^{+1} P_n(x) P_m(x)\, dx = \frac{2\pi}{2n+1} \delta_{mn}.$$

The Funk–Hecke theorem actually characterizes spherical functions. This result is generalized in Helgason [277, p. 439, Cor. 7.4]. And we shall also use this approach in later chapters. In order to generalize the Funk–Hecke theorem, one should view the spherical harmonic Y as a function on $G = SO(3)$ and replace the formula in Theorem 2.1.2 by

$$Y(I) \int_K Y(xky)\, dk = Y(x) \int_K Y(ky)\, dk \quad \text{for all } x, y \text{ in } G, \tag{2.19}$$

where dk is Haar measure on $K = SO(2)$ embedded in G as in Exercise 2.1.1, and $I =$ the identity matrix. Furthermore zonal spherical harmonics P are characterized by the integral equation

$$P(I) \int_K P(xky)\, dk = P(x)P(y) \quad \text{for all } x, y \text{ in } G. \tag{2.20}$$

This characterization of zonal spherical harmonics is due to Gelfand [201] in the general case.

Exercise 2.1.25 (The Group-Theoretical Version of the Funk–Hecke Theorem). Show that formula (2.19) implies the Funk–Hecke theorem by integrating (2.19) times $f(y)$ where $f : K\backslash G/K \to \mathbb{C}$.

Exercise 2.1.26 (The Convolution Theorem for $SO(3)$). Let f, g be integrable functions on $SO(3)$ with respect to Haar measure. Define the **convolution** of f and g by

$$(f*g)(x) = \int_{SO(3)} f(u)g(xu^{-1})\, du \quad \text{for } x \text{ in } SO(3).$$

Define the Fourier transform $\hat{f}(n)$ for $n = 0, 1, \ldots$ as in formula (2.18). Show that

$$\widehat{(f*g)}(n) = \hat{f}(n)\hat{g}(n) \quad \text{for all } n = 0, 1, 2, \ldots.$$

Hint. *Use the fact that A_n is a representation.*

Exercise 2.1.27 (The Fundamental Solution to the Heat Equation on the Sphere $S^2 \subset \mathbb{R}^3$). Given the initial heat distribution $f(\theta)$, solve the following initial value problem:

$$\Delta^* u(\theta, \phi, t) = u_t, \quad u(\theta, \phi, 0) = f(\theta), \quad t > 0.$$

Here Δ^* is the Laplacian on the sphere given by formula (2.10).

Answer. *$u = G_t * f$, where*

$$G_t(\theta, \phi) = G_t(\theta) = \sum_{n>0} c_n P_n(\cos\theta) \exp[-n(n+1)t].$$

The constants c_n are chosen so that $G_t \to \delta$ as $t \to 0^+$.

Note. G. Watson [720] considers various analogues of the Gaussian distribution for the sphere, in connection with various statistical problems such as that of studying the distribution of the unit normal vectors to the planes of all known comets. Another reference for group theory and statistics is the book by Diaconis [132] which includes an interesting story about testing for uniformity, Fisher, Jeffries, and continental drift on pages 170–171.

2.2 *O(3)* and $\mathbb{R}^3$: The Radon Transform

> It is interesting to consider the method by which algorithms have been “justified” mathematically in this field [computerized tomography]. While this method consists of mathematical

reasoning in a certain sense, the reasoning is far from rigorous. Approximations are introduced at many steps with only intuition as a guide to the error involved. We do not know of a single instance in which a tomographic algorithm has been justified in a truly rigorous sense. Thus, in contrast to some other workers in this field, we do not feel that one derivation is more rigorous than another, whether it is based on Radon's inversion formula, the Fourier inversion formula or any other foundation.

—From Shepp and Kruskal [588, p. 421].

2.2.1 Harmonic Analysis on $\mathbb{R}^3$ in Spherical Polar Coordinates and Spectral Measures

It is easy to see (cf. Exercise 2.1.1) that the Fourier transform on $\mathbb{R}^n$ commutes with rotation. This leads to a new formulation of harmonic analysis on $\mathbb{R}^n$ in terms of spherical harmonics. For simplicity, we shall consider only the case $n = 3$ here. The generalization to $\mathbb{R}^n$, n arbitrary, can be found in Coifman and Weiss [98] as well as in Stein and Weiss [636]. These results are due to Cauchy and Poisson for radial functions, and Bochner and Hecke in general (see Bochner [43, p. 235], and the article of Stein in J. M. Ash [15, pp. 104–105]).

Exercise 2.2.1 (The Fourier Transform on $\mathbb{R}^n$ Commutes with Rotation). If $g \in O(n)$ and $f : \mathbb{R}^n \to \mathbb{C}$, set $(L(g)f)(x) = f(gx)$ for $x \in \mathbb{R}^n$. Show that $L(g)\hat{f} = \widehat{L(g)f}$, when f is a sufficiently nice function, that the Fourier transform of f exists (as defined in Sect. 1.2 for various classes of functions).

Exercise 2.2.2 (*J*-Bessel Functions). Define the Bessel function of the first kind by the power series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \qquad \text{for } |\arg z| < \pi.$$

(a) Show that $y(z) = J_\nu(z)$ satisfies Bessel's equation:

$$y'' + (1/z)y' + \left(1 - (\nu/z)^2\right) y = 0.$$

(b) Show that $J_\nu(z)$ is represented by the integral formula:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{+1} (1-t^2)^{\nu-1/2} \exp(izt)\, dt,$$

when Re $\nu > -\frac{1}{2}$ and $|\arg z| < \pi$.

(c) Show that $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$.

(d) Prove that

$$J_{n+\frac{1}{2}}(z) = (-1)^n \sqrt{\frac{2}{\pi}} z^{n+\frac{1}{2}} \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z}, n = 0,\ 1, 2, \ldots.$$

These functions $J_{n+\frac{1}{2}}$ are often called **spherical Bessel functions**.

Exercise 2.2.3 (More Properties of Bessel Functions).

(a) **Asymptotics.** Show that

$$\begin{aligned} J_\nu(z) &\sim \Gamma(1+\nu)^{-1}(z/2)^\nu, && \text{as } z \to 0; \\ J_\nu(z) &\sim \sqrt{\tfrac{2}{\pi z}} \cos\left(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}\right), && \text{as } z \to \infty. \end{aligned}$$

(b) Show that $J_n(z) = (-1)^n J_{-n}(z)$, when $n = 0, 1, 2, \ldots$. Then show that if ν is not an integer, J_ν and $J_{-\nu}$ are linearly independent.

(c) **Functional Equation.** Prove that $J_\nu(-z) = \exp(\nu\pi i) J_\nu(z)$.

(d) **Addition Formula.** Set $R = (r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta)^{1/2}$ and prove that

$$J_{\frac{1}{2}}(R) = \sqrt{\frac{2R}{r_1 r_2}} \Gamma\left(\frac{1}{2}\right) \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) J_{m+\frac{1}{2}}(r_1) J_{m+\frac{1}{2}}(r_2) P_m(\cos\theta).$$

Theorem 2.2.1 (The Bochner–Hecke Formula). *Suppose that $Y(u)$, $u \in S^2$ is a surface spherical harmonic of degree n (as in Theorem 2.1.1 of Sect. 2.1) and let $f(ru) = g(r)Y(u)$ for $r \in \mathbb{R}^+$, $u \in S^2$, where $g : \mathbb{R}^+ \to \mathbb{C}$, is such that*

$$\int_0^\infty |g(r)| r^2\, dr < \infty.$$

Then the Euclidean Fourier transform as defined in Sect. 1.2 of f (assuming that f is nice enough for the transform to exist), is given by

$$\hat{f}(ru) = \frac{2\pi}{i^n \sqrt{r}} Y(u) \int_{s \in \mathbb{R}^+} g(s) s^{3/2} J_{n+\frac{1}{2}}(2\pi rs)\, ds.$$

Proof. Let dv denote the element of surface area on S^2. Then, using Theorem 2.1.2 of Sect. 2.1, we have

$$\begin{aligned} \hat{f}(ru) &= \int_{s \in \mathbb{R}^+} \int_{v \in S^2} \exp(-2\pi i r s\, {}^t u v) g(s) Y(v) s^2\, ds\, dv \\ &= 2\pi Y(u) \int_{s \in \mathbb{R}^+} g(s) s^2 \int_{-1}^{+1} \exp(-2\pi r s t) P_n(t)\, dt\, ds. \end{aligned}$$

The proof of this theorem is completed in Exercise 2.2.4. □

Exercise 2.2.4. Show that $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$ implies that

$$\int_{-1}^{+1} \exp(-2\pi i r s t) P_n(t)\, dt = \frac{(-i)^n}{\sqrt{rs}} J_{n+\frac{1}{2}}(2\pi r s).$$

Use integration by parts and Exercise 2.2.2. Then complete the proof of Theorem 2.2.1.

How does the formula of Bochner and Hecke fit into the general scheme of harmonic analysis on symmetric spaces? Why did the Bessel functions suddenly appear out of the blue? To understand this, one must view $\mathbb{R}^3$ as a symmetric space rather than just as an additive group. We explained this at the end of Chap. 1. It leads one to think of the full group of isometries of $\mathbb{R}^3$, namely, the Euclidean group $M(3,\mathbb{R})$ of rigid motions of $\mathbb{R}^3$. If we consider only the orientation-preserving motions G, then G is the semidirect product of the groups T and K, where T consists of translations and is isomorphic to $\mathbb{R}^3$, and $K = SO(3)$. One can view $M(3,\mathbb{R})$ as a group of 4×4 matrices of the form $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$, $A \in SO(3), b \in \mathbb{R}^3$. The matrix $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$ sends $x \in \mathbb{R}^3$ to $Ax+b$. Clearly one can identify $\mathbb{R}^3$ with G/K by mapping a coset $gK,\ g \in G$, to $g0 =$ the result of applying the motion g to the origin 0 in $\mathbb{R}^3$.

Exercise 2.2.5. Do an analogue of the Bochner–Hecke theorem for $\mathbb{R}^2$ for rotation-invariant functions f on $\mathbb{R}^2$.

Answer. *For nice rotation-invariant functions f on $\mathbb{R}^2$, one has*

$$\hat{f}(\rho u) = 2\pi \int_0^{\infty} f(r) J_0(2\pi\rho r) r dr.$$

The connection of Theorem 2.2.2 below with the representations of the Euclidean group (as well as the generalization to $\mathbb{R}^n$) is made in Vilenkin [704, Chap. 11] and Wawrzyńczyk [722]. The case of $\mathbb{R}^2$ is discussed in Dym and McKean [147, pp. 263–273]. See also Helgason [277, pp. 402–403], and Talman [655, Chap. 12].

Exercise 2.2.6 (The Eigenfunctions of the Euclidean Laplacian in Spherical Polar Coordinates). Use Exercise 2.1.2 of Sect. 2.1 to show that if $\Delta f = f_{xx} + f_{yy} + f_{zz} = \lambda f$, and $f(ru) = g(r)Y(u)$ for $r \in \mathbb{R}^+,\ u \in S^2$, and Y is a surface spherical harmonic of degree n, then

$$g(r) = \frac{1}{\sqrt{rt}} J_{n+\frac{1}{2}}(2\pi r t),\ \lambda = (2\pi t)^2.$$

Theorem 2.2.2 (Harmonic Analysis on $\mathbb{R}^3$ in Spherical Polar Coordinates). *Let $e_{n,m,t}(ru) = 2\pi Y_n^m(u)(rt)^{-1/2} J_{n+\frac{1}{2}}(2\pi r t)$ for $r \in \mathbb{R}^+,\ u \in S^2$, where $\{Y_n^m \mid n = 0,\ 1,2,\ldots, |m| \le n\}$ denotes a complete orthonormal set of surface spherical* *harmonics of degree n (as in Theorem 2.1.1 of Sect. 2.1). Then any f in* $L^2(\mathbb{R}^3)$ *has a Fourier expansion (converging in the* L^2*-norm) of the form*

$$f(x) = \sum_{n\geq 0}\sum_{|m|\leq n}\int_{t>0}\hat{f}(n,m,t)e_{n,m,t}(x)t^2\,dt,$$

where

$$\hat{f}(n,m,t) = \int_{\mathbb{R}^3} f(y)\overline{e_{n,m,t}(y)}\,dy.$$

Proof. It suffices to assume that $f(ru) = g(r)Y_n^m(u)$ for $r \in \mathbb{R}^+$, $u \in S^2$. Then we must show that

$$g(r) = (2\pi)^2\int_{s>0}\left(\int_{t>0} g(t)\frac{1}{\sqrt{st}}J_{n+\frac{1}{2}}(2\pi st)t^2\,dt\right)\frac{1}{\sqrt{rs}}J_{n+\frac{1}{2}}(2\pi rs)s^2\,ds. \quad (2.21)$$

This is done in the following exercise. □

Exercise 2.2.7. (a) Prove formula (2.21) by writing down the inversion formula for the ordinary Fourier transform (from Theorem 1.2.1 of Sect. 1.2) in spherical polar coordinates. Then use Theorem 2.2.1 to evaluate the inner Fourier transform. Finally, Theorem 2.1.2 of Sect. 2.1 and Exercise 2.2.4 should be used to complete the proof.

(b) Show that formula (2.21) is equivalent by change of variables to a special case of **Hankel's inversion formula**: (assuming $\nu > -\frac{1}{2}$ and $r > 0$) :

$$f(r) = \int_0^\infty yJ_\nu(yr)\int_0^\infty xJ_\nu(xy)f(x)\,dx\,dy. \quad (2.22)$$

Hankel's inversion formula (2.22) gives the spectral resolution of the singular Sturm–Liouville operator

$$(Lf)(x) = \frac{1}{x}\left(-(xf')' + \frac{\nu^2}{x}f\right) \quad \text{for } x \in (0,\infty) \quad (2.23)$$

This spectral resolution can be derived from a formula of Stieltjes, Stone, Kodaira, and Titchmarsh, which is itself a corollary of the Von Neumann Spectral Theorem for unbounded operators. Let us summarize the theory briefly, following Lang [387], Reed and Simon [540, Chap. 7], Stakgold [618, 619], and some unpublished notes of J. Korevaar. More details can be found in these references as well as Dunford and Schwartz [145], Gelfand and Vilenkin [207], Levitan and Sargsjan [415], Maurin [459], Naimark [490], Titchmarsh [679], Weyl [730, Vol. I, pp. 195–297], and Yosida [748]. In particular, Dunford and Schwartz [145, Vol. II, pp. 1333–1392, 1532–1533], provides an extremely careful discussion of spectral theory for singular differential operators, including the Bessel operator (2.23).

We want to consider linear, possibly unbounded, operators $T : D \to V$, where D is a dense subspace of a Hilbert space V. Define

$$D^* = \{y \in V | \text{ there is a } z \text{ in } V \text{ with } (Tx, y) = (x, z), \text{ for all } x \in D\}.$$

Define T^* = the **adjoint** of T by $(Tx, y) = (x, T^*y)$. We say that T is **self-adjoint** if $D = D^*$ and $T = T^*$. The **spectrum** $\sigma(T)$ of the operator T is defined to be

$$\sigma(T) = \left\{\lambda \in \mathbb{C} \;\middle|\; (T - \lambda I)^{-1} \text{ does not exist as a bounded linear operator on } V \right\}.$$

The **point or discrete spectrum** of T (or the set of **eigenvalues** of T) consists of $\lambda \in \mathbb{C}$ such that $(T - \lambda I)$ is not one-to-one. The **continuous spectrum** of T is the set of $\lambda \in \mathbb{C}$ such that $(T - \lambda I)$ is 1 - 1 and $D =$ range $(T - \lambda I)$ is dense in V but $(T - \lambda I)^{-1}$ is an unbounded linear operator with domain D.

The **von Neumann Spectral Theorem** says that if T is any self-adjoint operator (bounded or not), there is a Stieltjes integral representation

$$I = \int dE_\lambda, \quad T = \int \lambda dE_\lambda, \tag{2.24}$$

for some family of projection-valued measures dE_λ (cf. Reed and Simon [540, Chap. 7]). The integrals are over the spectrum of T, which is real, because T is self-adjoint. The spectral theorem implies that for polynomials $p(\lambda)$, $p(T) = \int p(\lambda) dE_\lambda$. It is possible to define $f(T)$ for continuous functions f on the spectrum of T and then this same formula holds upon replacing p by f.

In order to obtain the formula of Stieltjes, Stone, Kodaira, and Titchmarsh one must be able to compute the **Green's function** or **resolvent kernel** $G(\lambda; x, y)$ defined by

$$(T - \lambda I)^{-1} v(x) = \int G(\lambda; x, y) v(y)\, dy. \tag{2.25}$$

For if we have a convergent integral of the form

$$f(\mu) = \int_{-\infty}^{+\infty} (\lambda - \mu)^{-1} g(\lambda) d\lambda,$$

then at any point c of continuity of $g(\lambda)$, we have

$$\begin{aligned} &\tfrac{1}{2\pi i} \lim_{\varepsilon \to 0+} (f(c + i\varepsilon) - f(c - i\varepsilon)) \\ &\quad = \lim_{\varepsilon \to 0+} \left\{ \tfrac{1}{\pi} \int_{-\infty}^{+\infty} \tfrac{\varepsilon}{(\lambda - c)^2 + \varepsilon^2}\, g(\lambda)\, d\lambda \right\} = g(c). \end{aligned}$$

This is Poisson's integral formula for a half-plane (see Ahlfors [3, p. 171]). It follows that the spectral measure for the operator T is given by the **formula of Stieltjes, Stone, Kodaira, and Titchmarsh:**

$$2\pi i \frac{dE_\lambda}{d\lambda} = (T-(\lambda+i0)I)^{-1} - (T-(\lambda-i0)I)^{-1}, \tag{2.26}$$

assuming that E_λ is well-behaved. This formula is proved in Dunford and Schwartz [145, Vol. II], Lang [387, pp. 412–413], and Reed and Simon [540, p. 237]. Stieltjes [639] found the result in 1894. We will usually refer to formula (2.26) as the **Kodaira–Titchmarsh formula** for brevity.

In order to apply formula (2.26), we need a way to compute the Green's function $G(\lambda;x,y)$. We use the method given in Stakgold [618, 619]. Consider a **Sturm–Liouville operator which is singular** at a and b:

$$Lf = \frac{1}{w}\left(-(pf')' + qf\right), \quad a < x < b. \tag{2.27}$$

Here "singular" means that either the interval is infinite or the functions $w(x)$ or $q(x)$ blow up, or $p(x)$ vanishes at some point in $[a,b]$. The Hilbert space associated to (2.27) is $L^2([a,b],w)$ consisting of Lebesgue measurable functions f on $[a,b]$ such that

$$\int_a^b |f(x)|^2 w(x)\, dx < \infty.$$

The **inner product** for this weighted L^2-space with weight w is

$$(f,g) = \int_a^b f(x)\overline{g(x)}w(x)\, dx.$$

If one makes the correct assumptions about p,q,w and if one imposes the correct sort of boundary conditions at a and b, the operator L will be self-adjoint. In particular, p,q,w should be real and w should be positive.

The Green's function $G(\lambda;x,y)$ of formula (2.24) must satisfy

$$-\frac{\partial}{\partial x}\left(p\frac{\partial}{\partial x}G(\lambda;x,y)\right) + (q-\lambda w)G(\lambda;x,y) = \delta(x-y).$$

Pick c in (a,b). We want to choose two linearly independent solutions φ_λ and ψ_λ of $Lu = \lambda u$, as follows. Suppose that φ_λ solves $Lu = \lambda u$ and in addition φ_λ lies in $L^2([a,c],w)$. Let ψ_λ solve $Lu = \lambda u$ and $\psi_\lambda \in L^2([c,b],w)$. Then

$$G(\lambda;x,y) = \frac{1}{c_\lambda}\begin{cases} \varphi_\lambda(x)\psi_\lambda(y), & a < x < y < b \\ \varphi_\lambda(y)\psi_\lambda(x), & a < y < x < b \end{cases} \tag{2.28}$$

and

$$c_\lambda = \begin{vmatrix} \varphi_\lambda(x) & -\psi_\lambda(x) \\ p\varphi'_\lambda(x) & -p\psi'_\lambda(x) \end{vmatrix}.$$

*Example 2.2.1 (**Spectral Measure for the Hankel Transform Using the Kodaira–Titchmarsh Formula**).* Consider the Sturm–Liouville operator given by formula (2.23). Then formula (2.28) becomes

$$G(\lambda;x,y) = \frac{\pi i}{2}\begin{cases} J_\lambda(\lambda^{1/2}x)H_\nu^{(1)}(\lambda^{1/2}y), & 0<x<y<\infty \\ J_\lambda(\lambda^{1/2}y)H_\nu^{(1)}(\lambda^{1/2}x), & 0<y<x<\infty. \end{cases}$$

Here J_ν is the Bessel function of the first kind from Exercise 2.2.2 and $H_\nu^{(1)}$ is the Hankel function (or Bessel function of the third kind) defined by

$$H_\nu^1(z) = \frac{J_{-\nu}(z) - \exp(-\nu\pi i)J_\nu(z)}{i\sin(\nu\pi)}$$

when ν is not an integer. Take limits as $\nu \to n$ to define $H_n^{(1)}$ when n is an integer. It follows from Lebedev [401, pp. 112–113], that the determinant $c_\lambda = -2i/\pi$ in this case.

In order to check this formula for $G(\lambda;x,y)$, one needs the following asymptotic properties of the Bessel functions from Lebedev [401, Chap. 5], $(\nu > 0)$:

$$\begin{aligned} J_\nu(x) &\sim \left(\tfrac{x}{2}\right)^\nu \tfrac{1}{\Gamma(1+\nu)}, && \text{as } x\to 0, \\ J_\nu(x) &\sim \sqrt{\tfrac{2}{\pi x}}\cos\left(x - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}\right), && \text{as } x\to\infty, \\ H_\nu^{(1)}(x) &\sim -\tfrac{i}{\pi}\left(\tfrac{2}{x}\right)^\nu \Gamma(\nu), && \text{as } x\to 0, \\ H_\nu^{(1)}(x) &\sim \sqrt{\tfrac{2}{\pi x}}\exp\left[i\left(x - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}\right)\right], && \text{as } x\to\infty. \end{aligned} \tag{2.29}$$

Note that if $0<\nu<1$, both $J_\nu(\lambda^{1/2}x)$ and $H_\nu^{(1)}(\lambda^{1/2}x)$ are in $L^2([0,1],x)$. One says that 0 is then in the limit-circle case in Weyl's theory (cf. Stakgold [618,619]). If $\nu \geq 1$, then $J_\nu(\lambda^{1/2}x)$ is in $L^2([0,1],x)$ and $H_\nu^{(1)}(\lambda^{1/2}x) \notin L^2([0,1],x)$. One says that 0 is then in the limit-point case. Note that $J_\nu(\lambda^{1/2}x) \notin L^2([1,\infty],x)$ and $H_\nu^{(1)}(\lambda^{1/2}x) \in L^2([1,\infty],x)$ so that infinity is always in the limit-point case.

To compute the jump of $G(\lambda;x,y)$ required by formula (2.26), assume that $x<y$. Then for $\nu>0$,

$$\begin{aligned} &G(\lambda+i0;x,y) - G(\lambda-i0;x,y) \\ &\quad = \tfrac{i\pi}{2}\left(J_\nu(\lambda^{1/2}x)H_\nu^{(1)}(\lambda^{1/2}y) - J_\nu(-\lambda^{1/2}x)H_\nu^{(1)}(-\lambda^{1/2}y)\right). \end{aligned}$$

So we need the functional equations

$$\begin{aligned} J_\nu(-x) &= \exp(\nu\pi i) J_\nu(x), \\ H_\nu^{(1)}(-x) &= \exp(-\nu\pi i)\left(H_\nu^{(1)}(x) - 2J_\nu(x)\right). \end{aligned}$$

These formulas imply

$$G(\lambda + i0; x, y) - G(\lambda - i0; x, y) = i\pi J_\nu(\lambda^{1/2}x) J_\nu(\lambda^{1/2}y).$$

Thus we obtain from (2.24) and (2.26),

$$\frac{\delta(x-y)}{x} = \frac{1}{2}\int_0^\infty J_\nu(\lambda^{1/2}x) J_\nu(\lambda^{1/2}y)\, dy = \int_0^\infty J_\nu(\rho x) J_\nu(\rho y)\, \rho\, d\rho.$$

This is Hankel's integral theorem.

For higher-rank symmetric spaces, the method just used to compute the spectral measure proves rather unwieldy because the Green's functions are much more complicated. Thus we will formulate another method in Chap. 3 and Volume II [667]. For this method, one needs only the asymptotics and functional equations of the eigenfunctions appearing in the spectral expansion of the operator L, rather than the expression for the resolvent kernel, involving other eigenfunctions as well. The method is due to Harish-Chandra and it would be very interesting to derive Harish-Chandra's Plancherel measure from the spectral theorem in a similar way to that which gives formula (2.26).

One can also ask for the spectral decomposition of the operator in (2.23) on the finite interval $(0,a)$. The problem still has a singularity at 0. But on $(0,a)$, the spectral resolution of the operator L in (2.23) is given by a series rather than an integral, when $\nu > -1$:

$$f(x) = \sum_{m=1}^{\infty} c_m J_\nu(\alpha_{\nu m} x/a), \quad 0 \le x \le a, \quad \nu > -1, \tag{2.30}$$

where $\{\alpha_{\nu m}\}_{m \ge 1}$ is the set of all positive roots of $J_\nu(\alpha_{\nu m}) = 0$ and

$$c_m = 2a^{-2}[J_{\nu+1}(\alpha_{\nu m})]^{-2}\int_0^a f(r) J_\nu(\alpha_{\nu m} r/a) r\, dr.$$

The **Fourier–Bessel series expansion** (2.30) is derived in Titchmarsh [679, Vol. 1, pp. 81–86]. See also Stakgold [619, pp. 305–308, 313–315]. It is quite interesting to let a approach infinity in (2.30) and watch (2.30) approach (2.22) (cf. Morse and Feshbach [479, Vol. I, pp. 762–766]).

Exercise 2.2.8 (The Kontorovich–Lebedev Transform). Consider Bessel's equation with the role of the parameters interchanged:

$$-(xw')' + \mu x w - \lambda w/x = 0, \quad 0 < x < \infty.$$

Here $\mu > 0$ and λ is the eigenvalue. Show that the Green's function is

$$G(\lambda; x, y) = \begin{cases} I_{-i\sqrt{\lambda}}(\mu^{1/2}x)K_{-i\sqrt{\lambda}}(\mu^{1/2}y), & 0 < x < y < \infty, \\ I_{-i\sqrt{\lambda}}(\mu^{1/2}y)K_{-i\sqrt{\lambda}}(\mu^{1/2}x), & 0 < y < x < \infty, \end{cases}$$

where I and K are the **Bessel functions of imaginary argument**. The definitions are

$$\begin{aligned} I_\nu(z) &= \exp(-\nu\pi i/2)J_\nu(z\exp(\pi i/2)), \quad -\pi < \arg z < \pi/2, \\ K_\nu(z) &= \frac{\pi i}{2}\exp(\nu\pi i/2)H_\nu^{(1)}(z\exp(\pi i/2)), \quad -\pi < \arg z < \pi/2. \end{aligned}$$

See Chap. 3 and Lebedev [401] for more information on these functions.

Show that $K_{-\nu}(x) = K_\nu(x)$ and $I_{-\nu}(x) - I_\nu(x) = 2\sin(\nu\pi)K_\nu(x)/\pi$. Then prove the **Kontorovich–Lebedev inversion formula**:

$$x\delta(x-y) = \frac{2}{\pi^2}\int_0^\infty K_{i\nu}(\sqrt{\mu}x)K_{i\nu}(\sqrt{\mu}y)\ \nu\sinh(\pi\nu)\ d\nu.$$

This will be a central result in Sect. 3.1.

2.2.2 CAT Scanners and the Radon Transform

Modern X-ray scanners can reproduce the tissue density function $f(x),\ x \in \mathbb{R}^2$, for a plane slice of a person's head, for example. They operate by inverting the **Radon transform**, defined for $k \in \mathbb{R},\ u \in S^1$, by

$$Rf(k,u) = \int_{\substack{{}^t xu = k \\ x \in \mathbb{R}^2}} f(x)\ ds, \tag{2.31}$$

where the integral is over a line in the plane and ds is the element of arc length. The **inversion formula** for this transform goes back to Radon [528] in 1917. It says that

$$f(x) = -\frac{1}{\pi}\int_{q>0}\frac{1}{q}d\left(\frac{1}{2\pi}\int_{u\varepsilon S^1} Rf(q + {}^t xu, u)\ du\right). \tag{2.32}$$

Here the integral over q in $\mathbb{R}^+$ can be viewed as a Stieltjes integral or as a Cauchy principal value integral, assuming that f is continuous with compact support.

References for the Radon transform include Bracewell [62], Dym and McKean [147], Gelfand, Graev, and Vilenkin [204], Herman [290], Helgason [279], Louis [431, 432], Ludwig [435], Elena Prestini [522], Quinto [524], and Shepp and Kruskal [588]. In fact, Funk [187] proved the analogue of (2.32) for S^2 rather than

$\mathbb{S}^1$ one year earlier than Radon proved his result. Helgason [279] shows that a vast generalization of this theory is possible, viewing the Radon inversion formula as involving two dual integrations—one over the points in a hyperplane and the other over the hyperplanes through a point. The Radon transform appears in so many applications to signal processing that Matlab's Signal Processing Toolbox has a Radon transform command. We look at finite analogues of the Radon transform in [668, pp. 71–72]

The next sequence of exercises presents a derivation of Radon's inversion formula.

Exercise 2.2.9. (a) Suppose that $f:\mathbb{R}^2 \to \mathbb{C}$ is a Schwartz function. Show that the Fourier inversion formula can be written in the form

$$f(x) = \int_{u\in S^1} Ef({}^t ux, u)\, du,$$

where

$$Ef(t,u) = \frac{1}{2}\int_{r\in\mathbb{R}}\int_{k\in\mathbb{R}} |r| Rf(k,u) \exp[2\pi i r(t-k)]\, dk\, dr.$$

In the first formula du denotes the angle measure on the unit circle S^1. The Radon transform $Rf(k,u)$ is defined by (2.31).

(b) Let $\mathrm{abs}(r) = |r|,\ r\in\mathbb{R}$. Show that $\mathrm{abs}(r)$ is not the Fourier transform of a Lebesgue integrable function.

Hint. *Recall the Riemann–Lebesgue lemma. Note that if* $\widehat{\mathrm{abs}(r)}$ *were a function, then you could write:*

$$Ef(t,u) = \frac{1}{2}\int_{k\in\mathbb{R}} \widehat{\mathrm{abs}(t-k)} Rf(k,u)\, dk.$$

Exercise 2.2.10 (Derivatives and Fourier Transforms of Some Distributions). Define the Cauchy principal value integral by

$$PV\left(\int f(x)\, dx\right) = \lim_{\varepsilon\to 0}\left(\int_{|x|>\varepsilon} f(x)\, dx\right).$$

Define

$$(x^{-1},\varphi) = PV\int x^{-1}\varphi(x)\, dx$$

and

$$(x^{-2},\varphi) = PV\int x^{-2}(\varphi(x)-\varphi(0))\, dx$$

for test functions φ as in Sect. 1.1. Prove that in the sense of distributions (see Sect. 1.1) the following formulas are valid:

(a) $x^{-1} = (\log|x|)'$.
(b) $x^{-2} = -(x^{-1})'$.
(c) $\widehat{x^{-1}} = -i\pi\,\mathrm{sgn}(x)$, where $\mathrm{sgn}(x) = \mathrm{abs}(x)/x$.
(d) $\widehat{\mathrm{abs}(x)} = -(2\pi^2)^{-1}x^{-2}$.

Hint. *See Vladimirov [706, pp. 75, 86, 134], or Bracewell [61, p. 130].*

Exercise 2.2.11. Derive Radon's inversion formula from Exercises 2.2.9 and 2.2.10, using properties of Fourier transforms of distributions.

Note. The **Hilbert transform** of a function is

$$Hf(x) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{(x-t)}\,dt = -\frac{1}{\pi} f*(x^{-1}).$$

Thus part (c) of Exercise 2.2.10 implies that

$$\widehat{Hf(x)} = i\,\mathrm{sgn}(x)\hat{f}.$$

Because Radon's inversion formula (2.32) contains a derivative, it does not appear to be as useful for numerical calculation as the formula of Exercise 2.2.9. So those that have computed these transforms in practice have used approximations to $\widehat{\mathrm{abs}(x)}$ (cf. Herman [290, pp. 19ff], and Shepp and Kruskal [588]).

Exercise 2.2.12. Compute the Fourier transform of the following approximation to $\mathrm{abs}(x)$ for $x \in \mathbb{R}^2$:

$$f(x) = \begin{cases} |x|, & \text{if } |x| < A, \\ 0, & \text{otherwise.} \end{cases}$$

Compare the decay at infinity with that of $|x|^{-2}$ from part (d) of Exercise 2.2.10.

Note. You can write the Fourier transform in the plane as a Bessel transform, much as we did in Theorem 2.2.1 for the Fourier transform in 3-space. See Exercise 2.2.5.

Some History. In 1956, R. Bracewell used a method analogous to the Radon transform to study solar radiation. In the 1960s and 1970s CT scanners for medicine were developed independently by A. M. Cormack and G. N. Hounsfield. They were not based on the Radon transform. The Hounsfield algorithm was used in the first commercial CAT scanner made by EMI Central Research Labs. in the UK. The first patient brain scan was done in 1971. A. M. Cormack and G. N. Hounsfield shared the Nobel Prize in Medicine in 1979.

The four illustrations in Figs. 2.6 and 2.7 from the 1978 paper of Shepp and Kruskal [588] show a mathematical phantom on the bottom in Fig. 2.6 representing

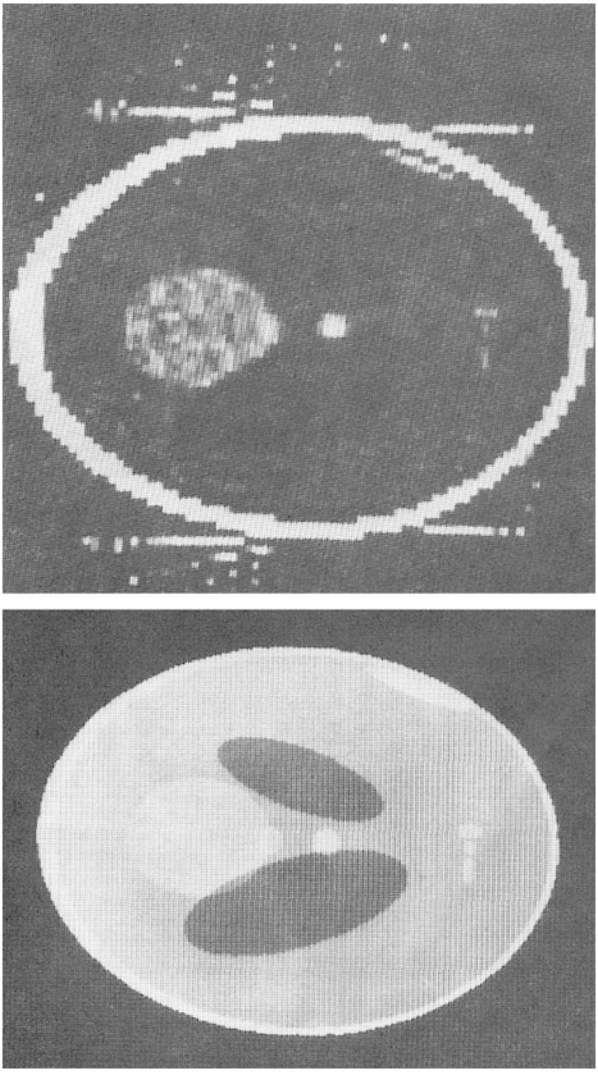

Fig. 2.6 On the bottom is a simulation of a human head using 11 ellipses. On the *top* is a reconstruction using the algorithm embodied in the first commercial machine (EMI Ltd.) from 180×160 strip projection data obtained by exact calculation from the image on the *bottom*. (From Shepp and Kruskal [588, pp. 422–423]. Reprinted by permission of *American Mathematics Monthly*)

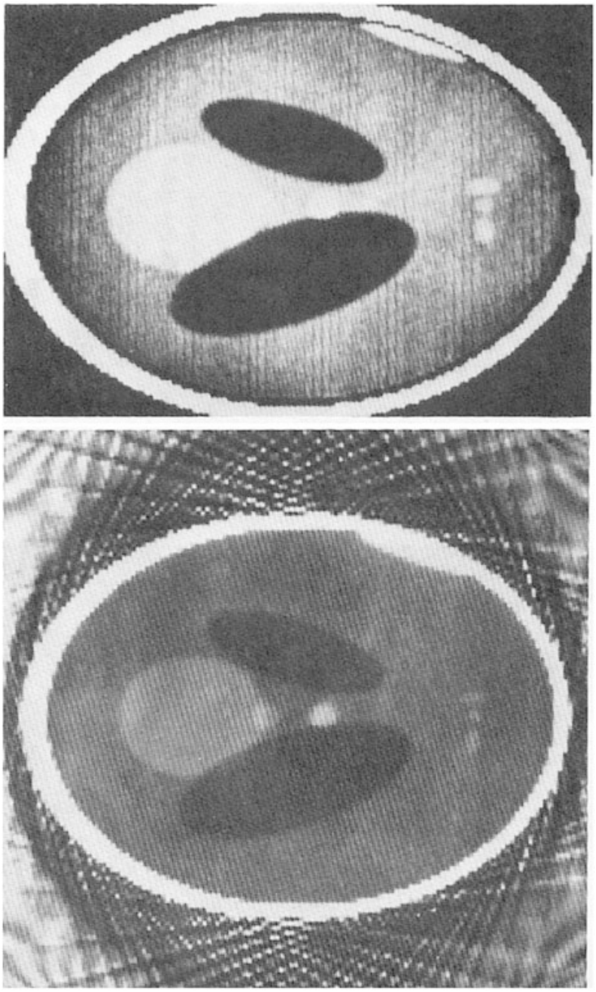

Fig. 2.7 On the bottom is a reconstruction from the *bottom image* in Fig. 2.6 using the Fourier based algorithm of Shepp. On the *top* is a reconstruction using the algorithm used in the 1970s EMI machine from 180×239 strip projection data obtained by exact calculation from the *bottom image* in Fig. 2.6. (From Shepp and Kruskal [588, p. 424]. Reprinted by permission of *American Mathematics Monthly*

a slice of a human head and reconstructions of this phantom by three different algorithms. The Hounsfield algorithm was used to produce the top part of Fig. 2.6. The bottom part of Fig. 2.7 was produced with a Fourier-based algorithm of Shepp. The top part of Fig. 2.7 was produced with an algorithm used by EMI Ltd. in the 1970s. Clearly progress was made.

A description of the progress made between the 1970s and 1980s is contained in the following quotation from *Science 214* (1981), p. 1327:

> When CT scanners first became commercially available about 8 years ago it took 5 minutes to scan a patient's head and 5 minutes for each computerized reconstruction of an image from the x-ray data. Now, because of advances in the design of the scanners and in computer technology, the newest machines can scan a head just 10 seconds and can reconstruct an image virtually instantaneously. According to Jay Thomas Payne of Abbott Northwestern Hospital in Minneapolis, the Mayo Clinic's first CT scanner, which is only 5 years old, has been relegated to the clinic's historical museum.

Of course, despite the progress in speed and accuracy of CT scanners, they still expose people to radiation. In a *New York Times* article from August 21, 2012, Jane E. Brody wrote the following.

> But it [radiation] also has a potentially serious medical downside: the ability to damage DNA and, 10 to 20 years later, to cause cancer. CT scans alone, which deliver 100 to 500 times the radiation associated with an ordinary X-ray and now provide three-fourths of Americans' radiation exposure, are believed to account for 1.5 percent of all cancers that occur in the United States.

Thus nuclear magnetic resonance tomography (NMR alias MRI) may be destined to be the tomography of the future. It uses magnetic fields rather than x-rays and thus is *presumably* less damaging to the body. Louis [431,432] considers applications of the three-dimensional Radon transform to NMR tomography. Spherical harmonics are used to improve the algorithm by studying the kernel (i.e., inverse image of 0) of the transform (ghosts). In 2003, the Nobel Prize for Physiology or Medicine went to Paul Lauterbur and Sir Peter Mansfields for their work on MRI. See Elena Prestini [522], Chap. 8, for more details on the subject. She begins the chapter with a quotation from a *New York Post* article from 1939 about a talk by I. I. Rabi explaining why NMR scanning works. The quote is: "We are all radio stations."

There are many other applications of the Radon transform; for example, in radio astronomy (see Bracewell's article in Herman [290, pp. 81–104]). And there are applications to partial differential equations; e.g., to find solutions of the wave equation (see Dym and McKean [147, pp. 137–139], and Helgason [279].

This concludes our discussion of harmonic analysis on the sphere. It would be possible to consider spherical analogues of many more results from Chap. 1. We shall leave this to the interested reader. For example, the group of isometries of the sphere is just $O(3)$. A discontinuous subgroup $\Gamma \subset O(3)$ has the property that any domain in the sphere can contain only finitely many points equivalent under Γ to any given point. There are very few such discontinuous Γ for the sphere or elliptic plane. They correspond to the regular polyhedra (see Hilbert and Cohn-Vossen [297, p. 242]). These are the analogues of the space groups considered in Sect. 1.4. It is also possible to discuss the analogue of the Poisson summation formula for $\Gamma\backslash S^2 \cong \Gamma\backslash G/K,\ G = SO(3),\ K = SO(2)$. We shall not do this here, since our main interest is the Selberg trace formula for noncompact fundamental domains (see Chap. 3). In fact, we never examined the Selberg trace formula when G is the Euclidean group either (but see Hejhal [265]).

There is work on fast computation of Fourier transforms on the sphere by Driscoll and Healy [140], for example. There are also finite analogues of Radon transforms (see [668]).

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