Chapter 2
Proofs on Unit Sphere Packings

Abstract The proofs presented in this chapter can be grouped as follows. We prove lower and upper estimates for the contact numbers of packings of n unit balls in Euclidean 3-space. One can regard this problem as a combinatorial relative of the Kepler problem on the densest unit sphere packings. Next, we give lower estimates for the surface volume of Voronoi cells in packings of unit balls in Euclidean d-space for all $d \geq 2$ and then we improve those estimates in dimensions $d \geq 8$. All these results imply upper bounds for the usual density of unit ball packings. Returning to the 3-dimensional Euclidean space we give lower bounds for the average surface area (resp., average edge curvature) of the cells in an arbitrary normal tiling with each cell holding a unit ball. On the one hand, it leads to a new version of the Kepler problem on unit sphere packings on the other hand, it generates a new relative of Kelvin's foam problem. Finally, we find sufficient conditions for sphere packings being uniformly stable, a property that holds for all densest lattice sphere packings up to dimension 8.

2.1 Proof of Theorem 1.1.6

Theorem 1.1.6.

$$(i)\ \ C(n) < 6n - 0.926n^{\frac{2}{3}} \textit{ for all } n \geq 2;$$
$$(ii)\ \ C_{fcc}(n) < 6n - \frac{3\sqrt[3]{18\pi}}{\pi}n^{\frac{2}{3}} = 6n - 3.665\ldots n^{\frac{2}{3}} \textit{ for all } n \geq 2;$$
$$(iii)\ \ 6n - \sqrt[3]{486}n^{\frac{2}{3}} < C_{fcc}(n) \leq C(n) \textit{ for all } n = \frac{k(2k^2+1)}{3} \textit{ with } k \geq 2.$$

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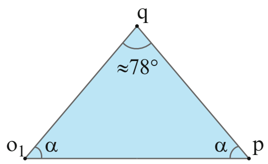

Fig. 2.1 The isosceles triangle $\triangle \mathbf{o}_1\mathbf{pq}$

2.1.1 An Upper Bound for Sphere Packings: Proof of (i)

The proof presented in this section follows the proof of (i) of Theorem 1.1 in [54] and as such it is based on the recent breakthrough results of Hales [120]. The details are as follows.

Let $\mathbf{B}$ denote the (closed) unit ball centered at the origin $\mathbf{o}$ of $\mathbb{E}^3$ and let $\mathcal{P} := \{\mathbf{c}_1 + \mathbf{B}, \mathbf{c}_2 + \mathbf{B}, \dots, \mathbf{c}_n + \mathbf{B}\}$ denote the packing of n unit balls with centers $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ in $\mathbb{E}^3$ having the largest number $C(n)$ of touching pairs among all packings of n unit balls in $\mathbb{E}^3$. ($\mathcal{P}$ might not be uniquely determined up to congruence in which case $\mathcal{P}$ stands for any of those extremal packings.) Now, let $\hat{r} := 1.58731$. The following statement shows the main property of $\hat{r}$ that is needed for our proof of Theorem 1.1.6.

Theorem 2.1.1. *Let $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_{13}$ be 13 different members of a packing of unit balls in $\mathbb{E}^3$. Assume that each ball of the family $\mathbf{B}_2, \mathbf{B}_3, \dots, \mathbf{B}_{13}$ touches $\mathbf{B}_1$. Let $\hat{\mathbf{B}}_i$ be the closed ball concentric with $\mathbf{B}_i$ having radius $\hat{r}$, $1 \le i \le 13$. Then the boundary $\mathrm{bd}(\hat{\mathbf{B}}_1)$ of $\hat{\mathbf{B}}_1$ is covered by the balls $\hat{\mathbf{B}}_2, \hat{\mathbf{B}}_3, \dots, \hat{\mathbf{B}}_{13}$, that is,*

$$\mathrm{bd}(\hat{\mathbf{B}}_1) \subset \cup_{j=2}^{13} \hat{\mathbf{B}}_j .$$

Proof. Let $\mathbf{o}_i$ be the center of the unit ball $\mathbf{B}_i$, $1 \le i \le 13$ and assume that $\mathbf{B}_1$ is tangent to the unit balls $\mathbf{B}_2, \mathbf{B}_3, \dots, \mathbf{B}_{13}$ at the points $\mathbf{t}_j \in \mathrm{bd}(\mathbf{B}_j) \cap \mathrm{bd}(\mathbf{B}_1), 2 \le j \le 13$.

Let α denote the measure of the angles opposite to the equal sides of the isosceles triangle $\triangle \mathbf{o}_1\mathbf{pq}$ with $\mathrm{dist}(\mathbf{o}_1, \mathbf{p}) = 2$ and $\mathrm{dist}(\mathbf{p}, \mathbf{q}) = \mathrm{dist}(\mathbf{o}_1, \mathbf{q}) = \hat{r}$, where $\mathrm{dist}(\cdot, \cdot)$ denotes the Euclidean distance between the corresponding two points. Clearly, $\cos\alpha = \frac{1}{\hat{r}}$ with $\alpha < \frac{\pi}{3}$ (Fig. 2.1).

Lemma 2.1.2. *Let $\mathbf{T}$ be the convex hull of the points $\mathbf{t}_2, \mathbf{t}_3, \dots, \mathbf{t}_{13}$. Then the radius of the circumscribed circle of each face of the convex polyhedron $\mathbf{T}$ is less than $\sin\alpha$.*

Proof. Let F be an arbitrary face of $\mathbf{T}$ with vertices $\mathbf{t}_j$, $j \in I_F \subset \{2, 3, \dots, 13\}$ and let $\mathbf{c}_F$ denote the center of the circumscribed circle of F. Clearly, the triangle $\triangle \mathbf{o}_1\mathbf{c}_F\mathbf{t}_j$ is a right triangle with a right angle at $\mathbf{c}_F$ and with an acute angle of measure β_F at $\mathbf{o}_1$ for all $j \in I_F$. We have to show that $\beta_F < \alpha$. We prove this by contradiction. Namely, assume that $\alpha \le \beta_F$. Then either $\frac{\pi}{3} < \beta_F$ or $\alpha \le \beta_F \le \frac{\pi}{3}$. First, let us take a closer look of the case $\frac{\pi}{3} < \beta_F$. Reflect the point $\mathbf{o}_1$ about the plane of F and label the point obtained by $\mathbf{o}_1'$ (Fig. 2.2).

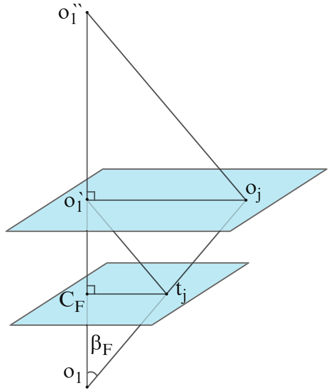

Fig. 2.2 The plane reflections to obtain $\mathbf{o}_1'$ and $\mathbf{o}_1''$

Clearly, the triangle $\triangle \mathbf{o}_1\mathbf{o}_1'\mathbf{o}_j$ is a right triangle with a right angle at $\mathbf{o}_1'$ and with an acute angle of measure β_F at $\mathbf{o}_1$ for all $j \in I_F$. Then reflect the point $\mathbf{o}_1$ about $\mathbf{o}_1'$ and label the point obtained by $\mathbf{o}_1''$ furthermore, let $\mathbf{B}_1''$ denote the unit ball centered at $\mathbf{o}_1''$. As $\frac{\pi}{3} < \beta_F$ therefore $\mathrm{dist}(\mathbf{o}_1, \mathbf{o}_1'') < 2$ and so, one can simply translate $\mathbf{B}_1''$ along the line $\mathbf{o}_1\mathbf{o}_1''$ away from $\mathbf{o}_1$ to a new position say, $\mathbf{B}_1'''$ such that it is tangent to $\mathbf{B}_1$. However, this would mean that $\mathbf{B}_1$ is tangent to 13 non-overlapping unit balls namely, to $\mathbf{B}_1''', \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_{13}$, clearly contradicting to the well-known fact [170] that this number cannot be larger than 12. Thus, we are left with the case when $\alpha \leq \beta_F \leq \frac{\pi}{3}$. By repeating the definitions of $\mathbf{o}_1'$, $\mathbf{o}_1''$, and $\mathbf{B}_1''$, the inequality $\beta_F \leq \frac{\pi}{3}$ implies in a straightforward way that the 14 unit balls $\mathbf{B}_1, \mathbf{B}_1'', \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_{13}$ form a packing in $\mathbb{E}^3$. Moreover, the inequality $\alpha \leq \beta_F$ yields that $\mathrm{dist}(\mathbf{o}_1, \mathbf{o}_1'') \leq 4\cos\alpha = \frac{4}{\hat{r}} = 2.51998\ldots < 2.52$. Finally, notice that the latter inequality contradicts to the following recent result of Hales [120].

Theorem 2.1.3. *Let $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{14}$ be 14 different members of a packing of unit balls in $\mathbb{E}^3$. Assume that each ball of the family $\mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_{13}$ touches $\mathbf{B}_1$. Then the distance between the centers of $\mathbf{B}_1$ and $\mathbf{B}_{14}$ is at least 2.52.*

This completes the proof of Lemma 2.1.2. □

Now, we are ready to prove Theorem 2.1.1. First, we note that by projecting the faces F of $\mathbf{T}$ from the center point $\mathbf{o}_1$ onto the sphere $\mathrm{bd}(\hat{\mathbf{B}}_1)$ we get a tiling of $\mathrm{bd}(\hat{\mathbf{B}}_1)$ into spherically convex polygons $\hat{F}$. Thus, it is sufficient to show that if F is an arbitrary face of $\mathbf{T}$ with vertices $\mathbf{t}_j$, $j \in I_F \subset \{2, 3, \ldots, 13\}$, then its central projection $\hat{F} \subset \mathrm{bd}(\hat{\mathbf{B}}_1)$ is covered by the closed balls $\hat{\mathbf{B}}_j$, $j \in I_F \subset \{2, 3, \ldots, 13\}$. Second, in order to achieve this it is sufficient to prove that the projection $\hat{\mathbf{c}}_F$ of the center $\mathbf{c}_F$ of the circumscribed circle of F from the center point $\mathbf{o}_1$ onto the sphere $\mathrm{bd}(\hat{\mathbf{B}}_1)$ is covered by each of the closed balls $\hat{\mathbf{B}}_j$, $j \in I_F \subset \{2, 3, \ldots, 13\}$. Indeed, if in the triangle $\triangle \mathbf{o}_1\mathbf{o}_j\hat{\mathbf{c}}_F$ the measure of the angle at $\mathbf{o}_1$ is denoted by β_F, then Lemma 2.1.2 implies in a straighforward way that $\beta_F < \alpha$. Hence, based on $\mathrm{dist}(\mathbf{o}_1, \mathbf{o}_j) = 2$ and $\mathrm{dist}(\mathbf{o}_1, \hat{\mathbf{c}}_F) = \hat{r}$, a simple comparison of the triangle $\triangle \mathbf{o}_1\mathbf{o}_j\hat{\mathbf{c}}_F$ with the triangle $\triangle \mathbf{o}_1\mathbf{p}\mathbf{q}$ yields that $\mathrm{dist}(\mathbf{o}_j, \hat{\mathbf{c}}_F) < \hat{r}$ holds for all $j \in I_F \subset \{2, 3, \ldots, 13\}$, finishing the proof of Theorem 2.1.1. □

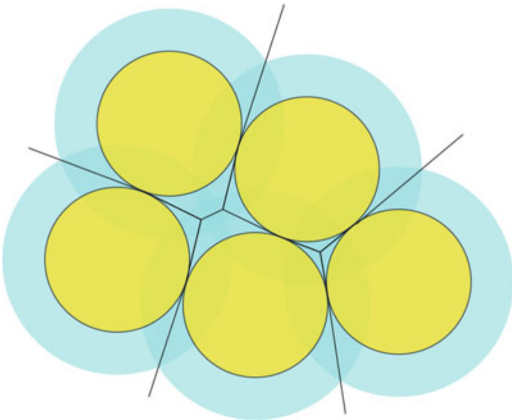

Fig. 2.3 Voronoi cells of a packing with *yellow* $\mathbf{c}_i + \mathbf{B}$'s and *blue* $\mathbf{c}_i + \hat{r}\mathbf{B}$'s

Next, let us take the union $\bigcup_{i=1}^{n} (\mathbf{c}_i + \hat{r}\mathbf{B})$ of the closed balls $\mathbf{c}_1 + \hat{r}\mathbf{B}, \mathbf{c}_2 + \hat{r}\mathbf{B}, \dots, \mathbf{c}_n + \hat{r}\mathbf{B}$ of radii $\hat{r}$ centered at the points $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ in $\mathbb{E}^3$.

Theorem 2.1.4.

$$\frac{n\mathrm{vol}_3(\mathbf{B})}{\mathrm{vol}_3\left(\bigcup_{i=1}^{n} (\mathbf{c}_i + \hat{r}\mathbf{B})\right)} < 0.7547,$$

where $\mathrm{vol}_3(\cdot)$ *refers to the* 3*-dimensional volume of the corresponding set.*

Proof. First, partition $\bigcup_{i=1}^{n} (\mathbf{c}_i + \hat{r}\mathbf{B})$ into truncated Voronoi cells as follows. Let $\mathbf{P}_i$ denote the Voronoi cell of the packing $\mathcal{P}$ assigned to $\mathbf{c}_i + \mathbf{B}$, $1 \leq i \leq n$, that is, let $\mathbf{P}_i$ stand for the set of points of $\mathbb{E}^3$ that are not farther away from $\mathbf{c}_i$ than from any other $\mathbf{c}_j$ with $j \neq i, 1 \leq j \leq n$. Then, recall the well-known fact (see for example, [99]) that the Voronoi cells $\mathbf{P}_i$, $1 \leq i \leq n$ just introduced form a tiling of $\mathbb{E}^3$. Based on this it is easy to see that the truncated Voronoi cells $\mathbf{P}_i \cap (\mathbf{c}_i + \hat{r}\mathbf{B})$, $1 \leq i \leq n$ generate a tiling of the non-convex container $\bigcup_{i=1}^{n} (\mathbf{c}_i + \hat{r}\mathbf{B})$ for the packing $\mathcal{P}$. Second, as $\sqrt{2} < \hat{r}$ therefore the following very recent result of Hales [120] (see Lemma 9.13 on p. 228) applied to the truncated Voronoi cells $\mathbf{P}_i \cap (\mathbf{c}_i + \hat{r}\mathbf{B})$, $1 \leq i \leq n$ implies the inequality of Theorem 2.1.4 in a straightforward way (Fig. 2.3).

Theorem 2.1.5. *Let $\mathcal{F}$ be an arbitrary (finite or infinite) family of non-overlapping unit balls in $\mathbb{E}^3$ with the unit ball $\mathbf{B}$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^3$ belonging to $\mathcal{F}$. Let $\mathbf{P}$ stand for the Voronoi cell of the packing $\mathcal{F}$ assigned to $\mathbf{B}$. Let $\mathbf{Q}$ denote a regular dodecahedron circumscribed $\mathbf{B}$ (having circumradius $\sqrt{3}\tan\frac{\pi}{5} = 1.2584\dots$). Finally, let $r := \sqrt{2} = 1.4142\dots$ and let $r\mathbf{B}$ denote the ball of radius r centered at the origin $\mathbf{o}$ of $\mathbb{E}^3$. Then*

$$\frac{\mathrm{vol}_3(\mathbf{B})}{\mathrm{vol}_3(\mathbf{P})} \leq \frac{\mathrm{vol}_3(\mathbf{B})}{\mathrm{vol}_3(\mathbf{P} \cap r\mathbf{B})} \leq \frac{\mathrm{vol}_3(\mathbf{B})}{\mathrm{vol}_3(\mathbf{Q})} < 0.7547.$$

This finishes the proof of Theorem 2.1.4. $\square$

The well-known isoperimetric inequality [155] applied to $\bigcup_{i=1}^{n}(\mathbf{c}_i+\hat{r}\mathbf{B})$ yields

Lemma 2.1.6.

$$36\pi\,\mathrm{vol}_3^2\left(\bigcup_{i=1}^{n}(\mathbf{c}_i+\hat{r}\mathbf{B})\right)\le \mathrm{svol}_2^3\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i+\hat{r}\mathbf{B})\right)\right),$$

where $\mathrm{svol}_2(\cdot)$ *refers to the* 2*-dimensional surface volume of the corresponding set.*

Thus, Theorem 2.1.4 and Lemma 2.1.6 generate the following inequality.

Corollary 2.1.7.

$$15.159805n^{\frac{2}{3}}<15.15980554\ldots n^{\frac{2}{3}}=\frac{4\pi}{(0.7547)^{\frac{2}{3}}}n^{\frac{2}{3}}$$
$$<\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i+\hat{r}\mathbf{B})\right)\right).$$

Now, assume that $\mathbf{c}_i+\mathbf{B}\in\mathcal{P}$ is tangent to $\mathbf{c}_j+\mathbf{B}\in\mathcal{P}$ for all $j\in T_i$, where $T_i\subset\{1,2,\ldots,n\}$ stands for the family of indices $1\le j\le n$ for which $\mathrm{dist}(\mathbf{c}_i,\mathbf{c}_j)=2$. Then let $\hat{S}_i:=\mathrm{bd}(\mathbf{c}_i+\hat{r}\mathbf{B})$ and let $\hat{\mathbf{c}}_{ij}$ be the intersection of the line segment $\mathbf{c}_i\mathbf{c}_j$ with $\hat{S}_i$ for all $j\in T_i$. Moreover, let $C_{\hat{S}_i}(\hat{\mathbf{c}}_{ij},\frac{\pi}{6})$ (resp., $C_{\hat{S}_i}(\hat{\mathbf{c}}_{ij},\alpha)$) denote the open spherical cap of $\hat{S}_i$ centered at $\hat{\mathbf{c}}_{ij}\in\hat{S}_i$ having angular radius $\frac{\pi}{6}$ (resp., α with $0<\alpha<\frac{\pi}{2}$ and $\cos\alpha=\frac{1}{\hat{r}}$). Clearly, the family $\{C_{\hat{S}_i}(\hat{\mathbf{c}}_{ij},\frac{\pi}{6}),\,j\in T_i\}$ consists of pairwise disjoint open spherical caps of $\hat{S}_i$; moreover,

$$\frac{\sum_{j\in T_i}\mathrm{svol}_2\left(C_{\hat{S}_i}(\hat{\mathbf{c}}_{ij},\frac{\pi}{6})\right)}{\mathrm{svol}_2\left(\cup_{j\in T_i}C_{\hat{S}_i}(\hat{\mathbf{c}}_{ij},\alpha)\right)}=\frac{\sum_{j\in T_i}\mathrm{Sarea}\left(C(\mathbf{u}_{ij},\frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j\in T_i}C(\mathbf{u}_{ij},\alpha)\right)},\tag{2.1}$$

where $\mathbf{u}_{ij}:=\frac{1}{2}(\mathbf{c}_j-\mathbf{c}_i)\in\mathbb{S}^2:=\mathrm{bd}(\mathbf{B})$ and $C(\mathbf{u}_{ij},\frac{\pi}{6})\subset\mathbb{S}^2$ (resp., $C(\mathbf{u}_{ij},\alpha)\subset\mathbb{S}^2$) denotes the open spherical cap of $\mathbb{S}^2$ centered at $\mathbf{u}_{ij}$ having angular radius $\frac{\pi}{6}$ (resp., α) and where $\mathrm{Sarea}(\cdot)$ refers to the spherical area measure on $\mathbb{S}^2$. Now, Molnár's density bound (Satz I in [148]) implies that

$$\frac{\sum_{j\in T_i}\mathrm{Sarea}\left(C(\mathbf{u}_{ij},\frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j\in T_i}C(\mathbf{u}_{ij},\alpha)\right)}<0.89332\,.\tag{2.2}$$

In order to estimate

$$\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i+\hat{r}\mathbf{B})\right)\right)$$

from above let us assume that m members of $\mathcal{P}$ have 12 touching neighbours in $\mathcal{P}$ and k members of $\mathcal{P}$ have at most 9 touching neighbours in $\mathcal{P}$. Thus, $n-m-k$ members of $\mathcal{P}$ have either 10 or 11 touching neighbours in $\mathcal{P}$. (Here we have used the well-known fact that $\tau_3 = 12$, that is, no member of $\mathcal{P}$ can have more than 12 touching neighbours.) Without loss of generality we may assume that $4 \le k \le n-m$.

First, we note that $\mathrm{Sarea}\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right) = 2\pi(1-\cos\frac{\pi}{6}) = 2\pi(1-\frac{\sqrt{3}}{2})$ and $\mathrm{svol}_2\left(C_{\hat{S}_i}(\hat{\mathbf{c}}_{ij}, \frac{\pi}{6})\right) = 2\pi(1-\frac{\sqrt{3}}{2})\hat{r}^2$. Second, recall Theorem 2.1.1 according to which if a member of $\mathcal{P}$ say, $\mathbf{c}_i + \mathbf{B}$ has exactly 12 touching neighbours in $\mathcal{P}$, then $\hat{S}_i \subset \bigcup_{j\in T_i}(\mathbf{c}_j + \hat{r}\mathbf{B})$. These facts together with (2.1) and (2.2) imply the following estimate.

Corollary 2.1.8. $\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i + \hat{r}\mathbf{B})\right)\right) < \frac{24.53902}{3}(n-m-k) + 24.53902k$.

Proof.

$$\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i + \hat{r}\mathbf{B})\right)\right)$$
$$< \left(4\pi\hat{r}^2 - \frac{10\cdot 2\pi(1-\frac{\sqrt{3}}{2})\hat{r}^2}{0.89332}\right)(n-m-k) + \left(4\pi\hat{r}^2 - \frac{3\cdot 2\pi(1-\frac{\sqrt{3}}{2})\hat{r}^2}{0.89332}\right)k$$
$$< 7.91956(n-m-k) + 24.53902k < \frac{24.53902}{3}(n-m-k) + 24.53902k\ .$$

□

Hence, Corollaries 2.1.7 and 2.1.8 yield in a straightforward way that

$$1.85335n^{\frac{2}{3}} - 3k < n-m-k\ . \tag{2.3}$$

Finally, as the number $C(n)$ of touching pairs in $\mathcal{P}$ is obviously at most

$$\frac{1}{2}\left(12n - (n-m-k) - 3k\right)\ ,$$

therefore (2.3) implies that

$$C(n) \le \frac{1}{2}\left(12n - (n-m-k) - 3k\right) < 6n - 0.926675n^{\frac{2}{3}} < 6n - 0.926n^{\frac{2}{3}},$$

finishing the proof of (i) in Theorem 1.1.6.

2.1.2 An Upper Bound for the fcc Lattice: Proof of (ii)

Although the idea of the proof of (ii) is similar to that of (i) they differ in the combinatorial counting part (see (2.9)) as well as in the density estimate for packings of spherical caps of angular radii $\frac{\pi}{6}$ (see (2.8)). Moreover, the proof of (ii) is based on the new parameter value $\bar{r} := \sqrt{2}$ (replacing $\hat{r} = 1.81383$). The details are as follows.

First, recall that if Λ_{fcc} denotes the face-centered cubic lattice with shortest non-zero lattice vector of length 2 in $\mathbb{E}^3$ and we place unit balls centered at each lattice point of Λ_{fcc}, then we get the fcc lattice packing of unit balls, labelled by $\mathcal{P}_{fcc}$, in which each unit ball is touched by 12 others such that their centers form the vertices of a cuboctahedron. (Recall that a cuboctahedron is a convex polyhedron with 8 triangular faces and 6 square faces having 12 identical vertices, with 2 triangles and 2 squares meeting at each, and 24 identical edges, each separating a triangle from a square. As such it is a quasiregular polyhedron, i.e. an Archimedean solid, being vertex-transitive and edge-transitive.) Second, it is well known (see [99] for more details) that the Voronoi cell of each unit ball in $\mathcal{P}_{fcc}$ is a rhombic dodecahedron (the dual of a cuboctahedron) of volume $\sqrt{32}$ (and of circumradius $\sqrt{2}$). Thus, the density of $\mathcal{P}_{fcc}$ is $\frac{\frac{4\pi}{3}}{\sqrt{32}} = \frac{\pi}{\sqrt{18}}$.

Now, let $\mathbf{B}$ denote the unit ball centered at the origin $\mathbf{o} \in \Lambda_{fcc}$ of $\mathbb{E}^3$ and let $\mathcal{P} := \{\mathbf{c}_1 + \mathbf{B}, \mathbf{c}_2 + \mathbf{B}, \dots, \mathbf{c}_n + \mathbf{B}\}$ denote the packing of n unit balls with centers $\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subset \Lambda_{fcc}$ in $\mathbb{E}^3$ having the largest number $C_{fcc}(n)$ of touching pairs among all packings of n unit balls being a sub-packing of $\mathcal{P}_{fcc}$. ($\mathcal{P}$ might not be uniquely determined up to congruence in which case $\mathcal{P}$ stands for any of those extremal packings.) As the Voronoi cell of each unit ball in $\mathcal{P}_{fcc}$ is contained in a ball of radius $\bar{r} = \sqrt{2}$ therefore, based on the corresponding decomposition of $\bigcup_{i=1}^{n}(\mathbf{c}_i + \bar{r}\mathbf{B})$ into truncated Voronoi cells, we get that

$$\frac{n\,\mathrm{vol}_3(\mathbf{B})}{\mathrm{vol}_3(\bigcup_{i=1}^{n}(\mathbf{c}_i + \bar{r}\mathbf{B}))} < \frac{\pi}{\sqrt{18}} = 0.7404\dots . \tag{2.4}$$

As a next step we apply the isoperimetric inequality [155]:

$$36\pi\,\mathrm{vol}_3^2\left(\bigcup_{i=1}^{n}(\mathbf{c}_i + \bar{r}\mathbf{B})\right) \leq \mathrm{svol}_2^3\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i + \bar{r}\mathbf{B})\right)\right). \tag{2.5}$$

Thus, (2.4) and (2.5) yield in a straightforward way that

$$15.3532\dots n^{\frac{2}{3}} = 4\sqrt[3]{18\pi}\,n^{\frac{2}{3}} < \mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n}(\mathbf{c}_i + \bar{r}\mathbf{B})\right)\right). \tag{2.6}$$

Now, assume that $\mathbf{c}_i + \mathbf{B} \in \mathcal{P}$ is tangent to $\mathbf{c}_j + \mathbf{B} \in \mathcal{P}$ for all $j \in T_i$, where $T_i \subset \{1, 2, \dots, n\}$ stands for the family of indices $1 \le j \le n$ for which $\mathrm{dist}(\mathbf{c}_i, \mathbf{c}_j) = 2$. Then let $\bar{S}_i := \mathrm{bd}(\mathbf{c}_i + \bar{r}\mathbf{B})$ and let $\bar{\mathbf{c}}_{ij}$ be the intersection of the line segment $\mathbf{c}_i\mathbf{c}_j$ with $\bar{S}_i$ for all $j \in T_i$. Moreover, let $C_{\bar{S}_i}(\bar{\mathbf{c}}_{ij}, \frac{\pi}{6})$ (resp., $C_{\bar{S}_i}(\bar{\mathbf{c}}_{ij}, \frac{\pi}{4})$) denote the open spherical cap of $\bar{S}_i$ centered at $\bar{\mathbf{c}}_{ij} \in \bar{S}_i$ having angular radius $\frac{\pi}{6}$ (resp., $\frac{\pi}{4}$). Clearly, the family $\{C_{\bar{S}_i}(\bar{\mathbf{c}}_{ij}, \frac{\pi}{6}), j \in T_i\}$ consists of pairwise disjoint open spherical caps of $\bar{S}_i$; moreover,

$$\frac{\sum_{j\in T_i} \mathrm{svol}_2\left(C_{\bar{S}_i}(\bar{\mathbf{c}}_{ij}, \frac{\pi}{6})\right)}{\mathrm{svol}_2\left(\cup_{j\in T_i} C_{\bar{S}_i}(\bar{\mathbf{c}}_{ij}, \frac{\pi}{4})\right)} = \frac{\sum_{j\in T_i} \mathrm{Sarea}\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j\in T_i} C(\mathbf{u}_{ij}, \frac{\pi}{4})\right)}, \tag{2.7}$$

where $\mathbf{u}_{ij} = \frac{1}{2}(\mathbf{c}_j - \mathbf{c}_i) \in \mathbb{S}^2$ and $C(\mathbf{u}_{ij}, \frac{\pi}{6}) \subset \mathbb{S}^2$ (resp., $C(\mathbf{u}_{ij}, \frac{\pi}{4}) \subset \mathbb{S}^2$) denotes the open spherical cap of $\mathbb{S}^2$ centered at $\mathbf{u}_{ij}$ having angular radius $\frac{\pi}{6}$ (resp., $\frac{\pi}{4}$). Now, the geometry of the cuboctahedron representing the 12 touching neighbours of an arbitrary unit ball in $\mathcal{P}_{fcc}$ implies in a straightforward way that

$$\frac{\sum_{j\in T_i} \mathrm{Sarea}\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right)}{\mathrm{Sarea}\left(\cup_{j\in T_i} C(\mathbf{u}_{ij}, \frac{\pi}{4})\right)} \le 6(1 - \frac{\sqrt{3}}{2}) = 0.8038\dots \tag{2.8}$$

with equality when 12 spherical caps of angular radius $\frac{\pi}{6}$ are packed on $\mathbb{S}^2$.

Finally, as $\mathrm{svol}_2\left(C(\mathbf{u}_{ij}, \frac{\pi}{6})\right) = 2\pi(1 - \cos\frac{\pi}{6})$ and $\mathrm{svol}_2\left(C_{\bar{S}_i}(\bar{\mathbf{c}}_{ij}, \frac{\pi}{6})\right) = 2\pi(1 - \frac{\sqrt{3}}{2})\bar{r}^2$ therefore (2.7) and (2.8) yield that

$$\mathrm{svol}_2\left(\mathrm{bd}\left(\bigcup_{i=1}^{n} \mathbf{c}_i + \bar{r}\mathbf{B}\right)\right) \le 4\pi\bar{r}^2 n - \frac{1}{6(1-\frac{\sqrt{3}}{2})} 2\left(2\pi\left(1 - \frac{\sqrt{3}}{2}\right)\bar{r}^2\right) C_{fcc}(n)$$

$$= 8\pi n - \frac{4\pi}{3} C_{fcc}(n)\,. \tag{2.9}$$

Thus, (2.6) and (2.9) imply that

$$4\sqrt[3]{18\pi}\, n^{\frac{2}{3}} < 8\pi n - \frac{4\pi}{3} C_{fcc}(n)\,. \tag{2.10}$$

From (2.10) the inequality $C_{fcc}(n) < 6n - \frac{3\sqrt[3]{18\pi}}{\pi} n^{\frac{2}{3}} = 6n - 3.665\dots n^{\frac{2}{3}}$ follows in a straightforward way for all $n \ge 2$. This completes the proof of (ii) in Theorem 1.1.6.

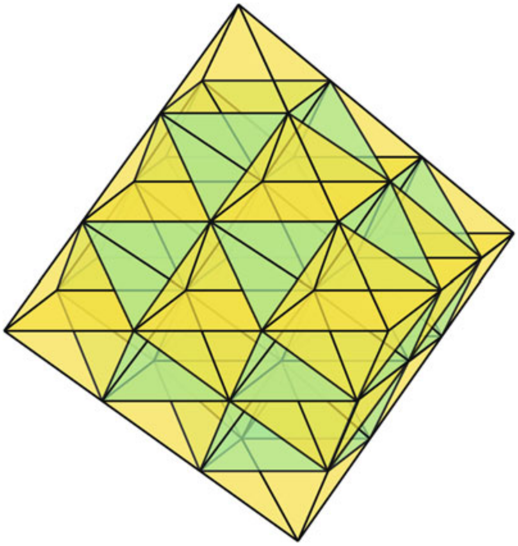

Fig. 2.4 The octahedral construction for $k = 4$

2.1.3 Octahedral Unit Sphere Packings: Proof of (iii)

It is rather easy to show that for any positive integer $k \geq 2$ there are $n(k) := \frac{2k^3+k}{3} = \frac{k(2k^2+1)}{3}$ lattice points of the face-centered cubic lattice Λ_{fcc} such that their convex hull is a regular octahedron $\mathbf{K} \subset \mathbb{E}^3$ of edge length $2(k-1)$ having exactly k lattice points along each of its edges (see Fig. 2.4 for $k = 4$).

Now, draw a unit ball around each lattice point of $\Lambda_{fcc} \cap \mathbf{K}$ and label the packing of the $n(k)$ unit balls obtained in this way by $\mathcal{P}_{fcc}(k)$. It is easy to check that if the center of a unit ball of $\mathcal{P}_{fcc}(k)$ is a relative interior point of an edge (resp., of a face) of $\mathbf{K}$, then the unit ball in question has 7 (resp., 9) touching neighbours in $\mathcal{P}_{fcc}(k)$. Last but not least, any unit ball of $\mathcal{P}_{fcc}(k)$ whose center is an interior pont of $\mathbf{K}$ has 12 touching neighbours in $\mathcal{P}_{fcc}(k)$. Next we note that the number of lattice points of Λ_{fcc} lying in the relative interior of the edges (resp., faces) of $\mathbf{K}$ is $12(k-2) = 12k - 24$ (resp., $8\left(\frac{1}{2}(k-3)^2 + \frac{1}{2}(k-3)\right) = 4(k-3)^2 + 4(k-3)$). Furthermore the number of lattice points of Λ_{fcc} in the interior of $\mathbf{K}$ is equal to $\frac{2}{3}(k-2)^3 + \frac{1}{3}(k-2)$. Thus, the contact number $C\left(\mathcal{P}_{fcc}(k)\right)$ of the packing $\mathcal{P}_{fcc}(k)$ is equal to

$$\frac{12}{2}\left(\frac{2}{3}(k-2)^3 + \frac{1}{3}(k-2)\right) + \frac{9}{2}\left(4(k-3)^2 + 4(k-3)\right) + \frac{7}{2}(12k-24) + \frac{24}{2}$$
$$= 4k^3 - 6k^2 + 2k.$$

As a result we get that

$$C\left(\mathcal{P}_{fcc}(k)\right) = 6n(k) - 6k^2. \tag{2.11}$$

Finally, as $\frac{2k^3}{3} < n(k)$ therefore $6k^2 < \sqrt[3]{486}n^{\frac{2}{3}}(k)$ and so, (2.11) implies (iii) of Theorem 1.1.6 in a straightforward way.

2.2 Proof of Theorem 1.2.4

Theorem 1.2.4. *The surface volume of any Voronoi cell in a packing of unit balls in* $\mathbb{E}^d$, $d \geq 2$ *is at least* $\frac{d\omega_d}{\sigma_d}$.

2.2.1 The Voronoi Star of Voronoi Cells

Without loss of generality we may assume that the d-dimensional unit ball $\mathbf{B} \subset \mathbb{E}^d$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ is one of the unit balls of the given unit ball packing in $\mathbb{E}^d$, $d \geq 2$. Let $\mathbf{V}$ be the Voronoi cell assigned to $\mathbf{B}$. We may assume that $\mathbf{V}$ is bounded; that is, $\mathbf{V}$ is a d-dimensional convex polytope in $\mathbb{E}^d$.

First, following [159], we dissect $\mathbf{V}$ into finitely many d-dimensional simplices as follows. Let F_i denote an arbitrary i-dimensional face of $\mathbf{V}$, $0 \leq i \leq d-1$. Let the chain $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ be called a *flag* of $\mathbf{V}$, and let $\mathcal{F}$ be the family of all flags of $\mathbf{V}$. Now, let $f \in \mathcal{F}$ be an arbitrary flag of $\mathbf{V}$ with the associated chain $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$. Then let $\mathbf{v}_i \in F_{d-i}$ be the point of F_{d-i} closest to $\mathbf{o}$, $1 \leq i \leq d$. Finally, let $\mathbf{V}_f := \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$, where $\mathrm{conv}(\cdot)$ stands for the convex hull of the given set. It is easy to see that the family $\mathcal{V} := \{\mathbf{V}_f \mid f \in \mathcal{F} \text{ and } \dim(\mathbf{V}_f) = d\}$ of d-dimensional simplices forms a tiling of $\mathbf{V}$ (i.e., $\cup_{\mathbf{V}_f \in \mathcal{V}} \mathbf{V}_f = \mathbf{V}$ and no two simplices of $\mathcal{V}$ have an interior point in common). This tiling is a rather special one, namely the d-dimensional simplices of $\mathcal{V}$ have $\mathbf{o}$ as a common vertex; moreover the union of their facets opposite to $\mathbf{o}$ is the boundary $\mathrm{bd}\mathbf{V}$ of $\mathbf{V}$. Finally, as shown in [159], for any $\mathbf{V}_f \in \mathcal{V}$ with $\mathbf{V}_f = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ we have that

$$\sqrt{\frac{2i}{i+1}} \leq \|\mathbf{v}_i\| = \mathrm{dist}\,(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_d\})\,, 1 \leq i \leq d, \tag{2.12}$$

where $\mathrm{dist}(\cdot,\cdot)$ (resp., $\|\cdot\|$) stands for the Euclidean distance function (resp., norm) in $\mathbb{E}^d$.

Second, we define the *Voronoi star* $\mathbf{V}^* \subset \mathbf{V}$ assigned to the Voronoi cell $\mathbf{V}$ as follows.

Definition 2.2.1. Let $\mathbf{V}_f \in \mathcal{V}$ with $\mathbf{V}_f = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$. Then let $\mathbf{v}_1^* := H \cap \mathrm{lin}\{\mathbf{v}_1\}$, where H denotes the hyperplane parallel to the hyperplane $\mathrm{aff}\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ and tangent to $\mathbf{B}$ such that it separates $\mathbf{o}$ from $\mathrm{aff}\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ (with $\mathrm{lin}(\cdot)$ and $\mathrm{aff}(\cdot)$ standing for the linear and affine hulls of the given sets in $\mathbb{E}^d$). Finally, let $\mathbf{V}_f^* := \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1^*, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ and let the Voronoi star $\mathbf{V}^*$ of $\mathbf{V}$ be defined as $\mathbf{V}^* := \cup_{\mathbf{V}_f \in \mathcal{V}} \mathbf{V}_f^*$.

It follows from Definition 2.2.1 and from (2.12) that the following inequalities and (surface) volume formula hold:

$$1 \leq \|\mathbf{v}_1^*\| = \operatorname{dist}\left(\mathbf{o}, \operatorname{conv}\{\mathbf{v}_1^*, \mathbf{v}_2, \ldots, \mathbf{v}_d\}\right) \leq \|\mathbf{v}_1\|, \tag{2.13}$$

$$\sqrt{\frac{2i}{i+1}} \leq \|\mathbf{v}_i\| = \operatorname{dist}\left(\mathbf{o}, \operatorname{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_d\}\right), 2 \leq i \leq d, \text{ and} \tag{2.14}$$

$$\operatorname{vol}_d(\mathbf{V}^*) = \frac{1}{d}\operatorname{svol}_{d-1}(\operatorname{bd}\mathbf{V}), \tag{2.15}$$

where $\operatorname{vol}_d(\cdot)$ (resp., $\operatorname{svol}_{d-1}(\cdot)$) refers to the d-dimensional (resp., $(d-1)$-dimensional) volume (resp., surface volume) measure.

2.2.2 *Estimating the Volume of a Voronoi Star from Below*

As an obvious corollary of (2.15), we find that Theorem 1.2.4 follows from the following theorem.

Theorem 2.2.2. $\operatorname{vol}_d(\mathbf{V}^*) \geq \frac{\omega_d}{\sigma_d}$.

Proof. The main tool of our proof is the following lemma of Rogers. (See [159] and [160] for the original version of the lemma, which is somewhat different from the equivalent version below. Also, for a strengthening we refer the interested reader to Lemma 2.3.11.)

Lemma 2.2.3. *Let* $\mathbf{W} := \operatorname{conv}\{\mathbf{o}, \mathbf{w}_1, \ldots, \mathbf{w}_d\}$ *be a* d*-dimensional simplex of* $\mathbb{E}^d$ *having the property that* $\operatorname{lin}\{\mathbf{w}_j - \mathbf{w}_i \mid i < j \leq d\}$ *is orthogonal to the vector* $\mathbf{w}_i$ *in* $\mathbb{E}^d$ *for all* $1 \leq i \leq d-1$ *(i.e., let* $\mathbf{W}$ *be a* d*-dimensional orthoscheme in* $\mathbb{E}^d$*). Moreover, let* $\mathbf{U} := \operatorname{conv}\{\mathbf{o}, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ *be a* d*-dimensional simplex of* $\mathbb{E}^d$ *such that* $\|\mathbf{u}_i\| = \operatorname{dist}(\mathbf{o}, \operatorname{conv}\{\mathbf{u}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_d\})$ *for all* $1 \leq i \leq d$*. If* $\|\mathbf{w}_i\| \leq \|\mathbf{u}_i\|$ *holds for all* $1 \leq i \leq d$*, then*

$$\frac{\operatorname{vol}_d(\mathbf{W})}{\operatorname{vol}_d(\mathbf{B} \cap \mathbf{W})} \leq \frac{\operatorname{vol}_d(\mathbf{U})}{\operatorname{vol}_d(\mathbf{B} \cap \mathbf{U})},$$

where $\mathbf{B}$ *stands for the* d*-dimensional unit ball centered at the origin* $\mathbf{o}$ *of* $\mathbb{E}^d$*.*

Now, let $\mathbf{W}$ be the orthoscheme of Lemma 2.2.3 with the additional property that $\|\mathbf{w}_i\| = \sqrt{\frac{2i}{i+1}}$ for all $1 \leq i \leq d$. Notice that a regular d-dimensional simplex of edge length 2 in $\mathbb{E}^d$ can be dissected into $(d+1)!$ d-dimensional simplices, each congruent to $\mathbf{W}$. This implies that

$$\sigma_d = \frac{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{W})}{\mathrm{vol}_d(\mathbf{W})}. \tag{2.16}$$

Finally, let $\mathbf{U} := \mathbf{V}_f^* = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1^*, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ for $\mathbf{V}_f \in \mathcal{V}$. Clearly, (2.13) and (2.14) show that $\mathbf{W}$ and $\mathbf{U}$, just introduced, satisfy the assumptions of Lemma 2.2.3. Thus, Lemma 2.2.3 and (2.16) imply that

$$\frac{1}{\sigma_d} \le \frac{\mathrm{vol}_d(\mathbf{V}_f^*)}{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{V}_f^*)}. \tag{2.17}$$

Hence, (2.17) yields that

$$\frac{\omega_d}{\sigma_d} \le \sum_{\mathbf{V}_f \in \mathcal{V}} \mathrm{vol}_d(\mathbf{B} \cap \mathbf{V}_f^*) \frac{\mathrm{vol}_d(\mathbf{V}_f^*)}{\mathrm{vol}_d(\mathbf{B} \cap \mathbf{V}_f^*)} = \sum_{\mathbf{V}_f \in \mathcal{V}} \mathrm{vol}_d(\mathbf{V}_f^*) = \mathrm{vol}_d(\mathbf{V}^*),$$

finishing the proof of Theorem 2.2.2. □

2.3 Proof of Theorem 1.2.5

Theorem 1.2.5. *The surface volume of any Voronoi cell in a packing of unit balls in the d-dimensional Euclidean space $\mathbb{E}^d$, $d \ge 8$ is at least $\frac{d\omega_d}{\hat{\sigma}_d}$; that is, the surface density of any unit sphere in its Voronoi cell in a unit sphere packing of $\mathbb{E}^d$, $d \ge 8$ is at most $\hat{\sigma}_d$. Thus, the volume of any Voronoi cell in a packing of unit balls in $\mathbb{E}^d$, $d \ge 8$ is at least $\frac{\omega_d}{\hat{\sigma}_d}$ and so, the (upper) density of any unit ball packing in $\mathbb{E}^d$, $d \ge 8$ is at most $\hat{\sigma}_d$ $(< \sigma_d)$.*

2.3.1 Metric Properties of Voronoi Cells

Let $\mathbf{P}$ be a bounded Voronoi cell, that is, a d-dimensional Voronoi polytope of a packing $\mathcal{P}$ of d-dimensional unit balls in $\mathbb{E}^d$. Without loss of generality we may assume that the unit ball $\mathbf{B} = \{\mathbf{x} \in \mathbb{E}^d \mid \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le 1\}$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ is one of the unit balls of $\mathcal{P}$ with $\mathbf{P}$ as its Voronoi cell. Then $\mathbf{P}$ is the intersection of finitely many closed halfspaces of $\mathbb{E}^d$ each of which is bounded by a hyperplane that is the perpendicular bisector of a line segment $\mathbf{ox}$ with $\mathbf{x}$ being the center of some unit ball of $\mathcal{P}$. Now, let F_{d-i} be an arbitrary $(d-i)$-dimensional face of $\mathbf{P}$, $1 \le i \le d$. Then clearly there are at least $i+1$ Voronoi cells of $\mathcal{P}$ which meet along the face F_{d-i}, that is, contain F_{d-i} (one of which is, of course, $\mathbf{P}$). Also, it is clear from the construction that the affine hull of centers of the unit balls sitting in all of these Voronoi cells is orthogonal to $\mathrm{aff} F_{d-i}$. Thus, there are unit balls of these Voronoi cells with centers $\{\mathbf{o}, \mathbf{x}_1, \ldots, \mathbf{x}_i\}$ such that $X = \mathrm{conv}\{\mathbf{o}, \mathbf{x}_1, \ldots, \mathbf{x}_i\}$ is an i-dimensional simplex and of course, $\mathrm{aff}X$ is orthogonal to $\mathrm{aff}F_{d-i}$. Hence, if $R(F_{d-i})$ denotes the radius of the $(i-1)$-dimensional sphere that passes through the vertices of X, then

$$R(F_{d-i}) = \mathrm{dist}(\mathbf{o}, \mathrm{aff}F_{d-i}), \text{ where } 1 \le i \le d.$$

As the following statements are well known and their proofs are relatively straightforward, we refer the interested reader to the relevant section in [32] for the details of those proofs.

Lemma 2.3.1. *If $F_{d-i-1} \subset F_{d-i}$ and $R(F_{d-i}) = R < \sqrt{2}$ for some $i, 1 \le i \le d-1$, then*

$$\frac{2}{\sqrt{4-R^2}} \le R(F_{d-i-1}).$$

Corollary 2.3.2. $\sqrt{\frac{2i}{i+1}} \le R(F_{d-i})$ *for all* $1 \le i \le d$.

Lemma 2.3.3. *If $R(F_{d-i}) < \sqrt{2}$ for some $i, 1 \le i \le d$, then the orthogonal projection of* $\mathbf{o}$ *onto* $\mathrm{aff}F_{d-i}$ *belongs to* $\mathrm{relint}F_{d-i}$ *and so* $R(F_{d-i}) = \mathrm{dist}(\mathbf{o}, F_{d-i})$.

2.3.2 Wedges of Types I, II, and III, and Truncated Wedges

Let $F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ be an arbitrary flag of the Voronoi polytope $\mathbf{P}$. Then let $\mathbf{r}_i \in F_{d-i}$ be the uniquely determined point of the $(d-i)$-dimensional face F_{d-i} of $\mathbf{P}$ that is closest to the center point $\mathbf{o}$ of $\mathbf{P}$; that is, let

$$\mathbf{r}_i \in F_{d-i} \text{ such that } \|\mathbf{r}_i\| = \min\{\|\mathbf{x}\| \mid \mathbf{x} \in F_{d-i}\}, \text{ where } 1 \le i \le d.$$

Definition 2.3.4. If the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_i$ are linearly independent in $\mathbb{E}^d$, then we call $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ the *i-dimensional Rogers simplex* assigned to the subflag $F_{d-i} \subset \cdots \subset F_{d-1}$ of the Voronoi polytope $\mathbf{P}$, where $1 \le i \le d$. If $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\} \subset \mathbb{E}^d$ is the d-dimensional Rogers simplex assigned to the flag $F_0 \subset \cdots \subset F_{d-1}$ of $\mathbf{P}$, then $\mathrm{conv}\{\mathbf{r}_{d-i}, \ldots, \mathbf{r}_d\}$ is called the *i-dimensional base* of the given d-dimensional Rogers simplex and $\mathrm{dist}(\mathbf{o}, \mathrm{aff}\{\mathbf{r}_{d-i}, \ldots, \mathbf{r}_d\}) = \mathrm{dist}(\mathbf{o}, \mathrm{aff}F_i) = R(F_i)$ is called the *height* assigned to the i-dimensional base, where $1 \le i \le d$.

Definition 2.3.5. The i-dimensional simplex $Y = \mathrm{conv}\{\mathbf{o}, \mathbf{y}_1, \ldots, \mathbf{y}_i\} \subset \mathbb{E}^d$ with vertices $\mathbf{y}_0 = \mathbf{o}, \mathbf{y}_1, \ldots, \mathbf{y}_i$ is called an *i-dimensional orthoscheme* if for each $j, 0 \le j \le i-1$ the vector $\mathbf{y}_j$ is orthogonal to the linear hull $\mathrm{lin}\{\mathbf{y}_k - \mathbf{y}_j \mid j+1 \le k \le i\}$, where $1 \le i \le d$.

It is shown in [159] that the union of the d-dimensional Rogers simplices of the Voronoi polytope $\mathbf{P}$ is the polytope $\mathbf{P}$ itself and their interiors are pairwise disjoint. This fact together with Corollary 2.3.2 and Lemma 2.3.3 imply the following metric properties of Rogers simplices in a straightforward way.

Lemma 2.3.6.

(1) If $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ *is an* i*-dimensional Rogers simplex assigned to the subflag* $F_{d-i} \subset \cdots \subset F_{d-1}$ *of the Voronoi polytope* $\mathbf{P}$*, then* $\sqrt{\frac{2j}{j+1}} \leq \|\mathbf{r}_j\|$ *for all* $1 \leq j \leq i$*, where* $1 \leq i \leq d$.

(2) If $F_{d-i} \subset \cdots \subset F_{d-1}$ *is a subflag of the Voronoi polytope* $\mathbf{P}$ *with* $R(F_{d-i}) < \sqrt{2}$*, then* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_i\}$ *is an* i*-dimensional Rogers simplex which is, in fact, an* i*-dimensional orthoscheme (in short, an* i*-dimensional Rogers orthoscheme) with the property that each* $\mathbf{r}_j \in \mathrm{relint} F_{d-j}, 1 \leq j \leq i$ *is the orthogonal projection of* $\mathbf{o}$ *onto* $\mathrm{aff} F_{d-j}$*, where* $1 \leq i \leq d$.

(3) If $F_2 \subset \cdots \subset F_{d-1}$ *is a subflag of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d, 3 \leq d$ *with* $R(F_2) < \sqrt{2}$*, then the union of the* 2*-dimensional bases of the* d*-dimensional Rogers simplices that contain the orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *is the (uniquely determined)* 2*-dimensional face* F_2 *of the Voronoi polytope* $\mathbf{P}$ *that is totally orthogonal to* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *at the point* $\mathbf{r}_{d-2}$ *and so,* $\|\mathbf{r}_{d-2}\| = \mathrm{dist}(\mathbf{o}, \mathrm{aff} F_2)$ *with* $\mathbf{r}_{d-2} \in \mathrm{relint} F_2$.

Now we are ready for the definitions of *wedges* and *truncated wedges*. Recall that for any 2-dimensional face F_2 of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ we have that $\sqrt{\frac{2(d-2)}{d-1}} \leq R(F_2)$ (Fig. 2.5).

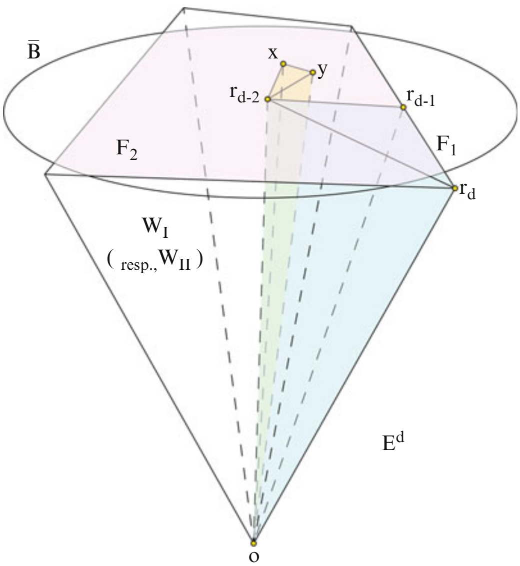

Fig. 2.5 Wedges $\mathbf{W}_I, \mathbf{W}_{II}$, and truncated wedges $\overline{\mathbf{W}}_I$, $\overline{\mathbf{W}}_{II}$

Definition 2.3.7.

(1) Let F_2 be a 2-dimensional face of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ with $\sqrt{\frac{2(d-2)}{d-1}} \leq R(F_2) < \sqrt{\frac{2(d-1)}{d}}$ and let $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ be any $(d-2)$-dimensional Rogers simplex with $\mathbf{r}_{d-2} \in \text{relint} F_2$. Then the union $\mathbf{W}_I$ of the d-dimensional Rogers simplices of $\mathbf{P}$ that contain the orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ is called a *wedge of type I (generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$*)*. F_2 is called the *2-dimensional base* of $\mathbf{W}_I$, and $\|\mathbf{r}_{d-2}\| = \text{dist}(\mathbf{o}, \text{aff} F_2)$ is the *height* of $\mathbf{W}_I$ assigned to the base F_2.

(2) Let F_2 be a 2-dimensional face of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ with $\sqrt{\frac{2(d-1)}{d}} \leq R(F_2) < \sqrt{\frac{2d}{d+1}}$ and let $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ be any $(d-2)$-dimensional Rogers simplex with $\mathbf{r}_{d-2} \in \text{relint} F_2$. Then the union $\mathbf{W}_{II}$ of the d-dimensional Rogers simplices of $\mathbf{P}$ that contain the orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ is called a *wedge of type II (generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$*)*. F_2 is called the *2-dimensional base* of $\mathbf{W}_{II}$, and $\|\mathbf{r}_{d-2}\| = \text{dist}(\mathbf{o}, \text{aff} F_2)$ is the *height* of $\mathbf{W}_{II}$ assigned to the base F_2.

(3) Let $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\}$ be the d-dimensional Rogers simplex assigned to the flag $F_0 \subset F_1 \cdots \subset F_{d-1}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$ with $\sqrt{\frac{2d}{d+1}} \leq R(F_2)$. Then $\mathbf{W}_{III} = \text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\}$ is called a *wedge of type III.*

At this point, it useful to recall, that for any vertex F_0 of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$ we have that $\sqrt{\frac{2d}{d+1}} \leq R(F_0)$.

Definition 2.3.8. Let $\overline{\mathbf{B}} = \left\{\mathbf{x} \in \mathbb{E}^d \mid \text{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq \sqrt{\frac{2d}{d+1}}\right\}$.

(1) If $\mathbf{W}_I$ is a wedge of type I with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$, then

$$\overline{\mathbf{W}}_I = \text{conv}\left((\overline{\mathbf{B}} \cap F_2) \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\}\right)$$

is called the *truncated wedge of type I* with the 2*-dimensional base* $\overline{\mathbf{B}} \cap F_2$ generated by the $(d-2)$-dimensional Rogers orthoscheme

$$\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}.$$

(2) If $\mathbf{W}_{II}$ is a wedge of type II with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\text{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$, then

$$\overline{\mathbf{W}}_{II} = \text{conv}\left((\overline{\mathbf{B}} \cap F_2) \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\}\right)$$

is called the *truncated wedge of type II* with the 2-*dimensional base* $\overline{\mathbf{B}} \cap F_2$ generated by the $(d-2)$-dimensional Rogers orthoscheme

$$\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}.$$

As the following claim can be proved by Lemma 2.3.6 in a straightforward way, we leave the relevant details to the reader.

Lemma 2.3.9.

(1) Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$. If the points $\mathbf{x}, \mathbf{y} \in \mathrm{aff} F_2$ are chosen so that the triangle $\triangle \mathbf{r}_{d-2}\mathbf{x}\mathbf{y}$ has a right angle at the vertex $\mathbf{x}$, then $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}, \mathbf{x}, \mathbf{y}\}$ is a d-dimensional orthoscheme. Moreover, if $\mathbf{z} \in \mathrm{aff} F_2$ is an arbitrary point, then $\mathrm{conv}\{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}, \mathbf{z}\}$ is a $(d-2)$-dimensional orthoscheme.

(2) Let $\mathbf{W}_I$ denote the wedge of type I with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o} = \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$. Let $Q_2 \subset \mathrm{aff} F_2$ and $Q_2^* \subset \mathrm{aff} F_2$ be compact convex sets with $\mathrm{relint} Q_2 \cap \mathrm{relint} Q_2^* = \emptyset$. If $K_2 = Q_2$ (resp., $K_2^* = Q_2^*$) and $K_j = \mathrm{conv}(K_{j-1} \cup \{\mathbf{r}_{d-j}\})$ (resp., $K_j^* = \mathrm{conv}(K_{j-1}^* \cup \{\mathbf{r}_{d-j}\})$) for $j = 3, \ldots, d$, then $K_d = \mathrm{conv}(Q_2 \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\})$ (resp., $K_d^* = \mathrm{conv}(Q_2^* \cup \{\mathbf{o} = \mathbf{r}_0, \ldots, \mathbf{r}_{d-3}\})$), moreover $\mathrm{relint} K_d \cap \mathrm{relint} K_d^* = \emptyset$. A similar statement holds for $\mathbf{W}_{II}$.

(3) Let $\mathbf{W}_I$ (resp., $\overline{\mathbf{W}}_I$) denote the wedge of type I (resp., truncated wedge of type I) with the 2-dimensional base F_2 (resp., $\overline{\mathbf{B}} \cap F_2$) which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o} = \mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 3$. If $K_2 = F_2$ (resp., $K_2 = \overline{\mathbf{B}} \cap F_2$) and $K_j = \mathrm{conv}(K_{j-1} \cup \{\mathbf{r}_{d-j}\})$ for $j = 3, \ldots, d$, then $K_d = \mathbf{W}_I$ (resp., $K_d = \overline{\mathbf{W}}_I$). Similar statements hold for $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$.

We close this section with the following important observation published in [32], and refer the interested reader to [32] for the details of the seven-page proof, which is based on Corollary 2.3.2 and Lemma 2.3.3.

Lemma 2.3.10. *Let $\overline{\mathbf{B}} \cap F_2$ be the 2-dimensional base of the type I truncated wedge $\overline{\mathbf{W}}_I$ (resp., type II truncated wedge $\overline{\mathbf{W}}_{II}$) in the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$ of dimension $d \geq 8$. Then the number of line segments of positive length in $\mathrm{relbd}(\overline{\mathbf{B}} \cap F_2)$ is at most 4.*

2.3.3 The Lemma of Comparison

Recall that $\mathbf{B} = \{\mathbf{x} \in \mathbb{E}^d \mid \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \leq 1\}$ and let

$$S = \{\mathbf{x} \in \mathbb{E}^d \mid \mathrm{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| = 1\}.$$

Then let $H \subset \mathbb{E}^d$ be a hyperplane disjoint from the interior of the unit ball $\mathbf{B}$ and let $Q \subset H$ be an arbitrary $(d-1)$-dimensional compact convex set. If $[\mathbf{o}, Q]$ denotes the convex cone $\mathrm{conv}(\{\mathbf{o}\} \cup Q)$ with apex $\mathbf{o}$ and base Q, then the *(volume) density* $\delta([\mathbf{o}, Q], \mathbf{B})$ of the unit ball $\mathbf{B}$ in the cone $[\mathbf{o}, Q]$ is defined as

$$\delta([\mathbf{o}, Q], B) = \frac{\mathrm{vol}_d([\mathbf{o}, Q] \cap \mathbf{B})}{\mathrm{vol}_d([\mathbf{o}, Q])},$$

where $\mathrm{vol}_d(\cdot)$ refers to the corresponding d-dimensional Euclidean volume measure. It is natural to introduce the following very similar notion. The *surface density* $\hat{\delta}([\mathbf{o}, Q], S)$ of the unit sphere S in the convex cone $[\mathbf{o}, Q]$ with apex $\mathbf{o}$ and base Q is defined by

$$\hat{\delta}([\mathbf{o}, Q], S) = \frac{\mathrm{Svol}_{d-1}([\mathbf{o}, Q] \cap S)}{\mathrm{vol}_{d-1}(Q)},$$

where $\mathrm{Svol}_{d-1}(\cdot)$ refers to the corresponding $(d-1)$-dimensional spherical volume measure.

If $h = \mathrm{dist}(\mathbf{o}, H)$, then clearly $h \cdot \delta([\mathbf{o}, Q], \mathbf{B}) = \hat{\delta}([\mathbf{o}, Q], S)$. We need the following statement, the first part of which is due to Rogers [159] and the second part of which has been proved by the author in [29].

Lemma 2.3.11. *Let* $\mathbf{U} = \mathrm{conv}\{\mathbf{o}, \mathbf{u}_1, \ldots, \mathbf{u}_d\}$ *be a* d*-dimensional orthoscheme in* $\mathbb{E}^d$ *and let* $\mathbf{V} = \mathrm{conv}\{\mathbf{o}, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ *be a* d*-dimensional simplex of* $\mathbb{E}^d$ *such that* $\|\mathbf{v}_i\| = \mathrm{dist}(\mathbf{o}, \mathrm{conv}\{\mathbf{v}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_d\})$ *for all* $1 \leq i \leq d-1$. *If* $1 \leq \|\mathbf{u}_i\| \leq \|\mathbf{v}_i\|$ *holds for all* $1 \leq i \leq d$, *then*

(1) $\delta(\mathbf{U}, \mathbf{B}) \geq \delta(\mathbf{V}, \mathbf{B})$ *and*
(2) $\hat{\delta}(\mathbf{U}, S) \geq \hat{\delta}(\mathbf{V}, S)$.

For the sake of completeness we mention the following statement that follows from Lemma 2.3.11 using the special decomposition of convex polytopes into Rogers simplices. Actually, the characterization of regular polytopes through the corresponding volume (resp., surface volume) inequality below was first observed by Böröczky and Máthéné Bognár [70] (resp., by the author [29]). (In fact, it is easy to see that the statement on surface volume implies the one on volume.) For more details on related problems we refer the interested reader to [69].

Corollary 2.3.12. *Let* $\mathbf{U}'$ *be a regular convex polytope in* $\mathbb{E}^d$ *with circumcenter* $\mathbf{o}$ *and let* s_i *denote the distance of an* i*-dimensional face of* $\mathbf{U}'$ *from* $\mathbf{o}$, $0 \leq i \leq d-1$. *If* $\mathbf{V}'$ *is an arbitrary convex polytope in* $\mathbb{E}^d$ *such that* $\mathbf{o} \in \mathrm{int}\mathbf{V}'$ *and the distance of any* i*-dimensional face of* $\mathbf{V}'$ *from* $\mathbf{o}$ *is at least* s_i *for all* $0 \leq i \leq d-1$, *then* $\mathrm{vol}_d(\mathbf{V}') \geq \mathrm{vol}_d(\mathbf{U}')$ *(resp.,* $\mathrm{svol}_{d-1}(\mathbf{V}') \geq \mathrm{svol}_{d-1}(\mathbf{U}')$*). Moreover, equality holds if and only if* $\mathbf{V}'$ *is congruent to* $\mathbf{U}'$ *and its circumcenter is* $\mathbf{o}$.

2.3.4 Volume Formulas for (Truncated) Wedges

Definition 2.3.13. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n, n \geq 1$ be points in $\mathbb{E}^d, d \geq 1$ and let $X \subset \mathbb{E}^d$ be an arbitrary convex set. If $X_0 = X$ and $X_m = \mathrm{conv}(\{\mathbf{x}_{n-(m-1)}\} \cup X_{m-1})$ for $m = 1, \ldots, n$, then we denote the final convex set X_n by

$$[\mathbf{x}_1, \ldots, \mathbf{x}_n, X].$$

Definition 2.3.14. Let $\mathbf{W}_I$ (resp., $\overline{\mathbf{W}}_I$) denote the wedge (resp., truncated wedge) of type I with the 2-dimensional base F_2 (resp., $\overline{\mathbf{B}} \cap F_2$) which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$. Then let

$$Q_I = [\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, F_2] \left(resp., \overline{Q}_I = [\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \overline{\mathbf{B}} \cap F_2]\right)$$

be called the $(d-1)$-*dimensional base* of the type I wedge $\mathbf{W}_I = [\mathbf{o}, Q_I]$ (resp., type I truncated wedge $\overline{\mathbf{W}}_I = [\mathbf{o}, \overline{Q}_I]$). Similarly, we define the $(d-1)$-dimensional bases Q_{II} and $\overline{Q}_{II}$ of $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$. Finally, let

$$h_1 = \|\mathbf{r}_1\|, h_2 = \|\mathbf{r}_2 - \mathbf{r}_1\|, \ldots, h_{d-2} = \|\mathbf{r}_{d-2} - \mathbf{r}_{d-3}\|.$$

Lemma 2.3.15. *Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 4$. Then we have the following volume formulas.*

(1) $\mathrm{vol}_{d-1}(Q_I) = \frac{2}{(d-1)!}\left(\prod_{i=2}^{d-2} h_i\right)\mathrm{vol}_2(F_2)$ *and*

(2) $\mathrm{vol}_d(\mathbf{W}_I) = \frac{2}{d!}\left(\prod_{i=1}^{d-2} h_i\right)\mathrm{vol}_2(F_2)$.

Similar formulas hold for the corresponding dimensional volumes of $\overline{Q}_I$, $\overline{\mathbf{W}}_I$, Q_{II}, $\mathbf{W}_{II}$, $\overline{Q}_{II}$, and $\overline{\mathbf{W}}_{II}$.

In general, if $K \subset \mathrm{aff} F_2$ is a convex domain, then

(3) $\mathrm{vol}_{d-1}([\mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]) = \frac{2}{(d-1)!}\left(\prod_{i=2}^{d-2} h_i\right)\mathrm{vol}_2(K)$ *and*

(4) $\mathrm{vol}_d([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, K]) = \frac{2}{d!}\left(\prod_{i=1}^{d-2} h_i\right)\mathrm{vol}_2(K)$.

Proof. The proof follows from Lemmas 2.3.6 and 2.3.9 in a straightforward way. □

2.3.5 The Integral Representation of Surface Density

The central notion of this section is the *limiting surface density* introduced as follows.

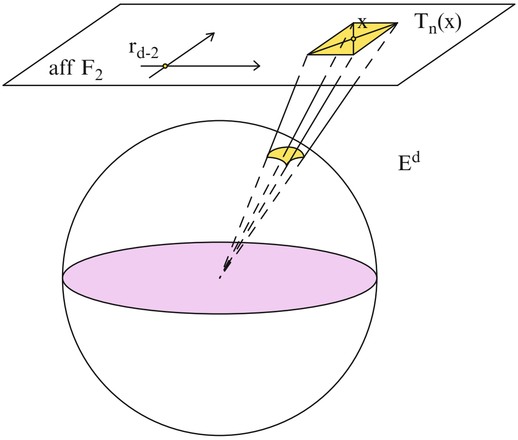

Fig. 2.6 The limiting surface density $\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)$

Definition 2.3.16. Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 4$. Then choose a coordinate system with two perpendicular axes in the plane $\mathrm{aff} F_2$ meeting at the point $\mathbf{r}_{d-2}$. Now, if $\mathbf{x}$ is an arbitrary point of the plane $\mathrm{aff} F_2$, then for a positive integer n let $T_n(\mathbf{x}) \subset \mathrm{aff} F_2$ denote the square centered at $\mathbf{x}$ having sides of length $\frac{1}{n}$ parallel to the fixed coordinate axes. Then *the limiting surface density* $\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)$ of the $(d-1)$-dimensional unit sphere S in the $(d-2)$-dimensional orthoscheme $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}]$ is defined by

$$\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S) = \lim_{n \to \infty} \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S).$$

Based on this we are able to give an integral representation of the surface density in a (truncated) wedge (Fig. 2.6).

Lemma 2.3.17. *Let $\mathbf{W}_I$ (resp., $\mathbf{W}_{II}$) denote the wedge of type I (resp., of type II) with the 2-dimensional base F_2 which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 4$.*

(1) If $\mathbf{x} \in \mathrm{aff} F_2$ and $\mathbf{y} \in \mathrm{aff} F_2$ are points such that $\|\mathbf{x}\| \leq \|\mathbf{y}\|$, then

$$\hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S) \geq \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{y}], S).$$

(2) For the surface densities of the unit sphere S in the wedge $\mathbf{W}_I$ and in the truncated wedge $\overline{\mathbf{W}}_I$ we have the following formulas.

$$\hat{\delta}(\mathbf{W}_I, S) = \frac{\mathrm{Svol}_{d-1}([\mathbf{o}, Q_I] \cap S)}{\mathrm{vol}_{d-1}(Q_I)} = \frac{1}{\mathrm{vol}_2(F_2)} \int_{F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$

and

$$\hat{\delta}(\overline{\mathbf{W}}_I, S) = \frac{\mathrm{Svol}_{d-1}([\mathbf{o}, \overline{Q}_I] \cap S)}{\mathrm{vol}_{d-1}(\overline{Q}_I)}$$

$$= \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \hat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, \mathbf{x}], S\right)\, d\mathbf{x},$$

where $d\mathbf{x}$ stands for the Euclidean area element in the plane $\mathrm{aff} F_2$*. Similar formulas hold for $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$.*

(3) In general, if $K \subset \mathrm{aff} F_2$ is a convex domain, then the surface density of the unit sphere S in the d-dimensional convex cone $[\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, K]$ with apex $\mathbf{o}$ and $(d-1)$-dimensional base $[\mathbf{r}_1, \dots, \mathbf{r}_{d-3}, K]$ can be computed as follows.

$$\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, K], S) = \frac{1}{\mathrm{vol}_2(K)} \int_K \hat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, \mathbf{x}], S\right)\, d\mathbf{x}.$$

Proof.(1) It is sufficient to look at the case $\|\mathbf{x}\| < \|\mathbf{y}\|$. (The case $\|\mathbf{x}\| = \|\mathbf{y}\|$ follows from this by standard limit procedure.) Then recall that

$$\hat{\delta}\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S\right) = h_1 \delta\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S\right)$$

and

$$\hat{\delta}\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S\right) = h_1 \delta\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S\right).$$

Thus, it is sufficient to show that if n is sufficiently large, then

$$\delta\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{x})], S\right) \geq \delta\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{y})], S\right).$$

This we can get as follows. We can approximate the d-dimensional convex cone $[\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{x})]$ (resp., $[\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, T_n(\mathbf{y})]$) arbitrarily close with a finite (but possibly large) number of non-overlapping d-dimensional orthoschemes each containing the $(d-3)$-dimensional orthoscheme $[\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}]$ as a face and each having all the edge lengths of the 3 edges going out from the vertex $\mathbf{o}$ and not lying on the face $[\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}]$ close to $\|\mathbf{x}\|$ (resp., $\|\mathbf{y}\|$) for n sufficiently large (see also Lemma 2.3.9). Thus, the claim follows from (1) of Lemma 2.3.11 rather easily.

(2),(3) It is sufficient to prove the corresponding formula for K.

A typical term of the Riemann–Lebesgue sum of

$$\frac{1}{\mathrm{vol}_2(K)} \int_K \hat{\delta}_{\lim}\left([\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-3}, \mathbf{x}], S\right)\, d\mathbf{x}$$

is equal to

$$\frac{1}{\mathrm{vol}_2(K)}\hat{\delta}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x}_m)],S\right)\mathrm{vol}_2(T_n(\mathbf{x}_m)), m\in M.$$

Using Lemma 2.3.15 this turns out to be equal to

$$\frac{\mathrm{vol}_{d-1}([\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x}_m)])}{\mathrm{vol}_{d-1}([\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K])}\hat{\delta}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x}_m)],S\right)$$
$$=\frac{\mathrm{Svol}_{d-1}([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x}_m)]\cap S)}{\mathrm{vol}_{d-1}([\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K])}.$$

Finally, as the union of the non-overlapping squares $T_n(\mathbf{x}_m), m\in M$ is a good approximation of the convex domain K in the plane affF_2 we get that

$$\sum_{m\in M}\frac{\mathrm{Svol}_{d-1}([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x}_m)]\cap S)}{\mathrm{vol}_{d-1}([\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K])}$$
$$=\frac{\sum_{m\in M}\mathrm{Svol}_{d-1}([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},T_n(\mathbf{x}_m)]\cap S)}{\mathrm{vol}_{d-1}([\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K])}$$

is a good approximation of

$$\frac{\mathrm{Svol}_{d-1}\left([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K]\cap S\right)}{\mathrm{vol}_{d-1}([\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K])}=\hat{\delta}([\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-3},K],S).$$

This completes the proof of Lemma 2.3.17. □

2.3.6 Truncation of Wedges Increases the Surface Density

Lemma 2.3.18. *Let* $\mathbf{W}_I$ *(resp.,* $\mathbf{W}_{II}$*) denote the wedge of type I (resp., of type* II*) with the* 2*-dimensional base* F_2 *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o},\mathbf{r}_1,\ldots,\mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P}\subset\mathbb{E}^d, d\geq 4$*. Then*

$$\hat{\delta}(\mathbf{W}_I,S)\leq\hat{\delta}(\overline{\mathbf{W}}_I,S)\ \left(\mathit{resp.,}\ \hat{\delta}(\mathbf{W}_{II},S)\leq\hat{\delta}(\overline{\mathbf{W}}_{II},S)\right).$$

Proof. Notice that (1) of Lemma 2.3.17 easily implies that if $0<\mathrm{vol}_2(F_2\setminus\overline{\mathbf{B}})$, then for any $\mathbf{x}^*\in F_2$ with $\|\mathbf{x}^*\|=\sqrt{\frac{2d}{d+1}}$ we have that

$$\frac{1}{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})} \int_{F_2 \setminus \overline{\mathbf{B}}} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$
$$\leq \hat{\delta}_{\lim}\big([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}^*], S\big)$$
$$\leq \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}.$$

Thus, if $0 < \mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})$, then (2) of Lemma 2.3.17 yields that

$$\hat{\delta}(\mathbf{W}_I, S) = \frac{1}{\mathrm{vol}_2(F_2)} \int_{F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$
$$= \frac{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$
$$+ \frac{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})} \int_{F_2 \setminus \overline{\mathbf{B}}} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$
$$\leq \frac{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$
$$+ \frac{\mathrm{vol}_2(F_2 \setminus \overline{\mathbf{B}})}{\mathrm{vol}_2(F_2)} \cdot \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x}$$
$$= \frac{1}{\mathrm{vol}_2(\overline{\mathbf{B}} \cap F_2)} \int_{\overline{\mathbf{B}} \cap F_2} \hat{\delta}_{\lim}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, \mathbf{x}], S)\, d\mathbf{x} = \hat{\delta}(\overline{\mathbf{W}}_I, S).$$

As the same method works for $\mathbf{W}_{II}$ and $\overline{\mathbf{W}}_{II}$ this completes the proof of Lemma 2.3.18. □

2.3.7 *Maximum Surface Density in Truncated Wedges of Type I*

Let $\overline{\mathbf{W}}_I$ denote the truncated wedge of type I with the 2-dimensional base $\overline{\mathbf{B}} \cap F_2$ which is generated by the $(d-2)$-dimensional Rogers orthoscheme $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 8$. By assumption F_2 is a 2-dimensional face of the Voronoi polytope $\mathbf{P}$ with

$$\sqrt{\frac{2(d-2)}{d-1}} \leq h = R(F_2) < \sqrt{\frac{2(d-1)}{d}}.$$

Let $G_0 \subset \mathrm{aff} F_2$ (resp., $G \subset \mathrm{aff} F_2$) denote the closed circular disk of radius

$$g_0(h) = \sqrt{\frac{2d}{d+1} - h^2}\left(\text{resp., } g(h) = \frac{2-h^2}{\sqrt{4-h^2}}\right)$$

centered at the point $\mathbf{r}_{d-2}$. It is easy to see that $G \subset \text{relint}G_0$ for all $\sqrt{\frac{2(d-2)}{d-1}} \leq h < \sqrt{\frac{2(d-1)}{d}}$. (Moreover $G = G_0$ for $h = \sqrt{\frac{2(d-1)}{d}}$.) Notice that $G_0 = \overline{\mathbf{B}} \cap \text{aff}F_2$, thus Corollary 2.3.2 implies that there is no vertex of the face F_2 belonging to the relative interior of G_0. Moreover, as $h = R(F_2) < \sqrt{2}$ Lemma 2.3.1 yields that $\frac{2}{\sqrt{4-h^2}} \leq R(F_1)$ holds for any side F_1 of the face F_2, hence $G \subset F_2$ and of course, $G \subset \overline{\mathbf{B}} \cap F_2 = G_0 \cap F_2$. Now, let $M \subset \text{aff}F_2$ be a square circumscribed about G. A straightforward computation yields that $\frac{g_0(h)}{g(h)}$ is a strictly decreasing function on the interval $\left[\sqrt{\frac{2(d-2)}{d-1}}, \sqrt{\frac{2(d-1)}{d}}\right)$ (i.e., $\frac{d}{dh}\left(\frac{g_0(h)}{g(h)}\right) < 0$ on the interval $\left(\sqrt{\frac{2(d-2)}{d-1}}, \sqrt{\frac{2(d-1)}{d}}\right)$) and

$$\frac{g_0\left(\sqrt{\frac{2(d-2)}{d-1}}\right)}{g\left(\sqrt{\frac{2(d-2)}{d-1}}\right)} = \sqrt{\frac{2d}{d+1}} < \sqrt{2}.$$

Thus, the vertices of the square M do not belong to G_0. Finally, as $d \geq 8$ Lemma 2.3.10 implies that there are at most four sides of the face F_2 that intersect the relative interior of G_0.

The following statement is rather natural from the point of view of the local geometry introduced above, however, its three-page proof based on Lemmas 2.3.11 and 2.3.17 published in [32] is a bit technical and so, for that reason we do not prove it here; instead we refer the interested reader to the proper section in [32] (Fig. 2.7).

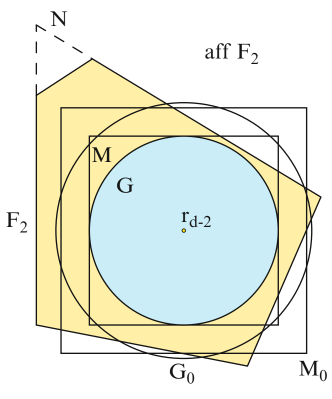

Fig. 2.7 Surface density bounded from above by $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S)$

Lemma 2.3.19. *Let* $\overline{\mathbf{W}}_I$ *denote the truncated wedge of type I with the 2-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 8$. *Then*

$$\hat{\delta}(\overline{\mathbf{W}}_I, S) \leq \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S).$$

It is clear from the construction that we can write $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S)$ as a function of $d-2$ variables, namely

$$\hat{\Delta}(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S),$$

where $\xi_1 = \|\mathbf{r}_1\|, \ldots, \xi_{d-3} = \|\mathbf{r}_{d-3}\|, \xi_{d-2} = \|\mathbf{r}_{d-2}\| = h$. Corollary 2.3.2 and the assumption on h imply that

$$m_1 = 1 \leq \xi_1, \ldots, m_i = \sqrt{\frac{2i}{i+1}} \leq \xi_i, \ldots, m_{d-3} = \sqrt{\frac{2(d-3)}{d-2}} \leq \xi_{d-3},$$

$$m_{d-2} = \sqrt{\frac{2(d-2)}{d-1}} \leq \xi_{d-2} = h < \sqrt{\frac{2(d-1)}{d}}.$$

Notice that if $\|\mathbf{r}_i\| = m_i$ for all $1 \leq i \leq d-2$, then $[\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M]$ can be dissected into four pieces each being congruent to $\mathbf{W}$ and therefore $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0 \cap M], S) = \hat{\sigma}_d$.

Lemma 2.3.20.

$$\hat{\Delta}(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) \leq \hat{\Delta}(m_1, \ldots, m_{d-3}, m_{d-2}) = \hat{\sigma}_d.$$

Proof. For any fixed $\xi_{d-2} = h$, (2) of Lemma 2.3.11 easily implies that

$$\hat{\Delta}(\xi_1, \ldots, \xi_{d-3}, h) \leq \hat{\Delta}(m_1, \ldots, m_{d-3}, h).$$

Finally, using Lemma 2.3.11 again, it is rather straightforward to show that the function $\hat{\Delta}(m_1, \ldots, m_{d-3}, h)$ as a function of h is decreasing on the interval $\left(\sqrt{\frac{2(d-2)}{d-1}}, \sqrt{\frac{2(d-1)}{d}}\right)$. From this it follows that

$$\hat{\Delta}(m_1, \ldots, m_{d-3}, h) \leq \hat{\Delta}(m_1, \ldots, m_{d-3}, m_{d-2}) = \hat{\sigma}_d,$$

finishing the proof of Lemma 2.3.20. □

Thus, Lemmas 2.3.19 and 2.3.20 yield the following immediate estimate.

Corollary 2.3.21. *Let* $\overline{\mathbf{W}}_I$ *denote the truncated wedge of type I with the 2-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 8$. *Then*

$$\hat{\delta}(\overline{\mathbf{W}}_I, S) \leq \hat{\sigma}_d .$$

2.3.8 Surface Density in Truncated Wedges of Type II

It is sufficient to prove the following statement.

Lemma 2.3.22. *Let* $\overline{\mathbf{W}}_{II}$ *denote the truncated wedge of type II with the 2-dimensional base* $\overline{\mathbf{B}} \cap F_2$ *which is generated by the* $(d-2)$*-dimensional Rogers orthoscheme* $\mathrm{conv}\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ *of the Voronoi polytope* $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 4$. *Then*

$$\hat{\delta}(\overline{\mathbf{W}}_{II}, S) \leq \hat{\sigma}_d .$$

Proof. By assumption F_2 is a 2-dimensional face of the Voronoi polytope $\mathbf{P}$ with

$$\sqrt{\frac{2(d-1)}{d}} \leq h = R(F_2) < \sqrt{\frac{2d}{d+1}}.$$

Let $G_0 \subset \mathrm{aff} F_2$ denote the closed circular disk of radius $g_0(h) = \sqrt{\frac{2d}{d+1} - h^2}$ centered at the point $\mathbf{r}_{d-2}$. As $h = R(F_2) < \sqrt{2}$, therefore Lemma 2.3.1 yields that

$$\sqrt{\frac{2d}{d+1}} \leq \frac{2}{\sqrt{4-h^2}} \leq R(F_1)$$

holds for any side F_1 of the face F_2. Thus,

$$\overline{\mathbf{B}} \cap F_2 = G_0$$

and so

$$\hat{\delta}(\overline{\mathbf{W}}_{II}, S) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S).$$

It is clear from the construction that we can write $\hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S)$ as a function of $d-2$ variables, namely

$$\hat{\Delta}^*(\xi_1, \ldots, \xi_{d-3}, \xi_{d-2}) = \hat{\delta}([\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-3}, G_0], S),$$

where $\xi_1 = \|\mathbf{r}_1\|, \dots, \xi_{d-3} = \|\mathbf{r}_{d-3}\|, \xi_{d-2} = \|\mathbf{r}_{d-2}\| = h$. Corollary 2.3.2 and the assumption on h imply that

$$m_1 = 1 \le \xi_1, \dots, m_i = \sqrt{\frac{2i}{i+1}} \le \xi_i, \dots, m_{d-3} = \sqrt{\frac{2(d-3)}{d-2}} \le \xi_{d-3},$$

$$m^*_{d-2} = \sqrt{\frac{2(d-1)}{d}} \le \xi_{d-2} = h < \sqrt{\frac{2d}{d+1}}.$$

For any fixed $\xi_{d-2} = h$, (2) of Lemma 2.3.11 easily implies that

$$\hat{\Delta}^*(\xi_1, \dots, \xi_{d-3}, h) \le \hat{\Delta}^*(m_1, \dots, m_{d-3}, h).$$

Finally, again applying (2) of Lemma 2.3.11 we immediately get that

$$\hat{\Delta}^*(m_1, \dots, m_{d-3}, h) \le \hat{\Delta}^*(m_1, \dots, m_{d-3}, m^*_{d-2}) \le \hat{\sigma}_d.$$

This completes the proof of Lemma 2.3.22. □

2.3.9 The Overall Estimate of Surface Density in Voronoi Cells

Let $\mathbf{P}$ be a d-dimensional Voronoi polytope of a packing $\mathcal{P}$ of d-dimensional unit balls in $\mathbb{E}^d, d \ge 8$. Without loss of generality we may assume that the unit ball $\mathbf{B} = \{\mathbf{x} \in \mathbb{E}^d \mid \operatorname{dist}(\mathbf{o}, \mathbf{x}) = \|\mathbf{x}\| \le 1\}$ centered at the origin $\mathbf{o}$ of $\mathbb{E}^d$ is one of the unit balls of $\mathcal{P}$ with $\mathbf{P}$ as its Voronoi cell. As before, let S denote the boundary of $\mathbf{B}$.

First, we dissect $\mathbf{P}$ into d-dimensional Rogers simplices. Then let $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ be one of these d-dimensional Rogers simplices assigned to the flag say, $F_0 \subset \cdots \subset F_{d-1}$ of $\mathbf{P}$. As $\mathbf{r}_i \in F_{d-i}, 1 \le i \le d$ it is clear that $\operatorname{aff}\{\mathbf{r}_{d-2}, \mathbf{r}_{d-1}, \mathbf{r}_d\} = \operatorname{aff} F_2$ and so

$$\operatorname{dist}(\mathbf{o}, \operatorname{aff}\{\mathbf{r}_{d-2}, \mathbf{r}_{d-1}, \mathbf{r}_d\}) = \operatorname{dist}(\mathbf{o}, \operatorname{aff} F_2) = R(F_2).$$

Notice that Corollary 2.3.2 implies that $\sqrt{\frac{2(d-2)}{d-1}} \le R(F_2)$.

Second, we group the d-dimensional Rogers simplices of $\mathbf{P}$ as follows.

1. If $\sqrt{\frac{2(d-2)}{d-1}} \le R(F_2) < \sqrt{\frac{2(d-1)}{d}}$, then we assign the Rogers simplex $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_d\}$ to the type I wedge $\mathbf{W}_I$ with the 2-dimensional base F_2 generated by the $(d-2)$-dimensional Rogers orthoscheme $\operatorname{conv}\{\mathbf{o}, \mathbf{r}_1, \dots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d, d \ge 8$.

2. If $\sqrt{\frac{2(d-1)}{d}} \leq R(F_2) < \sqrt{\frac{2d}{d+1}}$, then we assign the Rogers simplex conv$\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\}$ to the type II wedge $\mathbf{W}_{II}$ with the 2-dimensional base F_2 generated by the $(d-2)$-dimensional Rogers orthoscheme conv$\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_{d-2}\}$ of the Voronoi polytope $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 8$.
3. If $\sqrt{\frac{2d}{d+1}} \leq R(F_2)$, then we assign the Rogers simplex conv$\{\mathbf{o}, \mathbf{r}_1, \ldots, \mathbf{r}_d\}$ to itself as the type III wedge $\mathbf{W}_{III}$.

As the wedges of types I–III of the given Voronoi polytope $\mathbf{P}$ sit over the 2-skeleton of $\mathbf{P}$ and form a tiling of $\mathbf{P}$ it is clear that each d-dimensional Rogers simplex of $\mathbf{P}$ belongs to exactly one of them. As a result, in order to show that the surface density $\hat{\delta}(\mathbf{P}, S) = \frac{\mathrm{Svol}_{d-1}(S)}{\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{P})} = \frac{d\omega_d}{\mathrm{svol}_{d-1}(\mathrm{bd}\mathbf{P})}$ of the unit sphere S in the Voronoi polytope $\mathbf{P}$ is bounded from above by $\hat{\sigma}_d$, it is sufficient to prove the following inequalities.

$(\hat{1})$: $\hat{\delta}(W_I, S) \leq \hat{\sigma}_d$,
$(\hat{2})$: $\hat{\delta}(W_{II}, S) \leq \hat{\sigma}_d$,
$(\hat{3})$: $\hat{\delta}(W_{III}, S) \leq \hat{\sigma}_d$.

This final task is now easy. Namely, Lemma 2.3.18, Corollary 2.3.21, and Lemma 2.3.22 yield $(\hat{1})$ and $(\hat{2})$ in a straightforward way. Finally, $(\hat{3})$ follows with the help of (2) of Lemma 2.3.11 rather easily.

For the details of the proof of $\hat{\sigma}_d < \sigma_d$, based on the so-called "Lemma of Strict Comparison", we refer the interested reader to the proper section in [32].

This completes the proof of Theorem 1.2.5.

2.4 Proof of Theorem 1.3.3

Theorem 1.3.3. *Let $\mathcal{T}$ be an arbitrary normal tiling of $\mathbb{E}^3$. Then the average surface area of the cells in $\mathcal{T}$ is always at least $\frac{24}{\sqrt{3}}$, i.e., $\underline{s}(\mathcal{T}) \geq \frac{24}{\sqrt{3}} = 13.8564\ldots$.*

2.4.1 Average Surface Area of Cells in Normal Tilings of a Cube

First, we prove the following "compact" version of Theorem 1.3.3.

Theorem 2.4.1. *If the cube $\mathbf{C}$ is partitioned into the convex cells $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n$ each containing a unit ball in $\mathbb{E}^3$, then the sum of the surface areas of the n convex cells is at least $\frac{24}{\sqrt{3}}n$, i.e., $\sum_{i=1}^{n} \mathrm{sarea}(\mathbf{Q}_i) \geq \frac{24}{\sqrt{3}}n$.*

Proof. It is well known that the Brunn-Minkowski inequality implies the following inequality:

$$\mathrm{sarea}^2(\mathbf{Q}_i) \geq 3\mathrm{vol}(\mathbf{Q}_i)\mathrm{ecurv}(\mathbf{Q}_i)\ , \tag{2.18}$$

where $1 \leq i \leq n$. (For a proof we refer the interested reader to p. 287 in [99].) In what follows it is more proper to use the inner dihedral angles $\beta_e := \pi - \alpha_e$ and the relevant formula for the edge curvature:

$$\mathrm{ecurv}(\mathbf{Q}_i) = \sum_{e \in E(\mathbf{Q}_i)} L(e) \cot \frac{\beta_e}{2}\ . \tag{2.19}$$

As, by assumption, $\mathbf{Q}_i$ contains a unit ball therefore

$$\mathrm{vol}(\mathbf{Q}_i) \geq \frac{1}{3}\mathrm{sarea}(\mathbf{Q}_i)\ . \tag{2.20}$$

Hence, (2.18)–(2.20) imply in a straightforward way that

$$\mathrm{sarea}(\mathbf{Q}_i) \geq \sum_{e \in E(\mathbf{Q}_i)} L(e) \cot \frac{\beta_e}{2} \tag{2.21}$$

holds for all $1 \leq i \leq n$.

Now, let $s \subset \mathbf{C}$ be a closed line segment along which exactly k members of the family $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$ meet having inner dihedral angles $\beta_1, \beta_2, \ldots, \beta_k$. There are the following three possibilities:

(a) s is on an edge of the cube $\mathbf{C}$;
(b) s is in the relative interior either of a face of $\mathbf{C}$ or of a face of a convex cell in the family $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$;
(c) s is in the interior of $\mathbf{C}$ and not in the relative interior of any face of any convex cell in the family $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n\}$.

In each of the above cases we can make the following easy observations:

(a) $\beta_1 + \beta_2 + \cdots + \beta_k = \frac{\pi}{2}$ with $k \geq 1$;
(b) $\beta_1 + \beta_2 + \cdots + \beta_k = \pi$ with $k \geq 2$;
(c) $\beta_1 + \beta_2 + \cdots + \beta_k = 2\pi$ with $k \geq 3$.

As $y = \cot x$ is convex and decreasing over the interval $0 < x \leq \frac{\pi}{2}$ therefore the following inequalities must hold:

(a) $\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq k \cot \frac{\pi}{4k} \geq k$;
(b) $\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq k \cot \frac{\pi}{2k} \geq k$;
(c) $\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq k \cot \frac{\pi}{k} \geq \frac{1}{\sqrt{3}}k$.

In short, the following inequality holds in all three cases:

$$\cot \frac{\beta_1}{2} + \cot \frac{\beta_2}{2} + \cdots + \cot \frac{\beta_k}{2} \geq \frac{1}{\sqrt{3}}k\ . \tag{2.22}$$

Thus, by adding together the inequalities (2.21) for all $1 \le i \le n$ and using (2.22) we get that

$$\sum_{i=1}^{n} \operatorname{sarea}(\mathbf{Q}_i) \ge \frac{1}{\sqrt{3}} \sum_{i=1}^{n} \sum_{e \in E(\mathbf{Q}_i)} L(e) \; . \tag{2.23}$$

Finally, recall the elegant theorem of Besicovitch and Eggleston [23] claiming that the total edge length of any convex polyhedron containing a unit ball in $\mathbb{E}^3$ is always at least as large as the total edge length of a cube circumscribed a unit ball. This implies that

$$\sum_{e \in E(\mathbf{Q}_i)} L(e) \ge 24 \tag{2.24}$$

holds for all $1 \le i \le n$. Hence, (2.23) and (2.24) finish the proof of Theorem 2.4.1. □

2.4.2 *Average Surface Area of Cells in Normal Tilings*

Second, we take a closer look of the given normal tiling $\mathcal{T}$ and using the above proof of Theorem 2.4.1 we give a proof of Theorem 1.3.3. The details are as follows.

By assumption $D := \sup\{\operatorname{diam}(\mathbf{P}_i) | i = 1, 2, \ldots\} < \infty$. Thus, clearly each closed ball of radius D in $\mathbb{E}^3$ contains at least one of the convex polyhedra $\mathbf{P}_i, i = 1, 2, \ldots$ (forming the tiling $\mathcal{T}$ of $\mathbb{E}^3$). Now, let $\mathbf{C}_{L_N}, N = 1, 2, \ldots$ be an arbitrary sequence of cubes centered at the origin $\mathbf{o}$ with edges parallel to the coordinate axes of $\mathbb{E}^3$ and having edge length $L_N, N = 1, 2, \ldots$ with $\lim_{N\to\infty} L_n = \infty$. It follows that

$$0 < \liminf_{N\to\infty} \frac{\frac{4\pi}{3}\operatorname{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}}{\operatorname{vol}(\mathbf{C}_{L_N})} \le \limsup_{N\to\infty} \frac{\frac{4\pi}{3}\operatorname{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}}{\operatorname{vol}(\mathbf{C}_{L_N})} < 1. \tag{2.25}$$

Note that clearly

$$\frac{\operatorname{card}\{i | \mathbf{P}_i \cap \operatorname{bd}\mathbf{C}_{L_N} \ne \emptyset\}}{\operatorname{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \le \frac{\big(\operatorname{vol}(\mathbf{C}_{L_N+2D}) - \operatorname{vol}(\mathbf{C}_{L_N-2D})\big)\operatorname{vol}(\mathbf{C}_{L_N})}{\operatorname{vol}(\mathbf{C}_{L_N})\frac{4\pi}{3}\operatorname{card}\{i | \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \tag{2.26}$$

moreover,

$$\lim_{N\to\infty} \frac{\operatorname{vol}(\mathbf{C}_{L_N+2D}) - \operatorname{vol}(\mathbf{C}_{L_N-2D})}{\operatorname{vol}(\mathbf{C}_{L_N})} = 0. \tag{2.27}$$

Thus, (2.25)–(2.27) imply in a straightforward way that

$$\lim_{N\to\infty} \frac{\mathrm{card}\{i \,|\, \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_{L_N} \neq \emptyset\}}{\mathrm{card}\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} = 0\,. \tag{2.28}$$

Moreover, (2.21) yields that

$$\mathrm{sarea}(\mathbf{P}_i) \geq \mathrm{ecurv}(\mathbf{P}_i) = \sum_{e\in E(\mathbf{P}_i)} L(e)\cot\frac{\beta_e}{2} \tag{2.29}$$

holds for all $i = 1, 2, \ldots$. As a next step, using

$$\mathrm{sarea}(\mathbf{P}_i) = \mathrm{sarea}\,(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L) + \delta_i \tag{2.30}$$

and

$$\mathrm{ecurv}(\mathbf{P}_i) \geq \sum_{e\in E(\mathrm{bd}(\mathbf{P}_i\cap\mathbf{C}_L)\setminus\mathrm{bd}\mathbf{C}_L)} L(e)\cot\frac{\beta_e}{2} \tag{2.31}$$

(with $\mathrm{bd}(\cdot)$ denoting the boundary of the corresponding set) we obtain the following from (2.29):

$$\mathrm{sarea}\,(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L) + \delta_i \geq \sum_{e\in E(\mathrm{bd}(\mathbf{P}_i\cap\mathbf{C}_L)\setminus\mathrm{bd}\mathbf{C}_L)} L(e)\cot\frac{\beta_e}{2}\,, \tag{2.32}$$

where clearly $0 \leq \delta_i \leq \mathrm{sarea}(\mathbf{P}_i)$. Hence, (2.32) combined with (2.22) yields

Corollary 2.4.2.

$$f(L) := \sum_{\{i\,|\,\mathrm{int}\mathbf{P}_i\cap\mathbf{C}_L\neq\emptyset\}} \mathrm{sarea}\big(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L\big) + \sum_{\{i\,|\,\mathbf{P}_i\cap\mathrm{bd}\mathbf{C}_L\neq\emptyset\}} \delta_i$$

$$\geq g(L) := \frac{1}{\sqrt{3}} \sum_{\{i\,|\,\mathrm{int}\mathbf{P}_i\cap\mathbf{C}_L\neq\emptyset\}} \left(\sum_{e\in E(\mathrm{bd}(\mathbf{P}_i\cap\mathbf{C}_L)\setminus\mathrm{bd}\mathbf{C}_L)} L(e) \right)$$

Now, it is easy to see that

$$f(L) = \Delta(L) + \sum_{\{i\,|\,\mathbf{B}_i\subset\mathbf{C}_L\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L)\,, \tag{2.33}$$

where $0 \leq \Delta(L) \leq 2\sum_{\{i\,|\,\mathbf{P}_i\cap\mathrm{bd}\mathbf{C}_L\neq\emptyset\}} \mathrm{sarea}(\mathbf{P}_i)$.

Moreover, (2.24) implies that

$$g(L) \geq -\overline{\Delta}(L) + \sum_{\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_L\}} \frac{24}{\sqrt{3}}\,, \tag{2.34}$$

where $0 \leq \overline{\Delta}(L) \leq \sum_{\{i \,|\, \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\}} \sum_{e \in E(\mathbf{P}_i)} L(e)$.

Lemma 2.4.3.

$$A := \sup\{\mathrm{sarea}(\mathbf{P}_i) | i = 1, 2, \ldots\} < \infty$$

and

$$E := \sup\{\sum_{e \in E(\mathbf{P}_i)} L(e) | i = 1, 2, \ldots\} < \infty\,.$$

Proof. As $D = \sup\{\mathrm{diam}(\mathbf{P}_i) | i = 1, 2, \ldots\} < \infty$ therefore according to Jung's theorem [90] each $\mathbf{P}_i$ is contained in a closed ball of radius $\sqrt{\frac{3}{8}}D$ in $\mathbb{E}^3$. Thus, $A \leq \frac{3}{2}\pi D^2 < \infty$.

For a proof of the other claim recall that $\mathbf{P}_i$ contains the unit ball $\mathbf{B}_i$ centered at $\mathbf{o}_i$. If the number of faces of $\mathbf{P}_i$ is f_i, then $\mathbf{P}_i$ must have at least f_i neighbours (i.e., cells of $\mathcal{T}$ that have at least one point in common with $\mathbf{P}_i$) and as each neighbour is contained in the closed 3-dimensional ball of radius $2D$ centered at $\mathbf{o}_i$ therefore the number of neighbours of $\mathbf{P}_i$ is at most $(2D)^3 - 1$ and so, $f_i \leq 8D^3 - 1$. (Here, we have used the fact that each neighbour contains a unit ball and therefore its volume is larger than $\frac{4\pi}{3}$.) Finally, Euler's formula implies that the number of edges of $\mathbf{P}_i$ is at most $3f_i - 6 \leq 24D^3 - 9$. Thus, $E \leq 24D^4 - 9D < \infty$ (because the length of any edge of $\mathbf{P}_i$ is at most D). □

Thus, Corollary 2.4.2, (2.33), (2.34), and Lemma 2.4.3 imply the following inequality in a straightforward way.

Corollary 2.4.4.

$$\frac{2A\mathrm{card}\{i \,|\, \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\} + \sum_{\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_L\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L)}{\mathrm{card}\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_L\}}$$
$$\geq \frac{-E\mathrm{card}\{i \,|\, \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\} + \sum_{\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_L\}} \frac{24}{\sqrt{3}}}{\mathrm{card}\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_L\}}\,.$$

Finally, Corollary 2.4.4 and (2.28) yield that

$$\liminf_{N \to \infty} \frac{\sum_{\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_{L_N})}{\mathrm{card}\{i \,|\, \mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \geq \frac{24}{\sqrt{3}}\,,$$

finishing the proof of Theorem 1.3.3.

2.5 Proof of Theorem 1.3.5

Theorem 1.3.5. *Let $\mathcal{T}$ be an arbitrary normal tiling of $\mathbb{E}^3$. Then $\underline{s}(\mathcal{T}) \geq \underline{ec}(\mathcal{T}) \geq \frac{24}{\sqrt{3}} = 13.8564\ldots$. Moreover, if $\mathcal{T}$ is a Voronoi tiling of a unit ball packing in $\mathbb{E}^3$, then $\underline{ec}(\mathcal{T}) \geq \frac{2\pi^2}{6\sqrt{6}\arcsin\left(\frac{1}{\sqrt{3}}\right)-\pi\sqrt{6}} = 14.6176\ldots$.*

2.5.1 Average Edge Curvature of Cells in Normal Tilings

If $\mathcal{T}$ is an arbitrary normal tiling of $\mathbb{E}^3$ with the underlying packing $\mathcal{P}$ of the unit balls $\mathbf{B}_i, i = 1, 2, \ldots$, then (2.21) implies

$$\mathrm{sarea}(\mathbf{P}_i \cap \mathbf{C}_L) \geq \mathrm{ecurv}(\mathbf{P}_i \cap \mathbf{C}_L)$$

and therefore $\underline{s}(\mathcal{T}) \geq \underline{ec}(\mathcal{T})$ follows in a straightforward way. So, we are left to show that $\underline{ec}(\mathcal{T}) \geq \frac{24}{\sqrt{3}}$. In order to achieve that, we take a closer look of the given normal tiling $\mathcal{T}$ and using some of the properly modified estimates of the proof of Theorem 1.3.3 we derive the inequality $\underline{ec}(\mathcal{T}) \geq \frac{24}{\sqrt{3}}$. The details are as follows.

We start with the following immediate analogue of (2.32):

$$\mathrm{ecurv}(\mathbf{P}_i) = \delta_i^* + \sum_{e \in E(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L)} L(e) \cot \frac{\beta_e}{2}, \tag{2.35}$$

where $0 \leq \delta_i^* \leq \mathrm{ecurv}(\mathbf{P}_i) \leq \mathrm{sarea}(\mathbf{P}_i)$. If

$$\mathrm{ecurv}\big(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L\big) := \sum_{e \in E(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L)} L(e) \cot \frac{\beta_e}{2},$$

then (2.35) combined with (2.22) yields

Corollary 2.5.1.

$$f^*(L) := \sum_{\{i \mid \mathrm{int}\mathbf{P}_i \cap \mathbf{C}_L \neq \emptyset\}} \mathrm{ecurv}\big(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L\big) + \sum_{\{i \mid \mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\}} \delta_i^*$$

$$\geq g(L) = \frac{1}{\sqrt{3}} \sum_{\{i \mid \mathrm{int}\mathbf{P}_i \cap \mathbf{C}_L \neq \emptyset\}} \Bigg(\sum_{e \in E(\mathrm{bd}(\mathbf{P}_i \cap \mathbf{C}_L) \setminus \mathrm{bd}\mathbf{C}_L)} L(e) \Bigg)$$

Now, it is easy to see that

$$f^*(L) = \Delta^*(L) + \sum_{\{i\,|\mathbf{B}_i \subset \mathbf{C}_L\}} \mathrm{ecurv}(\mathbf{P}_i \cap \mathbf{C}_L)\ , \tag{2.36}$$

where $0 \leq \Delta^*(L) \leq 2\sum_{\{i\,|\mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\}} \mathrm{ecurv}(\mathbf{P}_i)$. Moreover, $g(L)$ must satisfy (2.34). Thus, Corollary 2.5.1, (2.36), (2.34), and Lemma 2.4.3 imply the following inequality in a straightforward way.

Corollary 2.5.2.

$$\frac{2A\mathrm{card}\{i\,|\mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\} + \sum_{\{i\,|\mathbf{B}_i \subset \mathbf{C}_L\}} \mathrm{ecurv}(\mathbf{P}_i \cap \mathbf{C}_L)}{\mathrm{card}\{i\,|\mathbf{B}_i \subset \mathbf{C}_L\}}$$
$$\geq \frac{-E\mathrm{card}\{i\,|\mathbf{P}_i \cap \mathrm{bd}\mathbf{C}_L \neq \emptyset\} + \sum_{\{i\,|\mathbf{B}_i \subset \mathbf{C}_L\}} \frac{24}{\sqrt{3}}}{\mathrm{card}\{i\,|\mathbf{B}_i \subset \mathbf{C}_L\}}\ .$$

Hence, Corollary 2.5.2 and (2.28) yield that

$$\liminf_{N\to\infty} \frac{\sum_{\{i\,|\mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \mathrm{ecurv}(\mathbf{P}_i \cap \mathbf{C}_{L_N})}{\mathrm{card}\{i\,|\mathbf{B}_i \subset \mathbf{C}_{L_N}\}} \geq \frac{24}{\sqrt{3}}\ ,$$

finishing the proof of the inequality $\underline{ec}(\mathcal{T}) \geq \frac{24}{\sqrt{3}}$.

2.5.2 Average Edge Curvature of Voronoi Cells in Unit Ball Packings

Finally, let $\mathcal{T}$ be the Voronoi tiling of a unit ball packing in $\mathbb{E}^3$ consisting of the Voronoi cells $\mathbf{P}_i, i = 1, 2, \ldots$ (each containing a unit ball). First, recall the inequality

$$\mathrm{ecurv}(\mathbf{P}_i) > 2\pi\ \mathrm{mwidth}(\mathbf{P}_i)\ , \tag{2.37}$$

where $i = 1, 2, \ldots$ and $\mathrm{mwidth}(\cdot)$ denotes the mean width of the corresponding set. (For more details on this inequality see p. 287 in [99] as well as the relevant discussion on p. 392 in [31].) Second, recall that according to a recent result of the author [31] the inequality

$$\mathrm{mwidth}(\mathbf{P}_i) \geq \frac{\pi}{6\sqrt{6}\arcsin\left(\frac{1}{\sqrt{3}}\right) - \pi\sqrt{6}} = 2.3264\ldots \tag{2.38}$$

holds for all $i = 1, 2, \ldots$. Thus, (2.37) and (2.38) yield

$$\mathrm{ecurv}(\mathbf{P}_i) > \frac{2\pi^2}{6\sqrt{6}\arcsin\left(\frac{1}{\sqrt{3}}\right) - \pi\sqrt{6}} = 14.6176\ldots$$

from which it follows in a straightforward way that $\underline{ec}(\mathcal{T}) \geq 14.6176\ldots$, finishing the proof of Theorem 1.3.5.

2.6 Proof of Theorem 1.5.3

Theorem 1.5.3. *Suppose that $\mathbb{E}^d, d \geq 2$ can be tiled face-to-face by congruent copies of finitely many convex polytopes $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_m$ such that the vertices and edges of that tiling form the vertex and edge system of the contact graph $G_{\mathcal{P}}$ of some ball packing $\mathcal{P}$ in $\mathbb{E}^d$. Assume that each $\mathbf{P}_i$ and the graph $G_{\mathcal{P}}$ restricted to the vertices of $\mathbf{P}_i$ (and regarded as a strut graph), satisfy the critical volume condition and assume that the bar framework $\overline{G}_{\mathcal{P}}$ (which is $G_{\mathcal{P}}$ with all the struts changed to bars) restricted to the vertices of $\mathbf{P}_i$ is infinitesimally rigid. Then the packing $\mathcal{P}$ is uniformly stable.*

2.6.1 The Signed Volume of Convex Polytopes

Definition 2.6.1. Let $\mathbf{P} := \mathrm{conv}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ be a d-dimensional convex polytope in $\mathbb{E}^d, d \geq 2$ with vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$. If $F := \mathrm{conv}\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_k}\}$ is an arbitrary face of $\mathbf{P}$, then the barycenter of F is

$$\mathbf{c}_F := \frac{1}{k}\sum_{j=1}^{k} \mathbf{p}_{i_j}. \tag{2.39}$$

Let $F_0 \subset F_1 \subset \cdots \subset F_l, 0 \leq l \leq d-1$ denote a sequence of faces, called a (partial) flag of $\mathbf{P}$, where F_0 is a vertex and F_{i-1} is a facet (a face one dimension lower) of F_i for $i = 1, \ldots, l$. Then the simplices of the form $\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_l}\}$ constitute a simplicial complex $\mathcal{C}_{\mathbf{P}}$ whose underlying space is the boundary of $\mathbf{P}$.

We regard all points in $\mathbb{E}^d$ as row vectors and use $\mathbf{q}^T$ for the column vector that is the transpose of the row vector $\mathbf{q}$. Moreover, $[\mathbf{q}_1, \ldots, \mathbf{q}_d]$ is the (square) matrix with the ith row $\mathbf{q}_i$. Choosing a$(d-1)$-dimensional simplex of $\mathcal{C}_{\mathbf{P}}$ to be positively oriented, one can check whether the orientation of an arbitrary $(d-1)$-dimensional simplex $\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}$ of $\mathcal{C}_{\mathbf{P}}$ (generated by the given sequence of its vertices), is positive or negative. Let $\mathrm{sign}\left(\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}\right)$ be equal to 1 (resp., -1) if the orientation of the $(d-1)$-dimensional simplex $\mathrm{conv}\{\mathbf{c}_{F_0}, \mathbf{c}_{F_1}, \ldots, \mathbf{c}_{F_{d-1}}\}$ is positive (resp., negative).

Definition 2.6.2. The signed volume $V(\mathbf{P})$ of $\mathbf{P}$ is defined as

$$\frac{1}{d!}\sum_{F_0\subset F_1\subset\cdots\subset F_{d-1}} \operatorname{sign}\left(\operatorname{conv}\{\mathbf{c}_{F_0},\mathbf{c}_{F_1},\ldots,\mathbf{c}_{F_{d-1}}\}\right)\det[\mathbf{c}_{F_0},\mathbf{c}_{F_1},\ldots,\mathbf{c}_{F_{d-1}}], \quad (2.40)$$

where the sum is taken over all flags of faces $F_0\subset F_1\subset\cdots\subset F_{d-1}$ of $\mathbf{P}$, and $\det[\cdot]$ is the determinant function.

The following is clear.

Lemma 2.6.3.

$$V(\mathbf{P})=\frac{1}{d!}\sum_{F_0\subset\cdots\subset F_{d-1}} \operatorname{sign}\left(\operatorname{conv}\{\mathbf{c}_{F_0},\mathbf{c}_{F_1},\ldots,\mathbf{c}_{F_{d-1}}\}\right)\mathbf{c}_{F_0}\wedge\mathbf{c}_{F_1}\wedge\cdots\wedge\mathbf{c}_{F_{d-1}},$$

where $\wedge$ stands for the wedge product of vectors. Moreover, one can choose the orientation of the boundary of $\mathbf{P}$ such that $V(\mathbf{P})=\operatorname{vol}_d(\mathbf{P})$, where $\operatorname{vol}_d(\cdot)$ refers to the d-dimensional volume measure in $\mathbb{E}^d$, $d\geq 2$.

2.6.2 *The Volume Force of Convex Polytopes*

We wish to compute the gradient of $V(\mathbf{P})$, where $\mathbf{P}=\operatorname{conv}\{\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\}$ is regarded as a function of its vertices $\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n$. To achieve this we consider an arbitrary path $\mathbf{p}(t)=\mathbf{p}+t\mathbf{p}'$ in the space of the configurations $\mathbf{p}:=(\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n)$, where $\mathbf{p}':=(\mathbf{p}'_1,\mathbf{p}'_2,\ldots,\mathbf{p}'_n)$ (Fig. 2.8). Based on Definitions 2.6.1, 2.6.2, and Lemma 2.6.3 we introduce $V(\mathbf{P}(t))$ as a function of t (with t being an arbitrary real with sufficiently small absolute value) via

$$\frac{1}{d!}\sum_{F_0\subset F_1\subset\cdots\subset F_{d-1}} \operatorname{sign}\left(\operatorname{conv}\{\mathbf{c}_{F_0}(t),\ldots,\mathbf{c}_{F_{d-1}}(t)\}\right)\det[\mathbf{c}_{F_0}(t),\ldots,\mathbf{c}_{F_{d-1}}(t)]$$
$$=\frac{1}{d!}\sum_{F_0\subset F_1\subset\cdots\subset F_{d-1}} \operatorname{sign}\left(\operatorname{conv}\{\mathbf{c}_{F_0}(t),\ldots,\mathbf{c}_{F_{d-1}}(t)\}\right)\mathbf{c}_{F_0}(t)\wedge\cdots\wedge\mathbf{c}_{F_{d-1}}(t),$$

where $\mathbf{c}_F(t):=\frac{1}{k}\sum_{j=1}^{k}\mathbf{p}_{i_j}(t)$ for any face $F=\operatorname{conv}\{\mathbf{p}_{i_1},\ldots,\mathbf{p}_{i_k}\}$ of $\mathbf{P}$. Clearly, $V(\mathbf{P}(0))=V(\mathbf{P})$. Moreover, evaluating the derivative $\frac{d}{dt}V(\mathbf{P}(t))$ of $V(\mathbf{P}(t))$ at $t=0$, collecting terms, and using the anticommutativity of the wedge product we get that

$$\frac{d}{dt}V(\mathbf{P}(t))\,|_{t=0}=\frac{1}{d!}\sum_{i=1}^{n}\mathbf{N}_i\wedge\mathbf{p}'_i, \quad (2.41)$$

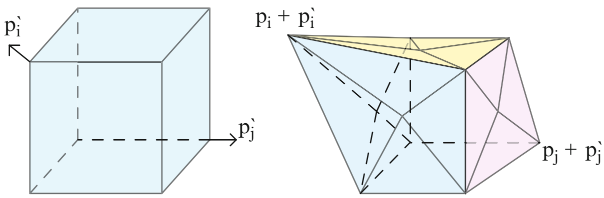

Fig. 2.8 Deformation of the convex polytope $\mathbf{P}$ under the motion given by $\mathbf{p}'$

where each $\mathbf{N}_i$ is some linear combination of wedge products of $d-1$ vectors $\mathbf{p}_j$ with $\mathbf{p}_j$ and $\mathbf{p}_i$ sharing a common face.

Definition 2.6.4. We call $\mathbf{N} := (\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)$ the volume force of the d-dimensional convex polytope $\mathbf{P} \subset \mathbb{E}^d$ with n vertices.

The following are some simple properties of the volume force. We leave the rather straightforward proofs to the reader.

Lemma 2.6.5. *Let* $\mathbf{N} := (\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)$ *be the volume force of the* d*-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ *with vertices* $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$. *Then the following hold.*

(1) Each $\mathbf{N}_i$ *is only a function of the vertices that share a face with* $\mathbf{p}_i$*, but not* $\mathbf{p}_i$ *itself.*
(2) Assume that the origin $\mathbf{o}$ *of* $\mathbb{E}^d$ *is the barycenter of* $\mathbf{P}$*; moreover, let* $T : \mathbb{E}^d \to \mathbb{E}^d$ *be an orthogonal linear map satisfying* $T(\mathbf{P}) = \mathbf{P}$*. If* $T(\mathbf{p}_i) = \mathbf{p}_j$*, then* $T(\mathbf{N}_i) = \mathbf{N}_j$*.*

For more details and examples on volume forces we refer the interested reader to the proper sections in [56].

2.6.3 Critical Volume Condition

Let $\mathbf{P} := \mathrm{conv}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$ be a d-dimensional convex polytope in $\mathbb{E}^d, d \geq 2$ with vertices $\mathbf{p} := (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$. Let G be a graph defined on this vertex set $\mathbf{p}$. Here, G may or may not consist of the edges of $\mathbf{P}$. We think of the edges of G as defining those pairs of vertices of $\mathbf{P}$ that are constrained not to get closer. In the terminology of the geometry of rigid tensegrity frameworks each edge of G is a *strut*. (For more information on rigid tensegrity frameworks and the basic terminology used there we refer the interested reader to [162].)

Let $\mathbf{p}' := (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ be an *infinitesimal flex* of $G(\mathbf{p})$, where $G(\mathbf{p})$ refers to the realization of G over the point configuration $\mathbf{p}$. That is, for each edge (strut) $\{i, j\}$ of G we have

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0, \tag{2.42}$$

where "$\cdot$" denotes the standard inner product (also called the "dot product") in $\mathbb{E}^d$.

Let e denote the number of edges of G. Then the *rigidity matrix* $R(\mathbf{p})$ of $G(\mathbf{p})$ is the $e \times nd$ matrix whose row corresponding to the edge $\{i, j\}$ of G consists of the coordinates of d-dimensional vectors within a sequence of n vectors such that all the coordinates are zero except maybe the ones that correspond to the coordinates of the vectors $\mathbf{p}_i - \mathbf{p}_j$ and $\mathbf{p}_j - \mathbf{p}_i$ listed on the ith and jth position. Another way to introduce $R(\mathbf{p})$ is the following. Let $f : \mathbb{E}^{nd} \to \mathbb{E}^e$ be the map defined by $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) \to (\dots, \|\mathbf{x}_i - \mathbf{x}_j\|^2, \dots)$. Then it is immediate that $\frac{1}{2}\frac{d}{d\mathbf{x}} f|_{\mathbf{x}=\mathbf{p}} = R(\mathbf{p})$. Now, we can rewrite the inequalities of (2.42) in terms of the rigidity matrix $R(\mathbf{p})$ of $G(\mathbf{p})$ (using the usual matrix multiplication applied to $R(\mathbf{p})$ and the indicated column vector) as follows,

$$R(\mathbf{p})(\mathbf{p}')^T \geq 0, \tag{2.43}$$

where the inequality is meant for each coordinate.

For each edge $\{i, j\}$ of G, let ω_{ij} be a scalar. We collect all such scalars into a single row vector called the *stress* $\omega := (\dots, \omega_{ij}, \dots)$ corresponding to the rows of the matrix $R(\mathbf{p})$. Append the volume force $\mathbf{N} := (\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_n)$ as the last row onto $R(\mathbf{p})$ to get a new matrix $\hat{R}(\mathbf{p})$, which we call the *augmented rigidity matrix*. So, when performing the matrix multiplication $\hat{R}(\mathbf{p})(\mathbf{p}')^T$, we find that the result is a column vector of length $e + 1$ having $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j)$ on the position corresponding to the edge $\{i, j\}$ of G, and having $\sum_{k=1}^{n} \mathbf{N}_k \cdot \mathbf{p}'_k$ on the $(e + 1)$st position. Also, it is easy to see that

$$(\omega, 1)\hat{R}(\mathbf{p}) = \left(\dots, \sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) + \mathbf{N}_i, \dots \right), \tag{2.44}$$

where each sum is taken over all $\mathbf{p}_j$ adjacent to $\mathbf{p}_i$ in G, and we collect d coordinates at a time.

Definition 2.6.6. Let $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_n)$ be the volume force of the d-dimensional convex polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ with vertices $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$. We say that the stress $\omega = (\dots, \omega_{ij}, \dots)$ resolves $\mathbf{N}$ if for each i we have that $\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) + \mathbf{N}_i = \mathbf{o}$ or, equivalently, $(\omega, 1)\hat{R}(\mathbf{p}) = \mathbf{o}$, where $\mathbf{o}$ denotes the zero vector.

Definition 2.6.7. The d-dimensional convex polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ and the graph G defined on the vertices of $\mathbf{P}$ satisfy the critical volume condition if the volume force $\mathbf{N}$ can be resolved by a stress $\omega = (\dots, \omega_{ij}, \dots)$ such that for each edge $\{i, j\}$ of G, $\omega_{ij} < 0$.

Theorem 2.6.8. *Let the d-dimensional convex polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ and the strut graph G, defined on the vertices of $\mathbf{P}$, satisfy the critical volume condition. Moreover, let $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \dots, \mathbf{p}'_n)$ be an infinitesimal flex of the strut framework $G(\mathbf{p})$ (i.e., let $\mathbf{p}'$ satisfy (2.42)). Then*

$$\frac{d}{dt}V\left(\mathbf{P}(t)\right)|_{t=0} = \frac{1}{d!}\sum_{i=1}^{n}\mathbf{N}_i \wedge \mathbf{p}'_i \geq 0$$

with equality if and only if $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ *for each edge* $\{i, j\}$ *of* G.

Proof. The assumptions, (2.44), the associativity of matrix multiplication, and (2.41) imply in a straightforward way that

$$0 = \mathbf{o} \cdot \mathbf{p}' = (\omega, 1)\hat{R}(\mathbf{p})(\mathbf{p}')^T = \sum_{\{i,j\}}\omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + \sum_{i=1}^{n}\mathbf{N}_i \cdot \mathbf{p}'_i$$

$$= \sum_{\{i,j\}}\omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + \sum_{i=1}^{n}\mathbf{N}_i \wedge \mathbf{p}'_i \leq \sum_{i=1}^{n}\mathbf{N}_i \wedge \mathbf{p}'_i = \frac{d}{dt}V\left(\mathbf{P}(t)\right)|_{t=0},$$

where $\mathbf{N}_i$ is regarded as a d-dimensional vector so that $\mathbf{N}_i \wedge \mathbf{p}'_i$ can be interpreted as the standard inner product $\mathbf{N}_i \cdot \mathbf{p}_i$, with appropriate identification of bases. We clearly get equality if and only if $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0$ for each edge $\{i, j\}$ of G. □

2.6.4 Strictly Locally Volume Expanding Convex Polytopes

The following definition recalls standard terminology from the theory of rigid tensegrity frameworks. (See [78] for more information.) Consider now just the *bar graph* $\overline{G}$, which is the graph G with all the struts changed to *bars*, and take its realization $\overline{G}(\mathbf{p})$ sitting over the point configuration $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$. (Here bars mean edges whose lengths are constrained not to change.) We say that the infinitesimal motion $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ is an *infinitesimal flex* of $\overline{G}(\mathbf{p})$ if for each edge (bar) $\{i, j\}$ of $\overline{G}$, we have

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0.$$

This is the same as saying $R(\mathbf{p})(\mathbf{p}')^T = \mathbf{o}$ for the rigidity matrix $R(\mathbf{p})$.

Definition 2.6.9. We say that $\mathbf{p}'$ is a trivial infinitesimal flex if $\mathbf{p}'$ is a (directional) derivative of an isometric motion of $\mathbb{E}^d$, $d \geq 2$. We say that $G(\mathbf{p})$ (resp., $\overline{G}(\mathbf{p})$) is infinitesimally rigid if $G(\mathbf{p})$ (resp., $\overline{G}(\mathbf{p})$) has only trivial infinitesimal flexes.

Theorem 2.6.10. *Let the* d*-dimensional convex polytope* $\mathbf{P} \subset \mathbb{E}^d$, $d \geq 2$ *and the strut graph* G*, defined on the vertices of* $\mathbf{P}$*, satisfy the critical volume condition and assume that the bar framework* $\overline{G}(\mathbf{p})$ *is infinitesimally rigid. Then*

$$\frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0} = \frac{1}{d!} \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i > 0$$

for every non-trivial infinitesimal flex $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ *of the strut framework* $G(\mathbf{p})$.

Proof. By Theorem 2.6.8 we have that $\frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0} = \frac{1}{d!} \sum_{i=1}^{n} \mathbf{N}_i \wedge \mathbf{p}'_i \geq 0$. If $\frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0} = 0$, then applying Theorem 2.6.8 again, $\mathbf{p}' = (\mathbf{p}'_1, \mathbf{p}'_2, \ldots, \mathbf{p}'_n)$ must be an infinitesimal flex of the bar framework $\overline{G}(\mathbf{p})$. However, then by the infinitesimal rigidity of $\overline{G}(\mathbf{p})$, this would imply that $\mathbf{p}'$ is trivial. Thus, $\frac{d}{dt} V(\mathbf{P}(t))\,|_{t=0} > 0$. □

The following definition leads us to the core part of this section.

Definition 2.6.11. Let $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ be a d-dimensional convex polytope and let G be a strut graph defined on the vertices $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ of $\mathbf{P}$. We say that $\mathbf{P}$ is strictly locally volume expanding over G, if there is an $\varepsilon > 0$ with the following property. For every $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$ satisfying

$$\|\mathbf{p}_i - \mathbf{q}_i\| < \varepsilon \text{ for all } i = 1, \ldots, n \tag{2.45}$$

and

$$\|\mathbf{p}_i - \mathbf{p}_j\| \leq \|\mathbf{q}_i - \mathbf{q}_j\| \text{ for each edge } \{i, j\} \text{ of } G, \tag{2.46}$$

we have $V(\mathbf{P}) \leq V(\mathbf{Q})$ (where $V(\mathbf{Q})$ is defined via (2.39) and (2.40) substituting $\mathbf{q}$ for $\mathbf{p}$) with equality only when $\mathbf{P}$ is congruent to $\mathbf{Q}$, where $\mathbf{Q}$ is the polytope generated by the simplices of the barycenters in (2.39) using $\mathbf{q}$ instead of $\mathbf{p}$.

Theorem 2.6.12. *Let the d-dimensional convex polytope $\mathbf{P} \subset \mathbb{E}^d, d \geq 2$ and the strut graph G, defined on the vertices of $\mathbf{P}$, satisfy the critical volume condition and assume that the bar framework $\overline{G}(\mathbf{p})$ is infinitesimally rigid. Then $\mathbf{P}$ is strictly locally volume expanding over G.*

Proof. The inequalities (2.46) define a semialgebraic set X in the space of all configurations $\{(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n) | \mathbf{q}_i \in \mathbb{E}^d, i = 1, \ldots, n\}$. Suppose there is no ε as in the conclusion. Add $V(\mathbf{P}) \geq V(\mathbf{Q})$ to the constraints defining X. By Wallace [179] (see [78]) there is an *analytic* path $\mathbf{p}(t) = (\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots, \mathbf{p}_n(t)), 0 \leq t < 1$, with $\mathbf{p}(0) = \mathbf{p}$ and $\mathbf{p}(t) \in X$, $\mathbf{p}(t)$ not congruent to $\mathbf{p}(0)$ for $0 < t < 1$. So,

$$\|\mathbf{p}_i - \mathbf{p}_j\| \leq \|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| \text{ for each edge } \{i, j\} \text{ of } G \text{ and} \tag{2.47}$$

$$V(\mathbf{P}) \geq V(\mathbf{P}(t)) \text{ for } 0 \leq t < 1. \tag{2.48}$$

Then after suitably adjusting $\mathbf{p}(t)$ by congruences (as in [78] as well as [80]) we can define

$$\mathbf{p}' := \frac{d^k \mathbf{p}(t)}{dt^k}\Big|_{t=0}$$

for the smallest k that makes $\mathbf{p}'$ a non-trivial infinitesimal flex. (Such k exists by the argument in [78] as well as [80]).

Because (2.47) holds we see that $\mathbf{p}'$ is a non-trivial infinitesimal flex of $G(\mathbf{p})$ and (2.48) implies that

$$\frac{d}{dt} V\left(\mathbf{P}(t)\right)|_{t=0} \leq 0.$$

But this contradicts Theorem 2.6.10, finishing the proof of Theorem 2.6.12. $\square$

2.6.5 From Critical Volume Condition to Uniform Stability

Here we start with the assumptions of Theorem 1.5.3 and apply Theorem 2.6.12 to each $\mathbf{P}_i$ and $G_{\mathcal{P}}$ restricted to the vertices of $\mathbf{P}_i$, $1 \leq i \leq m$. Then let $\varepsilon_0 > 0$ be the smallest $\varepsilon > 0$ guaranteed by the strict locally volume expanding property of Theorem 2.6.12. All but a finite number of tiles are fixed. The tiles that are free to move are confined to a region of fixed volume in $\mathbb{E}^d, d \geq 2$. Each $\mathbf{P}_i$ is strictly locally volume expanding, therefore the volume of each of the tiles must be fixed. But the strict condition implies that the motion of each tile must be an isometry. Because the tiling is face-to-face and the vertices are given by $G_{\mathcal{P}}$ we conclude inductively (on the number of tiles) that each vertex of $G_{\mathcal{P}}$ must be fixed. Thus, $\mathcal{P}$ is uniformly stable with respect to ε_0 introduced above, finishing the proof of Theorem 1.5.3.

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