Chapter 2
Basic Properties

2.1 Definition

In the literature, several definitions of elliptically contoured distributions can be found, e.g. see Anderson and Fang (1982b), Fang and Chen (1984), and Sutradhar and Ali (1989). We will use the following definition given in Gupta and Varga (1994b).

Definition 2.1. Let $\mathbf{X}$ be a random matrix of dimensions $p \times n$. Then, $\mathbf{X}$ is said to have a matrix variate elliptically contoured (m.e.c.) distribution if its characteristic function has the form

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr(i\mathbf{T}'\mathbf{M})\psi(tr(\mathbf{T}'\boldsymbol{\Sigma}\mathbf{T}\boldsymbol{\Phi}))$$

with $\mathbf{T}:\ p \times n$, $\mathbf{M}:\ p \times n$, $\boldsymbol{\Sigma}:\ p \times p$, $\boldsymbol{\Phi}:\ n \times n$, $\boldsymbol{\Sigma} \geq \mathbf{0}$, $\boldsymbol{\Phi} \geq \mathbf{0}$, and $\psi : [0,\infty) \to \mathbb{R}$.

This distribution will be denoted by $E_{p,n}(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Phi}, \psi)$.

Remark 2.1. If in Definition 2.1 $n = 1$, we say that $\mathbf{X}$ has a vector variate elliptically contoured distribution. It is also called multivariate elliptical distribution. Then the characteristic function of $\mathbf{X}$ takes on the form

$$\phi_{\mathbf{x}}(\mathbf{t}) = exp(i\mathbf{t}'\mathbf{m})\psi(\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}),$$

where $\mathbf{t}$ and $\mathbf{m}$ are p-dimensional vectors. This definition was given by many authors, e.g. Kelker (1970), Cambanis, Huang and Simons (1981) and Anderson and Fang (1987). In this case, in the notation $E_{p,n}(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Phi}, \psi)$, the index n can be dropped; that is, $E_p(\mathbf{m}, \boldsymbol{\Sigma}, \psi)$ will denote the distribution $E_{p,1}(\mathbf{m}, \boldsymbol{\Sigma}, \psi)$.

Remark 2.2. It follows from Definition 2.1 that $|\psi(t)| \leq 1$ for $t \in \mathbb{R}_0^+$.

The following theorem shows the relationship between matrix variate and vector variate elliptically contoured distributions.

A.K. Gupta et al., *Elliptically Contoured Models in Statistics and Portfolio Theory*,
DOI 10.1007/978-1-4614-8154-6_2,

Theorem 2.1. *Let* $\mathbf{X}$ *be a* $p \times n$ *random matrix and* $\mathbf{x} = vec(\mathbf{X}')$. *Then,* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ *if and only if* $\mathbf{x} \sim E_{pn}(vec(\mathbf{M}'), \Sigma \otimes \Phi, \psi)$.

PROOF: Note that $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ iff

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr(i\mathbf{T}'\mathbf{M})\psi(tr(\mathbf{T}'\Sigma\mathbf{T}\Phi)) \tag{2.1}$$

On the other hand, $\mathbf{x} \sim E_{pn}(vec(\mathbf{M}'), \Sigma \otimes \Phi, \psi)$ iff

$$\phi_{\mathbf{x}}(\mathbf{t}) = exp(i\mathbf{t}'vec(\mathbf{M}'))\psi(\mathbf{t}'(\Sigma \otimes \Phi)\mathbf{t}).$$

Let $\mathbf{t} = vec(\mathbf{T}')$. Then

$$\phi_{\mathbf{x}}(\mathbf{t}) = exp(i(vec(\mathbf{T}'))'vec(\mathbf{M}'))\psi((vec(\mathbf{T}'))'(\Sigma \otimes \Phi)vec(\mathbf{T}')). \tag{2.2}$$

Now, using Theorem 1.17, we can write

$$(vec(\mathbf{T}'))'vec(\mathbf{M}') = tr(\mathbf{T}'\mathbf{M}) \tag{2.3}$$

and

$$(vec(\mathbf{T}'))'(\Sigma \otimes \Phi)vec(\mathbf{T}') = tr(\mathbf{T}'\Sigma\mathbf{T}\Phi). \tag{2.4}$$

From (2.1), (2.2), (2.3), and (2.4) it follows that $\phi_{\mathbf{X}}(\mathbf{T}) = \phi_{\mathbf{x}}(vec(\mathbf{T}'))$. This completes the proof. ■

The next theorem shows that linear functions of a random matrix with m.e.c. distribution have elliptically contoured distributions also.

Theorem 2.2. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. *Assume* $\mathbf{C} : q \times m$, $\mathbf{A} : q \times p$, *and* $\mathbf{B} : n \times m$ *are constant matrices. Then,*

$$\mathbf{AXB} + \mathbf{C} \sim E_{q,m}(\mathbf{AMB} + \mathbf{C}, (\mathbf{A}\Sigma\mathbf{A}') \otimes (\mathbf{B}'\Phi\mathbf{B}), \psi).$$

PROOF: The characteristic function of $\mathbf{Y} = \mathbf{AXB} + \mathbf{C}$ can be written as

$$\begin{aligned}
\phi_{\mathbf{Y}}(\mathbf{T}) &= E(etr(i\mathbf{T}'\mathbf{Y})) \\
&= E(etr(i\mathbf{T}'(\mathbf{AXB} + \mathbf{C}))) \\
&= E(etr(i\mathbf{T}'\mathbf{AXB}))etr(i\mathbf{T}'\mathbf{C}) \\
&= E(etr(i\mathbf{BT}'\mathbf{AX}))etr(i\mathbf{T}'\mathbf{C}) \\
&= \phi_{\mathbf{X}}(\mathbf{A}'\mathbf{TB}')etr(i\mathbf{T}'\mathbf{C}) \\
&= etr(i\mathbf{BT}'\mathbf{AM})\psi(tr(\mathbf{BT}'\mathbf{A}\Sigma\mathbf{A}'\mathbf{TB}'\Phi))etr(i\mathbf{T}'\mathbf{C}) \\
&= etr(i\mathbf{T}'(\mathbf{AMB} + \mathbf{C}))\psi(tr(\mathbf{T}'(\mathbf{A}\Sigma\mathbf{A}')\mathbf{T}(\mathbf{B}'\Phi\mathbf{B}))).
\end{aligned}$$

This is the characteristic function of $E_{q,m}(\mathbf{AMB} + \mathbf{C}, (\mathbf{A}\Sigma\mathbf{A}') \otimes (\mathbf{B}'\Phi\mathbf{B}), \psi)$. ■

Corollary 2.1. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$*, and let* $\Sigma = \mathbf{AA}'$ *and* $\Phi = \mathbf{BB}'$ *be rank factorizations of* Σ *and* Φ*. That is,* $\mathbf{A}$ *is* $p \times p_1$ *and* $\mathbf{B}$ *is* $n \times n_1$ *matrix, where* $p_1 = rk(\Sigma)$*,* $n_1 = rk(\Phi)$*. Then,*

$$\mathbf{A}^-(\mathbf{X} - \mathbf{M})\mathbf{B}'^- \sim E_{p_1,n_1}(\mathbf{0}, \mathbf{I}_{p_1} \otimes \mathbf{I}_{n_1}, \psi)\,.$$

Conversely, if $\mathbf{Y} \sim E_{p_1,n_1}(\mathbf{0}, \mathbf{I}_{p_1} \otimes \mathbf{I}_{n_1}, \psi)$*, then*

$$\mathbf{AYB}' + \mathbf{M} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)\,.$$

with $\Sigma = \mathbf{AA}'$ *and* $\Phi = \mathbf{BB}'$*.*

PROOF: Let $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$, $\Sigma = \mathbf{AA}'$ and $\Phi = \mathbf{BB}'$ be the rank factorizations of Σ and Φ. Then, it follows from Theorem 2.2 that

$$\mathbf{A}^-(\mathbf{X} - \mathbf{M})\mathbf{B}'^- \sim E_{p_1,n_1}(\mathbf{0}, (\mathbf{A}^-\Sigma\mathbf{A}'^-) \otimes (\mathbf{B}^-\Phi\mathbf{B}'^-), \psi)\,.$$

Using Theorem 1.23, we get $\mathbf{A}^-\Sigma\mathbf{A}'^- = \mathbf{I}_{p_1}$ and $\mathbf{B}^-\Phi\mathbf{B}'^- = \mathbf{I}_{n_1}$, which completes the proof of the first part of the theorem. The second part follows directly from Theorem 2.2. ∎

If $\mathbf{x} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$, then it follows from Theorem 2.2 that $\mathbf{Gx} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$ for every $\mathbf{G} \in O(p)$. This gives rise to the following definition.

Definition 2.2. The distribution $E_p(\mathbf{0}, \mathbf{I}_p, \psi)$ is called spherical distribution.

A consequence of the definition of the m.e.c. distribution is that if $\mathbf{X}$ has m.e.c. distribution, then $\mathbf{X}'$ also has m.e.c. distribution. This is shown in the following theorem.

Theorem 2.3. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$*. Then,* $\mathbf{X}' \sim E_{n,p}(\mathbf{M}', \Phi \otimes \Sigma, \psi)$*.*

PROOF: We have

$$\begin{aligned}
\phi_{\mathbf{X}'}(\mathbf{T}) &= E(etr(i\mathbf{T}'\mathbf{X}')) \\
&= E(etr(i\mathbf{XT})) \\
&= E(etr(i\mathbf{TX})) \\
&= etr(i\mathbf{TM})\psi(tr(\mathbf{T}\Sigma\mathbf{T}'\Phi)) \\
&= etr(i\mathbf{T}'\mathbf{M}')\psi(tr(\mathbf{T}'\Phi\mathbf{T}\Sigma)).
\end{aligned}$$

This is the characteristic function of $\mathbf{X}' \sim E_{n,p}(\mathbf{M}', \Phi \otimes \Sigma, \psi)$. ∎

The question arises whether the parameters in the definition of a m.e.c. distribution are uniquely defined. The answer is they are not. To see this assume that a, b, and c are positive constants such that $c = ab$, $\Sigma_2 = a\Sigma_1$, $\Phi_2 = b\Phi_1$,

$\psi_2(z) = \psi_1\left(\frac{1}{c}z\right)$. Then, $E_{p,n}(\mathbf{M}, \Sigma_1 \otimes \Phi_1, \psi_1)$ and $E_{p,n}(\mathbf{M}, \Sigma_2 \otimes \Phi_2, \psi_2)$ define the same m.e.c. distribution. However, this is the only way that two formulae define the same m.e.c. distribution as shown in the following theorem.

Theorem 2.4. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}_1, \Sigma_1 \otimes \Phi_1, \psi_1)$ *and at the same time* $\mathbf{X} \sim E_{p,n}(\mathbf{M}_2, \Sigma_2 \otimes \Phi_2, \psi_2)$. *If* $\mathbf{X}$ *is nondegenerate, then there exist positive constants a, b, and c such that* $c = ab$, *and* $\mathrm{M}_2 = \mathrm{M}_1$, $\Sigma_2 = a\Sigma_1$, $\Phi_2 = b\Phi_1$, *and* $\psi_2(z) = \psi_1\left(\frac{1}{c}z\right)$.

PROOF: The proof follows the lines of Cambanis, Huang and Simons (1981). First of all, note that the distribution of $\mathbf{X}$ is symmetric about $\mathbf{M}_1$ as well as about $\mathbf{M}_2$. Therefore, $\mathbf{M}_1 = \mathbf{M}_2$ must hold. Let $\mathbf{M} = \mathbf{M}_1$. Let us introduce the following notations

$$\begin{aligned} \Sigma_l &= {}_l\sigma_{ij},\ i,j = 1,\ldots,p; \quad l = 1,2, \\ \Phi_l &= {}_l\phi_{ij}, i,j = 1,\ldots,n; \quad l = 1,2. \end{aligned}$$

Let $\mathbf{k}(a)$ denote the p-dimensional vector whose ath element is 1 and all the others are 0 and $\mathbf{l}(b)$ denote the n-dimensional vector whose bth element is 1 and all the others are 0. Since $\mathbf{X}$ is nondegenerate, it must have an element $x_{i_0 j_0}$ which is nondegenerate. Since $x_{i_0 j_0} = \mathbf{k}'(i_0)\mathbf{X}\mathbf{l}(j_0)$, from Theorem 2.2 we get

$$x_{i_0 j_0} \sim E_1(m_{i_0 j_0},\ {}_1\sigma_{i_0 i_0}\ {}_1\phi_{j_0 j_0}, \psi_1)$$

and

$$x_{i_0 j_0} \sim E_1(m_{i_0 j_0},\ {}_2\sigma_{i_0 i_0}\ {}_2\phi_{j_0 j_0}, \psi_2).$$

Therefore, the characteristic function of $x_{i_0 j_0} - m_{i_0 j_0}$ is

$$\begin{aligned} \phi(t) &= \psi_1(t^2\ {}_1\sigma_{i_0 i_0}\ {}_1\phi_{j_0 j_0}) \\ &= \psi_2(t^2\ {}_2\sigma_{i_0 i_0}\ {}_2\phi_{j_0 j_0}) \end{aligned} \tag{2.5}$$

with $t \in \mathbb{R}$.

Since, ${}_l\sigma_{i_0 i_0}$ and ${}_l\phi_{j_0 j_0}$ $(l = 1,2)$ are diagonal elements of positive semidefinite matrices, they cannot be negative, and since $x_{i_0 j_0}$ is nondegenerate, they cannot be zero either. So, we can define

$$c = \frac{{}_2\sigma_{i_0 i_0}\,{}_2\phi_{j_0 j_0}}{{}_1\sigma_{i_0 i_0}\,{}_1\phi_{j_0 j_0}}.$$

Then, $c > 0$ and $\psi_2(z) = \psi_1\left(\frac{1}{c}z\right)$ for $z \in [0, \infty)$.

We claim that $\Sigma_2 \otimes \Phi_2 = c(\Sigma_1 \otimes \Phi_1)$. Suppose this is not the case. Then, there exists $\mathbf{t} \in \mathbb{R}^{pn}$ such that $\mathbf{t}'(\Sigma_2 \otimes \Phi_2)\mathbf{t} \neq c\mathbf{t}'(\Sigma_1 \otimes \Phi_1)\mathbf{t}$. From Theorem 1.17, it follows that there exists $\mathbf{T}_0 \in \mathbb{R}^{p\times n}$ such that $tr(\mathbf{T}_0'\Sigma_2\mathbf{T}_0\Phi_2) \neq c\,tr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\Phi_1)$.

Define $\mathbf{T} = u\mathbf{T}_0$, $u \in I\!R$. Then, the characteristic function of $\mathbf{X} - \mathbf{M}$ at $u\mathbf{T}_0$ is

$$\psi_1(utr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1)) = \psi_2(uctr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1)).$$

On the other hand, the characteristic function of $\mathbf{X} - \mathbf{M}$ at $u\mathbf{T}_0$ can be expressed as $\psi_2(utr(\mathbf{T}_0'\Sigma_2\mathbf{T}_0\boldsymbol{\Phi}_2))$. So

$$\psi_2(uctr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1)) = \psi_2(utr(\mathbf{T}_0'\Sigma_2\mathbf{T}_0\boldsymbol{\Phi}_2)). \tag{2.6}$$

If $tr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1) = 0$ or $tr(\mathbf{T}_0'\Sigma_2\mathbf{T}_0\boldsymbol{\Phi}_2) = 0$, then from (2.6) we get that $\psi(u) = 0$ for every $u \in I\!R$. However, this is impossible since $\mathbf{X}$ is nondegenerate.

If $tr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1) \neq 0$ and $tr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1) \neq 0$, then define

$$d = c\frac{tr(\mathbf{T}_0'\Sigma_1\mathbf{T}_0\boldsymbol{\Phi}_1)}{tr(\mathbf{T}_0'\Sigma_2\mathbf{T}_0\boldsymbol{\Phi}_2)}.$$

Then, $d \neq 0$, $d \neq 1$, and from (2.6) we get $\psi_2(u) = \psi_2(du)$. By induction, we get

$$\psi_2(u) = \psi_2(d^n u) \quad \text{and} \quad \psi_2(u) = \psi_2\left(\left(\frac{1}{d}\right)^n u\right), \; n = 1, 2, \ldots.$$

Now either $d^n \to 0$ or $\left(\frac{1}{d}\right)^n \to 0$, and from the continuity of the characteristic function and the fact that $\psi_2(0) = 1$ it follows that $\psi_2(u) = 0$ for every $u \in I\!R$. However, this is impossible. So, we must have $\Sigma_2 \otimes \boldsymbol{\Phi}_2 = c(\Sigma_1 \otimes \boldsymbol{\Phi}_1)$. From Theorem 1.16 it follows that there exist $a > 0$ and $b > 0$ such that $\Sigma_2 = a\Sigma_1$, $\boldsymbol{\Phi}_2 = b\boldsymbol{\Phi}_1$, and $ab = c$. This completes the proof. ■

An important subclass of the class of the m.e.c. distributions is the class of matrix variate normal distributions.

Definition 2.3. The $p \times n$ random matrix $\mathbf{X}$ is said to have a matrix variate normal distribution if its characteristic function has the form

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr(i\mathbf{T}'\mathbf{M})etr\left(-\frac{1}{2}\mathbf{T}'\boldsymbol{\Sigma}\mathbf{T}\boldsymbol{\Phi}\right),$$

with $\mathbf{T} : p \times n$, $\mathbf{M} : p \times n$, $\boldsymbol{\Sigma} : p \times p$, $\boldsymbol{\Phi} : n \times n$, $\boldsymbol{\Sigma} \geq \mathbf{0}$, $\boldsymbol{\Phi} \geq \mathbf{0}$. This distribution is denoted by $N_{p,n}(\mathbf{M}, \boldsymbol{\Sigma} \otimes \boldsymbol{\Phi})$.

The next theorem shows that the matrix variate normal distribution can be used to represent samples taken from multivariate normal distributions.

Theorem 2.5. *Let* $\mathbf{X} \sim N_{p,n}(\mu\mathbf{e}_n', \Sigma \otimes \mathbf{I}_n)$, *where* $\mu \in I\!R^p$. *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *be the columns of* $\mathbf{X}$. *Then,* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *are independent identically distributed random vectors with common distribution* $N_p(\mu, \Sigma)$.

PROOF: Let $\mathbf{T} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n)$ be $p \times n$ matrix. Then

$$\begin{aligned}
\phi_{\mathbf{X}}(\mathbf{T}) &= etr\left(i \begin{pmatrix} \mathbf{t}_1' \\ \mathbf{t}_2' \\ \vdots \\ \mathbf{t}_n' \end{pmatrix} (\mu, \mu, \ldots, \mu) \right) etr\left(-\frac{1}{2} \begin{pmatrix} \mathbf{t}_1' \\ \mathbf{t}_2' \\ \vdots \\ \mathbf{t}_n' \end{pmatrix} \Sigma (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n) \right) \\
&= exp\left(i \sum_{j=1}^{n} \mathbf{t}_j' \mu \right) exp\left(-\frac{1}{2} \sum_{j=1}^{n} \mathbf{t}_j' \Sigma \mathbf{t}_j \right) \neg \\
&= \prod_{j=1}^{n} exp\left(i\mathbf{t}_j' \mu - \frac{1}{2} \mathbf{t}_j' \Sigma \mathbf{t}_j \right),
\end{aligned}$$

which shows that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are independent, each with distribution $N_p(\mu, \Sigma)$.

■

2.2 Probability Density Function

If $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ defines an absolutely continuous elliptically contoured distribution, Σ and Φ must be positive definite. Assume this is not the case. For example, $\Sigma \geq \mathbf{0}$ but Σ is not positive definite. Then, from Theorem 1.7, it follows that $\Sigma = \mathbf{GDG}'$ where $\mathbf{G} \in O(n)$, and $\mathbf{D}$ is diagonal and $d_{11} = 0$. Let $\mathbf{Y} = \mathbf{G}'(\mathbf{X} - \mathbf{M})$. Then, $\mathbf{Y} \sim E_{p,n}(\mathbf{0}, \mathbf{D} \otimes \Phi, \psi)$, and the distribution of $\mathbf{Y}$ is also absolutely continuous. On the other hand, $y_{11} \sim E_1(0, 0, \psi)$ so y_{11} is degenerate. But the marginal of an absolutely continuous distribution cannot be degenerate. Hence, we get a contradiction. So, $\Sigma > \mathbf{0}$ and $\Phi > \mathbf{0}$ must hold when the m.e.c. distribution is absolutely continuous.

The probability density function (p.d.f.) of a m.e.c. distribution is of a special form as the following theorem shows.

Theorem 2.6. *Let $\mathbf{X}$ be a $p \times n$ dimensional random matrix whose distribution is absolutely continuous. Then, $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ if and only if the p.d.f. of $\mathbf{X}$ has the form*

$$f(\mathbf{X}) = |\Sigma|^{-\frac{n}{2}} |\Phi|^{-\frac{p}{2}} h(tr((\mathbf{X} - \mathbf{M})' \Sigma^{-1} (\mathbf{X} - \mathbf{M}) \Phi^{-1})), \tag{2.7}$$

where h and ψ determine each other for specified p and n.

PROOF: I. First, we prove that if $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ and $E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ is absolutely continuous, then the p.d.f. of $\mathbf{X}$ has the form (2.7).

Step 1. Assume that $\mathbf{M} = \mathbf{0}$ and $\Sigma \otimes \Phi = \mathbf{I}_{pn}$. Then, $\mathbf{X} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi)$. We want to show that the p.d.f. of $\mathbf{X}$ depends on $\mathbf{X}$ only through $tr(\mathbf{X}'\mathbf{X})$. Let $\mathbf{x} = vec(\mathbf{X}')$. From Theorem 2.1 we know that $\mathbf{x} \sim E_{pn}(\mathbf{0}, \mathbf{I}_{pn}, \psi)$. Let $\mathbf{H} \in O(pn)$, then, in view of Theorem 2.2,

$$\mathbf{Hx} \sim E_{pn}(\mathbf{0}, \mathbf{HH}', \psi) = E_{pn}(\mathbf{0}, \mathbf{I}_{pn}, \psi).$$

Thus, the distribution of $\mathbf{x}$ is invariant under orthogonal transformation. Therefore, using Theorem 1.11, we conclude that the p.d.f. of $\mathbf{x}$ depends on $\mathbf{x}$ only through $\mathbf{x}'\mathbf{x}$. Let us denote the p.d.f. of $\mathbf{x}$ by $f_1(\mathbf{x})$. We have $f_1(\mathbf{x}) = h(\mathbf{x}'\mathbf{x})$. Clearly, h only depends on p, n, and ψ. It follows from Theorem 1.17, that $\mathbf{x}'\mathbf{x} = tr(\mathbf{X}'\mathbf{X})$. Thus, denoting the p.d.f. of $\mathbf{X}$ by $f(\mathbf{X})$, we get $f(\mathbf{X}) = h(tr(\mathbf{X}'\mathbf{X}))$.

Step 2. Now, let $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. From Corollary 2.1, it follows that $\mathbf{Y} = \Sigma^{-\frac{1}{2}}(\mathbf{X} - \mathbf{M})\Phi^{-\frac{1}{2}} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi)$. Therefore, if $g(\mathbf{Y})$ is the p.d.f. of $\mathbf{Y}$, then $g(\mathbf{Y}) = h(tr(\mathbf{Y}'\mathbf{Y}))$. The Jacobian of the transformation $\mathbf{Y} \to \mathbf{X}$ is $|\Sigma^{-\frac{1}{2}}|^n |\Phi^{-\frac{1}{2}}|^p$. So the p.d.f. of $\mathbf{X}$ is

$$\begin{aligned} f(\mathbf{X}) &= h\left(tr\left(\Phi^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})'\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}(\mathbf{X}-\mathbf{M})\Phi^{-\frac{1}{2}}\right)\right)|\Sigma|^{-\frac{n}{2}}|\Phi|^{-\frac{p}{2}} \\ &= |\Sigma|^{-\frac{n}{2}}|\Phi|^{-\frac{p}{2}} h\left(tr\left((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1}\right)\right). \end{aligned}$$

II. Next, we show that if a random matrix $\mathbf{X}$ has the p.d.f. of the form (2.7), then its distribution is elliptically contoured. That is, assume that the $p \times n$ random matrix $\mathbf{X}$ has the p.d.f.

$$f(\mathbf{X}) = |\Sigma|^{-\frac{n}{2}}|\Phi|^{-\frac{p}{2}} h\left(tr\left((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1}\right)\right),$$

then we want to show that $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. Let $\mathbf{Y} = \Sigma^{-\frac{1}{2}}(\mathbf{X} - \mathbf{M})\Phi^{-\frac{1}{2}}$. Then, the p.d.f. of $\mathbf{Y}$ is $g(\mathbf{Y}) = h(tr(\mathbf{Y}'\mathbf{Y}))$. Let $\mathbf{y} = vec(\mathbf{Y}')$. Then, the p.d.f. of $\mathbf{y}$ is $g_1(\mathbf{y}) = h(\mathbf{y}'\mathbf{y})$. The characteristic function of $\mathbf{y}$ is

$$\phi_{\mathbf{y}}(\mathbf{t}) = \int_{\mathbb{R}^{pn}} \exp(i\mathbf{t}'\mathbf{y})h(\mathbf{y}'\mathbf{y})d\mathbf{y},$$

where $\mathbf{t} \in \mathbb{R}^{pn}$.

Next, we prove that if $\mathbf{t}_1$ and $\mathbf{t}_2$ are vectors of dimension pn such that $\mathbf{t}_1'\mathbf{t}_1 = \mathbf{t}_2'\mathbf{t}_2$, then $\phi_{\mathbf{y}}(\mathbf{t}_1) = \phi_{\mathbf{y}}(\mathbf{t}_2)$. Using Theorem 1.11, we see that there exists $\mathbf{H} \in O(pn)$, such that $\mathbf{t}_1'\mathbf{H} = \mathbf{t}_2'$. Therefore,

$$\begin{aligned} \phi_{\mathbf{y}}(\mathbf{t}_2) &= \int_{\mathbb{R}^{pn}} \exp(i\mathbf{t}_2'\mathbf{y})h(\mathbf{y}'\mathbf{y})d\mathbf{y} \\ &= \int_{\mathbb{R}^{pn}} \exp(i\mathbf{t}_1'\mathbf{H}\mathbf{y})h(\mathbf{y}'\mathbf{y})d\mathbf{y}. \end{aligned}$$

Let $\mathbf{z} = \mathbf{H}\mathbf{y}$. The Jacobian of the transformation $\mathbf{y} \to \mathbf{z}$ is $|\mathbf{H}'|^{pn} = 1$. So

$$\begin{aligned} \int_{\mathbb{R}^{pn}} \exp(i\mathbf{t}_1'\mathbf{H}\mathbf{y})h(\mathbf{y}'\mathbf{y})d\mathbf{y} &= \int_{\mathbb{R}^{pn}} \exp(i\mathbf{t}_1'\mathbf{z})h(\mathbf{z}'\mathbf{H}\mathbf{H}'\mathbf{z})d\mathbf{z} \\ &= \int_{\mathbb{R}^{pn}} \exp(i\mathbf{t}_1'\mathbf{z})h(\mathbf{z}'\mathbf{z})d\mathbf{z} \\ &= \phi_{\mathbf{y}}(\mathbf{t}_1). \end{aligned}$$

This means that $\phi_{\mathbf{y}}(\mathbf{t}_1) = \phi_{\mathbf{y}}(\mathbf{t}_2)$. Therefore, $\phi_{\mathbf{y}}(\mathbf{t})$ is a function of $\mathbf{t}'\mathbf{t}$, which implies that $\phi_{\mathbf{Y}}(\mathbf{T})$ is a function of $tr(\mathbf{T}'\mathbf{T})$. Therefore, there exists a function ψ such that $\phi_{\mathbf{Y}}(\mathbf{T}) = \psi(tr(\mathbf{T}'\mathbf{T}))$. That is, $\mathbf{y} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi)$. Using Corollary 2.1, we get $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. ■

Next we prove a lemma which will be useful for further study of m.e.c. distributions.

Lemma 2.1. *Let f be a function $f : A \times \mathbb{R}^p \to \mathbb{R}^q$, where A can be any set. Assume there exists a function $g : A \times \mathbb{R} \to \mathbb{R}^q$ such that $f(a, \mathbf{x}) = g(a, \mathbf{x}'\mathbf{x})$ for any $a \in A$ and $\mathbf{x} \in \mathbb{R}^p$. Then, we have*

$$\int_{\mathbb{R}^p} f(a, \mathbf{x}) d\mathbf{x} = \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty r^{p-1} g(a, r^2) dr$$

for any $a \in A$.

PROOF: Let $\mathbf{x} = (x_1, x_2, \ldots, x_p)'$ and introduce the polar coordinates

$$\begin{aligned}
x_1 &= r sin\theta_1 sin\theta_2 \ldots sin\theta_{p-2} sin\theta_{p-1} \\
x_2 &= r sin\theta_1 sin\theta_2 \ldots sin\theta_{p-2} cos\theta_{p-1} \\
x_3 &= r sin\theta_1 sin\theta_2 \ldots cos\theta_{p-2} \\
&\vdots \\
x_{p-1} &= r sin\theta_1 cos\theta_2 \\
x_p &= r cos\theta_1 \,,
\end{aligned}$$

where $r > 0$, $0 < \theta_i < \pi$, $i = 1, 2, \ldots, p-2$, and $0 < \theta_{p-1} < 2\pi$. Then, the Jacobian of the transformation $(x_1, x_2, \ldots, x_p) \to (r, \theta_1, \theta_2, \ldots, \theta_{p-1})$ is

$$r^{p-1} sin^{p-2}\theta_1 sin^{p-3}\theta_2 \ldots sin\theta_{p-2}.$$

We also have $\mathbf{x}'\mathbf{x} = r^2$. Thus,

$$\begin{aligned}
&\int_{\mathbf{x} \in \mathbb{R}^p} f(a, \mathbf{x}) d\mathbf{x} = \int_{\mathbf{x} \in \mathbb{R}^p} g(a, \mathbf{x}'\mathbf{x}) d\mathbf{x} \\
&= \int_0^\infty \int_0^\pi \int_0^\pi \ldots \int_0^{2\pi} g(a, r^2) r^{p-1} sin^{p-2}\theta_1 sin^{p-3}\theta_2 \ldots sin\theta_{p-2} d\theta_{p-1} \ldots d\theta_2 d\theta_1 dr \\
&= \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty r^{p-1} g(a, r^2) dr.
\end{aligned}$$
■

The next theorem is due to Fang, Kotz, and Ng (1990).

Theorem 2.7. *Let $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a measurable function. Then, there exists a constant c such that*

$$cg(tr(\mathbf{X}'\mathbf{X})), \quad \mathbf{X} \in \mathbb{R}^{p\times n}$$

is the p.d.f. of the $p \times n$ random matrix $\mathbf{X}$ if and only if

$$0 < \int_0^\infty r^{pn-1} g(r^2) dr < \infty.$$

Moreover, the relationship between g and c is given by

$$c = \frac{\Gamma\left(\frac{pn}{2}\right)}{2\pi^{\frac{pn}{2}} \int_0^\infty r^{np-1} g(r^2) dr}.$$

PROOF: By definition, $cg(tr(\mathbf{X}'\mathbf{X})), \mathbf{X} \in \mathbb{R}^{p\times n}$, is the p.d.f. of a $p \times n$ random matrix $\mathbf{X}$ iff $cg(\mathbf{y}'\mathbf{y}), \mathbf{y} \in \mathbb{R}^{pn}$ is the p.d.f. of a pn-dimensional random vector $\mathbf{y}$. On the other hand, $cg(\mathbf{y}'\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^{pn}$ is the p.d.f. of a pn-dimensional random vector $\mathbf{y}$ iff

$$\int_{\mathbb{R}^{pn}} cg(\mathbf{y}'\mathbf{y}) d\mathbf{y} = 1\,.$$

From Lemma 2.1, we get

$$\int_{\mathbb{R}^{pn}} cg(\mathbf{y}'\mathbf{y}) d\mathbf{y} = c \frac{2\pi^{\frac{pn}{2}}}{\Gamma\left(\frac{pn}{2}\right)} \int_0^\infty r^{pn-1} g(r^2) dr.$$

Hence, we must have

$$0 \leq \int_0^\infty r^{pn-1} g(r^2) dr < \infty.$$

and

$$c = \frac{\Gamma\left(\frac{pn}{2}\right)}{2\pi^{\frac{pn}{2}} \int_0^\infty r^{np-1} g(r^2) dr}. \qquad \blacksquare$$

2.3 Marginal Distributions

Using Theorem 2.2, we can derive the marginal distributions of a m.e.c. distribution.

Theorem 2.8. *Let $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$, and partition $\mathbf{X}$, $\mathbf{M}$, and Σ as*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{pmatrix}, \quad \text{and} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ *is* $q \times n$, $\mathbf{M}_1$ *is* $q \times n$, *and* $\boldsymbol{\Sigma}_{11}$ *is* $q \times q$, $1 \leq q < p$. *Then,*

$$\mathbf{X}_1 \sim E_{q,n}(\mathbf{M}_1, \boldsymbol{\Sigma}_{11} \otimes \boldsymbol{\Phi}, \psi).$$

PROOF: Let $\mathbf{A} = (\mathbf{I}_q, \mathbf{0})$ be of dimensions $q \times p$. Then, $\mathbf{AX} = \mathbf{X}_1$, and from Theorem 2.2, we obtain $\mathbf{X}_1 \sim E_{q,n}\left((\mathbf{I}_q, \mathbf{0})\mathbf{M}, \left((\mathbf{I}_q, \mathbf{0})\Sigma \begin{pmatrix} \mathbf{I}_q \\ \mathbf{0} \end{pmatrix}\right) \otimes \boldsymbol{\Phi}, \psi\right)$, i.e. $\mathbf{X}_1 \sim E_{q,n}(\mathbf{M}_1, \boldsymbol{\Sigma}_{11} \otimes \boldsymbol{\Phi}, \psi)$. ■

If we partition $\mathbf{X}$ vertically, we obtain the following result.

Theorem 2.9. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \boldsymbol{\Phi}, \psi)$, *and partition* $\mathbf{X}$, $\mathbf{M}$, *and* $\boldsymbol{\Phi}$ *as*

$$\mathbf{X} = \left(\mathbf{X}_1, \mathbf{X}_2\right), \quad \mathbf{M} = \left(\mathbf{M}_1, \mathbf{M}_2\right), \quad \textit{and} \quad \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ *is* $p \times m$, $\mathbf{M}_1$ *is* $p \times m$, *and* $\boldsymbol{\Phi}_{11}$ *is* $m \times m$, $1 \leq m < n$. *Then,*

$$\mathbf{X}_1 \sim E_{p,m}(\mathbf{M}_1, \Sigma \otimes \boldsymbol{\Phi}_{11}, \psi). \tag{2.8}$$

PROOF: From Theorem 2.3, it follows that

$$\mathbf{X}' = \begin{pmatrix} \mathbf{X}'_1 \\ \mathbf{X}'_2 \end{pmatrix} \sim E_{n,p}\left(\begin{pmatrix} \mathbf{M}'_1 \\ \mathbf{M}'_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{21} & \boldsymbol{\Phi}_{22} \end{pmatrix} \otimes \Sigma, \psi\right)$$

Then (2.8) follows directly from Theorem 2.8. ■

Theorem 2.10. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \boldsymbol{\Phi}, \psi)$, *then* $x_{ij} \sim E_1(m_{ij}, \sigma_{ii}\phi_{jj}, \psi)$.

PROOF: The result follows from Theorems 2.8 and 2.9. ■

Remark 2.3. It follows from Theorems 2.8 and 2.9 that if $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \boldsymbol{\Phi}, \psi)$ and $\mathbf{Y}$ is a $q \times m$ submatrix of $\mathbf{X}$, then $\mathbf{Y}$ also has m.e.c. distribution; $\mathbf{Y} \sim E_{q,m}(\mathbf{M}^*, \Sigma^* \otimes \boldsymbol{\Phi}^*, \psi)$.

2.4 Expected Value and Covariance

In this section, the first two moments of a m.e.c. distribution will be derived. In Chap. 3, moments of higher orders will also be obtained.

Theorem 2.11. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \boldsymbol{\Phi}, \psi)$.

(a) If $\mathbf{X}$ *has finite first moment, then* $E(\mathbf{X}) = \mathbf{M}$.
(b) If $\mathbf{X}$ *has finite second moment, then* $Cov(\mathbf{X}) = c\Sigma \otimes \boldsymbol{\Phi}$, *where* $c = -2\psi'(0)$.

PROOF: Step 1. First, let us assume $\mathbf{M} = \mathbf{0}$ and $\Sigma \otimes \Phi = \mathbf{I}_{pn}$. Then, $\mathbf{X} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi)$.

(a) In view of Theorem 2.2, we have

$$(-\mathbf{I}_p)\mathbf{X} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi).$$

Therefore, $E(\mathbf{X}) = E(-\mathbf{X})$, and $E(\mathbf{X}) = \mathbf{0}$.

(b) Let $\mathbf{x} = vec(\mathbf{X}')$. Then $\mathbf{x} \sim E_{pn}(\mathbf{0}, \mathbf{I}_{pn}, \psi)$. The characteristic function of $\mathbf{x}$ is $\phi_{\mathbf{x}}(\mathbf{t}) = \psi(\mathbf{t}'\mathbf{t})$, where $\mathbf{t} = (t_1, \ldots, t_{pn})'$. Then,

$$\frac{\partial \phi_{\mathbf{x}}(\mathbf{t})}{\partial t_i} = \frac{\partial \psi\left(\sum_{l=1}^{pn} t_l^2\right)}{\partial t_i} = 2t_i \psi'\left(\sum_{l=1}^{pn} t_l^2\right).$$

So,

$$\frac{\partial^2 \phi_{\mathbf{x}}(\mathbf{t})}{\partial t_i^2} = 2\psi'\left(\sum_{l=1}^{pn} t_l^2\right) + 4t_i^2 \psi''\left(\sum_{l=1}^{pn} t_l^2\right).$$

and if $i \neq j$, then

$$\frac{\partial^2 \phi_{\mathbf{x}}(\mathbf{t})}{\partial t_j \partial t_i} = 4t_i t_j \psi''\left(\sum_{l=1}^{pn} t_l^2\right).$$

Therefore,

$$\left.\frac{\partial^2 \phi_{\mathbf{x}}(\mathbf{t})}{\partial t_i^2}\right|_{\mathbf{t}=\mathbf{0}} = 2\psi'(0) \quad \text{and} \quad \left.\frac{\partial^2 \phi_{\mathbf{x}}(\mathbf{t})}{\partial t_i \partial t_j}\right|_{\mathbf{t}=\mathbf{0}} = 0 \quad \text{if} \quad i \neq j.$$

Thus, $Cov(\mathbf{x}) = -2\psi'(0)\mathbf{I}_{pn}$.

Step 2. Now, let $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. Let $\Sigma = \mathbf{A}\mathbf{A}'$ and $\Phi = \mathbf{B}\mathbf{B}'$ be the rank factorizations of Σ and Φ. Then, from Corollary 2.1, it follows that $\mathbf{Y} = \mathbf{A}^{-}(\mathbf{X} - \mathbf{M})\Phi^{'-} \sim E_{p_1,n_1}(\mathbf{0}, \mathbf{I}_{p_1} \otimes \mathbf{I}_{n_1}, \psi)$ and $\mathbf{X} = \mathbf{A}\mathbf{Y}\mathbf{B}'$. Using Step 1 ,we get the following results:

(a) $E(\mathbf{Y}) = \mathbf{0}$. Hence $E(\mathbf{X}) = \mathbf{A}\mathbf{0}\mathbf{B}' + \mathbf{M} = \mathbf{M}$.

(b) Let $\mathbf{x} = vec(\mathbf{X}')$, $\mathbf{y} = vec(\mathbf{Y}')$, and $\mu = vec(\mathbf{M}')$. Then $\mathbf{x} = (\mathbf{A} \otimes \mathbf{B})\mathbf{y} + \mu$, and $Cov(\mathbf{y}) = -2\psi'(0)\mathbf{I}_{pn}$ and so

$$\begin{aligned} Cov(\mathbf{x}) &= -2\psi'(0)(\mathbf{A} \otimes \mathbf{B})\mathbf{I}_{pn}(\mathbf{A}' \otimes \mathbf{B}') \\ &= -2\psi'(0)(\mathbf{A}\mathbf{A}') \otimes (\mathbf{B}\mathbf{B}') \\ &= -2\psi'(0)\Sigma \otimes \Phi. \end{aligned}$$

■

Corollary 2.2. *With the conditions of Theorem 2.11, the ith ($i = 1,\dots,p$) column of the matrix* $\mathbf{X}$ *has the covariance matrix* $\phi_{ii}\Sigma$ *and the jth row* ($j = 1,\dots,n$) *has the covariance matrix* $\sigma_{jj}\Phi$.

Corollary 2.3. *With the conditions of Theorem 2.11,*

$$Corr(x_{ij}, x_{kl}) = \frac{\sigma_{ik}\phi_{jl}}{\sqrt{\sigma_{ii}\sigma_{kk}\phi_{jj}\phi_{ll}}},$$

that is, the correlations between two elements of the matrix $\mathbf{X}$*, depend only on* Σ *and* Φ *but not on* ψ.

PROOF: From Theorem 2.11, we get $Cov(x_{ij}, x_{kl}) = c\sigma_{ik}\phi_{jl}$, $Var(x_{ij}) = c\sigma_{ii}\phi_{jj}$, and $Var(x_{kl}) = c\sigma_{kk}\phi_{ll}$, where $c = -2\psi'(0)$. Therefore

$$\begin{aligned} Corr(x_{ij}, x_{kl}) &= \frac{c\sigma_{ik}\phi_{jl}}{\sqrt{c^2\sigma_{ii}\sigma_{kk}\phi_{jj}\phi_{ll}}} \\ &= \frac{\sigma_{ik}\phi_{jl}}{\sqrt{\sigma_{ii}\sigma_{kk}\phi_{jj}\phi_{ll}}}. \end{aligned}$$ ■

2.5 Stochastic Representation

In Cambanis, Huang, and Simons (1981) the stochastic representation of vector variate elliptically contoured distribution was obtained using a result of Schoenberg (1938). This result was extended to m.e.c. distributions by Anderson and Fang (1982b). Shoenberg's result is given in the next theorem.

Theorem 2.12. *Let* ψ *be a real function* $\psi : [0,\infty) \to I\!R$. *Then,* $\psi(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in I\!R^k$ *is the characteristic function of a k-dimensional random variable* $\mathbf{x}$*, if and only if* $\psi(u) = \int_0^\infty \Omega_k(r^2u)dF(r)$, $u \geq 0$, *where* F *is a distribution function on* $[0,\infty)$ *and* $\Omega_k(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in I\!R^k$ *is the characteristic function of the k-dimensional random variable* $\mathbf{u}_k$ *which is uniformly distributed on the unit sphere in* $I\!R^k$. *Moreover,* $F(r)$ *is the distribution function of* $r = (\mathbf{x}'\mathbf{x})^{\frac{1}{2}}$.

PROOF: Let us denote the unit sphere in $I\!R^k$ by S_k:

$$S_k = \{\mathbf{x}|\mathbf{x} \in I\!R^k; \mathbf{x}'\mathbf{x} = 1\},$$

and let A_k be the surface area of S_k i.e.

$$A_k = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}.$$

First, assume $\psi(u) = \int_0^\infty \Omega_k(r^2 u)dF(r)$. Let r be a random variable with distribution function $F(r)$, and let $\mathbf{u}_k$ be independent of r and uniformly distributed on S_k. Define $\mathbf{x} = r\mathbf{u}_k$. Then, the characteristic function of $\mathbf{x}$ is

$$\begin{aligned}
\phi_{\mathbf{x}}(\mathbf{t}) &= E(exp(i\mathbf{t}'\mathbf{x})) \\
&= E(exp(i\mathbf{t}'r\mathbf{u}_k)) \\
&= E\{E(exp(i\mathbf{t}'r\mathbf{u}_k)|r)\} \\
&= \int_0^\infty E(exp(i\mathbf{t}'r\mathbf{u}_k)|r=y)dF(y) \\
&= \int_0^\infty \phi_{\mathbf{u}_k}(y\mathbf{t})dF(y) \\
&= \int_0^\infty \Omega_k(y^2\mathbf{t}'\mathbf{t})dF(y).
\end{aligned}$$

Therefore, $\psi(\mathbf{t}'\mathbf{t}) = \int_0^\infty \Omega_k(y^2\mathbf{t}'\mathbf{t})dF(y)$ is indeed the characteristic function of the k-dimensional random vector $\mathbf{x}$. Moreover,

$$F(y) = P(r \le y) = P((r^2)^{\frac{1}{2}} \le y) = P(((r\mathbf{u}_k)'(r\mathbf{u}_k))^{\frac{1}{2}} \le y) = P((\mathbf{x}'\mathbf{x})^{\frac{1}{2}} \le y).$$

Conversely, assume $\psi(\mathbf{t}'\mathbf{t})$ is the characteristic function of a k-dimensional random vector $\mathbf{x}$. Let $G(\mathbf{x})$ be the distribution function of $\mathbf{x}$. Let $d\omega_k(\mathbf{t})$ denote the integration on S_k. We have $\psi(u) = \psi(u\mathbf{t}'\mathbf{t})$ for $\mathbf{t}'\mathbf{t} = 1$, and therefore we can write

$$\begin{aligned}
\psi(u) &= \frac{1}{A_k}\int_{S_k} \psi(u\mathbf{t}'\mathbf{t})d\omega_k(\mathbf{t}) \\
&= \frac{1}{A_k}\int_{S_k} \phi_{\mathbf{x}}(\sqrt{u}\mathbf{t})d\omega_k(\mathbf{t}) \\
&= \frac{1}{A_k}\int_{S_k}\int_{\mathbb{R}^m} \exp(i\sqrt{u}\mathbf{t}'\mathbf{x})dG(\mathbf{x})d\omega_k(\mathbf{t}) \\
&= \int_{\mathbb{R}^m}\left(\frac{1}{A_k}\int_{S_k} \exp(i\sqrt{u}\mathbf{x}'\mathbf{t})d\omega_k(\mathbf{t})\right)dG(\mathbf{x}) \\
&= \int_{\mathbb{R}^m} \Omega_k\left((\sqrt{u}\mathbf{x})'(\sqrt{u}\mathbf{x})\right)dG(\mathbf{x}) \\
&= \int_{\mathbb{R}^m} \Omega_k(u\mathbf{x}'\mathbf{x})dG(\mathbf{x}) \\
&= \int_0^\infty \Omega_k(uy^2)dF(y),
\end{aligned}$$

where $F(y) = P((\mathbf{x}'\mathbf{x})^{\frac{1}{2}} \le y)$. ■

Now, we can derive the stochastic representation of a m.e.c. distribution.

Theorem 2.13. *Let* $\mathbf{X}$ *be a* $p \times n$ *random matrix. Let* $\mathbf{M}$ *be* $p \times n$, Σ *be* $p \times p$, *and* Φ *be* $n \times n$ *constant matrices,* $\Sigma \geq \mathbf{0}$, $\Phi \geq \mathbf{0}$, $rk(\Sigma) = p_1$, $rk(\Phi) = n_1$. *Then,*

$$\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi) \tag{2.9}$$

if and only if

$$\mathbf{X} \approx \mathbf{M} + r\mathbf{AUB}', \tag{2.10}$$

where $\mathbf{U}$ *is* $p_1 \times n_1$ *and* $vec(\mathbf{U}')$ *is uniformly distributed on* $S_{p_1 n_1}$, r *is a nonnegative random variable,* r *and* $\mathbf{U}$ *are independent,* $\Sigma = \mathbf{AA}'$, *and* $\Phi = \mathbf{BB}'$ *are rank factorizations of* Σ *and* Φ. *Moreover,* $\psi(u) = \int_0^\infty \Omega_{p_1 n_1}(r^2 u) dF(r)$, $u \geq 0$, *where* $\Omega_{p_1 n_1}(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{p_1 n_1}$ *denotes the characteristic function of* $vec(\mathbf{U}')$, *and* $F(r)$ *denotes the distribution function of* r. *The expression,* $\mathbf{M} + r\mathbf{AUB}'$, *is called the stochastic representation of* $\mathbf{X}$.

PROOF: First, assume $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. Then, it follows from Corollary 2.1, that $\mathbf{Y} = \mathbf{A}^-(\mathbf{X} - \mathbf{M})\mathbf{B}'^- \sim E_{p_1,n_1}(\mathbf{0}, \mathbf{I}_{p_1} \otimes \mathbf{I}_{n_1}, \psi)$. Thus,

$$\mathbf{y} = vec(\mathbf{Y}') \sim E_{p_1 n_1}(\mathbf{0}, \mathbf{I}_{p_1 n_1}, \psi).$$

So, $\psi(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{p_1 n_1}$ is a characteristic function and from Theorem 2.12, we get

$$\psi(u) = \int_0^\infty \Omega_{p_1 n_1}(y^2 u) dF(y), \quad u \geq 0,$$

which means that $\mathbf{y} \approx r\mathbf{u}$, where r is nonnegative with distribution function $F(y)$, $\mathbf{u}$ is uniformly distributed on $S_{p_1 n_1}$, and r and $\mathbf{u}$ are independent. Therefore, we can write $\mathbf{y} \approx r\mathbf{u}$, where $\mathbf{u} = vec(\mathbf{U}')$. Now, using Corollary 2.1 again, we get

$$\mathbf{X} \approx \mathbf{AYB}' + \mathbf{M} \approx \mathbf{M} + r\mathbf{AUB}'.$$

Conversely, suppose $\mathbf{X} \approx \mathbf{M} + r\mathbf{AUB}'$. Let $\mathbf{u} = vec(\mathbf{U}')$. Define

$$\psi(u) = \int_0^\infty \Omega_{p_1 n_1}(y^2 u) dF(y),$$

where $F(y)$ is the distribution function of r, $u \geq 0$. Then, it follows from Theorem 2.12, that $\psi(\mathbf{t}'\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{p_1 n_1}$ is the characteristic function of $r\mathbf{u}$. So $r\mathbf{u} \sim E_{p_1 n_1}(\mathbf{0}, \mathbf{I}_{p_1 n_1}, \psi)$ and hence,

$$r\mathbf{U} \sim E_{p_1 n_1}(\mathbf{0}, \mathbf{I}_{p_1} \otimes \mathbf{I}_{n_1}, \psi).$$

Therefore,

$$\mathbf{X} \approx \mathbf{M} + r\mathbf{AUB}' \sim E_{p,n}(\mathbf{M}, (\mathbf{AA}') \otimes (\mathbf{BB}'), \psi) = E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi). \quad \blacksquare$$

It may be noted that the stochastic representation is not uniquely defined. We can only say the following.

Theorem 2.14. $\mathbf{M}_1 + r_1\mathbf{A}_1\mathbf{U}\mathbf{B}_1'$ *and* $\mathbf{M}_2 + r_2\mathbf{A}_2\mathbf{U}\mathbf{B}_2'$, *where* $\mathbf{U}$ *is* $p_1 \times n_1$, *are two stochastic representations of the same* $p \times n$ *dimensional nondegenerate m.e.c. distribution if and only if* $\mathbf{M}_1 = \mathbf{M}_2$, *and there exist* $\mathbf{G} \in O(p_1)$, $\mathbf{H} \in O(n_1)$, *and positive constants* a, b, *and* c *such that* $ab = c$, $\mathbf{A}_2 = a\mathbf{A}_1\mathbf{G}$, $\mathbf{B}_2 = b\mathbf{B}_1\mathbf{H}$, *and* $r_2 = \frac{1}{c}r_1$.

PROOF: The "if" part is trivial. Conversely, let $\mathbf{X} \approx \mathbf{M}_1 + r_1\mathbf{A}_1\mathbf{U}\mathbf{B}_1'$ and $\mathbf{X}_2 \approx \mathbf{M}_2 + r_2\mathbf{A}_2\mathbf{U}\mathbf{B}_2'$. Then

$$\mathbf{X} \sim E_{p,n}(\mathbf{M}_1, (\mathbf{A}_1\mathbf{A}_1') \otimes (\mathbf{B}_1\mathbf{B}_1'), \psi_1)$$

and

$$\mathbf{X} \sim E_{p,n}(\mathbf{M}_2, (\mathbf{A}_2\mathbf{A}_2') \otimes (\mathbf{B}_2\mathbf{B}_2'), \psi_2),$$

where $\psi_i(u) = \int_0^\infty \Omega_{p_1 n_1}(y^2 u) dF_i(y)$, and $F_i(y)$ denotes the distribution function of r_i, $i = 1, 2$.

It follows, from Theorem 2.4, that $\mathbf{M}_1 = \mathbf{M}_2$, and there exist $a^2 > 0$, $b^2 > 0$, and $c^2 > 0$ such that $a^2 b^2 = c^2$, $\mathbf{A}_2\mathbf{A}_2' = a^2\mathbf{A}_1\mathbf{A}_1'$, $\mathbf{B}_2\mathbf{B}_2' = b^2\mathbf{B}_1\mathbf{B}_1'$, and $\psi_2(z) = \psi_1\left(\frac{1}{c^2}z\right)$. Now, from Theorem 1.11, it follows that there exist $\mathbf{G} \in O(p_1)$ and $\mathbf{H} \in O(n_1)$ such that $\mathbf{A}_2 = a\mathbf{A}_1\mathbf{G}$, and $\mathbf{B}_2 = b\mathbf{B}_1\mathbf{H}$. Since, $\psi_2(z) = \psi_1\left(\frac{1}{c^2}z\right)$, we have

$$\begin{aligned}
\psi_2(z) &= \int_0^\infty \Omega_{p_1 n_1}(y^2 z) dF_2(y) \\
&= \psi_1\left(\frac{z}{c^2}\right) \\
&= \int_0^\infty \Omega_{p_1 n_1}\left(y^2 \frac{z}{c^2}\right) dF_1(y) \\
&= \int_0^\infty \Omega_{p_1 n_1}\left(\left(\frac{y}{c}\right)^2 z\right) dF_1(y) \\
&= \int_0^\infty \Omega_{p_1 n_1}(t^2 z) dF_1(ct) .
\end{aligned}$$

Therefore $F_2(y) = F_1(cy)$, and

$$P(r_2 < y) = P(r_1 < cy) = P\left(\frac{r_1}{c} < y\right) .$$

Hence, $r_2 = \frac{1}{c}r_1$. ∎

Remark 2.4. It follows, from Theorem 2.13, that $\mathbf{U}$ does not depend on ψ. On the other hand, if p_1 and n_1 are fixed, ψ and r determine each other.

Remark 2.5. Let $E_{p,n}(\mathbf{0},\Sigma\otimes\Phi,\psi)$ and $r\mathbf{AUB}'$ be the stochastic representation of $\mathbf{X}$. Then, $\mathbf{A}^-\mathbf{XB}'^- \approx r\mathbf{U}$, and $tr((\mathbf{A}^-\mathbf{XB}'^-)'(\mathbf{A}^-\mathbf{XB}'^-)) \approx tr(r^2\mathbf{U}'\mathbf{U})$. Now,

$$\begin{aligned}
tr((\mathbf{A}^-\mathbf{XB}'^-)'(\mathbf{A}^-\mathbf{XB}'^-)) &= tr(\mathbf{B}^-\mathbf{X}'\mathbf{A}'^-\mathbf{A}^-\mathbf{XB}'^-)\\
&= tr(\mathbf{X}'\mathbf{A}'^-\mathbf{A}^-\mathbf{XB}'^-\mathbf{B}^-)\\
&= tr(\mathbf{X}'\Sigma^-\mathbf{X}\Phi^-)\,.
\end{aligned}$$

Here we used $\mathbf{A}'^-\mathbf{A}^- = \Sigma^-$, which follows from Theorem 1.23. On the other hand, $tr(\mathbf{U}'\mathbf{U}) = 1$. Therefore, we get $r^2 \approx tr(\mathbf{X}'\Sigma^-\mathbf{X}\Phi^-)$.

If an elliptically contoured random matrix is nonzero with probability one, then the terms of the stochastic representation can be obtained explicitly. First we introduce the following definition.

Definition 2.4. Let $\mathbf{X}$ be a $p\times n$ matrix. Then its norm, denoted by $\|\mathbf{X}\|$, is defined as

$$\|\mathbf{X}\| = \left(\sum_{i=1}^{p}\sum_{j=1}^{n} x_{ij}^2\right)^{\frac{1}{2}}.$$

That is, $\|\mathbf{X}\| = (tr(\mathbf{X}'\mathbf{X}))^{\frac{1}{2}}$, and if $n=1$, then we have $\|\mathbf{x}\| = (\mathbf{x}'\mathbf{x})^{\frac{1}{2}}$.

The proof of the following theorem is based on Muirhead (1982).

Theorem 2.15. *Let* $\mathbf{X}\sim E_{p,n}(\mathbf{0},\mathbf{I}_p\otimes\mathbf{I}_n,\psi)$ *with* $P(\mathbf{X}=\mathbf{0})=0$. *Then,* $\mathbf{X} = \|\mathbf{X}\|\frac{\mathbf{X}}{\|\mathbf{X}\|}$, $P(\|\mathbf{X}\|>0)=1$, $vec\left(\frac{\mathbf{X}'}{\|\mathbf{X}\|}\right)$ *is uniformly distributed on* S_{pn}, *and* $\|\mathbf{X}\|$ *and* $\frac{\mathbf{X}}{\|\mathbf{X}\|}$ *are independent. That is,* $\|\mathbf{X}\|\frac{\mathbf{X}}{\|\mathbf{X}\|}$ *is the stochastic representation of* $\mathbf{X}$.

PROOF: Since $\mathbf{X}=\mathbf{0}$ iff $tr(\mathbf{X}'\mathbf{X})=0$, $P(\|\mathbf{X}\|>0)=1$, follows so we can write $\mathbf{X} = \|\mathbf{X}\|\frac{\mathbf{X}}{\|\mathbf{X}\|}$. Define $\mathbf{x} = vec(\mathbf{X}')$. Then, $\mathbf{x}\sim E_{pn}(\mathbf{0},\mathbf{I}_{pn},\psi)$ and $\|\mathbf{X}\| = \|\mathbf{x}\|$. Hence, $\mathbf{x} = \|\mathbf{x}\|\frac{\mathbf{x}}{\|\mathbf{x}\|}$.

Let $T(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, and $\mathbf{G}\in O(pn)$. Then, we get $\mathbf{Gx}\sim E_{pn}(\mathbf{0},\mathbf{I}_{pn},\psi)$, so $\mathbf{x}\approx\mathbf{Gx}$ and $T(\mathbf{x})\approx T(\mathbf{Gx})$. On the other hand,

$$T(\mathbf{Gx}) = \frac{\mathbf{Gx}}{\|\mathbf{Gx}\|} = \frac{\mathbf{Gx}}{\|\mathbf{x}\|} = \mathbf{G}T(\mathbf{x})\,.$$

Hence, $T(\mathbf{x})\approx\mathbf{G}T(\mathbf{x})$. However, the uniform distribution is the only one on S_{pn} which is invariant under orthogonal transformation. So, $T(\mathbf{x})$ is uniformly distributed on S_{pn}.

Now, we define a measure μ on S_{pn}. Fix $B \subset I\!R_0^+$ Borel set. Let $A \subset S_{pn}$ be a Borel set. Then,

$$\mu(A) = P(T(\mathbf{x}) \in A | \|\mathbf{X}\| \in B).$$

Since $\mu(I\!R^{pn}) = 1$, μ is a probability measure on S_{pn}.

Let $\mathbf{G} \in O(pn)$. Then, $\mathbf{G}^{-1}\mathbf{x} \approx \mathbf{x}$, and we have

$$\begin{aligned}
\mu(\mathbf{G}A) &= P(T(\mathbf{x}) \in \mathbf{G}A | \|\mathbf{x}\| \in B) \\
&= P(\mathbf{G}^{-1}T(\mathbf{x}) \in A | \|\mathbf{x}\| \in B) \\
&= P(T(\mathbf{G}^{-1}\mathbf{x}) \in A | \|\mathbf{x}\| \in B) \\
&= P(T(\mathbf{G}^{-1}\mathbf{x}) \in A | \|\mathbf{G}^{-1}\mathbf{x}\| \in B) \\
&= P(T(\mathbf{x}) \in A | \|\mathbf{x}\| \in B) \\
&= \mu(A).
\end{aligned}$$

Thus, $\mu(A)$ is a probability measure on S_{pn}, invariant under orthogonal transformation, therefore, it must be the uniform distribution. That is, it coincides with the distribution of $T(\mathbf{x})$. So, $\mu(A) = P(T(\mathbf{x}) \in A)$, from which it follows that

$$P(T(\mathbf{x}) \in A | \|\mathbf{x}\| \in B) = P(T(\mathbf{x}) \in A).$$

Therefore, $T(\mathbf{x})$ and $\|\mathbf{x}\|$ are independently distributed. Returning to the matrix notation, the proof is completed. ■

Muirhead (1982) has given the derivation of the p.d.f. of r in the case when $\mathbf{x} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$ and $\mathbf{x}$ is absolutely continuous. Now for the elliptically contoured random matrices, the following theorem can be stated.

Theorem 2.16. *Let $\mathbf{X} \sim E_{p,n}(\mathbf{0}, \Sigma \otimes \Phi, \psi)$ and $r\mathbf{AUB}'$ be a stochastic representation of $\mathbf{X}$. Assume $\mathbf{X}$ is absolutely continuous and has the p.d.f.*

$$f(\mathbf{X}) = |\Sigma|^{-\frac{n}{2}} |\Phi|^{-\frac{p}{2}} h(tr(\mathbf{X}'\Sigma^{-1}\mathbf{X}\Phi^{-1})).$$

Then, r is also absolutely continuous and has the p.d.f.

$$g(r) = \frac{2\pi^{\frac{pn}{2}}}{\Gamma\left(\frac{pn}{2}\right)} r^{pn-1} h(r^2), \quad r \geq 0.$$

PROOF: Step 1. First we prove the theorem for $n = 1$. Then, $\mathbf{x} \sim E_p(\mathbf{0}, \Sigma, \psi)$ and so

$$\mathbf{y} = \mathbf{A}^{-1}\mathbf{x} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi).$$

Therefore $\mathbf{y}$ has the p.d.f. $h(\mathbf{y}'\mathbf{y})$. Let us introduce polar coordinates:

$$
\begin{aligned}
y_1 &= r sin\theta_1 sin\theta_2 \dots sin\theta_{p-2} sin\theta_{p-1} \\
y_2 &= r sin\theta_1 sin\theta_2 \dots sin\theta_{p-2} cos\theta_{p-1} \\
y_3 &= r sin\theta_1 sin\theta_2 \dots cos\theta_{p-2} \\
&\vdots \\
y_{p-1} &= r sin\theta_1 cos\theta_2 \\
y_p &= r cos\theta_1 \,,
\end{aligned}
$$

where $r > 0$, $0 < \theta_i < \pi$, $i = 1,2,\dots,p-2$, and $0 < \theta_{p-1} < 2\pi$. We want to express the p.d.f. of $\mathbf{y}$ in terms of $r, \theta_1, \dots, \theta_{p-1}$. The Jacobian of the transformation $(y_1, y_2, \dots, y_p) \to (r, \theta_1, \dots, \theta_{p-1})$ is $r^{p-1} sin^{p-2}\theta_1 sin^{p-3}\theta_2 \dots sin\theta_{p-2}$. On the other hand, $\mathbf{y}'\mathbf{y} = r^2$. Therefore, the p.d.f. of $(r, \theta_1, \dots, \theta_{p-1})$ is

$$h(r^2) r^{p-1} sin^{p-2}\theta_1 sin^{p-3}\theta_2 \dots sin\theta_{p-2}.$$

Consequently, the p.d.f. of r is

$$
\begin{aligned}
g(r) &= r^{p-1} h(r^2) \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} sin^{p-2}\theta_1 sin^{p-3}\theta_2 \dots sin\theta_{p-2} d\theta_1 d\theta_2 \dots d\theta_{p-2} d\theta_{p-1} \\
&= r^{p-1} h(r^2) \frac{2\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \,.
\end{aligned}
$$

Step 2. Now let $\mathbf{X} \sim E_{p,n}(\mathbf{0}, \Sigma \otimes \Phi, \psi)$ and $\mathbf{X} \approx r\mathbf{AUB}'$. Define $\mathbf{x} = vec(\mathbf{X}')$, and $\mathbf{u} = vec(\mathbf{U}')$. Then, $\mathbf{x} \sim E_{pn}(\mathbf{0}, \Sigma \otimes \Phi, \psi)$, $\mathbf{x}$ has p.d.f. $\frac{1}{|\Sigma \otimes \Phi|} h(\mathbf{x}'(\Sigma \otimes \Phi)^{-1}\mathbf{x})$, and $\mathbf{x} \approx r(\mathbf{A} \otimes \mathbf{B})\mathbf{u}$. Using Step 1 we get the following as the p.d.f. of r,

$$g(r) = r^{pn-1} h(r^2) \frac{2\pi^{\frac{pn}{2}}}{\Gamma\left(\frac{pn}{2}\right)} . \qquad \blacksquare$$

The stochastic representation is a major tool in the study of m.e.c. distributions. It will often be used in further discussion.

Cambanis, Huang and Simons (1981), and Anderson and Fang (1987) derived the relationship between the stochastic representation of a multivariate elliptically contoured distribution and the stochastic representation of its marginals. This result is given in the next theorem.

Theorem 2.17. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi)$ *with stochastic representation* $\mathbf{X} \approx r\mathbf{U}$. *Let* $\mathbf{X}$ *be partitioned into*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{pmatrix},$$

where $\mathbf{X}_i$ *is* $p_i \times n$ *matrix,* $i = 1, \ldots, m$. *Then,*

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{pmatrix} \approx \begin{pmatrix} rr_1\mathbf{U}_1 \\ rr_2\mathbf{U}_2 \\ \vdots \\ rr_m\mathbf{U}_m \end{pmatrix},$$

where r, $(r_1, r_2, \ldots, r_m)$, $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_m$ *are independent,* $r_i \geq 0$, $i = 1, \ldots, m$, $\sum_{i=1}^m r_i^2 = 1$,

$$(r_1^2, r_2^2, \ldots, r_{m-1}^2) \sim D\left(\frac{p_1 n}{2}, \frac{p_2 n}{2}, \ldots, \frac{p_{m-1} n}{2}; \frac{p_m n}{2}\right), \tag{2.11}$$

and $vec(\mathbf{U}_i')$ *is uniformly distributed on* $S_{p_i n}$, $i = 1, 2, \ldots, m$.

PROOF: Since $\mathbf{X} \approx r\mathbf{U}$, we have

$$\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_m \end{pmatrix} \approx r\mathbf{U},$$

where r and $\mathbf{U}$ are independent. Thus it suffices to prove that

$$\mathbf{U} \approx \begin{pmatrix} r_1\mathbf{U}_1 \\ r_2\mathbf{U}_2 \\ \vdots \\ r_m\mathbf{U}_m \end{pmatrix}.$$

Note that $\mathbf{U}$ does not depend on ψ, so we can choose $\psi(z) = \exp\left(-\frac{z}{2}\right)$, which means $\mathbf{X} \sim N_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n)$. It follows that $\mathbf{X}_i \sim N_{p_i,n}(\mathbf{0}, \mathbf{I}_{p_i} \otimes \mathbf{I}_n)$ and $\mathbf{X}_i$'s are mutually independent, $i = 1, \ldots, m$.

Now,

$$\mathbf{U} \approx \frac{\mathbf{X}}{\|\mathbf{X}\|} = \left(\frac{\mathbf{X}_1'}{\|\mathbf{X}\|}, \frac{\mathbf{X}_2'}{\|\mathbf{X}\|}, \ldots, \frac{\mathbf{X}_m'}{\|\mathbf{X}\|}\right)'.$$

From Theorem 2.15 it follows that $\mathbf{X}_i = \|\mathbf{X}_i\| \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}$, where $\|\mathbf{X}_i\|$ and $\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}$ are independent and $vec\left(\frac{\mathbf{X}_i'}{\|\mathbf{X}_i\|}\right) \approx \mathbf{u}_{p_i n}$ which is uniformly distributed on $S_{p_i n}$. Since, $\mathbf{X}_i$'s are independent, $\|\mathbf{X}_i\|$ and $\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}$ are mutually independent, $i = 1, 2, \ldots, m$. Therefore, we get

$$\mathbf{U} \approx \left(\frac{\|\mathbf{X}_1\|}{\|\mathbf{X}\|} \frac{\mathbf{X}_1'}{\|\mathbf{X}_1\|}, \frac{\|\mathbf{X}_2\|}{\|\mathbf{X}\|} \frac{\mathbf{X}_2'}{\|\mathbf{X}_2\|}, \ldots, \frac{\|\mathbf{X}_m\|}{\|\mathbf{X}\|} \frac{\mathbf{X}_m'}{\|\mathbf{X}_m\|} \right)'.$$

Define $r_i = \frac{\|\mathbf{X}_i\|}{\|\mathbf{X}\|}$, and $\mathbf{U}_i = \frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}$, $i = 1, 2, \ldots, m$. Since $\|\mathbf{X}\| = \left(\sum_{i=1}^m \|\mathbf{X}_i\|^2\right)^{\frac{1}{2}}$, r_i's are functions of $\|\mathbf{X}_1\|, \|\mathbf{X}_2\|, \ldots, \|\mathbf{X}_m\|$. Hence, $(r_1, r_2, \ldots, r_m)$, $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_m$ are independent. Moreover, $\|\mathbf{X}_i\|^2 = tr(\mathbf{X}_i'\mathbf{X}_i) \sim \chi^2_{p_i n}$ and $\|\mathbf{X}_i\|^2$'s are independent. Now, it is known that

$$\left(\frac{\|\mathbf{X}_1\|^2}{\sum_{i=1}^m \|\mathbf{X}_i\|^2}, \frac{\|\mathbf{X}_2\|^2}{\sum_{i=1}^m \|\mathbf{X}_i\|^2}, \ldots, \frac{\|\mathbf{X}_m\|^2}{\sum_{i=1}^m \|\mathbf{X}_i\|^2} \right) \sim D\left(\frac{p_1 n}{2}, \frac{p_2 n}{2}, \ldots, \frac{p_{m-1} n}{2}; \frac{p_m n}{2} \right)$$

(see Johnson and Kotz, 1972). Consequently, $(r_1^2, r_2^2, \ldots, r_{m-1}^2)$ has the distribution (2.11). ■

Corollary 2.4. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{0}, \mathbf{I}_p \otimes \mathbf{I}_n, \psi)$ *with stochastic representation* $\mathbf{X} \approx r\mathbf{U}$. *Let* $\mathbf{X}$ *be partitioned into*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix},$$

where $\mathbf{X}_1$ *is* $q \times n$ *matrix,* $1 \le q < p$. *Then,* $\begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \approx \begin{pmatrix} rr_1\mathbf{U}_1 \\ rr_2\mathbf{U}_2 \end{pmatrix}$, *where* r, (r_1, r_2), $\mathbf{U}_1, \mathbf{U}_2$ *are independent,* $r_i \ge 0$, $i = 1, 2$, $r_1^2 + r_2^2 = 1$, *and* $r_1^2 \sim B\left(\frac{qn}{2}, \frac{(p-q)n}{2}\right)$. *Also* $vec(\mathbf{U}_1')$ *is uniformly distributed on* S_{qn} *and* $vec(\mathbf{U}_2')$ *is uniformly distributed on* $S_{(p-q)n}$.

2.6 Conditional Distributions

First, we derive the conditional distribution for the vector variate elliptically contoured distribution. We will follow the lines of Cambanis, Huang, and Simons (1981). The following lemma will be needed in the proof.

Lemma 2.2. *Let* x *and* y *be one-dimensional nonnegative random variables. Assume that* y *is absolutely continuous with probability density function* $g(y)$. *Denote the distribution function of* x *by* $F(x)$. *Define* $z = xy$. *Then,* z *is absolutely continuous on* $\mathbb{R}^+$ *with p.d.f.*

$$h(z) = \int_0^\infty \frac{1}{x} g\left(\frac{z}{x}\right) dF(x). \tag{2.12}$$

If $F(0) = 0$, then z is absolutely continuous on $\mathbb{R}_0^+$, and if $F(0) > 0$, then z has an atom of size $F(0)$ at zero. Moreover, a conditional distribution of x given z is

$$P(x \leq x_0 | z = z_0) = \begin{cases} \frac{1}{h(z_0)} \int_{(0,w]} \frac{1}{x_0} g\left(\frac{z_0}{x_0}\right) dF(x_0) & \text{if } x_0 \geq 0,\ z_0 > 0, \text{ and } h(z_0) \neq 0 \\ 1 & \text{if } x_0 \geq 0, \text{ and } z_0 = 0 \\ & \text{or } x_0 \geq 0,\ z_0 > 0 \text{ and } h(z_0) = 0 \\ 0 & \text{if } x_0 < 0. \end{cases} \tag{2.13}$$

PROOF:

$$\begin{aligned} P(0 < z \leq z_0) &= P(0 < xy \leq z_0) \\ &= \int_0^\infty P(0 < xy \leq z_0 | x = x_0) dF(x_0) \\ &= \int_0^\infty P(0 < y \leq \frac{z_0}{x_0}) dF(x_0) \\ &= \int_0^\infty \int_{\left(0, \frac{z_0}{x_0}\right]} g(y) dy dF(x_0). \end{aligned}$$

Let $t = x_0 y$. Then, $y = \frac{t}{x_0}$, $dy = \frac{1}{x_0} dt$ and

$$\begin{aligned} \int_0^\infty \int_{\left(0, \frac{z_0}{x_0}\right]} g(y) dy dF(x_0) &= \int_0^\infty \int_{(0,z_0]} \frac{1}{x_0} g\left(\frac{t}{x_0}\right) dt dF(x_0) \\ &= \int_{(0,z_0]} \int_0^\infty \frac{1}{x_0} g\left(\frac{t}{x_0}\right) dF(x_0) dt \end{aligned}$$

and this proves (2.12).

Since, y is absolutely continuous, $P(y = 0) = 0$. Hence,

$$P(\chi_{\{0\}}(z) = \chi_{\{0\}}(x)) = 1. \tag{2.14}$$

Therefore, if $F(0) = 0$, then $P(z = 0) = 0$, and so z is absolutely continuous on R_0^+. If $F(0) > 0$, then $P(z = 0) = F(0)$ and thus z has an atom of size $F(0)$ at zero.

Now, we prove (2.13). Since $x \geq 0$, we have $P(x \leq x_0) = 0$ if $x_0 < 0$. Hence, $P(x \leq x_0 | z) = 0$ if $x_0 < 0$. If $x_0 \geq 0$, we have to prove that the function $P(x \leq x_0 | z)$ defined under (2.13) satisfies

$$\int_{[0,r]} P(x \leq x_0 | z) dH(z) = P(x \leq x_0, z \leq r),$$

where $H(z)$ denotes the distribution function of z and $r \geq 0$. Now,

$$\begin{aligned}\int_{[0,r]} P(x \leq x_0|z)dH(z) &= P(x \leq x_0|z=0)H(0) + \int_{(0,r]} P(x \leq x_0|z)dH(z) \\ &= H(0) + \int_{(0,r]} \frac{1}{h(z)} \left(\int_{(0,x_0]} \frac{1}{x} g\left(\frac{z}{x}\right) dF(x) \right) h(z)dz \\ &= H(0) + \int_{(0,x_0]} \int_{(0,r]} \frac{1}{x} g\left(\frac{z}{x}\right) dz dF(x).\end{aligned}$$

Let $u = \frac{z}{x}$. Then, $J(u \to z) = x$, and so

$$\int_{(0,r]} \frac{1}{x} g\left(\frac{z}{x}\right) dz = \int_{\left(0,\frac{r}{x}\right]} g(u)du = P\left(0 < y \leq \frac{r}{x}\right).$$

Hence,

$$\begin{aligned}H(0) + \int_{(0,x_0]} \int_{(0,r]} \frac{1}{x} g\left(\frac{z}{x}\right) dz dF(x) &= H(0) + \int_{(0,x_0]} P\left(0 < y \leq \frac{r}{x} \middle| x = x_0\right) dF(x_0) \\ &= H(0) + P\left(0 < x \leq x_0, 0 < y \leq \frac{r}{x}\right) \\ &= H(0) + P(0 < x \leq x_0, 0 < xy \leq r) \\ &= P(z=0) + P(0 < x \leq x_0, 0 < z \leq r) \\ &= P(x=0, z=0) + P(0 < x \leq x_0, 0 < z \leq r) \\ &= P(0 \leq x \leq x_0, 0 \leq z \leq r) \\ &= P(x \leq x_0, z \leq r),\end{aligned}$$

where we used (2.14). ■

Now, we obtain the conditional distribution for spherical distributions.

Theorem 2.18. *Let* $\mathbf{x} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$ *with stochastic representation* $r\mathbf{u}$. *Let us partition* $\mathbf{x}$ *as* $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$, *where* $\mathbf{x}_1$ *is q-dimensional* $(1 \leq q < p)$. *Then, the conditional distributionæ of* $\mathbf{x}_1$ *given* $\mathbf{x}_2$ *is* $(\mathbf{x}_1|\mathbf{x}_2) \sim E_q\left(\mathbf{0}, I_q, \psi_{\|\mathbf{x}_2\|^2}\right)$, *and the stochastic representation of* $(\mathbf{x}_1|\mathbf{x}_2)$ *is* $r_{\|\mathbf{x}_2\|^2}\mathbf{u}_1$, *where* $\mathbf{u}_1$ *is q-dimensional. The distribution of* $r_{\|\mathbf{x}_2\|^2}$ *is given by*

$$a) \qquad P\left(r_{a^2} \leq y\right) = \frac{\int_{(a,\sqrt{a^2+y^2}]} (w^2 - a^2)^{\frac{q}{2}-1} w^{-(p-2)} dF(w)}{\int_{(a,\infty)} (w^2 - a^2)^{\frac{q}{2}-1} w^{-(p-2)} dF(w)} \tag{2.15}$$

for $y \geq 0$ *if* $a > 0$ *and* $F(a) < 1$,

$$b) \qquad P(r_{a^2} = 0) = 1 \quad \text{if} \quad a = 0 \quad \text{or} \quad F(a) = 1\,. \tag{2.16}$$

Here F denotes the distribution function of r.

PROOF: From Corollary 2.4, we have the representation

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \approx \begin{pmatrix} rr_1\mathbf{u}_1 \\ rr_2\mathbf{u}_2 \end{pmatrix}.$$

Using the independence of r, r_1, $\mathbf{u}_1$ and $\mathbf{u}_2$, we get

$$\begin{aligned}(\mathbf{x}_1|\mathbf{x}_2) &\approx (rr_1\mathbf{u}_1|rr_2\mathbf{u}_2 = \mathbf{x}_2)\\ &= (rr1\mathbf{u}_1|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2)\\ &= (rr_1|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2)\mathbf{u}_1,\end{aligned}$$

and defining $r_0 = (rr_1|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2)$, we see that r and $\mathbf{u}_1$ are independent; therefore, $(\mathbf{x}_1|\mathbf{x}_2)$ has a spherical distribution.

Next, we show that

$$(rr_1|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2) \approx \left((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}} = \|\mathbf{x}_2\|\right).$$

If $r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2$, then $\left(r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2\right)' r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \|\mathbf{x}_2\|^2$ and therefore, $r^2(1-r_1^2) = \|\mathbf{x}_2\|^2$. Hence, we get $r^2 - r^2r_1^2 = \|\mathbf{x}_2\|^2$, thus $r^2r_1^2 = r^2 - \|\mathbf{x}_2\|^2$ and $rr_1 = (r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}$. Therefore,

$$(rr_1|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2) \approx ((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2).$$

If $\mathbf{x}_2 = \mathbf{0}$, then $\|\mathbf{x}_2\| = 0$, and using the fact that $\mathbf{u}_1 \neq \mathbf{0}$ we get

$$\begin{aligned}((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2) &= ((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}} = 0)\\ &= ((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}} = \|\mathbf{x}_2\|).\end{aligned}$$

If $\mathbf{x}_2 \neq \mathbf{0}$, then we can write

$$\begin{aligned}&((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \mathbf{x}_2)\\ &= \left((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}}|r(1-r_1^2)^{\frac{1}{2}}\mathbf{u}_2 = \|\mathbf{x}_2\|\frac{\mathbf{x}_2}{\|\mathbf{x}_2\|}\right)\end{aligned}$$

$$= \left((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}} | r(1-r_1^2)^{\frac{1}{2}} = \|\mathbf{x}_2\| \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \right)$$
$$= \left((r^2 - \|\mathbf{x}_2\|^2)^{\frac{1}{2}} | r(1-r_1^2)^{\frac{1}{2}} = \|\mathbf{x}_2\| \right),$$

where we used the fact that r, r_1, and $\mathbf{u}_2$ are independent. Since $1-r_1^2 \sim B\left(\frac{p-q}{2}, \frac{q}{2}\right)$ its p.d.f. is

$$b(t) = \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p-q}{2}\right)} t^{\frac{p-q}{2}-1}(1-t)^{\frac{q}{2}-1}, \quad 0 < t < 1.$$

Hence, the p.d.f. of $(1-r_1^2)^{\frac{1}{2}}$ is

$$g(y) = \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p-q}{2}\right)} (y^2)^{\frac{p-q}{2}-1}(1-y^2)^{\frac{q}{2}-1} 2y$$
$$= \frac{2\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p-q}{2}\right)} y^{p-q-1}(1-y^2)^{\frac{q}{2}-1}, \quad 0 < y < 1.$$

Using Lemma 2.2 we obtain a conditional distribution of r given $r(1-r_1^2)^{\frac{1}{2}} = a$.

$$P(r \le u | r(1-r_1^2)^{\frac{1}{2}} = a) \tag{2.17}$$
$$= \begin{cases} \frac{1}{h(a)} \int_{(0,u]} \frac{1}{w} \frac{2\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p-q}{2}\right)} \left(\frac{a}{w}\right)^{p-q-1} \left(1 - \frac{a^2}{w^2}\right)^{\frac{q}{2}-1} dF(w) & \text{if } u \ge 0,\ a > 0, \text{ and } h(a) \ne 0 \\ 1 & \text{if } u \ge 0, \text{ and } a = 0 \\ & \text{or } u \ge 0,\ a > 0 \text{ and } h(a) = 0 \\ 0 & \text{if } u < 0. \end{cases}$$

where

$$h(a) = \int_{(a,\infty)} \frac{1}{w} \frac{2\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p-q}{2}\right)} \left(\frac{a}{w}\right)^{p-q-1} \left(1 - \frac{a^2}{w^2}\right)^{\frac{q}{2}-1} dF(w).$$

Now,

$$\frac{\int_{(a,u]} \frac{1}{w} \frac{2\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p-q}{2}\right)} \left(\frac{a}{w}\right)^{p-q-1} \left(1 - \frac{a^2}{w^2}\right)^{\frac{q}{2}-1} dF(w)}{h(a)}$$
$$= \frac{\int_{(a,u]} \left(w^2 - a^2\right)^{\frac{q}{2}-1} w^{-(1+p-q-1+q-2)} dF(w)}{\int_{(a,\infty)} (w^2 - a^2)^{\frac{q}{2}-1} w^{-(1+p-q-1+q-2)} dF(w)}$$

$$= \frac{\int_{(a,u]} \left(w^2 - a^2\right)^{\frac{q}{2}-1} w^{-(p-2)} dF(w)}{\int_{(a,\infty)} \left(w^2 - a^2\right)^{\frac{q}{2}-1} w^{-(p-2)} dF(w)} . \tag{2.18}$$

We note that $h(a) = 0$ if and only if

$$\int_{(a,\infty)} \left(w^2 - a^2\right)^{\frac{q}{2}-1} w^{-(p-2)} dF(w) , \tag{2.19}$$

and since $\left(w^2 - a^2\right)^{\frac{q}{2}-1} w^{-(p-2)} > 0$ for $w > a$, we see that (2.19) is equivalent to $F(a) = 1$. Therefore, $h(a) = 0$ if and only if $F(a) = 1$.

(a) If $a > 0$ and $F(a) < 1$, then for $r \geq 0$ we have

$$\begin{aligned} P(r_{a^2} \leq y) &= P((r^2 - a^2)^{\frac{1}{2}} \leq y | r(1 - r_1^2)^{\frac{1}{2}} = a) \\ &= P(r \leq (y^2 + a^2)^{\frac{1}{2}} | r(1 - r_1^2)^{\frac{1}{2}} = a) . \end{aligned} \tag{2.20}$$

From (2.17), (2.18), and (2.20) we have

$$\begin{aligned} P(r_{a^2} \leq y) &= P(r \leq (y^2 + a^2)^{\frac{1}{2}} | r(1 - r_1^2)^{\frac{1}{2}} = a) \\ &= \frac{\int_{(a,\sqrt{a^2+y^2}]} \left(w^2 - a^2\right)^{\frac{q}{2}-1} w^{-(p-2)} dF(w)}{\int_{(a,\infty)} (w^2 - a^2)^{\frac{q}{2}-1} w^{-(p-2)} dF(w)} . \end{aligned}$$

(b) If $r \geq 0$ and $a = 0$ or $r \geq 0$, $a > 0$ and $F(a) = 1$, then from (2.17) we get

$$P(r \leq (y^2 + a^2)^{\frac{1}{2}} | r(1 - r_1^2)^{\frac{1}{2}} = a) = 1 .$$

Take $y = 0$, then we get

$$P(r_{a^2} \leq 0) = P(r \leq a | r(1 - r_1^2)^{\frac{1}{2}} = a) = 1. \tag{2.21}$$

Now, since $r_{a^2} \geq 0$, (2.21) implies $P(r_{a^2} = 0) = 1$. ■

In order to derive the conditional distribution for the multivariate elliptical distribution we need an additional lemma.

Lemma 2.3. *Let* $\mathbf{x} \sim E_p(\mathbf{m}, \Sigma, \psi)$ *and partition* $\mathbf{x}$, $\mathbf{m}$, Σ *as*

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} , \quad \mathbf{m} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} , \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} ,$$

where $\mathbf{x}_1$*,* $\mathbf{m}_1$ *are q-dimensional vectors and* Σ_{11} *is* $q \times q$*,* $1 \leq q < p$*. Let* $\mathbf{y} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$ *and partition* $\mathbf{y}$ *as* $\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}$*, where* $\mathbf{y}_1$ *is q-dimensional. Define* $\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-}\Sigma_{21}$ *and let* $\Sigma_{11\cdot 2} = \mathbf{A}\mathbf{A}'$ *and* $\Sigma_{22} = \mathbf{A}_2\mathbf{A}_2'$ *be rank factorizations of* $\Sigma_{11\cdot 2}$ *and* Σ_{22}*. Then*

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \approx \begin{pmatrix} \mathbf{m}_1 + \mathbf{A}\mathbf{y}_1 + \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{y}_2 \\ \mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 \end{pmatrix}.$$

PROOF: Since $\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$, we have

$$\begin{aligned}
&\begin{pmatrix} \mathbf{m}_1 + \mathbf{A}\mathbf{y}_1 + \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{y}_2 \\ \mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{A} & \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} \\
&\sim E_p\left(\begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{A} & \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2 \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix} \begin{pmatrix} \mathbf{A}' & \mathbf{0} \\ \mathbf{A}_2'\Sigma_{22}^{-}\Sigma_{21} & \mathbf{A}_2' \end{pmatrix}, \psi \right) \\
&= E_p\left(\mathbf{m}, \begin{pmatrix} \mathbf{A}\mathbf{A}' + \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{A}_2'\Sigma_{22}^{-}\Sigma_{21} & \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{A}_2' \\ \mathbf{A}_2\mathbf{A}_2'\Sigma_{22}^{-}\Sigma_{21} & \mathbf{A}_2\mathbf{A}_2' \end{pmatrix}, \psi \right) \\
&= E_p\left(\mathbf{m}, \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-}\Sigma_{22}\Sigma_{22}^{-}\Sigma_{21} & \Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} \\ \Sigma_{22}\Sigma_{22}^{-}\Sigma_{21} & \mathbf{A}_2\mathbf{A}_2' \end{pmatrix}, \psi \right). \qquad (2.22)
\end{aligned}$$

Now we prove that $\Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} = \Sigma_{12}$. If Σ_{22} is of the form $\Sigma_{22} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where $\mathbf{L}$ is a nonsingular, diagonal $s \times s$ matrix, then Σ_{12} must be of the form $\Sigma_{12} = (\mathbf{K}, \mathbf{0})$ where $\mathbf{K}$ is $q \times s$. Indeed, otherwise there would be numbers i and j such that $1 \leq i \leq q$ and $q + s \leq j \leq p$ and $\sigma_{ij} \neq 0$, $\sigma_{jj} = 0$. Since $\Sigma_{22} \geq \mathbf{0}$, we must have $\begin{vmatrix} \sigma_{ii} & \sigma_{ji} \\ \sigma_{ij} & \sigma_{jj} \end{vmatrix} \geq 0$. However with $\sigma_{ij} \neq 0$, $\sigma_{jj} = 0$, we have $\begin{vmatrix} \sigma_{ii} & \sigma_{ji} \\ \sigma_{ij} & \sigma_{jj} \end{vmatrix} = -\sigma_{ij}^2 < 0$ which is a contradiction. Therefore, $\Sigma_{12} = (\mathbf{K}, \mathbf{0})$.

Let Σ_{22}^{-} be partitioned as

$$\Sigma_{22}^{-} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

By the definition of a generalized inverse matrix, we must have

$$\begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

which gives

$$\begin{pmatrix} \mathbf{LAL} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

So $\mathbf{LAL} = \mathbf{L}$, and since $\mathbf{L}$ is nonsingular, we get $\mathbf{A} = \mathbf{L}^{-1}$. Thus $\Sigma_{22}^{-} = \begin{pmatrix} \mathbf{L}^{-1} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$. Then,

$$\begin{aligned} \Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} &= \left(\mathbf{K}, \mathbf{0}\right) \begin{pmatrix} \mathbf{L}^{-1} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ &= \left(\mathbf{K}, \mathbf{0}\right) \begin{pmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{CL} & \mathbf{0} \end{pmatrix} \\ &= \left(\mathbf{K}, \mathbf{0}\right) \\ &= \Sigma_{12}. \end{aligned}$$

If Σ_{22} is not of the form $\Sigma_{22} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, then there exists a $\mathbf{G} \in O(p-q)$ such that $\mathbf{G}\Sigma_{22}\mathbf{G}' = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Now, define

$$\begin{aligned} \Sigma^* &= \begin{pmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{G}' \end{pmatrix} \\ &= \begin{pmatrix} \Sigma_{11} & \Sigma_{12}\mathbf{G}' \\ \mathbf{G}\Sigma_{21} & \mathbf{G}\Sigma_{22}\mathbf{G}' \end{pmatrix}. \end{aligned}$$

Then, we must have

$$\Sigma_{12}\mathbf{G}'(\mathbf{G}\Sigma_{22}\mathbf{G}')^{-}(\mathbf{G}\Sigma_{22}\mathbf{G}') = \Sigma_{12}\mathbf{G}'$$

That is, $\Sigma_{12}\mathbf{G}'\mathbf{G}\Sigma_{22}^{-}\mathbf{G}'\mathbf{G}\Sigma_{22}\mathbf{G}' = \Sigma_{12}\mathbf{G}'$, which is equivalent to $\Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} = \Sigma_{12}$. Using $\Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} = \Sigma_{12}$ in (2.21), we have

$$\begin{aligned} \begin{pmatrix} \mathbf{m}_1 + \mathbf{A}\mathbf{y}_1 + \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{y}_2 \\ \mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 \end{pmatrix} &= E_p\left(\mathbf{m}, \begin{pmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-}\Sigma_{21} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \psi\right) \\ &= E_p\left(\mathbf{m}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \psi\right) \\ &= E_p(\mathbf{m}, \Sigma, \psi) \end{aligned}$$

which is the distribution of $\mathbf{x}$. ■

Next, we give the conditional distribution of the multivariate elliptical distribution in two different forms.

Theorem 2.19. *Let* $\mathbf{x} \sim E_p(\mathbf{m}, \Sigma, \psi)$ *with stochastic representation* $\mathbf{m} + r\mathbf{A}\mathbf{u}$. *Let* F *be the distribution function of* r. *Partition* $\mathbf{x}$, $\mathbf{m}$, Σ *as*

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{x}_1$, $\mathbf{m}_1$ *are* q*-dimensional vectors and* Σ_{11} *is* $q \times q$, $1 \le q < p$. *Assume* $rk(\Sigma_{22}) \ge 1$. *Then, a conditional distribution of* $\mathbf{x}_1$ *given* $\mathbf{x}_2$ *is*

$$(\mathbf{x}_1|\mathbf{x}_2) \sim E_q(\mathbf{m}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2), \Sigma_{11\cdot 2}, \psi_{q(\mathbf{x}_2)}),$$

where

$$\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-}\Sigma_{21}, \quad q(\mathbf{x}_2) = (\mathbf{x}_2 - \mathbf{m}_2)'\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2),$$

and

$$\psi_{q(\mathbf{x}_2)}(u) = \int_0^\infty \Omega_q(r^2 u) dF_{q(\mathbf{x}_2)}(r), \tag{2.23}$$

where

(a)

$$F_{q(\mathbf{x}_2)}(r) = \frac{\int_{(\sqrt{q(\mathbf{x}_2)}, \sqrt{q(\mathbf{x}_2)+r^2}]} (w^2 - q(\mathbf{x}_2))^{\frac{q}{2}-1} w^{-(p-2)} dF(w)}{\int_{(\sqrt{q(\mathbf{x}_2)}, \infty)} (w^2 - q(\mathbf{x}_2))^{\frac{q}{2}-1} w^{-(p-2)} dF(w)} \tag{2.24}$$

for $r \ge 0$ *if* $q(\mathbf{x}_2) > 0$ *and* $F(\sqrt{q(\mathbf{x}_2)}) < 1$, *and*

(b)

$$F_{q(\mathbf{x}_2)}(r) = 1 \quad \textit{for } r \ge 0 \textit{ if } q(\mathbf{x}_2) = 0 \textit{ and } F(\sqrt{q(\mathbf{x}_2)}) = 1. \tag{2.25}$$

PROOF: From Lemma 2.3, we get

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \approx \begin{pmatrix} \mathbf{m}_1 + \mathbf{A}\mathbf{y}_1 + \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{y}_2 \\ \mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 \end{pmatrix},$$

where $\mathbf{A}\mathbf{A}' = \Sigma_{11\cdot 2}$ and $\mathbf{A}_2\mathbf{A}_2' = \Sigma_{22}$ are rank factorizations of $\Sigma_{11\cdot 2}$, Σ_{22}, and $\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} \sim E_p(\mathbf{0}, \mathbf{I}_p, \psi)$. Thus,

$$\begin{aligned} (\mathbf{x}_1|\mathbf{x}_2) &\approx (\mathbf{m}_1 + \mathbf{A}\mathbf{y}_1 + \Sigma_{12}\Sigma_{22}^{-}\mathbf{A}_2\mathbf{y}_2 | \mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 = \mathbf{x}_2) \\ &= \mathbf{m}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{A}_2\mathbf{y}_2|\mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 = \mathbf{x}_2) + \mathbf{A}(\mathbf{y}_1|\mathbf{m}_2 + \mathbf{A}_2\mathbf{y}_2 = \mathbf{x}_2) \end{aligned}$$

$$= \mathbf{m}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{A}_2\mathbf{y}_2|\mathbf{A}_2\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{m}_2) + \mathbf{A}(\mathbf{y}_1|\mathbf{A}_2\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{m}_2)$$
$$= \mathbf{m}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2) + \mathbf{A}(\mathbf{y}_1|\mathbf{A}_2^{-}\mathbf{A}_2\mathbf{y}_2 = \mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2)). \quad (2.26)$$

Now $\mathbf{A}_2^{-}\mathbf{A}_2 = \mathbf{I}_{rk(\Sigma_{22})}$, and hence we get

$$(\mathbf{y}_1|\mathbf{A}_2^{-}\mathbf{A}_2\mathbf{y}_2 = \mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2)) = (\mathbf{y}_1|\mathbf{y}_2 = \mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2)). \quad (2.27)$$

From Theorem 2.18, we get

$$(\mathbf{y}_1|\mathbf{y}_2 = \mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2)) \sim E_q(\mathbf{0}, \mathbf{I}_q, \psi_{q(\mathbf{x}_2)}),$$

where

$$\begin{aligned} q(\mathbf{x}_2) &= (\mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2))'\mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2) \\ &= (\mathbf{x}_2 - \mathbf{m}_2)'\mathbf{A}_2^{-\prime}\mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2) \\ &= (\mathbf{x}_2 - \mathbf{m}_2)'\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2) \end{aligned}$$

and $\psi_{q(\mathbf{x}_2)}$ is defined by (2.23)–(2.25). Thus,

$$\begin{aligned} \mathbf{A}(\mathbf{y}_1|\mathbf{y}_2 = \mathbf{A}_2^{-}(\mathbf{x}_2 - \mathbf{m}_2)) &\sim E_q(\mathbf{0}, \mathbf{A}\mathbf{A}', \psi_{q(\mathbf{x}_2)}) \\ &= E_q(\mathbf{0}, \Sigma_{11\cdot 2}, \psi_{q(\mathbf{x}_2)}). \end{aligned} \quad (2.28)$$

Finally, from (2.26), (2.27), and (2.28) we get

$$(\mathbf{x}_1|\mathbf{x}_2) \sim E_q(\mathbf{m}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2), \Sigma_{11\cdot 2}, \psi_{q(\mathbf{x}_2)}). \quad \blacksquare$$

Another version of the conditional distribution is given in the following theorem.

Theorem 2.20. *Let* $\mathbf{x} \sim E_p(\mathbf{m}, \Sigma, \psi)$ *with stochastic representation* $\mathbf{m} + r\mathbf{A}\mathbf{u}$. *Let F be the distribution function of r. Partition* $\mathbf{x}$, $\mathbf{m}$, Σ *as*

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}, \quad \mathbf{m} = \begin{pmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{x}_1$, $\mathbf{m}_1$ *are q-dimensional vectors and* Σ_{11} *is* $q \times q$, $1 \le q < p$. *Assume* $rk(\Sigma_{22}) \ge 1$.

Let S denote the subspace of $\mathbb{R}^{p-q}$ *defined by the columns of* Σ_{22}; *that is,* $\mathbf{y} \in S$, *if there exists* $\mathbf{a} \in \mathbb{R}^{p-q}$ *such that* $\mathbf{y} = \Sigma_{22}\mathbf{a}$.

Then, a conditional distribution of $\mathbf{x}_1$ *given* $\mathbf{x}_2$ *is*

(a) $(\mathbf{x}_1|\mathbf{x}_2) \sim E_q(\mathbf{m}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2), \Sigma_{11\cdot 2}, \psi_{q(\mathbf{x}_2)})$ *for* $\mathbf{x}_2 \in \mathbf{m}_2 + S$, *where* $q(\mathbf{x}_2) = (\mathbf{x}_2 - \mathbf{m}_2)'\Sigma_{22}^{-}(\mathbf{x}_2 - \mathbf{m}_2)$ *and* $\psi_{q(\mathbf{x}_2)}$ *is defined by (2.23)–(2.25).*

(b) $(\mathbf{x}_1|\mathbf{x}_2) = \mathbf{m}_1$ *for* $\mathbf{x}_2 \notin \mathbf{m}_2 + S$.

PROOF: It suffices to prove that $P(\mathbf{x}_2 \notin \mathbf{m}_2 + S) = 0$ since $(\mathbf{x}_1|\mathbf{x}_2)$ can be arbitrarily defined for $\mathbf{x}_2 \in B$ where B is of measure zero. However, $P(\mathbf{x}_2 \notin \mathbf{m}_2 + S) = 0$ is equivalent to $P(\mathbf{x}_2 \in \mathbf{m}_2 + S) = 1$; that is, $P(\mathbf{x}_2 - \mathbf{m}_2 \in S) = 1$. Now, $\mathbf{x}_2 \sim E_{p-q}(\mathbf{m}_2, \Sigma_{22}, \psi)$ and so $\mathbf{x}_2 - \mathbf{m}_2 \sim E_{p-q}(\mathbf{0}, \Sigma_{22}, \psi)$.

Let $k = rk(\Sigma_{22})$. Let $\mathbf{G} \in O(p-q)$ such that $\mathbf{G}\Sigma_{22}\mathbf{G}' = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, where $\mathbf{L}$ is a diagonal and nonsingular $k \times k$ matrix and define $\mathbf{y} = \mathbf{G}(\mathbf{x}_2 - \mathbf{m}_2)$. Then,

$$\mathbf{y} \sim E_{p-q}\left(\mathbf{0}, \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \psi\right). \tag{2.29}$$

Partition $\mathbf{y}$ as $\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}$, where $\mathbf{y}_1$ is $k \times 1$. We have

$$\begin{aligned}
P(\mathbf{x}_2 - \mathbf{m}_2 \in S) &= P(\mathbf{x}_2 - \mathbf{m}_2 = \Sigma_{22}\mathbf{a} \quad \text{with} \quad \mathbf{a} \in \mathbb{R}^{p-q}) \\
&= P(\mathbf{G}(\mathbf{x}_2 - \mathbf{m}_2) = \mathbf{G}\Sigma_{22}\mathbf{G}'\mathbf{G}\mathbf{a} \quad \text{with} \quad \mathbf{a} \in \mathbb{R}^{p-q}) \\
&= P\left(\mathbf{y} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{b} \quad \text{with} \quad \mathbf{b} \in \mathbb{R}^{p-q}\right) \\
&= P\left(\mathbf{y} = \begin{pmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \quad \text{with} \quad \mathbf{b}_1 \in \mathbb{R}^k, \mathbf{b}_2 \in \mathbb{R}^{p-q-k}\right) \\
&= P\left(\mathbf{y} = \begin{pmatrix} \mathbf{L}\mathbf{b}_1 \\ \mathbf{0} \end{pmatrix} \quad \text{with} \quad \mathbf{b}_1 \in \mathbb{R}^k\right) \\
&= P\left(\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{0} \end{pmatrix} \quad \text{with} \quad \mathbf{c} \in \mathbb{R}^k\right) \\
&= P(\mathbf{y}_2 = \mathbf{0}).
\end{aligned}$$

Now, it follows from (2.29) that $\mathbf{y}_2 \sim E_{p-q-k}(\mathbf{0}, \mathbf{0}, \psi)$ and so $P(\mathbf{y}_2 = \mathbf{0}) = 1$. Therefore, $P(\mathbf{x}_2 - \mathbf{m}_2 \in S) = 1$. ■

Now we can derive the conditional distribution for m.e.c. distributions.

Theorem 2.21. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ *with stochastic representation* $\mathbf{M} + r\mathbf{A}\mathbf{U}\mathbf{B}'$. *Let* F *be the distribution function of* r. *Partition* $\mathbf{X}$, $\mathbf{M}$, Σ *as*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{pmatrix}, \quad \textit{and} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{X}_1$ *is* $q \times n$, $\mathbf{M}_1$ *is* $q \times n$, *and* Σ_{11} *is* $q \times q$, $1 \le q < p$. *Assume* $rk(\Sigma_{22}) \ge 1$.

Let S denote the subspace of $\mathbb{R}^{(p-q)n}$ defined by the columns of $\Sigma_{22} \otimes \Phi$; that is, $\mathbf{y} \in S$, if there exists $\mathbf{b} \in \mathbb{R}^{(p-q)n}$ such that $\mathbf{y} = (\Sigma_{22} \otimes \Phi)\mathbf{b}$. Then, a conditional distribution of $\mathbf{X}_1$ given $\mathbf{X}_2$ is

1. $(\mathbf{X}_1|\mathbf{X}_2) \sim E_{q,n}(\mathbf{M}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{X}_2 - \mathbf{M}_2), \Sigma_{11\cdot 2} \otimes \Phi, \psi_{q(\mathbf{X}_2)})$
for $vec(\mathbf{X}_2') \in vec(\mathbf{M}_2') + S$, where $q(\mathbf{X}_2) = tr((\mathbf{X}_2 - \mathbf{M}_2)'\Sigma_{22}^{-}(\mathbf{X}_2 - \mathbf{M}_2)\Phi^{-})$,

$$\psi_{q(\mathbf{X}_2)}(u) = \int_0^\infty \Omega_{qn}(r^2u)dF_{q(\mathbf{X}_2)}(r)\,, \tag{2.30}$$

where

(a)

$$F_{a^2}(r) = \frac{\int_{(a,\sqrt{a^2+r^2}]}(w^2-a^2)^{\frac{qn}{2}-1}w^{-(pn-2)}dF(w)}{\int_{(a,\infty)}(w^2-a^2)^{\frac{qn}{2}-1}w^{-(pn-2)}dF(w)} \tag{2.31}$$

for $r \geq 0$ if $a > 0$ and $F(a) < 1$, and

(b)

$$F_{a^2}(r) = 1 \quad \text{for } r \geq 0 \text{ if } a = 0 \text{ and } F(a) = 1\,. \tag{2.32}$$

2. $(\mathbf{X}_1|\mathbf{X}_2) = \mathbf{M}_1$ *for $vec(\mathbf{X}_2') \notin vec(\mathbf{M}_2') + S$.*

PROOF: Define $\mathbf{x} = vec(\mathbf{X}')$, $\mathbf{x}_1 = vec(\mathbf{X}_1')$, $\mathbf{x}_2 = vec(\mathbf{X}_2')$, $\mathbf{m} = vec(\mathbf{M}')$, $\mathbf{m}_1 = vec(\mathbf{M}_1')$, and $\mathbf{m}_2 = vec(\mathbf{M}_2')$. Then $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$ and $\mathbf{x} \sim E_{pn}(\mathbf{m}, \Sigma \otimes \Phi, \psi)$. Now apply Theorem 2.20.

(1) If $\mathbf{x}_2 \in vec(\mathbf{M}_2') + S$, we have

$$(\mathbf{x}_1|\mathbf{x}_2) \sim E_{qn}\Big(\mathbf{m}_1 + (\Sigma_{12} \otimes \Phi)(\Sigma_{22} \otimes \Phi)^{-}(\mathbf{x}_2 - \mathbf{m}_2),$$
$$(\Sigma_{11} \otimes \Phi) - (\Sigma_{12} \otimes \Phi)(\Sigma_{22} \otimes \Phi)^{-}(\Sigma_{21} \otimes \Phi), \psi_{q(\mathbf{x}_2)}\Big) \tag{2.33}$$

where

$$\begin{aligned} q(\mathbf{x}_2) &= (\mathbf{x}_2 - \mathbf{m}_2)'(\Sigma_{22} \otimes \Phi)^{-}(\mathbf{x}_2 - \mathbf{m}_2) \\ &= (vec((\mathbf{X}_2 - \mathbf{M}_2)'))'(\Sigma_{22}^{-} \otimes \Phi^{-})(vec(\mathbf{X}_2 - \mathbf{M}_2)') \\ &= tr((\mathbf{X}_2 - \mathbf{M}_2)'\Sigma_{22}^{-}(\mathbf{X}_2 - \mathbf{M}_2)\Phi^{-})\,. \end{aligned}$$

From (2.23) and (2.24) we get (2.30) and (2.31). Since $\mathbf{x}_2 \in \mathbf{m}_2 + S$, there exists $\mathbf{b} \in I\!R^{(p-q)n}$ such that $\mathbf{x}_2 - \mathbf{m}_2 = (\Sigma_{22} \otimes \boldsymbol{\Phi})\mathbf{b}$. Then, we have

$$\begin{aligned}(\Sigma_{12} \otimes \boldsymbol{\Phi})(\Sigma_{22} \otimes \boldsymbol{\Phi})^{-}(\mathbf{x}_2 - \mathbf{m}_2) &= (\Sigma_{12} \otimes \boldsymbol{\Phi})(\Sigma_{22} \otimes \boldsymbol{\Phi})^{-}(\Sigma_{22} \otimes \boldsymbol{\Phi})\mathbf{b}\\ &= (\Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} \otimes \boldsymbol{\Phi}\boldsymbol{\Phi}^{-}\boldsymbol{\Phi})\mathbf{b}\\ &= (\Sigma_{12}\Sigma_{22}^{-}\Sigma_{22} \otimes \boldsymbol{\Phi})\mathbf{b}\\ &= (\Sigma_{12}\Sigma_{22}^{-} \otimes \mathbf{I}_n)(\Sigma_{22} \otimes \boldsymbol{\Phi})\mathbf{b}\\ &= (\Sigma_{12}\Sigma_{22}^{-} \otimes \mathbf{I}_n)(\mathbf{x}_2 - \mathbf{m}_2)\,.\end{aligned}$$

We also have

$$\begin{aligned}&(\Sigma_{11} \otimes \boldsymbol{\Phi}) - (\Sigma_{12} \otimes \boldsymbol{\Phi})(\Sigma_{22} \otimes \boldsymbol{\Phi})^{-}(\Sigma_{21} \otimes \boldsymbol{\Phi})\\ &= (\Sigma_{11} \otimes \boldsymbol{\Phi}) - (\Sigma_{12}\Sigma_{22}^{-}\Sigma_{21}) \otimes \boldsymbol{\Phi}\\ &= (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-}\Sigma_{21}) \otimes \boldsymbol{\Phi}\,.\end{aligned}$$

Therefore, (2.33) can be written as

$$(\mathbf{x}_1|\mathbf{x}_2) \sim E_{qn}(\mathbf{m}_1 + (\Sigma_{12}\Sigma_{22}^{-} \otimes \mathbf{I}_n)(\mathbf{x}_2 - \mathbf{m}_2), \Sigma_{11\cdot 2} \otimes \boldsymbol{\Phi}, \psi_{q(\mathbf{x}_2)}).$$

Hence,

$$(\mathbf{X}_1|\mathbf{X}_2) \sim E_{q,n}(\mathbf{M}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{X}_2 - \mathbf{M}_2), \Sigma_{11\cdot 2} \otimes \boldsymbol{\Phi}, \psi_{q(\mathbf{X}_2)}).$$

(2) If $\mathbf{x}_2 \notin vec(\mathbf{M}_2') + S$, we get $(\mathbf{x}_1|\mathbf{x}_2) = \mathbf{m}_1$, so $(\mathbf{X}_1|\mathbf{X}_2) = \mathbf{M}_1$. ∎

Corollary 2.5. *With the notation of Theorem 2.21, we have*

$$1 - F(w) = K_{a^2} \int_{(\sqrt{w^2-a^2},\infty)} (r^2 + a^2)^{\frac{pn}{2}-1} r^{-(qn-2)} dF_{a^2}(r)\,, w \geq a\,,$$

where $K_{a^2} = \int_{(a,\infty)} (w^2 - a^2)^{\frac{qn}{2}-1} w^{-(pn-2)} dF(w)$.

PROOF: From (2.31) we get,

$$dF_{a^2}(r) = \frac{1}{K_{a^2}} (r^2)^{\frac{qn}{2}-1} (r^2 + a^2)^{-\frac{pn-2}{2}} \frac{dw}{dr} dF(w)\,,$$

where $r^2 + a^2 = w^2$. Hence,

$$dF(w) = K_{a^2} r^{-(qn-2)} (r^2 + a^2)^{\frac{pn}{2}-1} \frac{dr}{dw} dF_{a^2}(r)\,,$$

where $a < w \leq \sqrt{a^2+r^2}$. Therefore,

$$1-F(w) = K_{a^2}\int_{(\sqrt{w^2-a^2},\infty)}(r^2+a^2)^{\frac{pn}{2}-1}r^{-(qn-2)}dF_{a^2}(r)\,, w \geq a\,. \qquad \blacksquare$$

Theorem 2.22. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ *with stochastic representation* $\mathbf{M} + r\mathbf{AUB}'$. *Let* F *be the distribution function of* r *and* $\mathbf{X}$, $\mathbf{M}$, Σ *be partitioned as in Theorem 2.21.*

(1) If $vec(\mathbf{X}_2') \in vec(\mathbf{M}_2') + S$, *where* S *is defined in Theorem 2.21, and*

(a) If $\mathbf{X}$ *has finite first moment, then*

$$E(\mathbf{X}_1|\mathbf{X}_2) = \mathbf{M}_1 + \Sigma_{12}\Sigma_{22}^{-}(\mathbf{X}_2 - \mathbf{M}_2)\,.$$

(b) If $\mathbf{X}$ *has finite second moment, then*

$$Cov(\mathbf{X}_1|\mathbf{X}_2) = c_1\Sigma_{11\cdot 2} \otimes \Phi\,,$$

where $c_1 = -2\psi'_{q(\mathbf{X}_2)}(0)$, *and* $\psi_{q(\mathbf{X}_2)}$ *is defined by (2.30), (2.31) and (2.32).*

(2) If $vec(\mathbf{X}_2') \notin vec(\mathbf{M}_2') + S$, *then* $E(\mathbf{X}_1|\mathbf{X}_2) = \mathbf{M}_1$, $Cov(\mathbf{X}_1|\mathbf{X}_2) = \mathbf{0}$.

PROOF: It follows from Theorems 2.11 and 2.21. $\blacksquare$

The next theorem shows that if the distribution of $\mathbf{X}$ is absolutely continuous, then the constant c_1 in Theorem 2.22 can be obtained in a simple way. This was shown by Chu (1973), but his proof applies only to a subclass of absolutely continuous distributions. The following proof, however, works for all absolutely continuous distributions.

Theorem 2.23. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ *and* $\mathbf{X}$, $\mathbf{M}$, Σ *be partitioned as in Theorem 2.21. Assume the distribution of* $\mathbf{X}$ *is absolutely continuous and it has finite second moment.*

Let

$$f_2(\mathbf{X}_2) = \frac{1}{|\Sigma_{22}|^{\frac{n}{2}}|\Phi|^{\frac{p-q}{2}}}h_2\left(tr\left((\mathbf{X}_2-\mathbf{M}_2)'\Sigma_{22}^{-1}(\mathbf{X}_2-\mathbf{M}_2)\Phi^{-1}\right)\right)$$

be the p.d.f. of the submatrix $\mathbf{X}_2$. *Then,*

$$Cov(\mathbf{X}_1|\mathbf{X}_2) = \frac{\int_r^\infty h_2(z)dz}{2h_2(r)}\Sigma_{11\cdot 2} \otimes \Phi\,,$$

where $r = tr\left((\mathbf{X}_2-\mathbf{M}_2)'\Sigma_{22}^{-1}(\mathbf{X}_2-\mathbf{M}_2)\Phi^{-1}\right)$.

PROOF: Step 1. First we prove the theorem for the case $n = 1$, $\mathbf{m} = \mathbf{0}$. From Theorems 2.21 and 2.22, we conclude that $Cov(\mathbf{x}_1|\mathbf{x}_2) = c_1\Sigma_{11\cdot 2}$, where c_1 is determined by p, q, and $\mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2$. Hence, c_1 does not depend on Σ_{11} and Σ_{12}. Thus without loss of generality, we can assume that $\Sigma_{11} = \mathbf{I}_q$ and $\Sigma_{12} = \mathbf{0}$. Then $\Sigma_{11\cdot 2} = \mathbf{I}_q$. Let $\mathbf{x}_1 = (x_1, x_2 \ldots x_q)'$, and $f_1(x_1, \mathbf{x}_2) = |\Sigma_{22}|^{-\frac{1}{2}} h_1(x_1^2 + \mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2)$ be the joint p.d.f. of x_1 and $\mathbf{x}_2$. Then,

$$\begin{aligned} c_1 = Var(x_1|\mathbf{x}_2) &= \frac{\int_{-\infty}^{\infty} x_1^2 f_1(x_1, \mathbf{x}_2) dx_1}{f_2(\mathbf{x}_2)} \\ &= \frac{\int_{-\infty}^{\infty} x_1^2 h_1(x_1^2 + \mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2) dx_1}{h_2(\mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2)} \\ &= 2\frac{\int_0^{\infty} x_1^2 h_1(x_1^2 + \mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2) dx_1}{h_2(\mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2)}. \end{aligned} \tag{2.34}$$

Now, $f_2(\mathbf{x}_2) = \int_{-\infty}^{\infty} f_1(x_1, \mathbf{x}_2) dx_1$, hence

$$\begin{aligned} h_2(\mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2) &= \int_{-\infty}^{\infty} h_1(x_1^2 + \mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2) dx_1 \\ &= 2\int_0^{\infty} h_1(x_1^2 + \mathbf{x}_2'\Sigma_{22}^{-1}\mathbf{x}_2) dx_1 \,. \end{aligned}$$

So, $h_2(z) = 2\int_0^{\infty} h_1(x_1^2 + z) dx_1$, for $z \geq 0$. Hence, for $u \geq 0$, we get

$$\begin{aligned} \int_u^{\infty} h_2(z) dz &= 2\int_u^{\infty} \int_0^{\infty} h_1(x_1^2 + z) dx_1 dz \\ &= 2\int_0^{\infty} \int_0^{\infty} \chi(z \geq u) h_1(x_1^2 + z) dx_1 dz. \end{aligned}$$

Let $w = \sqrt{x_1^2 + z - u}$. Then, $w^2 = x_1^2 + z - u$ and so $J(z \to w) = 2w$. Hence,

$$\begin{aligned} \int_u^{\infty} h_2(z) dz &= 2\int_0^{\infty} \int_0^{\infty} \chi(w^2 - x_1^2 + u \geq u) h_1(w^2 + u) 2w dx_1 dw \\ &= 4\int_0^{\infty} \int_0^{\infty} \chi(w^2 \geq x_1^2) w h_1(w^2 + u) dx_1 dw \\ &= 4\int_0^{\infty} \int_0^{w} w h_1(w^2 + u) dx_1 dw \\ &= 4\int_0^{\infty} \left(w h_1(w^2 + u) \int_0^{w} dx_1 \right) dw \\ &= 4\int_0^{\infty} w h_1(w^2 + u) w dw \\ &= 4\int_0^{\infty} w^2 h_1(w^2 + u) dw. \end{aligned} \tag{2.35}$$

Now from (2.34) and (2.35), we get

$$c_1 = \frac{\int_u^\infty h_2(z)dz}{2h_2(u)}, \quad \text{where} \quad u = \mathbf{x}_2' \Sigma_{22}^{-1} \mathbf{x}_2 \,.$$

Step 2. Next, let $n = 1$, $\mathbf{m} \neq \mathbf{0}$, and $\mathbf{y} = \mathbf{x} - \mathbf{m}$. Then,

$$Cov(\mathbf{y}_1|\mathbf{y}_2) = \frac{\int_u^\infty h_2(z)dz}{2h_2(u)} \Sigma_{11\cdot 2}\,,$$

where $u = \mathbf{y}_2' \Sigma_{22}^{-1} \mathbf{y}_2$. Therefore,

$$\begin{aligned} Cov(\mathbf{x}_1|\mathbf{x}_2) &= Cov(\mathbf{y}_1 + \mathbf{m}_1|\mathbf{y}_2 = \mathbf{x}_2 - \mathbf{m}_2) \\ &= \frac{\int_u^\infty h_2(z)dz}{2h_2(u)} \Sigma_{11\cdot 2}\,, \end{aligned}$$

where $u = (\mathbf{x}_2 - \mathbf{m}_2)' \Sigma_{22}^{-1} (\mathbf{x}_2 - \mathbf{m}_2)$.

Step 3. Finally, let $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$. Define $\mathbf{x} = vec(\mathbf{X}')$. Now, for $\mathbf{x}$ we can use Step 2. Therefore,

$$Cov(\mathbf{X}_1|\mathbf{X}_2) = \frac{\int_r^\infty h_2(z)dz}{2h_2(r)} \Sigma_{11\cdot 2} \otimes \Phi\,,$$

where $r = tr\left((\mathbf{X}_2 - \mathbf{M}_2)' \Sigma_{22}^{-1} (\mathbf{X}_2 - \mathbf{M}_2) \Phi^{-1}\right)$. ■

2.7 Examples

In this section we give some examples of the elliptically contoured distributions. We also give a method to generate elliptically contoured distributions.

2.7.1 One-Dimensional Case

Let $p = n = 1$. Then, the class $E_1(m, \sigma, \psi)$, coincides with the class of one-dimensional distributions which are symmetric about a point. More precisely, $x \sim E_1(m, \sigma, \psi)$ if and only if $P(x \leq r) = P(x \geq m - r)$ for every $r \in I\!R$. Some examples are: uniform, normal, Cauchy, double exponential, Student's t-distribution, and the distribution with the p.d.f.

$$f(x) = \frac{\sqrt{2}}{\pi\sigma\left(1+\left(\frac{x}{\sigma}\right)^4\right)}, \quad \sigma > 0\,.$$

2.7.2 Vector Variate Case

The definitions and results here are taken from Fang, Kotz and Ng (1990). Let $p > 1$ and $n = 1$.

2.7.2.1 Multivariate Uniform Distribution

The p-dimensional random vector $\mathbf{u}$ is said to have a multivariate uniform distribution if it is uniformly distributed on the unit sphere in $I\!R^p$.

Theorem 2.24. *Let $x = (x_1, x_2, \ldots, x_p)'$ have a p-variate uniform distribution. Then the p.d.f. of $(x_1, x_2, \ldots, x_{p-1})$ is*

$$\frac{\Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}}}\left(1-\sum_{i=1}^{p-1} x_i^2\right)^{-\frac{1}{2}}, \quad \sum_{i=1}^{p-1} x_i^2 < 1\,.$$

PROOF: It follows from Theorem 2.17 that $\left(x_1^2, x_2^2, \ldots, x_{p-1}^2\right) \sim D\left(\frac{1}{2}, \ldots, \frac{1}{2}; \frac{1}{2}\right)$. Hence, the p.d.f. of $\left(x_1^2, x_2^2, \ldots, x_{p-1}^2\right)$ is

$$\frac{\Gamma\left(\frac{p}{2}\right)}{\left(\Gamma\left(\frac{1}{2}\right)\right)^p}\prod_{i=1}^{p-1}(x_i^2)^{-\frac{1}{2}}\left(1-\sum_{i=1}^{p-1} x_i^2\right)^{-\frac{1}{2}}$$

Since the Jacobian of the transformation $(x_1, x_2, \ldots, x_{p-1}) \to (|x_1|, |x_2|, \ldots, |x_{p-1}|)$ is $2^{p-1}\prod_{i=1}^{p-1}|x_i|$, the p.d.f. of $(|x_1|, |x_2|, \ldots, |x_{p-1}|)$ is

$$\frac{2^{p-1}\Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}}}\left(1-\sum_{i=1}^{p-1} x_i^2\right)^{-\frac{1}{2}}$$

Now because of the spherical symmetry of $\mathbf{x}$, the p.d.f. of $(x_1, x_2, \ldots, x_{p-1})$ is the p.d.f. of $(|x_1|, |x_2|, \ldots, |x_{p-1}|)$ divided by 2^{p-1}; that is

$$\frac{\Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}}}\left(1-\sum_{i=1}^{p-1} x_i^2\right)^{-\frac{1}{2}}, \quad \sum_{i=1}^{p-1} x_i^2 < 1\,. \qquad \blacksquare$$

2.7.2.2 Symmetric Kotz Type Distribution

The p-dimensional random vector $\mathbf{x}$ is said to have a symmetric Kotz type distribution with parameters q, r, $s \in I\!R$, μ: p-dimensional vector, Σ: $p \times p$ matrix, $r > 0$, $s > 0$, $2q + p > 2$, and $\Sigma > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{x}) = \frac{sr^{\frac{2q+p-2}{2s}} \Gamma\left(\frac{p}{2}\right)}{\pi^{\frac{p}{2}} \Gamma\left(\frac{2q+p-2}{2s}\right) |\Sigma|^{\frac{1}{2}}} [(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)]^{q-1} \exp\{-r[(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)]^{s}\}.$$

As a special case, take $q = s = 1$ and $r = \frac{1}{2}$. Then, we get the multivariate normal distribution with p.d.f.

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left\{-\frac{(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)}{2}\right\},$$

and its characteristic function is

$$\phi_{\mathbf{x}}(\mathbf{t}) = \exp(i\mathbf{t}'\mu) \exp\left(-\frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\right). \tag{2.36}$$

The multivariate normal distribution is denoted by $N_p(\mu, \Sigma)$.

Remark 2.6. The distribution, $N_p(\mu, \Sigma)$, can be defined by its characteristic function (2.36). Then, Σ does not have to be positive definite; it suffices to assume that $\Sigma \geq \mathbf{0}$.

2.7.2.3 Symmetric Multivariate Pearson Type II Distribution

The p-dimensional random vector $\mathbf{x}$ is said to have a symmetric multivariate Pearson type II distribution with parameters $q \in I\!R$, μ: p-dimensional vector, Σ: $p \times p$ matrix with $q > -1$, and $\Sigma > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{x}) = \frac{\Gamma\left(\frac{p}{2} + q + 1\right)}{\pi^{\frac{p}{2}} \Gamma(q+1) |\Sigma|^{\frac{1}{2}}} (1 - (\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu))^{q},$$

where $(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) \leq 1$.

2.7.2.4 Symmetric Multivariate Pearson Type VII Distribution

The p-dimensional random vector $\mathbf{x}$ is said to have a symmetric multivariate Pearson type VII distribution with parameters q, $r \in I\!R$, μ: p-dimensional vector, Σ: $p \times p$ matrix with $r > 0$, $q > \frac{p}{2}$, and $\Sigma > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{x}) = \frac{\Gamma(q)}{(\pi r)^{\frac{p}{2}} \Gamma\left(q - \frac{p}{2}\right) |\Sigma|^{\frac{1}{2}}} \left(1 + \frac{(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)}{r}\right)^{-q}. \tag{2.37}$$

As a special case, when $q = \frac{p+r}{2}$, $\mathbf{x}$ is said to have a multivariate t-distribution with r degrees of freedom and it is denoted by $Mt_p(r, \mu, \Sigma)$.

Theorem 2.25. *The class of symmetric multivariate Pearson type VII distributions coincides with the class of multivariate t-distributions.*

PROOF: Clearly, the multivariate t-distribution is Pearson type VII distribution. We only have to prove that all Pearson type VII distributions are multivariate t-distributions.

Assume $\mathbf{x}$ has p.d.f. (2.37). Define

$$r_0 = 2\left(q - \frac{p}{2}\right) \quad \text{and} \quad \Sigma_0 = \Sigma \frac{r}{r_0}.$$

Then, $Mt_p(r_0, \mu, \Sigma_0)$ is the same distribution as the one with p.d.f. (2.37). ■

The special case of multivariate t-distribution when $r = 1$; that is, $Mt_p(1, \mu, \Sigma)$ is called multivariate Cauchy distribution.

2.7.2.5 Symmetric Multivariate Bessel Distribution

The p-dimensional random vector $\mathbf{x}$ is said to have a symmetric multivariate Bessel distribution with parameters $q, r \in I\!R$, μ: p-dimensional vector, Σ: $p \times p$ matrix with $r > 0$, $q > -\frac{p}{2}$, and $\Sigma > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{x}) = \frac{[(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)]^{\frac{q}{2}}}{2^{q+p-1}\pi^{\frac{p}{2}} r^{p+q} \Gamma\left(q + \frac{p}{2}\right) |\Sigma|^{\frac{1}{2}}} K_q\left(\frac{[(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)]^{\frac{1}{2}}}{r}\right),$$

where $K_q(z)$ is the modified Bessel function of the third kind; that is $K_q(z) = \frac{\pi}{2} \frac{I_{-q}(z) - I_q(z)}{sin(q\pi)}$, $|arg(z)| < \pi$, q is integer and

$$I_q(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+q+1)} \left(\frac{z}{2}\right)^{q+2k}, \quad |z| < \infty, \quad |arg(z)| < \pi.$$

If $q = 0$ and $r = \frac{\sigma}{\sqrt{2}}$, $\sigma > 0$, then $\mathbf{x}$ is said to have a multivariate Laplace distribution.

2.7.2.6 Symmetric Multivariate Logistic Distribution

The p-dimensional random vector $\mathbf{x}$ is said to have an elliptically symmetric logistic distribution with parameters μ: p-dimensional vector, Σ: $p \times p$ matrix with $\Sigma > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{x}) = \frac{c}{|\Sigma|^{\frac{1}{2}}} \frac{\exp\{-(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\}}{(1+\exp\{-(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\})^2}$$

with

$$c = \frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \int_0^\infty z^{\frac{p}{2}-1} \frac{e^{-z}}{(1+e^{-z})^2} dz.$$

2.7.2.7 Symmetric Multivariate Stable Law

The p-dimensional random vector $\mathbf{x}$ is said to follow a symmetric multivariate stable law with parameters q, $r \in I\!R$, μ: p-dimensional vector, Σ: $p \times p$ matrix with $0 < q \leq 1$, $r > 0$, and $\Sigma \geq \mathbf{0}$ if its characteristic function is

$$\phi_{\mathbf{x}}(\mathbf{t}) = \exp(i\mathbf{t}'\mu - r(\mathbf{t}'\Sigma\mathbf{t})^q).$$

2.7.3 General Matrix Variate Case

The matrix variate elliptically contoured distributions listed here are the matrix variate versions of the multivariate distributions given in Sect. 2.7.2. Let $p \geq 1$ and $n \geq 1$.

2.7.3.1 Matrix Variate Symmetric Kotz Type Distribution

The $p \times n$ random matrix $\mathbf{X}$ is said to have a matrix variate symmetric Kotz type distribution with parameters q, r, $s \in I\!R$, $\mathbf{M} : p \times n$, $\Sigma : p \times p$, $\Phi : n \times n$ with $r > 0$, $s > 0$, $2q + pn > 2$, $\Sigma > \mathbf{0}$, and $\Phi > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{X}) = \frac{sr^{\frac{2q+pn-2}{2s}} \Gamma\left(\frac{pn}{2}\right)}{\pi^{\frac{pn}{2}} \Gamma\left(\frac{2q+pn-2}{2s}\right) |\Sigma|^{\frac{n}{2}} |\Phi|^{\frac{p}{2}}} [tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})]^{q-1}$$
$$\times \exp\{-r[tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})]^s\}.$$

If we take $q = s = 1$ and $r = \frac{1}{2}$, we obtain the p.d.f. of the absolutely continuous matrix variate normal distribution:

$$f(\mathbf{X}) = \frac{1}{(2\pi)^{\frac{pn}{2}}|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} etr\left\{-\frac{(\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1}}{2}\right\}.$$

The characteristic function of this distribution is

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr(i\mathbf{T}'\mathbf{M})etr\left(-\frac{1}{2}\mathbf{T}'\Sigma\mathbf{M}\Phi\right). \tag{2.38}$$

Remark 2.7. If we define $N_{p,n}(\mathbf{M}, \Sigma \otimes \Phi)$ through its characteristic function (2.38), then $\Sigma > \mathbf{0}$ is not required, instead it suffices to assume that $\Sigma \geq \mathbf{0}$.

2.7.3.2 Matrix Variate Pearson Type II Distribution

The $p \times n$ random matrix $\mathbf{X}$ is said to have a matrix variate symmetric Pearson type II distribution with parameters $q \in I\!R$, $\mathbf{M} : p \times n$, $\Sigma : p \times p$, $\Phi : n \times n$ with $q > -1$, $\Sigma > \mathbf{0}$, and $\Phi > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{X}) = \frac{\Gamma\left(\frac{pn}{2}+q+1\right)}{\pi^{\frac{pn}{2}}\Gamma(q+1)|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}}(1 - tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1}))^{q},$$

where $tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1}) \leq 1$.

2.7.3.3 Matrix Variate Pearson Type VII Distribution

The $p \times n$ random matrix $\mathbf{X}$ is said to have a matrix variate symmetric Pearson type VII distribution with parameters $q, r \in I\!R$, $\mathbf{M} : p \times n$, $\Sigma : p \times p$, $\Phi : n \times n$ with $r > 0$, $q > \frac{pn}{2}$, $\Sigma > \mathbf{0}$, and $\Phi > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{X}) = \frac{\Gamma(q)}{(\pi r)^{\frac{pn}{2}}\Gamma\left(q-\frac{pn}{2}\right)|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}}\left(1 + \frac{tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})}{r}\right)^{-q}. \tag{2.39}$$

Particularly, when $q = \frac{pn+r}{2}$, $\mathbf{X}$ is said to have a matrix variate t-distribution with r degrees of freedom and it is denoted by $Mt_{p,n}(r, \mathbf{M}, \Sigma \otimes \Phi)$. It follows, from Theorem 2.25, that the class of matrix variate symmetric Pearson type VII distributions coincides with the class of matrix variate t-distributions.

When $r = 1$, in the definition of matrix variate t-distribution, i.e., $Mt_{p,n}(1, \mathbf{M}, \Sigma \otimes \Phi)$, then $\mathbf{X}$ is said to have a matrix variate Cauchy distribution.

2.7.3.4 Matrix Variate Symmetric Bessel Distribution

The $p \times n$ random matrix $\mathbf{X}$ is said to have a matrix variate symmetric Bessel distribution with parameters q, $r \in I\!R$, $\mathbf{M} : p \times n$, $\boldsymbol{\Sigma} : p \times p$, $\boldsymbol{\Phi} : n \times n$ with $r > 0$, $q > -\frac{pn}{2}$, $\boldsymbol{\Sigma} > \mathbf{0}$, and $\boldsymbol{\Phi} > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{X}) = \frac{[tr((\mathbf{X}-\mathbf{M})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Phi}^{-1})]^{\frac{1}{2}}}{2^{q+pn-1}\pi^{\frac{pn}{2}} r^{pn+q}\Gamma\left(q+\frac{pn}{2}\right)|\boldsymbol{\Sigma}|^{\frac{n}{2}}|\boldsymbol{\Phi}|^{\frac{p}{2}}} \times K_q\left(\frac{[tr((\mathbf{X}-\mathbf{M})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Phi}^{-1})]^{\frac{1}{2}}}{r}\right),$$

where $K_q(z)$ is the modified Bessel function of the third kind as defined in Sect. 2.7.2.5. For $q = 0$ and $r = \frac{\sigma}{\sqrt{2}}$, $\sigma > 0$, this distribution is known as the matrix variate Laplace distribution.

2.7.3.5 Matrix Variate Symmetric Logistic Distribution

The $p \times n$ random matrix $\mathbf{X}$ is said to have a matrix variate symmetric logistic distribution with parameters $\mathbf{M} : p \times n$, $\boldsymbol{\Sigma} : p \times p$, $\boldsymbol{\Phi} : n \times n$ with $\boldsymbol{\Sigma} > \mathbf{0}$, and $\boldsymbol{\Phi} > \mathbf{0}$ if its p.d.f. is

$$f(\mathbf{X}) = \frac{c}{|\boldsymbol{\Sigma}|^{\frac{n}{2}}|\boldsymbol{\Phi}|^{\frac{p}{2}}} \frac{etr(-(\mathbf{X}-\mathbf{M})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Phi}^{-1})}{(1+etr(-(\mathbf{X}-\mathbf{M})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Phi}^{-1}))^2}$$

with

$$c = \frac{\pi^{\frac{pn}{2}}}{\Gamma\left(\frac{pn}{2}\right)} \int_0^\infty z^{\frac{pn}{2}-1} \frac{e^{-z}}{(1+e^{-z})^2} dz.$$

2.7.3.6 Matrix Variate Symmetric Stable Law

The $p \times n$ random matrix $\mathbf{X}$ is said to follow a matrix variate symmetric stable law with parameters q, $r \in I\!R$, $\mathbf{M} : p \times n$, $\boldsymbol{\Sigma} : p \times p$, $\boldsymbol{\Phi} : n \times n$ with $0 < q \leq 1$, $r > 0$, $\boldsymbol{\Sigma} \geq \mathbf{0}$, and $\boldsymbol{\Phi} \geq \mathbf{0}$ if its characteristic function is

$$\phi_{\mathbf{X}}(\mathbf{T}) = etr\left(i\mathbf{T}'\mathbf{M}\right)\exp\left(-r(tr(\mathbf{T}'\boldsymbol{\Sigma}\mathbf{M}\boldsymbol{\Phi}))^q\right).$$

2.7.4 Generating Elliptically Contoured Distributions

If we have a m.e.c. distribution, based on it we can easily generate other m.e.c. distributions. For vector variate elliptical distributions, this is given in Muirhead (1982).

Theorem 2.26. *Let* $\mathbf{X} \sim E_{p,n}(\mathbf{M}, \Sigma \otimes \Phi, \psi)$ *have the p.d.f.*

$$f(\mathbf{X}) = \frac{1}{|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} h(tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})).$$

Suppose $G(z)$ *is a distribution function on* $(0,\infty)$*. Let*

$$g(\mathbf{X}) = \frac{1}{|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} \int_0^\infty z^{-\frac{pn}{2}} h\left(\frac{1}{z} tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})\right) dG(z)\,.$$

Then, $g(\mathbf{X})$ *is also the p.d.f. of a m.e.c. distribution.*

PROOF: Clearly, $g(\mathbf{X}) \geq 0$. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^{p\times n}} g(\mathbf{X}) d\mathbf{X} \\
&= \int_{\mathbb{R}^{p\times n}} \frac{1}{|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} \int_0^\infty z^{-\frac{pn}{2}} h\left(\frac{1}{z} tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})\right) dG(z) d\mathbf{X} \\
&= \int_0^\infty \int_{\mathbb{R}^{p\times n}} \frac{1}{|z\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} h\left(tr((\mathbf{X}-\mathbf{M})'(z\Sigma)^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})\right) d\mathbf{X} dG(z) \\
&= \int_0^\infty 1 dG(z) = 1\,.
\end{aligned}$$

Hence, $g(\mathbf{X})$ is a p.d.f. Let $r(w) = \int_0^\infty z^{-\frac{pn}{2}} h\left(\frac{w}{z}\right) dG(z)$. Then,

$$g(\mathbf{X}) = \frac{1}{|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} r(tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})).$$

Therefore, $g(\mathbf{X})$ is the p.d.f. of an elliptically contoured distribution. ■

Corollary 2.6. *Let* $h(u) = (2\pi)^{-\frac{pn}{2}} \exp\left(-\frac{u}{2}\right)$ *in Theorem 2.26. Then, for any distribution function* $G(z)$ *on* $(0,\infty)$*,*

$$g(\mathbf{X}) = \frac{1}{(2\pi)^{\frac{pn}{2}}|\Sigma|^{\frac{n}{2}}|\Phi|^{\frac{p}{2}}} \int_0^\infty z^{-\frac{pn}{2}} \exp\left(-\frac{1}{2z} tr((\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\Phi^{-1})\right) dG(z)\,.$$

defines the p.d.f. of a m.e.c. distribution. In this case, the distribution of $\mathbf{X}$ *is said to be a mixture of normal distributions.*

In particular, if $G(1) = 1 - \varepsilon$ and $G(\sigma^2) = \varepsilon$ with $0 < \varepsilon < 1$, $\sigma^2 > 0$, we obtain the ε-contaminated matrix variate normal distribution. It has the p.d.f.

$$f(\mathbf{X}) = \frac{1}{(2\pi)^{\frac{pn}{2}} |\Sigma|^{\frac{n}{2}} |\boldsymbol{\Phi}|^{\frac{p}{2}}} \left[(1-\varepsilon) etr\left(-\frac{1}{2}(\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Phi}^{-1} \right) \right.$$
$$\left. + \frac{\varepsilon}{\sigma^{pn}} etr\left(-\frac{1}{2\sigma^2}(\mathbf{X}-\mathbf{M})'\Sigma^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Phi}^{-1} \right) \right].$$

http://www.springer.com/978-1-4614-8153-9

Elliptically Contoured Models in Statistics and Portfolio Theory
Gupta, A.K.; Varga, T.; Bodnar, T.
2013, XX, 321 p. 7 illus., Hardcover
ISBN: 978-1-4614-8153-9