(b)
$$\int_{[0,z]} f(u)W(du)\int_{[0,z]} g(u)W(du) = \int_{[0,z]} f(x,y)\left(\int_0^x\int_0^t g(\alpha,\beta)d\alpha d\beta\right)W(dx,dy)$$
$$+\int_{[0,z]} g(x,y)\left(\int_0^x\int_0^t f(\alpha,\beta)W(d\alpha,d\beta)\right)dx,dy.$$

Proof To prove the first relation we apply (1.5). We obtain

$$\begin{aligned}
&\int_{[0,z]} f\left(z'\right)W\left(dz'\right)\int_{[0,z]} g\left(z'\right)W\left(dz'\right)\\
&\quad=\frac{1}{2}\left(\int_{[0,z]}\left(f\left(z'\right)+g\left(z'\right)\right)W\left(dz'\right)\right)^2\\
&\qquad-\frac{1}{2}\left(\int_{[0,z]} f\left(z'\right)W\left(dz'\right)\right)^2-\frac{1}{2}\left(\int_{[0,z]} g\left(z'\right)W\left(dz'\right)\right)^2\\
&\quad=\int_0^t\int_0^s\left(\int_0^u\int_0^s(f(\lambda,\mu)+g(\lambda,\mu))W(d\lambda,d\mu)\right)(f(u,v)+g(u,v))W(du,dv)\\
&\qquad+\frac{1}{2}\int_{[0,z]}\left(f\left(z'\right)+g\left(z'\right)\right)^2dz'-\int_0^t\int_0^s\left(\int_0^u\int_0^s f(\lambda,\mu)W(d\lambda,d\mu)\right)f(u,v)W(du,dv)\\
&\qquad-\frac{1}{2}\int_{[0,z]} f^2\left(z'\right)dz'-\int_0^t\int_0^s\left(\int_0^u\int_0^s g(\lambda,\mu)W(d\lambda,d\mu)\right)g(u,v)W(du,dv)-\frac{1}{2}\int_{[0,z]} g^2\left(z'\right)dz'\\
&\quad=\int_{[0,z]} f(u)g(u)du+\int_{[0,z]} f(x,y)\left(\int_0^x\int_0^t g(\alpha,\beta)W(d\alpha,d\beta)\right)W(dx,dy)\\
&\qquad+\int_{[0,z]} g(x,y)\left(\int_0^x\int_0^t f(\alpha,\beta)W(d\alpha,d\beta)\right)W(dx,dy).
\end{aligned}$$

Thus, (a) is proved. The property (b) can be proved similarly, using (1.5) for the right-hand side part. □

Corollary 1.3 *Under the conditions of Theorem 1.17 we have*

$$E\int_{[0,z]} f(u)W(du)\int_{[0,z]} g(u)W(du) = \int_{[0,z]} Ef(u)g(u)du.$$

Below we give a more general version of the Ito formula.

Theorem 1.20 [30, 31] *Let the conditions of Theorem 1.14 be satisfied for the random field* $(\xi(z), \Im_z)$. *Then*

$$\begin{aligned}
f(\xi(z)) = & f(0) + \int_{[0,z]} f'(\xi(u))\phi(u)du + \int_{[0,z]} f'(\xi(u))\sigma(u)W(du) \\
& + \int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \phi(x,\beta)dx \int_0^\beta \phi(\alpha,y)dyd\alpha d\beta \\
& + \int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \phi(x,\beta)dx \int_0^\beta \sigma(\alpha,y)W(d\alpha,dy)d\beta \\
& + \int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \sigma(x,\beta)W(dx,d\beta) \int_0^\beta \phi(\alpha,y)dyd\alpha \\
& + \int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \sigma(x,\beta)W(dx,d\beta) \int_0^\beta \sigma(\alpha,y)W(d\alpha,dy) \\
& + \frac{1}{2}\int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \phi(x,\beta)dx \int_0^\beta \sigma^2(\alpha,y)dyd\alpha d\beta \\
& + \frac{1}{2}\int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \sigma(x,\beta)W(dx,d\beta) \int_0^\beta \sigma^2(\alpha,y)dyd\alpha \\
& + \frac{1}{2}\int_{[0,z]} f''(\xi(\alpha,\beta)\int_0^\alpha \sigma^2(x,\beta)dx \int_0^\beta \sigma(\alpha,y)W(d\alpha,dy)d\beta \\
& + \frac{1}{4}\int_{[0,z]} f'''(\xi(\alpha,\beta)\int_0^\alpha \sigma^2(x,\beta)dx \int_0^\beta \sigma^2(\alpha,y)dyd\alpha d\beta
\end{aligned}$$

holds true P *-almost surely for any bounded four times differentiable function* f(z)*.*

As an example of the application of the Ito formula, we consider below the generalization of the Lévy-Doob theorem.

Theorem 1.21 [25, 42, 64] *If* μ(z) *is a square integrable strong martingale with* Eμ $= 0$*,* $E(\mu(z,z']/\Im_z^*) = (t'-t)(s'-s)$*, then* μ(z) *is a Wiener martingale.*

Proof We apply the Ito formula to the function $\exp\left\{i\alpha\mu_{(z_0,z]}\right\}$. Denote by $j(\alpha,z)$ the characteristic function of the conditional distribution $\mu(z_0,\ z]$, $z \geq z_0$,

$$j(\alpha,z) = E\left(\exp\left\{i\alpha\mu_{(z_0,z]}\right\}/\Im_{z_0}^*\right).$$

We obtain

$$j(\alpha, z) = 1 - \int\limits_{(z_0,z]} \left(\frac{\alpha^2}{2} j(\alpha, z') + \frac{\alpha^4}{4} j(\alpha, z')(t' - t_0)(s' - s_0) \right) dt' ds'.$$

Let $\mu_1(z) := \mu(z_0,\ z]$, $z \geq z_0$. Therefore,

$$E\left(\int\limits_{(z_0,z]} j(\alpha, z')\mu(dt', s')\mu(t', ds')/\Im^*_{z_0} \right) = \int\limits_{(z_0,z]} j(\alpha, z')\mu_1(dt', s')\mu 1(t', ds').$$

This equation is equivalent to the following one:

$$\frac{\partial^2 j}{\partial t \partial s} = -\left(\frac{\alpha^2}{2} - \frac{\alpha^4}{4}(t - t_0)(s - s_0) \right) j, \quad j(t_0, s) = j(t, s_0) = 1,$$

whose solution is $j(\alpha, z) = e^{-\frac{\alpha^2}{2}(t-t_0)(s-s_0)}$. Thus, the random variable $\mu(z_0,\ z]$ does not depend on the σ-algebra $\Im^*_{z_0}$ and has the Gaussian distribution with mean 0 and variance $(t - t_0)(s - s_0)$. □

1.3 Stochastic Measures and Integrals for Nonrandom Functions

In applications, a lot of models use the integrals of the form $\int f(z)\xi(dz)$, where $f(z)$ is a nonrandom function, and $\xi(z)$ is a random field. Since in general $\xi(z)$ is not expected to be of bounded variation, such an integral cannot be treated in the Stieltjes or Lebesgue-Stieltjes sense. Nevertheless, it can be determined in such a way that it inherits the properties of the Lebesgue-Stieltjes integral.

Following [28], we consider the probability space $(\Omega, \Im, P)$, $L_2 = L_2(\Omega, \Im, P)$, S is some set, $\Re$ is the semiring generated by sub-sets of S. For any $\Delta \in \Re$ consider the complex-valued variable $\mu(\Delta)$ satisfying the following conditions:

(1) $\mu(\Delta) \in L_2(\Omega, \Im, P)$, $\mu(\varnothing) = 0$,
(2) $\mu(\Delta_1 \cup \Delta_2) = \mu(\Delta_1) + \mu(\Delta_1)$ (mod P), if $\Delta_1 \cap \Delta_2 = \varnothing$,
(3) $E\mu(\Delta_1)\overline{\mu(\Delta_2)} = m(\Delta_1 \cap \Delta_2)$,
where $m(\Delta)$ is some set function on $\Re$.

We call the family of random variables $\{\mu(\Delta)\}$, $\Delta \in \Re$, the elementary orthogonal stochastic measure, and respective function $m(\Delta)$ its structure function. In this context orthogonality means that

(4) if $\Delta_1 \cap \Delta_2 = \varnothing$, then the variables $\mu(\Delta_1)$ and $\mu(\Delta_2)$ are orthogonal.

By definition, the function $m(\Delta)$ is nonnegative: $m(\Delta) = E|\mu(\Delta)|^2 \geq 0$, $m(\emptyset) = 0$, and additive: if $\Delta_1 \cap \Delta_2 = \emptyset$, then

$$\begin{aligned} m(\Delta_1 \cup \Delta_2) &= E|\mu(\Delta_1) + \mu(\Delta_1)|^2 = m(\Delta_1) + m(\Delta_2) + 2m(\Delta_1 \cap \Delta_2) \\ &= m(\Delta_1) + m(\Delta_2). \end{aligned}$$

Denote by $L_0\{\mathfrak{R}\}$ the class of all simple functions $f(x)$, $f(x) = \sum_{k=1}^{n} c_k \chi_{\Delta_k}(x)$, $\Delta_k \in \mathfrak{R}$, $k = 1, \ldots, n$, where n is any natural number, and $\chi_A(x)$ is the indicator of a set A. For functions $f \in L_0\{\mathfrak{R}\}$ we define the stochastic integral with respect to the elementary stochastic measure μ by the formula

$$I := \int f(x)\mu(dx) = \sum_{k=1}^{n} c_k \mu(\Delta_k). \tag{1.6}$$

Since $\mathfrak{R}$ is a semiring, any function from $L_0\{\mathfrak{R}\}$ can be represented as linear combination of indicators of sets from $\mathfrak{R}$. Let $f, g \in L_0\{\mathfrak{R}\}$, and put

$$g(x) = \sum_{k=1}^{n} d_k \chi_{\Delta_k}(x),$$

where $\Delta_k \cap \Delta_l = \emptyset$ if $k \neq l$.

It follows from the orthogonality of the measure that

$$E\left(\int f(x)\mu(dx)\overline{\int g(x)\mu(dx)}\right) = \sum_{k=1}^{n} c_k \overline{d}_k m(\Delta_k).$$

Suppose that the elementary measure m is sub-additive, and hence can be extended to a full measure on $(E, \aleph, m)$. Then $L_0\{\mathfrak{R}\}$ is a linear subspace of the Hilbert space $L_2\{m\} = L_2(E, \aleph, m)$, and $L_2\{\mathfrak{R}\}$ is the closure of $L_0\{\mathfrak{R}\}$ in the topology generated by the scalar product $(f, g) = \int f(x)\overline{g(x)}\, m(dx)$. At the same time,

$$E\left(\int f(x)\mu(dx)\overline{\int g(x)\mu(dx)}\right) = \int f(x)\overline{g(x)}\, m(dx) \tag{1.7}$$

for any pair of functions $f, g \in L_0\{\mathfrak{R}\}$.

Denote by $\widetilde{L}_0\{\mu\}$ the linear span of the family of random variables $\{\mu(\Delta)\}, \Delta \in \mathfrak{R}$ (that is, $\widetilde{L}_0\{\mu\}$ is the set of random variables represented in the form (1.6)), and denote by $\widetilde{L}_2\{\mu\}$ the closure of $\widetilde{L}_0\{\mu\}$ in Hilbert space $L_2(\Omega, \mathfrak{I}, P)$. Relation (1.6) sets the isometric correspondence $I = \psi(f)$ between $L_0\{\mathfrak{R}\}$ and $\widetilde{L}_0\{\mu\}$, which can be extended [63] to the isometric correspondence between $L_2\{\mathfrak{R}\}$ and $\widetilde{L}_2\{\mu\}$. If $I = \psi(f), f \in L_2\{\mathfrak{R}\}$, we define

$$I = \psi(f) = \int f(x)\mu(dx) \tag{1.8}$$

and call the random variable I the stochastic integral of the function $f(x)$ with respect to the measure μ. The properties of such an integral are given in the theorem below.

Theorem 1.22 [25]

(a) for a simple function the stochastic integral can be defined by (1.6);
(b) for any f, $g \in L_2(E,\aleph,m)$ equality (1.7) holds true;
(c) $\int[\alpha f(x) + \beta g(x)]\mu(dx) = \alpha \int f(x)\mu(dx) + \beta \int g(x)\mu(dx)$;
(d) for any sequence of functions $f_n \in L_2(E,\aleph,m)$ such that

$$\int |f(x) - f_n(x)|^2 m(dx) \to 0, \quad n \to \infty, \tag{1.9}$$

we have $\int \mathrm{f(x)\mu(dx)} = \mathit{lim} \int \mathrm{f_n(x)\mu(dx)}$.

Remark 1.13 In particular, if $f_n(x)$, $n \geq 1$, is a sequence of simple functions, $f_n(x) = \sum_{k=1}^{m_n} c_k^{(n)} \chi_{\Delta_k^{(n)}}(x)$, $\Delta_k^{(n)} \in \mathfrak{R}$, $n = 1, 2, \ldots$, and (1.9) is satisfied, then

$$\int f(x)\mu(dx) = \lim \sum_{k=1}^{m_n} c_k^{(n)} \mu\left(\Delta_k^{(n)}\right).$$

The existence of the sequence of simple functions which approximates an arbitrary $L_2(E,\aleph,m)$ -function, was proved, for instance, in [35]. Thus, the stochastic integral can be considered as the mean square limit of appropriate integral sums.

Denote by $\aleph_0$ the class of all sets $A \in \aleph$, for which $m(\Delta) < \infty$. Define a random function $\widetilde{\mu}(A)$ of the set A:

$$\widetilde{\mu}(A) = \int \chi_A(x)\mu(dx) = \int_A \mu(dx). \tag{1.10}$$

This function possesses the following properties:

(a) $\widetilde{\mu}(A)$ is well defined on sets from $\aleph_0$

(b) if $A_n \in \aleph_0$, $n = 1, 2, \ldots$, $A_0 = \bigcup_{n=1}^{\infty} A_n$, $A_k \cap A_l = \varnothing$, $k \neq l$, then $\widetilde{\mu}(A_0) = \sum_{n=1}^{\infty} \widetilde{\mu}(A_n)$ (mod P);

(c) $E\widetilde{\mu}(A)\widetilde{\mu}(B) = m(A \cap B)$, A, $B \in \aleph_0$;
(d) $\widetilde{\mu}(\Delta) = \mu(\Delta)$ as $\Delta \in \Re$.

A random set function which satisfies the conditions (a), (b), (c) is called the stochastic orthogonal measure. Property (d) means that $\widetilde{\mu}(A)$ is the extension of the elementary stochastic measure μ.

Corollary 1.4 *If the structure function of an elementary stochastic measure* μ *is sub-additive, then* μ *can be extended to a stochastic measure.*

Remark 1.14 Since $\widetilde{L}_2\{\mu\} = \widetilde{L}_2\{\widetilde{\mu}\}$,

$$\int f(x)\mu(dx) = \int f(x)\widetilde{\mu}(dx).$$

In the future, we identify a stochastic integral with respect to the elementary orthogonal measure μ with sub-additive structure function and the stochastic integral with respect to the measure $\widetilde{\mu}$, defined in (1.10).

Let μ be the orthogonal stochastic measure with structure function m, which is the full measure on $\{E, \aleph\}$ and $g(z) \in \widetilde{L}_2\{m\}$. Put

$$\lambda(A) = \int \chi_A(x)g(x)\mu(dx), A \in \aleph$$

Then

$$E\lambda(A)\overline{\lambda(B)} = \int \chi_A(x)\chi_B(x)g^2(x)m(dx) = \int_{A \cap B} |g(x)|^2 m(dx).$$

If we introduce a new measure

$$l(A) = \int_A |g(x)|^2 m(dx)$$

on $\aleph$, then $\lambda(A)$ will be the orthogonal stochastic measure with the structure function $l(A)$, $A \in \aleph$.

Lemma 1.3 *If* $f \in \widetilde{L}_2\{l\}$, *then* $fg \in \widetilde{L}_2\{m\}$ *and* $\int f(x)\lambda(dx) = \int f(x)g(x)\mu(dx)$.

Proof The statement of the lemma is obvious for simple functions $f(x) = \sum_k c_k \chi_{A_k}(x)$, $A_k \in \aleph$. If $f_k(x)$ is a fundamental sequence of simple functions in $\widetilde{L}_2\{l\}$, then

$$\left\|\int f_n(x)\lambda(dx) - \int f_{n+m}(x)\lambda(dx)\right\|^2 = \int \left|f_n(x) - f_{n+m}(x)\right|^2 l(dx)$$
$$= \int \left|f_n(x) - f_{n+m}(x)\right|^2 |g(x)|^2 m(dx).$$

That is, the sequence $f_n(x)g(x)$ is fundamental in $\widetilde{L}_2\{m\}$. Passing to the limit as $n \to \infty$ in

$$\int f_n(x)\lambda(dx) = \int f_n(x)g(x)\mu(dx),$$

we obtain the statement of the lemma. □

Lemma 1.4 *If* $\lambda(A) = \int \chi_A(x)g(x)\mu(dx)$ *and* $g\in\widetilde{L}_2\{m\}$, *then for any* $A \in \aleph_0$

$$\mu(A) = \int \frac{1}{g(x)}\chi_A(x)\lambda(dx).$$

Proof First, $g(x) = 0$ on the set of l -measure 0, implying $\dfrac{1}{g(x)} \neq \infty \pmod{l}$. Moreover, if $A \in \aleph_0$, then

$$\int \frac{1}{|g(x)|^2}\chi_A(x)l(dx) = \int_A \frac{1}{|g(x)|^2}|g(x)|^2 m(dx) = m(A) < \infty.$$

From Lemma 1.3 we derive

$$\int \frac{1}{g(x)}\chi_A(x)l(dx) = \int \frac{1}{g(x)}\chi_A(x)g(x)\mu(dx) = \mu(A).$$ □

Let D be a rectangle (finite or infinite) on the plane, $\mathcal{B}$ is the σ-algebra of Lebesgue measurable subsets of D, and l is the Lebesgue measure. Suppose that the function $g(z,x)$ is $\mathcal{B} \times \aleph$ -measurable, $g(z,x)\in\widetilde{L}_2\{l \times m\}$ and $g(z,x)\in\widetilde{L}_2\{m\}$ for arbitrary $z \in D$. Consider the stochastic integral

$$\xi(z) = \int g(z,x)\mu(dx) \tag{1.11}$$

which is well defined for any z with probability 1.

Lemma 1.5 *The stochastic integral (1.11) can be determined as a function of* z *in such a way that the random field* ξ(z) *is measurable.*

Proof If $g(z,x) = \sum c_k\chi_{B_k}(z)\chi_{A_k}(x)$, $B_k \in \mathcal{B}$, $A_k \in \aleph$, then $\xi(z) = \sum c_k\chi_{B_k}(z)$ $\mu(A_k)$ is $\mathcal{B} \times \Im$ -measurable function of variables $z \in D$ and $\omega \in \Im$. In the general case we can build a sequence of simple functions $g_n(z,x)$ such that

$$\int\int |g(z,x) - g_n(z,x)|^2 m(dx)dz \to 0, \text{ as } n \to \infty.$$

Let $\xi_n(z)$ be a sequence of random fields constructed as in (1.11) with $g = g_n$. Then there exists a field $\widetilde{\xi}(z)$ such that $\int E\left|\widetilde{\xi}(z) - \xi_n(z)\right|^2 dz \to 0$ as $n \to \infty$, and $\widetilde{\xi}(z)$ is a $\mathcal{B}_k \in \mathcal{B} \times \Im$ -measurable function.

On the other hand,

$$\int E|\xi(z) - \xi_n(z)|^2 dz = \iint |g(z,x) - g_n(z,x)|^2 m(dx)dz \to 0,$$

implying that the random fields $\xi(z)$ and $\widetilde{\xi}(z)$ coincide for almost all z. Put

$$\hat{\xi}(z) = \begin{cases} \widetilde{\xi}(z), & P\left\{\xi(z) \neq \widetilde{\xi}(z)\right\} = 0, \\ \xi(z), & P\left\{\xi(z) \neq \widetilde{\xi}(z)\right\} > 0. \end{cases}$$

The field $\hat{\xi}(z)$ is measurable (since $\hat{\xi}(z)$ differs from a $B_k \in \mathcal{B} \times \Im$ -measurable function $\widetilde{\xi}(z)$ only on a set of measure 0) and is stochastically equivalent to $\xi(z)$. □

Everywhere below we assume that the random field determined by a stochastic integral of the form (1.11) and satisfying the conditions of Lemma 1.3 is measurable.

Lemma 1.6 *If* g(z,x) *and* h(z) *are Borel functions,* I = *[*a$_1$,b$_1$*]* × *[*a$_2$,b$_2$*]* ⊂ D,

$$\int_I \int_{-\infty}^{\infty} |g(z,x)|^2 dz m(dx) < \infty, \int_I |h(z)|^2 dz < \infty,$$

and μ *is an orthogonal stochastic measure on* $\mathcal{B}$*, then*

$$\int_I h(z) \int_{-\infty}^{\infty} g(z,x)\mu(dx)dz = \int_{-\infty}^{\infty} g_1(x)\mu(dx), \tag{1.12}$$

where $g_1(x) = \int_I h(z)g(z,x)dz.$

Proof Let us estimate the square of the mean in the left-hand side of (1.12). We have

$$\int_I \int_I h(z_1)\overline{h(z_2)} \int_{-\infty}^{\infty} g(z_1,x)\overline{g(z_2,x)}\, m(dx)dz_1dz_2$$
$$= \int_{-\infty}^{\infty} \left| \int_I h(z)g(z,x)dz \right|^2 m(dx)$$
$$\le \int_I |h(z)|^2 dz \int_{-\infty}^{\infty} \int_I |g(z,x)|^2 dzm(dx).$$

We obtain similar inequality for the right-hand side of relation (1.12) as well. Thus, we can take in (1.12) a sequence $g_n(z,x)$ converging in $\widetilde{L}_2\{l \otimes m\}$, and pass to the limit. Moreover, the set of functions $g(z,x)$, for which (1.12) holds true, is linear and contains all functions of the form $g(z,x) = \sum c_k \chi_{B_k}(z)\chi_{A_k}(x)$, Hence, it contains all functions of $L_2\{l \otimes m\}$. □

Lemma 1.7 *If the conditions of Lemma 1.3 hold true for each bounded* I ⊂ D, I = *[*a_1,b_1*]* × *[*a_2,b_2*], and the integral*

$$\int_{R^2} h(z)g(z,x)dz = \lim_{\substack{a_i \to -\infty \\ b_i \to \infty}} \int_I h(z)g(z,x)dz, \quad i = 1,2,$$

exists in the sense of convergence in L_2{m}*, then*

$$\int_{R^2} h(z) \int_{-\infty}^{\infty} g(z,x)\mu(dx)dz = \int_{-\infty}^{\infty} g_1(x)\mu(dx), \quad g_1(x) = \int_I h(z)g(z,x)dz.$$

The proof follows immediately from the fact that $\int_{R^2} h(z) \int_{-\infty}^{\infty} g(z,x)\mu(dx)dz$ *is the mean square limit of (1.12), and one can pass to the limit under the sign of the stochastic integral in the right-hand side of (1.12).*

Chapter 2
Stochastic Differential Equations on the Plane

In this chapter we investigate diffusion-type fields and Ito fields on the plane, two-parameter version of the Girsanov theorem, weak and strong solutions of stochastic differential equations on the plane, and the probability measures generated by stochastic fields. The results presented in this chapter are published in [10, 12, 14, 16, 25, 42, 44, 45, 47, 48, 65, 71].

2.1 Ito and Diffusion-Type Stochastic Fields

Denote by $\left(C[0,T]^2, \mathscr{B}\right)$ the space of continuous functions on $[0,T]^2$. We denote by $\|\cdot\|$ the uniform norm in $C[0,T]^2$ and by $\|\cdot\|_z$ —the uniform norm in $C[0,z]^2$. In this chapter we suppose that the Cairoli-Walsh condition F4 is fulfilled. Let $f \in C[0,T]^2$ and assume in addition that satisfies the boundary conditions $f(0,s) = f(t,0) = 0$. In the space $\left(C[0,T]^2, \mathscr{B}\right)$ we also define the σ-algebras $\mathscr{B} = \sigma\left\{f(z), z \in [0,T]^2\right\}$ and $\mathscr{B}_z = \sigma\{f(z'), z' \le z\}$.

Definition 2.1 We say that a measurable with respect to both variables functional $\varphi(z,f)$, $z \in [0,T]^2$, $f \in C[0,T]^2$, does not depend on the future, if for any z the functional $\varphi(z,f)$ is $\mathscr{B}_z$-measurable.

Definition 2.2 A continuous stochastic field $(\xi(z), \Im_z)$ defined on the probability space $(\Omega, \Im, P)$ is called

(a) An Ito field (with respect to the Wiener field $(W(z), \Im_z)$), if there exist random fields $(A(z), \Im_z)$ and $(B(z), \Im_z)$, $z \in [0,T]^2$, such that

$$\int_{[0,T]^2} |A(z)| dz < \infty \quad \text{and} \quad \int_{[0,T]^2} B^2(z) dz < \infty \quad P\text{-a.s.,} \tag{2.1}$$

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and for any $z \in [0,T]^2$

$$\xi(z) = \int_{[0,z]} A(u)du + \int_{[0,z]} B(u)W(du)$$

with probability 1.

(b) A diffusion-type field (with respect to the Wiener field $(W(z), \Im_z)$), if there exist functionals $a(z,f)$ and $b(z,f)$, $z \in [0,T]^2$, $f \in C[0,T]^2$, which do not depend on the future,

$$\int_{[0,T]^2} |a(z,\xi)|dz < \infty \quad \text{and} \quad \int_{[0,T]^2} b^2(z,\xi)dz < \infty \quad P\text{-a.s.},$$

and for any $z \in [0,T]^2$

$$\xi(z) = \int_{[0,z]} a(u,\xi)du + \int_{[0,z]} b(u,\xi)W(du)$$

with probability 1.

Let $\overline{f}^{-1}(z) := \begin{cases} f^{-1}(z), & f(z) \neq 0, \\ 0, & f(z) = 0. \end{cases}$

Often Ito fields can be represented as diffusion-type fields, probably with respect to another Wiener field.

Theorem 2.1 [14, 16] *Assume that $(\xi(z), \Im_z)$ is an Ito field and that stochastic fields $(A(z), \Im_z)$ and $(B(z), \Im_z)$, $z \in [0,T]^2$, satisfy condition (2.1) as well as the conditions below:*

(a) For almost all $z \in [0,T]^2$ the inequality $B^2(z) > 0$ holds true with probability 1.

(b) $\int_{[0,T]^2} E|A(z)|dz < \infty$.

(c) $\int_{[0,T]^2} E\left|\overline{B}^{-1}(z)\right||A(z)|dz < \infty$.

Then in the space $\left(C[0,T]^2, \mathcal{B}\right)$ there exist measurable functionals $\alpha(z,f)$ and $\beta(z,f)$,

$$\alpha(z,\xi) = E\left(A(z)/\Im_z^\xi\right), \beta(z,\xi) = \left(E\left(B^2(z)/\Im_z^\xi\right)\right)^{1/2},$$

$\Im_z^\xi = \sigma\left\{\xi(z'), z' \in [0,T]^2, z' \leq z\right\}$ *and a Wiener field $\left(\hat{W}(z), \Im_z^\xi\right)$, such that*

$$\xi(z) = \int_{[0,z]} \alpha(u,\xi)du + \int_{[0,z]} \beta(u,\xi)\hat{W}(du)$$

with probability 1.

Proof On the space $\left(C[0,T]^2, \mathcal{B}\right)$ we construct a stochastic field $\zeta(z,\omega)$, $z \in [0,T]^2$, as $\zeta(z,\omega) = E\left(A(z)/\Im_z^\xi\right)$. Since the field is $\zeta(z,\omega)$ obviously measurable, $\Im_z^\xi$-adapted, then from condition (b) we obtain for any $z \in [0,T]^2$ (see [52] for the similar trick)

$$\{(\omega, z' \leq z) : \xi(z',\omega) \in \mathrm{B}\} \in \Im_z \times \mathcal{B}_z,$$

where B is a Borel set in $R, \mathcal{B}_z$ is a σ-algebra of Borel sets on $[0,z]$. This means that the field $\zeta(z,\omega)$, $z \in [0,T]^2$, has a progressively measurable modification. Therefore, without loss of generality, we assume that the field $\zeta(z)$ is progressively measurable. Then for any $z' \in [0,T]^2$ the stochastic field $\zeta(z \wedge z', \omega)$ (as a function of z) is measurable in the space $\left([0,T]^2 \times \Omega,\ \mathcal{B}_z \times \Im_z^\xi,\ \mu \times P\right)$, where μ is the Lebesgue measure on $[0,z]$. Therefore there exists a $\Im_z^\xi$-adapted functional $\alpha_{z_0}(z,f)$ such that

$$\mu \times P\{(z,\omega) : \zeta(\min(z,z_0),\omega) \neq \alpha_{z_0}(z,\xi(\omega))\} = 0.$$

For any $n \geq 1$ we put $z_{k,j,n} = (2^{-n}k,\ 2^{-n}j)$ and

$$I_{k,j,n} = \left(z_{k-1,j-1,n}, z_{k,j-1,n}\right] \times \left(z_{k,j-1,n}, z_{k,j,n}\right].$$

Then we build piecewise constant functions

$$\alpha^{(n)}(z,f) = \alpha_0(0,f)\chi_{\{0\}}(z) + \sum_{k=1}^{2^n}\sum_{j=1}^{2^n} \alpha_{z_{k,j,n}}\chi_{I_{k,j,n}}(z),$$

and put

$$\alpha(z,f) = \lim_{n\to\infty} \alpha^{(n)}(z,f).$$

For any $z \in [0,T]^2$ the functional $\alpha(z,f)$ is $\mathcal{B}_z$-measurable. Moreover, for any $\varepsilon > 0$

$$\begin{aligned}\{(z,\omega) : |\alpha(z,\xi(\omega)) - \zeta(z,\omega)| > \varepsilon\} &\subset \left\{(z,\omega) : \left|\alpha(z,\xi(\omega)) - \alpha^{(n)}(z,\xi(\omega))\right| > \varepsilon/2\right\} \\ &\cup \left\{(0,\omega) : \left|\alpha^0(z,\xi(\omega)) - \zeta(z,\omega)\right| > \varepsilon/2\right\} \\ &\cup \left\{\bigcup_{k=1}^{2^n}\left\{(z,\omega) : z \in I_{k,j,n}, \left|\alpha_{z_{k,j,n}}(z,\xi(\omega)) - \zeta\left(\min\left(z, z_{k,j,n}\right)\right)\right| > \varepsilon/2\right\}\right\}.\end{aligned}$$

Therefore,

$$\mu \times P\{(z,\omega) : |\alpha(z,\xi(\omega)) - \zeta(z,\omega)| > \varepsilon\}$$
$$\leq \mu \times P\left\{(z,\omega) : \left|\alpha(z,\xi(\omega)) - \alpha^{(n)}(z,\xi(\omega))\right| > \varepsilon/2\right\}.$$

As $\alpha(z,f) = \overline{\lim}_{n\to\infty}\alpha^{(n)}(z,f)$, there exists a subsequence $\{n_j\}$, $j = 1, 2, \ldots$ such that

$$\lim_{n_j\to\infty} \mu \times P\left\{(z,\omega) : \left|\alpha(z,\xi(\omega)) - \alpha^{(n_j)}(z,\xi(\omega))\right| > \varepsilon/2\right\} = 0$$

Hence, for any $\varepsilon > 0$

$$\lim_{n_j\to\infty} \mu \times P\{(z,\omega) : |\alpha(z,\xi(\omega)) - \zeta(z,\omega)| > \varepsilon/2\} = 0.$$

Now we need to construct the functional $\beta(z,\xi)$. For this purpose we consider the partition of the set $(0,z]$, such that $0 = t_0{}^{(n)} \leq t_1{}^{(n)} \leq \ldots \leq t_n{}^{(n)} = t$, and $0 = s_0{}^{(n)} \leq s_1{}^{(n)} \leq \ldots \leq s_n{}^{(n)} = s$. Let $z_{ij}{}^{(n)} = (t_i{}^{(n)}, s_j{}^{(n)})$, $I_{i,j,n} = (t_i{}^{(n)}, t_{i+1}{}^{(n)}] \times (s_j{}^{(n)}, s_{j+1}{}^{(n)}]$ and assume that diam $I_{i,j,n} \to 0$, $n \to \infty$. The increments on these intervals satisfy

$$\sum_{i,j=0}^{n-1} \left(\xi\left(z_{ij}^{(n)}, z_{i+1,j+1}^{(n)}\right]\right)^2 = \sum_{i,j=0}^{n-1} \left(\int_{I_{i,j,n}} A(u)du\right)^2 + \sum_{i,j=0}^{n-1} \left(\int_{I_{i,j,n}} B(u)W(du)\right)^2$$
$$+ 2\sum_{i,j=0}^{n-1} \left(\int_{I_{i,j,n}} B(u)W(du)\right) \times \left(\int_{I_{i,j,n}} A(u)du\right).$$

Performing easy calculations we obtain

$$\sum_{i,j=0}^{n-1} \left(\int_{I_{i,j,n}} A(u)du\right)^2 \leq \max_{i,j}\left|\int_{I_{i,j,n}} A(u)du\right|\left|\int_{[0,T]^2} A(u)du\right| \to 0, \quad n\to\infty$$

and

$$\left|\sum_{i,j=0}^{n-1} \left(\int_{I_{i,j,n}} B(u)W(du)\right) \times \left(\int_{I_{i,j,n}} A(u)du\right)\right|$$
$$\leq \max_{i,j}\left|\int_{I_{i,j,n}} B(u)W(du)\right| \times \left|\int_{[0,T]^2} A(u)du\right| \to 0, \quad n\to\infty.$$

Taking into account Theorem 1.17 we get

$$\sum_{i,j=0}^{n-1}\left(\int_{I_{i,j,n}} B(u)W(du)\right)^2 = \sum_{i,j=0}^{n-1}\left(\int_{I_{i,j,n}} B^2(u)du\right)$$
$$+2\sum_{i,j=0}^{n-1}\left(\int_{I_{i,j,n}}\int_{\left[z_{ij}^{(n)},z'\right]} B(u)W(du)B(z')W(dz')\right)^2$$
$$=\int_{[0,z]} B^2(u)du + 2\int_{[0,z]} f_n(u)B(u)W(du),$$

where $f_n(z')$ is defined by

$$f_n(z') = \int_{\left[z_{ij}^{(n)},z,'\right]} B(u)W(du), \quad t_i^{(n)} \le t' \le t_{i+1}^{(n)}, \quad s_j^{(n)} \le s' \le s_{j+1}^{(n)},$$

and

$$\int_{[0,T]^2} f_n^2(u)B^2(u)du \le \max_{i,j}\ \sup_{I_{ij}} f_n^2(u)\int_{[0,T]^2} B^2(u)du,$$

Therefore,

$$\sum_{i,j=0}^{n-1}\left(\xi\left(z_{ij}^{(n)}, z_{i+1,j+1}^{(n)}\right]\right)^2 \to \int_{[0,z]} B^2(u)du \text{ as } n\to\infty \text{ with probability 1.}$$

Thus, for any $z \in [0,T]^2$ the stochastic field $\int_{[0,z]}B^2(u)du$ is $\mathfrak{I}_z^\xi$-measurable, and repeating the calculations for deriving the expression for α, it is easy to show the existence of β.

Consider now the stochastic field $\left(\hat{W}(z), \mathfrak{I}_z^\xi\right)$, $z \in [0,T]^2$, given by

$$\hat{W}(z) = \int_{[0,z]} \overline{\beta}^{-1}(u,\xi)\xi(du) - \int_{[0,z]} \overline{\beta}^{-1}(u,\xi)\alpha(u,\xi)du$$

and show that it is a Wiener field. By definition of ξ we have

$$\hat{W}(z) = \int_{[0,z]} \overline{\beta}^{-1}(u,\xi)B(u)W(du) + \int_{[0,z]} \overline{\beta}^{-1}(u,\xi)(A(u) - \alpha(u,\xi))du,$$

where the existence of these integrals is guaranteed by conditions (c) and (2.1). Condition (a) guarantees that $B(z)\overline{B}^{-1}(z) = 1$ P-a.s. For any $0 \le t \le t' \le T$, $0 \le s \le s' \le T$, $-\infty < \theta < \infty$ we have by the Ito formula

$$
\begin{aligned}
&\exp\left\{i\theta\left(\hat{W}(t,s') - \hat{W}(t,s)\right)\right\} \\
&\quad = 1 + i\theta \int_{[0,t]\times[s,s']} \exp\left\{i\theta\left(\hat{W}(t,y) - \hat{W}(t,s)\right)\right\}\overline{\beta}^{-1}(x,y,\xi)B(x,y)W(dx,dy) \\
&\qquad + i\theta \int_{[0,t]\times[s,s']} \exp\left\{i\theta\left(\hat{W}(t,y) - \hat{W}(t,s)\right)\right\}\overline{\beta}^{-1}(x,y,\xi)[A(x,y) - \alpha(x,y,\xi)]dxdy \\
&\qquad - \frac{\theta^2}{2}\int_{[0,t]\times[s,s']} \exp\left\{i\theta\left(\hat{W}(t,y) - \hat{W}(t,s)\right)\right\}dxdy
\end{aligned}
$$

Then, taking the conditional expectation, we get

$$
E\left(\exp\left\{i\theta\left(\hat{W}(t,s') - \hat{W}(t,s)\right)\right\}/\mathfrak{I}_z^{\xi 2}\right) = \exp\left(-\frac{\theta^2}{2}t(s'-s)\right).
$$

Similarly,

$$
E\left(\exp\left\{i\theta\left(\hat{W}(t',s) - \hat{W}(t,s)\right)\right\}/\mathfrak{I}_z^{\xi 1}\right) = \exp\left(-\frac{\theta^2}{2}(t'-t)s\right).
$$

Therefore, the stochastic field $\left(\hat{W}(z), \mathfrak{I}_z^{\xi}\right)$ is a Wiener field (see Theorem 1.3).

Now we prove that $\xi(z)$ is a diffusion-type field with respect to $\left(\hat{W}(z), \mathfrak{I}_z^{\xi}\right)$. It is easy to see that

$$
\begin{aligned}
\int_{[0,z]} \beta(u,\xi)\hat{W}(du) &= \int_{[0,z]} \beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\xi(du) - \int_{[0,z]} \beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\alpha(u,\xi)du \\
&= \xi(z) - \xi(0) - \int_{[0,z]} \alpha(u,\xi)du + \eta(z)
\end{aligned}
$$

where

$$
\eta(z) = \int_{[0,z]} \left[1 - \beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\right][\xi(du) - \alpha(u,\xi)du]
$$

Now we show that $\eta(z)$ is a martingale.

$$
\begin{aligned}
\eta(z) = &\int_{[0,z]} \left[1 - \beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\right]B(u)W(du) \\
&+ \int_{[0,z]} \left[1 - \beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\right][A(u) - \alpha(u,\xi)]du
\end{aligned}
$$

Condition (a) guarantees that

$$
\left[1 - \beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\right]B(u) = 0
$$

and

$$E\left(\int_{[0,z]}\left[1-\beta(u,\xi)\overline{\beta}^{-1}(u,\xi)\right]B(u)W(du)\right)^2=0.$$

Further, recall that

$$E\left\{(A(z')-\alpha((z'),\xi))/\Im_z^\xi\right\}=E\left\{\left(A(z')-E\left[A(z')/\Im_z^\xi\right]\right)/\Im_z^\xi\right\}=0,$$

a.s. for arbitrary $z\le z'$, implying $E\{\eta(z')/\Im_z^\xi\}=\eta(z)$. Conditions (a)–(c) guarantee that

$$P\left(\int_{[0,T]^2}\left|\left[1-\beta((z),\xi)\overline{\beta}^{-1}((z),\xi)\right][A(z)-\alpha((z),\xi)]\right|dz<\infty\right)=1.$$

Denote $\varphi(z):=\left[1-\beta((z),\xi)\overline{\beta}^{-1}((z),\xi)\right][A(z)-\alpha((z),\xi)]$ and put

$$\tau_N:=\inf\left\{t\le t':\int_0^t\int_0^{s'}|\varphi(z)|dz\ge N\right\}\quad\text{and}\quad\tau_N:=t'\quad\text{if}\quad\int_0^{t'}\int_0^{s'}|\varphi(z)|dz<N.$$

Further, let $\chi^{(N)}(t):=\chi_{\{t\le\tau_N\}}$ and $\varphi^{(N)}(t):=\int_0^t\int_0^{s'}\chi^{(N)}(x)\varphi(x,y)dxdy$. The one-parameter stochastic process $\varphi^{(N)}(t)$, $t\le t'$ with filtration $\Im_t^{(N)}=\Im_{(t\wedge\tau_N,\,s')}$ is the square integrable martingale, hence (see [54])

$$E\left(\varphi^{(N)}(t)\right)^2=\lim_{n\to\infty}\sum_{i=0}^{n-1}E\left(\varphi^{(N)}(t_{i+1})-\varphi_i^{(N)}(t)\right)^2$$

where $0=t_0<\ldots<t_n\le t'$ and $\max_i|t_{i+1}-t_i|\to 0$ as $n\to\infty$. We have

$$\begin{aligned}
E\left(\varphi^{(N)}(t)\right)^2&=\lim_{n\to\infty}\sum_{i=0}^{n-1}E\left(\int_{t_i}^{t_{i+1}}\int_0^{s'}\chi^{(N)}(x)\varphi(x,y)dxdy\right)^2\\
&\le\underline{\lim}_{n\to\infty}E\left\{\max_{i\le n-1}\int_{t_i}^{t_{i+1}}\int_0^{s'}\chi^{(N)}(x)|\varphi(x,y)|dxdy\sum_{i=0}^{n-1}E\int_{t_i}^{t_{i+1}}\int_0^{s'}\chi^{(N)}(x)|\varphi(x,y)|dxdy\right\}\\
&\le\underline{\lim}_{n\to\infty}E\left\{\max_{i\le n-1}\int_{t_i}^{t_{i+1}}\int_0^{s'}\chi^{(N)}(x)|\varphi(x,y)|dxdy\int_0^{t'}\int_0^{s'}\chi^{(N)}(x)|\varphi(x,y)|dxdy\right\}\\
&\le N\,\underline{\lim}_{n\to\infty}E\left\{\max_{i\le n-1}\int_{t_i}^{t_{i+1}}\int_0^{s'}\chi^{(N)}(x)|\varphi(x,y)|dxdy\right\}
\end{aligned}$$

But $\max_{i \le n-1} \int_{t_i}^{t_{i+1}} \int_0^{s'} \chi^{(N)}(x) |\varphi(x, y)| dxdy \le N,$ and converges to zero as $n \to \infty$ almost surely. Hence, $E(\varphi^{(N)}(t))^2 = 0$ and $\varphi(t,s') = 0$ almost surely for any $t \le t'$ and arbitrary z'. Similarly one can show that $\varphi(t',s) = 0$ almost surely for any $s \le s'$ and arbitrary z'. Thus,

$$\xi(z) = \xi(0) + \int_{[0,z]} \alpha(u, \xi) du + \int_{[0,z]} \beta(u, \xi) \hat{W}(du). \qquad \square$$

2.2 Strong Solution of Stochastic Differentiation Equations

Assume that the stochastic process $\xi(z)$ is given by the equation

$$\xi(dz) = a(z, \xi) dz + b(z, \xi) W(dz), \tag{2.2}$$

where $W(z)$ is a standard Wiener field taking values in R, coefficients $a(z,f)$ and $b(z,f)$ are defined on $z \in [0,T]^2, f \in C[0,T]^2$, and the boundary conditions are given by $\xi(t,0) = \varphi(t)$, $\xi(0,s) = \psi(s)$, $\varphi(0) = \psi(0) = \xi(0)$.

Suppose that the coefficients $a(z,f)$ and $b(z,f)$ in (2.2) are $\mathcal{B}_z$-measurable and linearly bounded, i.e. there exists a nonrandom constant c such that $\|a(\cdot)\|_z + \|b(\cdot)\|_z \le c(1 + \|f\|_z)$. (Here $\|\cdot\|$ is a uniform norm in the space of continuous functions $C[0,T]^2$, $\|\cdot\|_z$ is a uniform norm in $C[0,z]$).

Definition 2.3 The function $\xi(z)$, $z \in [0,T]^2$, defined on the probability space $(\Omega, \Im, P)$, is called a strong solution to (2.2) with boundary conditions $\varphi(t)$ and $\psi(s)$, if

(a) $\xi(z)$ is $\Im_z$-adapted and its realizations are continuous with probability 1.
(b) For any z the equality

$$\xi(z) = \varphi(t) + \psi(s) - \xi(0) + \int_{(0,z]} a(u, \xi(u)) du + \int_{(0,z]} b(u, \xi(u)) W(du)$$

holds with probability 1.

Consider the equation

$$\xi(z) = \varphi(z) + \int_{(0,z]} a(u, \xi(u)) du + \int_{(0,z]} b(u, \xi(u)) W(du), \tag{2.3}$$

where the stochastic field $\varphi(z)$ is $\Im_z$-adapted and for any z, z', $z \le z'$, the increment $W(z, z']$ does not depend on the σ-algebra $\Im_z^*$. Let us consider a bit more general equation than (2.2):

$$S(z) = S(\xi, z) = \varphi(z) + \int_{(o,z]} a(u, \xi(\cdot))du + \int_{(o,z]} b(u, \xi(\cdot))W(du).$$

If the stochastic field $\xi(z)$ is continuous and $\Im_z$-adapted, then the field $S(z)$ is defined for any $z \in [0,T]^2$, and its sample functions are continuous. Assume that the stochastic field $\varphi \in \overline{\mathbf{B}}_2$ and $\xi \in \mathbf{B}_2$. Then

$$\|S(\xi,\cdot)\|_z^2 \le 3\left(\|\varphi(\cdot)\|_z^2 + \int_{(0,z]} c\left(1 + \|\xi(\cdot)\|_u^2\right)du + \left\| \int_{(0,z]} b(u,\xi(\cdot))du \right\|_z^2 \right),$$

and from Theorem 1.17 we obtain

$$E\|S(\xi,\cdot)\|_z^2 \le 3E\|\varphi(\cdot)\|_z^2 + 51c\left(1 + \int_{(0,z]} E\|\xi(\cdot)\|_u^2 du\right).$$

Similarly, it is easy to obtain from Theorem 1.17 that

$$E\|S(\xi,\cdot) - \varphi(\cdot)\|_z^2 \le 34c\left(\int_{(0,z]} \left(1 + E\|\xi(\cdot)\|_u^2\right)du\right). \tag{2.4}$$

Definition 2.4 We say that the function $a(z,f)$, where $z \in [0,T]^2$ and $f \in C[0,T]^2$, satisfies the Lipschitz condition with some constant value L, if

$$\|a(\cdot, g(\cdot)) - a(\cdot, h(\cdot))\|_z \le L\|g(\cdot) - h(\cdot)\|_z$$

for any $z \in [0,T]^2$ and g, $h \in C[0,T]^2$.

Remark 2.1 [23] If the coefficients of (2.3) satisfy the Lipschitz condition, then

$$E\|S(\xi,\cdot) - S(\zeta,\cdot)\|_z^2 \le 34L^2\left(\int_{(0,z]} \|\xi(\cdot) - \zeta(\cdot)\|_u^2 du\right).$$

Theorem 2.2 *[25, 42, 64, 65, 71] Assume that the functions $a(z,f)$ and $b(z,f)$ are $\mathcal{B}_z$-measurable, satisfy the Lipschitz condition, and the function $\varphi(z) \in \overline{\mathbf{B}}_2$ is continuous. Then (2.3) has a solution $\xi \in \overline{\mathbf{B}}_2$, which is unique in the class $\mathbf{B}_2(\Im_z)$.*

Proof We take an arbitrary function $\xi_0(z) \in \overline{\mathbf{B}}_2(\Im_z)$ and construct the sequence $\xi_n(z)$ by induction: $\xi_{n+1}(z) = S(\xi_n, z)$, $n = 0, 1, 2, \ldots$ Hence, $\xi_n(z) \in \overline{\mathbf{B}}_2(\Im_z)$ and

$$E\|\xi_{n+1}(\cdot)-\xi_n(\cdot)\|_z^2 \le 34L^2\left(\int_{(0,z]}\|\xi_n(\cdot)-\xi_{n-1}(\cdot)\|_u^2 du\right).$$

Put $a := \int_{[0,T]^2} E\|\xi_1(\cdot)-\xi_0(\cdot)\|_u^2 du$. From the previous inequality we derive

$$E\|\xi_{n+1}(\cdot)-\xi_n(\cdot)\|_z^2 \le \frac{a\left(34L^2\right)^n}{(n!)^2} t^n s^n.$$

Let $\beta_n := \|\xi_{n+1}(\cdot) - \xi_n(\cdot)\|$. Therefore,

$$\sum_{n=1}^{\infty} P\left\{\beta_n > \frac{1}{n^2}\right\} \le \frac{a\left(34L^2\right)^n n^4}{(n!)^2}.$$

By the Borel–Cantelli lemma the series $\sum_{n=0}^{\infty}\beta_n$ converge with probability 1, implying that the series $\sum_{n=0}^{\infty}(\xi_{n+1}(z)-\xi_n(z))$ converge uniformly on $[0,T]^2$ with probability 1. The stochastic field $\xi(z)$ is $\mathfrak{I}_z$-adapted and its sample functions are continuous with probability 1.

Observe that

$$E\|\xi_{n+1}(\cdot)-\xi_n(\cdot)\|^2 \le \sum_{k=n}^{n+m+1}\frac{1}{k^2}\sum_{k=n}^{n+m-1} Ek^2\|\xi_{k+1}(\cdot)-\xi_k(\cdot)\|^2$$
$$\le \sum_{k=n}^{n+m+1}\frac{1}{k^2}\sum_{k=n}^{n+m-1}\frac{a\left(34L^2\right)^k k^4}{(k!)^4} \to 0.$$

Therefore $\xi \in \overline{\mathbf{B}}_2$ and $E\|\xi(\cdot) - \xi_n(\cdot)\|^2 \to 0$ as $n \to \infty$. Recall that inequality (1.8) holds true, and $\xi_{n+1}(z) = S(\xi_n, z)$. Passing to the limit as $n \to \infty$ we obtain $\xi = S(\xi)$, which proves that the solution of (2.3) exists. Assume now that (2.3) has two solutions in the class $\mathbf{B}_2(\mathfrak{I}_z)$, which we denote, respectively, by η_1 and η_2, and function $\varphi(z) \in \overline{\mathbf{B}}_2$. Let N be a positive variable,

$$I_N(z) = \begin{cases} 1, & \max\left(\|\eta_1\|_z, \|\eta_2\|_z\right) \le N \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$
\begin{aligned}
|I_N(\eta_1(z)-\eta_2(z))|^2 &\le 2I_N(z)\left|\int_{(0,z]}(a(u,\eta_1(u)-a(u,\eta_2(u))du\right|^2 \\
&+2I_N(z)\left|\int_{(0,z]}\big(b\big(u,\eta_1(u)-b\big(u,\eta_2(u)\big)W(du)\right|^2 \\
&\le 2\left|\int_{(0,z]}I_N(u)\big(a\big(u,\eta_1(u)-a\big(u,\eta_2(u)\big)du\right|^2 \\
&+2\left|\int_{(0,z]}I_N(u)\big(b\big(u,\eta_1(u)-b\big(u,\eta_2(u)\big)W(du)\right|^2,
\end{aligned}
$$

which implies

$$E\|I_N(\eta_1(\cdot)-\eta_2(\cdot))\|_z^2 \le 34L^2\int_{(0,z]}EI_N(u)\|\eta_1(\cdot)-\eta_2(\cdot)\|_u^2 du$$

and, taking into account that $I_N\|\eta_1(\cdot)-\eta_2(\cdot)\| \le 2N$, we obtain

$$E\|I_N(\eta_1(\cdot)-\eta_2(\cdot))\|_z^2 \le 34L^2 4N^2 ts.$$

Thus, for any n

$$E\|I_N(\eta_1(\cdot)-\eta_2(\cdot))\|_z^2 \le 4N^2\big(34L^2\big)^n\frac{t^n s^n}{(n!)^2}.$$

Passing to the limit as $n\to\infty$ we obtain that for any $N>0$ $I_N\|\eta_1(\cdot)-\eta_2(\cdot)\|=0$ with probability 1. Thus, $\eta_1(z)=\eta_2(z)$ for any $z\in[0,T]^2$ with probability 1, and therefore, under the conditions of our theorem the solution to (2.2) is unique. □

Remark 2.2 Assume that $\phi(z)=\varphi(t)+\psi(s)-\xi(0)$ and $\varphi(z)\in\overline{\mathbf{B}}_2$. Then the unique solution to (2.3) exists under weaker assumptions, i.e. when the coefficients $a(z,f)$ and $b(z,f)$ satisfy the Lipschitz condition on the subset of $C[0,T]^2$, consisting of the functions with boundary values $\xi(t,0)=\varphi(t)$ and $\xi(0,s)=\psi(s)$.

2.3 Generalized Girsanov Theorem for Stochastic Fields on the Plane

In this section we present results published in [10, 42, 45, 47]. To investigate Ito fields on the plane one often needs to transform the main probability measure P. In some cases it is possible to change the probability measure in such a way that the Ito field transforms into the Wiener field.

Denote

$$\varsigma(z,z',\varphi)=\int_{[z,z']}\varphi(u)W(du)-\frac{1}{2}\int_{[z,z']}\varphi^2(u)du$$

and $\zeta_\varphi(z)=\exp\{\varsigma(0,z,\varphi)\}$, $z\le z'$, z, $z'\in[0,T]^2$, where $(\varphi(z),\mathfrak{I}_z)$ is a random field satisfying the condition

$$P\left\{\int_{[0,T]^2}\varphi^2(u)du<\infty\right\}=1. \tag{2.5}$$

It is easy to see that application of the Ito formula (Theorem 1.14) to the function e^x and the field $\zeta(z)$ yields

$$\zeta_\varphi(z)=1+\int_{[0,z]}\zeta_\varphi(t',s)\varphi(t',s')W(dz').$$

Lemma 2.1 *Assume that the stochastic field $(\varphi(z),\mathfrak{I}_z)$ satisfies condition (2.5). Then $E\zeta_\varphi(z)\le 1$, and $E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}^*_z\big)\le 1$ almost surely.*

Proof Put

$$w_N(x)=\begin{cases}1, & |x|\le N,\\ 0, & |x|>N,\end{cases}$$

and

$$\eta_N(z)=w_N\left(\sup_{u\in[0,z]}\zeta_\varphi(u)\right)\zeta(z).$$

Obviously, $\zeta_\varphi(z)=\lim_{N\to\infty}\eta_N(z)$ almost surely. If $\varphi(z)$ is bounded, then $E\eta_N(z)=1$. Passing to the limit as $N\to\infty$ we obtain from the Fatou lemma that $E\zeta(z)\le 1$. Let $\varphi(z)=c(\omega)$, where $c(\omega)$ is the $\mathfrak{I}_z$-measurable random value. Then

$$\begin{aligned}E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}^*_z\big)&=E\left(\exp\left\{c(\omega)W(z,z']-\frac{1}{2}c^2(\omega)(t'-t)(s'-s)\right\}/\mathfrak{I}^*_z\right)\\&=\exp\left\{-\frac{1}{2}c^2(\omega)(t'-t)(s'-s)\right\}\\&\quad\times\exp\left\{\frac{1}{2}c^2(\omega)(t'-t)(s'-s)\right\}=1,\end{aligned}$$

Therefore, the equality $E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}_z^*\big)=1$ holds true for any bounded piecewise constant field $\varphi(z)$. Approximating the stochastic function $\varphi(z)$ by a sequence of bounded piecewise constant fields, we derive in the general case from the Fatou lemma that $E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}_z^*\big)\le 1$. □

Remark 2.3 Under conditions of Lemma 2.1, the stochastic field $\big(\zeta_\varphi(z),\mathfrak{I}_z\big)$ is a (nonnegative) super-martingale.

Lemma 2.2 *Assume that all conditions of Lemma 2.1 hold true and*

$$E\exp\left\{\int_{[0,T]^2}\varphi(u)W(du)-\frac{1}{2}\int_{[0,T]^2}\varphi^2(u)du\right\}=1. \tag{2.6}$$

Then
$E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}_z^*\big)=1$ *almost surely.*

Proof From Lemma 2.1 we obtain

$$\begin{aligned}1=E\zeta(T,T)\le E\zeta\big(z\big)&=E\exp\big\{\varsigma(z,z',\varphi)+\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}\\&=E\exp\big\{\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}_z^*\big).\end{aligned}$$

Put

$$B_n=\left\{\omega:E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}_z^*\big)\le 1-\frac{1}{n}\right\},\quad n=1,2,\ldots$$

Then

$$\begin{aligned}1\le&\int_{\Omega\backslash B_n}\exp\big\{\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}P(d\omega)\\&+\left(1-\frac{1}{n}\right)\int_{B_n}\exp\big\{\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}P(d\omega)\\=&E\exp\big\{\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}\\&-\frac{1}{n}\int_{B_n}\exp\big\{\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}P(d\omega).\end{aligned}$$

Further, it follows from Lemma 2.1 that

$$E\exp\big\{\varsigma\big(0,(t,s'),\varphi+\varsigma((t,0),(t',s),\varphi)\big\}\le 1,$$

implying $P(B_n)=0$ for any $n>0$. Therefore,

$$E\big(\exp\{\varsigma(z,z',\varphi)\}/\mathfrak{I}_z^*\big)=1.$$ □

Remark 2.4 Under conditions of Lemma 2.2, the stochastic field $\left(\zeta_\varphi(z), \Im_z\right)$ is a martingale.

Since $\zeta(z)$ is a nonnegative martingale, one can define on the probability space $(\Omega, \Im)$ the new probability measure $\widetilde{P}\,(d\omega) = \exp\left\{\zeta_\varphi(T,T)\right\}P(d\omega)$. We denote by $\widetilde{E}$ the expectation with respect to this new measure $\widetilde{P}$.

Lemma 2.3 *Assume that condition (2.6) is fulfilled. Then for any $\Im_{z'}$-measurable random variable $\eta(\omega)$, we have $\widetilde{E}\,|\eta| < \infty$ with probability 1, and*

$$\widetilde{E}\left(\eta/\Im_z^*\right) = E\left(\eta\exp\left\{\varsigma\left(z,z',\varphi\right\}/\Im_z^*\right), \quad z \le z' \quad z,z' \in [0,T]^2.$$

Proof It is enough to show that for any bounded $\Im_z^*$-measurable random variable $\gamma(\omega)$ one has

$$\widetilde{E}\left\{\gamma\widetilde{E}\left(\eta/\Im_z^*\right)\right\} = \widetilde{E}\left\{\gamma E\left(\eta\exp\left\{\varsigma\left(z,z',\varphi\right\}/\Im_z^*\right)\right\}.$$

Indeed, $\widetilde{E}\left\{\gamma\widetilde{E}\left(\eta/\Im_z^*\right)\right\} = \widetilde{E}\,\gamma\eta$. On the other hand, by Lemma 2.2 we obtain

$$\begin{aligned}
&\widetilde{E}\left\{\gamma E\left(\eta\exp\left\{\varsigma\left(z,z',\varphi\right\}/\Im_z^*\right)\right\} = E\gamma E\left(\eta\exp\left\{\varsigma\left(z,z',\varphi\right\}/\Im_z^*\right)\zeta(T,T)\\
&\quad = E\left\{\gamma E\left(\eta\exp\left\{\varsigma\left(z,z',\varphi\right\}/\Im_z^*\right)\left[\exp\{\varsigma(0,(T,s),\varphi) + \varsigma(0,(t,T),\varphi)\}\right]\right\}\\
&\quad = E(\gamma\eta\left[\exp\{\varsigma((t,0),(T,s),\varphi) + \varsigma(0,(t,T),\varphi) + \varsigma(z,z',\varphi)\}\right])\\
&\quad = E(\gamma\eta\left[\exp\{\varsigma((t,0),(T,s),\varphi) + \varsigma(0,(t,T),\varphi) + \varsigma(z,z',\varphi)\}\right])\\
&\qquad \times E\left[\exp\{\varsigma((t',s),(T,s'),\varphi)\}/\Im_z^*\right]\\
&\quad = E(\gamma\eta\left[\exp\{\varsigma((t,0),(T,s),\varphi) + \varsigma(0,(t,T),\varphi) + \varsigma(z,z',\varphi) + \varsigma((t',s),(T,s'),\varphi)\}\right])\\
&\qquad \times E\left[\exp\{\varsigma(z',(T,T),\varphi)\}/\Im_z^*\right] = E\gamma\eta\zeta(T,T) = \widetilde{E}\,\gamma\eta. \qquad \square
\end{aligned}$$

Lemma 2.4 *[54] Let $\xi_n \ge 0$, $n = 1, 2, \ldots$ be the sequence of random variables such that $\xi_n \to \xi$ in probability as $\mathrm{n} \to \infty$. If $\mathrm{E\xi_n} = \mathrm{E\xi} = const$, then* $\lim_{n\to\infty} E|\xi_n - \xi| = 0$.

Lemma 2.5 *Assume that $\phi^{(N)}(z)$ is the sequence of $\Im_z$-measurable functions such that $\phi^{(N)}(z) \to \phi(z)$ in probability as $N \to \infty$. If*

$$E\exp\left\{\varsigma\left(z,z',\phi^{(N)}\right)\right\} = E\exp\{\varsigma(z,z',\phi)\} = 1,$$

then

$$\lim_{N\to\infty} E\left|\exp\left\{\varsigma\left(z,z',\phi^{(N)}\right)\right\} - \exp\{\varsigma(z,z',\phi)\}\right| = 0.$$

The proof follows directly from Lemma 2.4.

Theorem 2.3 *(Girsanov theorem). Suppose that the random field* $(\varphi(z), \Im_z)$ *satisfies (2.5) and condition (2.6) is fulfilled. Then the random field* $(\xi(z), \Im_z)$ *given by*

$$\xi(z) = W(z) - \int_{[0,z]} \varphi(u)du,$$

is the Wiener field on the probability space $\left(\Omega, \Im, \widetilde{P}\right)$ *with respect to the flow* $(\Im_z)$ *and probability measure* $\widetilde{P}(d\omega) = Ee^{\zeta(T,T)}P(d\omega)$.

Proof To prove this theorem we construct the sequence of piecewise constant fields $(\varphi_N(z), \Im_z)$, $N = 1, 2, \ldots$, such that

$$\lim_{N\to\infty} \int_{[0,T]^2} (\varphi(u) - \varphi_N(u))^2 du = 0 \text{ a.s.},$$

$$\lim_{N\to\infty} \exp\{\varsigma(z, z', \varphi_N)\} = \exp\{\varsigma(z, z', \varphi)\} \text{ a.s.}$$

From Lemma 1.9,

$$E\exp\{\varsigma(0, (T,T), \varphi_N\} = 1.$$

We prove that for any $N > 0$ the field $(\xi_N(z), \Im_z)$ defined by

$$\xi_N(z) = W(z) - \int_{[0,z]} \varphi_N(u)du,$$

is a Wiener field with respect to the probability measure $\widetilde{P}^N(d\omega) = \exp\{\zeta_{\varphi_N}(T,T)\}$ $P(d\omega)$. Since the field $(\varphi_N(z), \Im_z)$ is bounded,

$$\widetilde{E}_N \exp\{\theta\xi_N(z, z']\} < \infty.$$

Moreover, for any θ

$$\begin{aligned}
&\widetilde{E}_N\left(\exp\left\{\theta\xi_N(z,z'] - \frac{\theta^2}{2}(t'-t)(s'-s)\right\} \Big/ \Im_z^*\right)\\
&= E\left(\exp\left\{\theta\xi_N(z,z'] - \frac{\theta^2}{2}(t'-t)(s'-s) + \varsigma(z,z',\phi_N)\right\} \Big/ \Im_z^*\right)\\
&= E\left(\exp\left\{\theta W(z,z'] - \theta\int_{[z,z']} \phi_N(u)du - \frac{\theta^2}{2}(t'-t)(s'-s) + \varsigma(z,z',\phi_N)\right\} \Big/ \Im_z^*\right)\\
&= E\left(\exp\left\{\varsigma(z,z',\phi_N) - \theta\int_{[z,z']} \phi_N(u)du\right\} \Big/ \Im_z^*\right)\\
&= E\left(\exp\{\varsigma(z,z',\phi_N)(\theta+\phi_N)\}/\Im_z^*\right).
\end{aligned}$$

The random field $(\varphi_N(z) + \theta, \mathfrak{I}_z)$ is bounded and piecewise constant, which implies by Lemma 2.2

$$E\big(\exp\{\varsigma(z, z', \varphi_N)(\theta + \varphi_N)\}/\mathfrak{I}_z^*\big) = 1.$$

Therefore, by Lemmas 2.3 for any c

$$\begin{aligned}\widetilde{E}_N\big(\exp\{ic\xi_N(z, z'] + \varsigma(z, z', \phi_N)\}/\mathfrak{I}_z^*\big) &= \widetilde{E}_N\big(\exp\{ic\xi_N(z, z']\}/\mathfrak{I}_z^*\big)\\ &= \exp\left\{-\frac{c^2}{2}(t' - t)(s' - s)\right\}.\end{aligned} \tag{2.7}$$

The sequence $\exp\{ic\xi_N(z, z']\}$, $N = 1, 2, \ldots$, is bounded and converges almost surely to $\exp\{ic\xi(z, z']\}$. Note also that $E \exp\{\varsigma(z,z',\varphi_N)\} = 1$. Thus, by Lemma 2.4

$$E \exp\{\varsigma(z, z', \phi)\} = 1,$$

and

$$\lim_{N\to\infty} E|\exp\{\varsigma(z, z', \varphi_N)\} - \exp\{\varsigma(z, z', \varphi)\}| = 0.$$

Passing to the limit as $N \to \infty$ under expectation in (2.7), we obtain

$$\begin{aligned}\widetilde{E}\left(\exp\{ic\xi(z, z'] + \varsigma(z, z', \varphi)\}/\mathfrak{I}_z^*\right) &= \widetilde{E}\left(\exp\{ic\xi(z, z']\}/\mathfrak{I}_z^*\right)\\ &= \exp\left\{-\frac{c^2}{2}(t' - t)(s' - s)\right\}.\end{aligned}$$

Thus, the random field $(\xi(z), \mathfrak{I}_z)$ is a Wiener field with respect to the probability measure $\widetilde{P}$. □

Definition 2.5 Introduce in the space $C[0,T]^2$ the measure μ_ξ, corresponding to the random field ξ by the following rule: for any $A\in\mathcal{B}$ put $\mu_\xi(A) = P\{\omega : \xi(\omega) \in A\}$.

For some applications it is important to know the sufficient conditions for the existence of μ_ξ and μ_W, as well as the Radon–Nikodym derivatives of these measures.

Theorem 2.4 *Let ξ be a random Ito field of the form*

$$\xi(z) = W(z) + \int_{[0,z]} \varphi(u)du,$$

and let the conditions

$$P\left\{\int_{[0,T]^2}\varphi^2(u)du<\infty\right\}=1$$

and

$$E\exp\left\{-\int_{[0,T]^2}\varphi(u)W(du)-\frac{1}{2}\int_{[0,T]^2}\varphi^2(u)du\right\}=1$$

hold true. Then the measure μ_ξ *generated by* ξ *is equivalent to the measure* μ_W, *corresponding to a standard Wiener field, and*

$$\frac{d\mu_W}{d\mu_\xi}(\xi)=E\left[\exp\left\{-\int_{[0,T]^2}\varphi(u)W(du)-\frac{1}{2}\int_{[0,T]^2}\varphi^2(u)du\right\}\bigg/\mathfrak{I}^\xi\right]P\text{-a.s.},$$

where $\mathfrak{I}^\xi=\sigma\left\{\xi(z),z\in[0,T]^2\right\}$. *If* $(\xi(z),\mathfrak{I}_z)$ *is a diffusion-type field, then*

$$\frac{d\mu_W}{d\mu_\xi}(\xi)=\exp\left\{-\int_{[0,T]^2}\varphi(u,\xi)W(du)+\frac{1}{2}\int_{[0,T]^2}\varphi^2(u,\xi)du\right\},$$

$$\frac{d\mu_\xi}{d\mu_W}(W)=\exp\left\{\int_{[0,T]^2}\varphi(u,W)W(du)-\frac{1}{2}\int_{[0,T]^2}\varphi^2(u,W)du\right\}.$$

Proof Put

$$\zeta(z)=\exp\left\{-\int_{[0,z]^2}\varphi(u)W(du)-\frac{1}{2}\int_{[0,z]^2}\varphi^2(u)du\right\}.$$

From our assumptions, we have $E\zeta(T,T)=1$, and thus $(\zeta(z),\mathfrak{I}_z)$ is a supermartingale. Denote by $\widetilde{P}$ a measure on the probability space $(\Omega,\mathfrak{I})$ with the property $d\widetilde{P}(\omega)=\zeta(T,T,\omega)dP$. By Theorem 2.3, the stochastic field $(\xi(z),\mathfrak{I}_z)$, $z\in[0,T]^2$ is a Wiener field with respect to the measure $\widetilde{P}$, which implies one has for any $A\in B_T$

$$\mu_W(A)=\widetilde{P}(\xi\in A)=\int_{\{\omega:\xi\in A\}}\zeta(T,T,\omega)dP=\int_{\{\omega:\xi\in A\}}E\left\{\zeta(T,T,\omega)/\mathfrak{I}_T^\xi\right\}dP.$$

Since the random variable $E\left\{\zeta(T,T,\omega)/\mathfrak{I}^\xi_{(T,T)}\right\}$ is $\mathfrak{I}^\xi_{(T,T)}$-measurable, there exists a $\mathscr{B}_{(T,T)}$-measurable nonnegative function $\psi(x)$, such that

$$E\left\{\zeta(T,T,\omega)/\mathfrak{I}^\xi_{(T,T)}\right\}=\psi(\xi(\omega)).$$

Thus,

$$\mu_W(A) = \int_{\{\omega:\xi\in A\}} \psi(\xi(\omega))dP(\omega) = \int_A \psi(x)d\mu_\xi(x).$$

From the last statement we obtain $\mu_W \ll \mu_\xi$ and $\frac{d\mu_W}{d\mu_\xi}(\xi) = \psi(\xi)\ \mu_\xi$-almost surely, implying $\frac{d\mu_W}{d\mu_\xi}(\xi) = E\Big\{\zeta(T,T,\omega)/\mathfrak{I}_T^\xi\Big\}$ P-almost surely.

Let us show now that $\mu_\xi \ll \mu_W$. Note that since $P\left\{\int_{[0,T]^2}\varphi(u)W(du) < \infty\right\} = 1$, $\frac{d\widetilde{P}}{dP}(\omega) = \zeta(T,T,\omega)$ and $P\{\zeta(T,T,\omega) = 0\} = 0$. Hence, $\frac{dP}{d\widetilde{P}}(\omega) = \zeta^{-1}(T,T,\omega)$.

Therefore, since $\widetilde{P}\{\omega : \xi\in A\} = \mu_W(A)$,

$$\begin{aligned}
\mu_\xi(A) = P\big(\xi\in A\big) &= \int_{\{\omega:\xi\in A\}} \zeta^{-1}(T,T,\omega)d\widetilde{P}(\omega) \\
&= \int_{\{\omega:\xi\in A\}} \widetilde{E}\left\{\zeta^{-1}(T,T,\omega)/\mathfrak{I}^\xi_{(T,T)}\right\}d\widetilde{P}(\omega) \\
&= \int_A \widetilde{E}\left\{\zeta^{-1}(T,T,\omega)/\mathfrak{I}^\xi_{(T,T)}\right\}_{\xi=x} d\mu_W(x).
\end{aligned}$$

As a result, we obtain $\mu_\xi \ll \mu_W$ and

$$\frac{d\mu_\xi}{d\mu_W}(W) = \widetilde{E}\left\{\zeta^{-1}(T,T,\omega)/\mathfrak{I}_T^\xi\right\} P\text{-a.s.} \qquad \square$$

Remark 2.5 Usually the assumption

$$E\exp\left\{-\int_{[0,T]^2}\varphi(z)W(dz) - \frac{1}{2}\int_{[0,T]^2}\varphi^2(z)dz\right\} = 1$$

is quite hard to verify. Following [54], it is easy to see that if for any $\delta > 0$

$$E\exp\left\{\left(\frac{1}{2}+\delta\right)\int_{[0,T]^2}\varphi^2(z)dz\right\} < \infty,$$

then the first condition of the Theorem 2.4 is also fulfilled.

Definition 2.6 We say that the stochastic differential equation $\xi(dz) = a(z,\xi)dz + b(z,\xi)W(dz)$ with boundary conditions $\varphi(t)$ and $\psi(s)$ has a weak solution if there exist a probability space $(\Omega,\mathfrak{I},P)$, a non-decreasing system of σ-algebras $(\mathfrak{I}_z)$, $z \in [0,T]^2$, a continuous random field $(\xi(z),\mathfrak{I}_z)$, and a standard Wiener field $(W(z),\mathfrak{I}_z)$, such that

(a) $\int_{[0,T]^2} |a(z,\xi)|dz < \infty\, P$-a.s.

(b) $\int_{[0,T]^2} b^2(z,\xi)dz < \infty\, P$-a.s.

(c) For any $z \in [0,T]^2$

$$\xi(z) = \varphi(t) + \psi(s) - \xi(0) + \int_{(o,z]} a(u,\xi(u))du + \int_{(o,z]} b(u,\xi(u))W(du)$$

with probability 1.

In fact, a weak solution is a set of objects $(\Omega, \Im, \Im_z, P, W(z), \xi(z))$.

Definition 2.7 The stochastic differential equation of type (2.2) has a unique weak solution if for any two solutions $(\Omega, \Im, \Im_z, P, W(z), \xi(z))$ and $\left(\widetilde{\Omega}, \widetilde{\Im}, \widetilde{\Im}_z, \widetilde{P}, \widetilde{W}(z), \widetilde{\xi}(z)\right)$ distributions of fields $\xi(z)$ and $\widetilde{\xi}(z)$ coincide, i.e. $\mu_{\xi}(A) = \widetilde{\mu}_{\widetilde{\xi}}(A)$ for any $A \in \mathcal{B}$.

We find necessary and sufficient conditions for the existence of a unique weak solution in the case $b(z,\xi) \equiv 1$.

Theorem 2.5 *[47] Assume that $a(z,f)$, where $z \in [0,T]^2$ and $f \in C[0,T]^2$, is some measurable functional that does not depend on future, and $\int_{[0,T]^2} a^2(z,f)dz < \infty$. Then the equation*

$$\xi(dz) = a(z,\xi)dz + W(dz) \tag{2.8}$$

has a unique weak solution if and only if there exists a Wiener field $\left(W'(z), \Im'_z\right)$ on some probability space $(\Omega', \Im', P')$, such that

$$E' \exp\left\{ \int_{[0,T]^2} \varphi(u,W')W'(du) - \frac{1}{2}\int_{[0,T]^2} \varphi^2(u,W)du \right\} = 1.$$

Here E' denotes expectation with respect to the measure P'.

Proof Suppose that a weak solution to this equation exists, i.e. for a set of objects $(\Omega, \Im, \Im_z, P, W(z), \xi(z))$, and $z \in [0,T]^2$, we have

$$\xi(z) = \int_{(o,z]} a(u,\xi(u))du + W(z)\, P\text{-a.s.}$$

Note that $\int_{[0,T]^2} a^2(z,\xi)dz < \infty$ and $\int_{[0,T]^2} a^2(z,W)dz < \infty$ P-almost surely, and therefore by Theorem 2.4 the measures corresponding to the fields ξ and W are P-almost surely equivalent, and

$$\frac{d\mu_\xi}{d\mu_W}(W) = \exp\left\{\int_{[0,T]^2} a(u, W(u))W(du) - \frac{1}{2}\int_{[0,T]^2} a^2(u, W(u))du\right\}.$$

Thus, the conditions of the theorem are fulfilled for the standard Wiener field $(W(z), \Im_z)$ from the definition of the weak solution.

Suppose now that the conditions of the theorem hold true. Then the random field

$$\widetilde{W}(z) = W'(z) - \int_{[0,z]} a(u, W'(u))du, \quad z \in [0, T]^2$$

is a Wiener field with respect to the set of σ-algebras $\left(\Im_z', z \in [0, T]^2\right)$, and the probability measure $\widetilde{P}$ is given by

$$\widetilde{P}(d\omega) = \exp\left\{\int_{[0,T]^2} a(u, W'(u))W'(du) - \frac{1}{2}\int_{[0,T]^2} a^2(u, W'(u))du\right\} P'(d\omega).$$

Therefore, the set of objects $\left(\Omega', \Im', \Im'_z, \widetilde{P}, \widetilde{W}(z), W'(z)\right)$ is a weak solution to (2.8). Moreover, the measure μ_ξ corresponding to any weak solution of (2.8) is equivalent to the measure μ_W. Since the Radon–Nikodym derivative depends only on the functional $a(z,f)$, this solution is unique. □

When dealing with some problems which involve Ito and diffusion-type fields it is convenient to work with a field which admits the following representation:

$$\xi(z) = 1 + \int_{[0,z]} \gamma(t_1, s)W(dz_1), \tag{2.9}$$

where $(\gamma(z), \Im_z)$ is a random field such that

$$P\left\{\int_{[0,T]^2} \gamma^2(u)du < \infty\right\} = 1. \tag{2.10}$$

Lemma 2.6 *Assume that the random field $\xi(z)$ is given by (2.9) and satisfies*

$$P\left\{\xi(z) \geq 0, z \in [0, T]^2\right\} = 1.$$

Then $E\xi(z) \leq 1$.
If in addition

$$E\xi((T, T)) = 1, \tag{2.11}$$

then the random field $(\xi(z), \Im_z)$ is a martingale.

The proof is similar to the proof of Lemma 2.2.

Define on the probability space $(\Omega, \Im)$ a new probability measure $\widetilde{P}$ by

$$d\widetilde{P}(\omega) = \xi((T,T),\omega)dP(\omega).$$

Lemma 2.7 *Suppose that the stochastic field ξ satisfies condition (2.9). Then*

$$\widetilde{P}\left(\inf_{z\in[0,T]^2}\xi(z)=0\right)=0.$$

Proof By definition $\widetilde{P}$ we have

$$\widetilde{P}\left(\inf_{z\in[0,T]^2}\xi(z)=0\right)=\int\limits_{\{\omega:\inf\xi(z)=0\}}\xi(T,T)dP(\omega).$$

Put $D^0=\{z:\xi(z)=0\}$. Then $\{\omega:\inf_{z\in D}\xi(z)=0\}=\left\{\omega:z\in D^0\right\}$, and therefore,

$$\widetilde{P}\left(\inf_{z\in[0,T]^2}\xi(z)=0\right)=\int\limits_{\left\{\omega:z\in D^0\right\}}E(\xi(T,T)/\Im_z)dP(\omega)=0. \qquad \square$$

Lemma 2.8 *Suppose that conditions (2.9) and (2.10) are satisfied, and let $\alpha=\alpha(\omega)$ be an $\Im^*_{z'}$-measurable random variable with $E|\alpha|<\infty$. Then for $z\in[0,T]^2$, $z\le z'$,*

(a) $\widetilde{E}\left(\alpha/\Im^2_z\right)=\overline{\xi}^{-1}(T,s)E\left(\alpha\xi(T,s')/\Im^2_z\right)$.

(b) $\widetilde{E}\left(\alpha/\Im^1_z\right)=\overline{\xi}^{-1}(t,T)E\left(\alpha\xi(t',T)/\Im^1_z\right)$.

Proof Let $\lambda=\lambda(\omega)$ be a bounded $\Im^2_z$-measurable random variable, $0\le s\le s'\le T$. Then

$$\begin{aligned}\widetilde{E}(\alpha\lambda)&=\widetilde{E}\left(\lambda\widetilde{E}\left(\alpha/\Im^2_z\right)\right)=E\left(\lambda\widetilde{E}\left[\alpha/\Im^2_z\right]\xi(T,T)\right)\\&=E\left(\lambda\widetilde{E}\left[\alpha/\Im^2_z\right]E\left[\xi(T,T)/\Im^2_z\right]\right)=E\left(\lambda\widetilde{E}\left[\alpha/\Im^2_z\right]\xi(T,s)\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{E}(\alpha\lambda)&=E(\lambda\alpha\xi(T,T))=E\left(\lambda\alpha E\left[\xi(T,T)/\Im^2_z\right]\right)\\&=E(\alpha\lambda\xi(T,s'))=E\left(\lambda\widetilde{E}\left[\alpha\xi(T,s')/\Im^2_z\right]\right).\end{aligned}$$

Therefore, $\xi(T,s)\widetilde{E}\left[\alpha/\Im_z^2\right] = E\left(\alpha\xi(T,s')/\Im_z^2\right)$ with P- and $\widetilde{P}$ -probability 1. Since $\widetilde{P}\{\xi(z)>0\}=1$, we have $\widetilde{P}\left\{\overline{\xi}^{-1}(z)=\xi^{-1}(z)\right\}=1$, and

$$\widetilde{E}\left(\alpha/\Im_z^2\right) = \overline{\xi}^{-1}(T,s)E\left(\alpha\xi(T,s')/\Im_z^2\right)$$

for all $z \le z'$, which proves the assertion (a) The proof of (b) is similar. □

Theorem 2.6 *(generalized Girsanov theorem) Let $(\xi(z),\Im_z)$ be a random field of the form (2.9) and assume that conditions (2.10) and (2.11) are fulfilled. Then the random field $\widetilde{W} = \left(\widetilde{W}(z),\Im_z\right)$, z $\in$ [0,T]^2, given by*

$$\widetilde{W}(z) = W(z) - \int_{[0,z]} \overline{\xi}^{-1}(t',s)\gamma(z')dz' \tag{2.12}$$

is a Wiener field on the probability space $\left(\Omega,\Im,\widetilde{P}\right)$ with respect to the set of σ-algebras $\Im_z$ and the measure $\widetilde{P}$.

Proof Since $\widetilde{P}\{\xi(z)=0\}=0$ and $\widetilde{P}\left\{\overline{\xi}^{-1}(z)=\xi^{-1}(z)\right\}=1$, the field $\overline{\xi}^{-1} = \left(\overline{\xi}^{-1}(z),\Im_z\right)$, $z \in [0,T]^2$ has continuous sample paths and, therefore,

$$\widetilde{P}\left\{\sup_{z\in[0,T]^2} \overline{\xi}^{-1}(z) < \infty\right\} = 1.$$

Moreover, the measure $\widetilde{P}$ is absolutely continuous with respect to the measure P, and

$$\widetilde{P}\left\{\int_{[0,T]^2} \gamma^2(u)du < \infty\right\} = 1.$$

Observe that

$$\int_{[0,T]^2} \left(\overline{\xi}^{-1}(z)\gamma(z)\right)^2 dz \le \sup_{z\in[0,T]^2}\left(\overline{\xi}^{-1}(z)\right)^2 \int_{[0,T]^2}\left(\overline{\xi}^{-1}(z)\gamma(z)\right)^2 dz.$$

Hence, the integral in (2.12) is well defined. To prove the theorem, it suffices to show that for any $z=(t,s) \in [0,T]^2$ the random processes $\left\{\widetilde{W}(x,s),\Im^1_{(x,s)},\ 0\le x\le T\right\}$ and $\left\{\widetilde{W}(t,y),\Im^2_{(t,y)},\ 0\le y\le T\right\}$ are one-parameter Wiener processes with parameters s and t, respectively.

Fix t, $0 \le t \le T$. Assume that for some constant values c_1 and c_2 we have

$$P\left(0 < c_1 \le \inf_{z \in D} \xi(z) \le \sup_{z \in D} \xi(z) \le c_2 < \infty\right) = 1, \tag{2.13}$$

and

$$E\left(\int_{[0,T]^2} \gamma^2(u)du\right) < \infty. \tag{2.14}$$

Put

$$\varsigma(t,y,s) = \exp\left\{i\theta\left(\widetilde{W}(t,y) - \widetilde{W}(t,s)\right)\right\}.$$

Lemma (1.5) implies that

$$E\left[\varsigma/\Im_z^2\right] = \overline{\xi}^{-1}(T,s)E\left[\varsigma\xi(T,y)/\Im_z^2\right]\widetilde{P} - \text{a.s.}$$

Therefore, using the Ito formula (Theorem 1.16), we get

$$\begin{aligned}&\varsigma(t,y,s)\,\xi(T,y) - \varsigma(t,s,s)\,\xi(T,s)\\ &= \int_0^t \int_s^y \varsigma(t,v,s)\xi(T,v)\overline{\xi}^{-1}(t,s)\gamma(u,v)W(du,dv).\end{aligned}$$

Conditions (2.13) and (2.14) ensure the existence of all integrals in the above equation. Taking the conditional expectation with respect to the σ-algebra $\Im_z^2$ we see that

$$\overline{\xi}^{-1}(T,s)E\left[\varsigma\xi(T,y)/\Im_z^2\right] = 1 - \frac{\theta^2}{2}\int_0^t \int_s^y \overline{\xi}^{-1}(T,s)\,E\left(\varsigma(t,v,s)\xi(T,v)/\Im_z^2\right)du,dv$$

with P- and $\widetilde{P}$-probability 1, whence

$$\overline{\xi}^{-1}(T,s)E\left[\varsigma\xi(T,y)/\Im_z^2\right] = \exp\left\{-\frac{\theta^2}{2}t(y-s)\right\}.$$

Therefore,

$$E\exp\left\{i\theta\left(\widetilde{W}(t,y) - \widetilde{W}(t,s)\right)\right\} = \exp\left\{-\frac{\theta^2}{2}t(y-s)\right\}$$

$\widetilde{P}$-almost surely.

Now suppose that conditions (2.13) and (2.14) are violated. For every integer n introduce the sets

$$D^n = \left\{ z \in [0,T]^2 : \int_{[0,z]} \gamma^2(t',s)dz' + \left(\inf_{z' \le z} \xi(z') \right)^{-1} + \sup_{z' \le z} \xi(z') \le n \right\},$$

which form the sequence of stopping regions $D^n \subset D^{n+1} \subset \ldots$. Since

$$P\left\{ \int_{[0,T]^2} \gamma^2(t',s)dz' + \sup_{z \in [0,T]^2} \xi(z) \le \infty \right\} = 1$$

and

$$\widetilde{P}\left\{ \inf_{z \in [0,T]^2} \xi(z) > 0 \right\} = 1, \tag{2.15}$$

and the measure $\widetilde{P}$ is absolutely continuous with respect to P, we conclude that $D^n \uparrow [0,T]^2$ as $n \to \infty$ $\widetilde{P}$-almost surely.

Put $\gamma_n(z) = \gamma(z)\chi_{z \in D^n}$,

$$\xi_n(z) = 1 + \int_{[0,z]} \gamma_n(t',s)W(dz'),$$

$$\widetilde{W}_n(z) = W(z) - \int_{[0,z]} \overline{\xi}_n^{-1}(t',s)\gamma_n(t',s)dz'.$$

The random field $\xi_n = (\xi_n(z), \mathfrak{I}_z)$, $z \in [0,T]^2$, is a martingale, $E\xi_n(T,T) = 1$, and satisfies conditions (2.12) and (2.13) with $c_1 = 1/n$ and $c_2 = n$. Define the probability measure $\widetilde{P}_n$ by the equality $d\widetilde{P}_n(\omega) = \xi_n(T,T)P(\omega)$. From above,

$$\widetilde{E}_n\left[\exp\left\{ i\theta\left(\widetilde{W}_n(t,y) - \widetilde{W}_n(t,s) \right) \right\} \Big/ \mathfrak{I}_z^2 \right] = \exp\left\{ -\frac{\theta^2}{2} t(y-s) \right\},$$

where $0 \le s \le y \le T$ and $\widetilde{E}_n$ is the expectation with respect to the measure $\widetilde{P}_n$. Since

$$\widetilde{E}_n\left[\exp\left\{ i\theta\left(\widetilde{W}_n(t,y) - \widetilde{W}_n(t,s) \right) \right\} \Big/ \mathfrak{I}_z^2 \right] \to \widetilde{E}\left[\exp\left\{ i\theta\left(\widetilde{W}(t,y) - \widetilde{W}(t,s) \right) \right\} \Big/ \mathfrak{I}_z^2 \right]$$

in $\widetilde{P}$ a.s. $n \to \infty$, it suffices to verify that

$$\overline{\lim_{n\to\infty}}\, \widetilde{E}\left|\widetilde{E}_n\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\Big/\mathfrak{I}_z^2\right]\right.$$
$$\left.-\widetilde{E}\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\Big/\mathfrak{I}_z^2\right]\right|=0.$$

For each n the measure $\widetilde{P}_n$ is equivalent to the measure P, that is, $\dfrac{d\widetilde{P}_n}{dP}=\dfrac{1}{\xi_n(T,T)}$. Applying Lemma 2.8 and performing simple transformations, we see that

$$\widetilde{E}_n\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\Big/\mathfrak{I}_z^2\right]$$
$$=E\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\xi(T,y)\,\overline{\xi}^{-1}(T,s)\right],$$

$$\widetilde{E}_n\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\Big/\mathfrak{I}_z^2\right]$$
$$=E\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\frac{\xi_n(T,y)}{\xi_n(T,s)}\right],$$

with $\widetilde{P}$-probability 1. Therefore,

$$\widetilde{E}\left|\widetilde{E}_n\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\Big/\mathfrak{I}_z^2\right]\right.$$
$$\left.-\widetilde{E}\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\Big/\mathfrak{I}_z^2\right]\right|$$
$$\le \widetilde{E}\left|E\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\frac{\xi_n(T,y)}{\xi_n(T,s)}\right].\right.$$
$$\left.-\left[\exp\left\{i\theta\left(\widetilde{W}_n(t,y)-\widetilde{W}_n(t,s)\right)\right\}\xi(T,y)\,\overline{\xi}^{-1}(T,s)\right]\right|$$
$$\le \widetilde{E}\,E\left[\left|\frac{\xi_n(T,y)}{\xi_n(T,s)}-\xi(T,y)\,\overline{\xi}^{-1}(T,s)\right|\Big/\mathfrak{I}_z^2\right]$$
$$=E\left|\xi(T,s)\,\xi_n(T,y)\,\overline{\xi}_n^{-1}(T,s)-\xi(T,s)\,\xi(T,y)\,\xi^*(T,s)\right|.$$

Introduce the stopping region $D^0=\left\{z\in[0,T]^2:\ \inf_{z'\le z}\xi(T,s')>0\right\}$. It follows from condition (2.15) that the above-defined stopping regions D^n converge to D^0 as $n\to\infty$ with P-probability 1. Thus, $\xi(T,s)=\xi_0(T,s)$ for all $0\le s\le T$, where $\xi_0(z)$ is defined in the same way as $\xi_n(z)$.

Now we shall to verify that

$$\lim_{n\to\infty}\!\overline{m}\ \xi_0(T,s)\,\xi_n(T,y)\,\overline{\xi}_n^{-1}(T,s)=\xi_0(T,s)\,\xi_0(T,y)\,\overline{\xi}_0^{-1}(T,s). \tag{2.16}$$

By continuity of $\xi(z)$, this equality is valid provided that

$$\xi_0(T,s)\,\overline{\xi}_n^{-1}(T,s) = \xi_0(T,s)\,\overline{\xi}_0^{-1}(T,s)P\text{-a.s.}$$

If the point (T,s) does not belong to D^0, then $\xi_0(T,s)\,\xi_n^{+}(T,s) = 0$ for all n. Since on the set D^0 we have $\inf_n \xi_n(T,s) > 0$, the equality (2.16) holds true. Hence,

$$\xi(T,s)\,\xi_n(T,y)\,\overline{\xi}_n^{-1}(T,s) \to \xi(T,s)\,\xi(T,y)\,\overline{\xi}^{-1}(T,s), \quad n\to\infty\, P\text{-a.s.}$$

Next,

$$E\Big(\xi(T,s)\,\overline{\xi}^{-1}(T,s)\,\xi(T,y)\Big) = E\Big(\xi(T,s)\,\overline{\xi}^{-1}(T,s)\,E\big[\xi(T,y)/\mathfrak{I}_{(T,s)}\big]\Big) = 1,$$

$$E\big(\xi(T,s)\,\xi_n(T,y)\,\xi_n^{-1}(T,s)\big) = E\big(\xi(T,s)\,\xi_n^{-1}(T,s)\,E\left[\xi_n(T,y)/\mathfrak{I}_{(T,s)}\right]\big) = 1.$$

Thus,

$$\varlimsup_{n\to\infty} E\Big|\xi_0(T,s)\,\xi_n(T,y)\,\overline{\xi}_n^{-1}(T,s) - \xi(T,s)\,\xi(T,y)\,\overline{\xi}^{-1}(T,s)\Big| = 0,$$

which implies the required assertion. □

To summarize, we have shown that under the conditions of the theorem $\left(\widetilde{W}(z), \mathfrak{I}_z^2\right)$, $0 \le t \le T$, is a one-parameter martingale for any fixed s. The assertion that $\left(\widetilde{W}(z), \mathfrak{I}_z^1\right)$ is a one-parameter Wiener $\widetilde{P}$-martingale with parameter t can be proved in the same fashion by applying part (b) of Lemma 2.8.

2.4 Some Properties of Measures Corresponding to Random Fields on the Plane

In this section we present results published in [12, 14, 16, 42, 47, 48]. Let $(\xi(z), \mathfrak{I}_z)$ be a continuous random diffusion-type field, given by

$$\xi(z) = \int_{[0,z]} \varphi(u,\xi)du + W(z), \tag{2.17}$$

for any $z \in [0,T]^2$, where $\varphi(z,f)$, $f \in C[0,T]^2$ is some measurable functional independent of the future, and the condition

$$P\left\{\int_{[0,T]^2} |\varphi(u,\xi)|du < \infty\right\} = 1$$

holds true. In this section we study the properties of the random field $\zeta(z,W) = \frac{d\mu_\xi}{d\mu_W}(z,W)$.

Define $\Im_z^W := \sigma\{\omega : \ W(z'), \ z' \le z\}$, where the σ-algebras $\Im_z^W$ are completed by the sets from initial σ-algebra $\mathfrak{F}$ with P-measure 0.

It is easy to see that the field $(\zeta(z,W),\Im_z^W)$ is a martingale. Indeed, assume that $z < z'$ and that $\lambda(W)$ is some bounded $\Im_z^W$-measurable random variable. Then

$$E\lambda(W)\zeta(z',W) = \int \lambda(x)\frac{d\mu_\xi}{d\mu_W}(z',x)d\mu_W(x) = \int \lambda(x)d\mu_\xi(z',x)$$
$$= \int \lambda(x)\zeta(z,x)d\mu_W(z,x),$$

implying $E\big(\zeta\big(z,z'\big](W)/\Im_z^W\big) = 0$. Further we assume that the field $\big(\zeta(z,W),\Im_z^W\big)$ satisfies the stronger condition, namely, that it is a strong martingale:

$$E\big(\zeta\big(z,z'\big](W)/\Im_z^{W*}\big) = 0. \tag{2.18}$$

Theorem 2.7 *Let* $(\xi(z),\Im_z)$, $z \in [0,T]^2$, $\xi(0) = 0$, *be a diffusion-type random field (2.17). Assume that condition (2.18) is fulfilled, and*

$$P\left\{\int_{[0,T]^2} \varphi^2(u,\xi)du < \infty\right\} = 1.$$

Then the field $\zeta(z,W)$, $z \in [0,T]^2$ *is the unique solution to the equation*

$$\zeta(z,W) = 1 + \int_{[0,z]} \zeta(t',s)\varphi(z')W(dz'). \tag{2.19}$$

Proof It follows that $E\zeta((T,T),W) = 1$ from the definition of $\zeta(z,W)$. Then, condition (2.18) allows to apply Theorem 2.6 (the Girsanov theorem) to $\zeta(z,W)$. By this theorem the random field $\zeta(z,W)$ is continuous with probability 1 and can be represented as

$$\zeta(z,W) = 1 + \int_{[0,z]} \gamma(t_1,s)W(dz_1),$$

where $\gamma(z)$ is a $\Im_z$-measurable stochastic field with

$$E\left(\int_{[0,T]^2} \gamma^2(z)dz\right) < \infty.$$

Therefore, we can introduce a new probability measure $\widetilde{P}$ by $d\widetilde{P}(\omega) = \zeta((T,T),W)dP(\omega)$. We consider on the probability space $\left(\Omega, \Im, \widetilde{P}\right)$ a random field $\widetilde{W} = \left(\widetilde{W}(z), \Im_z^W\right)$, $z \in [0,T]^2$,

$$\widetilde{W}(z) = W(z) - \int_{[0,z]} B(z', \omega)dz',$$

$$B(z', \omega) = \overline{\zeta}^{-1}(t', s)\gamma(t', s').$$

It is a Wiener random field, and

$$\widetilde{P}\left\{\int_{[0,T]^2} B^2(z)dz < \infty\right\} = 1.$$

Thus, by Theorem 1.14 there exists a functional $\beta(z,f)$, where $f \in C[0,T]^2$, such that for almost all $z \in [0,T]^2$ we have $B(z) = \beta(z,W)$ and

$$\widetilde{W}(z) = W(z) - \int_{[0,z]} \beta(z', W)dz'$$

with probability 1. Since the measure μ_ξ is absolutely continuous with respect to the measure μ_W, we have

$$\begin{aligned}
&P\left\{\int_{[0,T]^2} \beta^2(\mathrm{z}, \xi)\, dz < \infty\right\} = \mu_\xi\left\{x : \int_{[0,T]^2} \beta^2(\mathrm{z}, x)\, dz < \infty\right\} \\
&\quad = \int \chi_{\int_{[0,T]^2} \beta^2(z,x)dz < \infty}(x)d\mu_\xi(x) = \int \chi_{\int_{[0,T]^2} \beta^2(z,x)dz < \infty}(x)\zeta((T,T),x)d\mu_W(x) \\
&\quad = \widetilde{P}\left\{\int_{[0,T]^2} \beta^2(z,\xi)dz < \infty\right\} = 1
\end{aligned}$$

Define on the probability space $(\Omega, \Im, P)$ a random field $\hat{W} = \left(\hat{W}(z), \Im_z^\xi\right)$, $z \in [0,T]^2$, by

$$\hat{W}(z,f) = f(z) - \int_{[0,z]} \beta(z',f)dz',$$

where $f \in C[0,T]^2$. If $f = \xi$, then $\hat{W} = \left(\hat{W}(z), \Im_z^\xi\right)$ is a Wiener field. Indeed, let $\lambda(\xi)$ be a bounded random $\Im_z^\xi$-measurable variable. Then for any $z' > z$

$$E\lambda(\xi)\exp\left\{i\theta\hat{W}\left(z,z'\right]\right\} = E\left(\lambda(\xi)E\left[\exp\left\{i\theta\hat{W}\left(z,z'\right]\right\}/\Im_z^\xi\right]\right).$$

On the other hand,

$$
\begin{aligned}
E\lambda(\xi)\exp\{i\theta\hat{W}(z,z']\} &= \int \lambda(f)\exp\{i\theta f(z,z']\}d\mu_\xi(f) \\
&= \int \lambda(f)\exp\{i\theta f(z,z']\}\zeta((T,T),f)d\mu_W(f) \\
&= \int \lambda(W)\exp\left\{i\theta\widetilde{W}(z,z']\right\}\zeta((T,T),W)dP \\
&= \widetilde{E}\,\lambda(W)\exp\left\{i\theta\widetilde{W}(z,z']\right\} \\
&= \widetilde{E}\,\lambda(W)\exp\left\{-\frac{\theta^2}{2}(t'-t)(s'-s)\right\} \\
&= \exp\left\{-\frac{\theta^2}{2}(t'-t)(s'-s)\right\}E\lambda(\xi),
\end{aligned}
$$

whence

$$
E\left(\lambda(\xi)E\left[\exp\{i\theta\hat{W}(z,z']\}/\mathfrak{I}_z^\xi\right]\right) = \exp\left\{-\frac{\theta^2}{2}(t'-t)(s'-s)\right\}
$$

as $E\left[\lambda(\xi)/\mathfrak{I}_z^\xi\right] = 1$. Therefore,

$$
\hat{W}(z) - W(z) = \int_{[0,z]}(\varphi(z',\xi)-\beta(z',\xi))dz',
$$

where $\left(\hat{W}(z),\mathfrak{I}_z^\xi\right)$ and $\left(W(z),\mathfrak{I}_z^\xi\right)$ are two Wiener fields. This means that $\hat{W}(z) - W(z) = 0$ with probability 1. Hence, for almost all $z \in [0,T]^2$ we have $P\{\varphi(z,\xi) = \beta(z,\xi)\} = 1$, and taking into account that $P\{\zeta(z,\xi) = 0\} = 0$, we get

$$
\zeta(z,W)\varphi(z,W) = \zeta(z,W)\beta(z,W).
$$

for almost all $z \in [0,T]^2$ with probability 1.

By definition, $\beta(z',\omega) = \overline{\zeta}^{-1}(t',s,W)\gamma(t',s',W)$, whence

$$
\begin{aligned}
&P\left\{\int_{[0,T]^2}(\zeta(t',T,W)\varphi(z',W))^2dz' < \infty\right\} \\
&\quad = P\left\{\int_{[0,T]^2}\left(\zeta(t',T,W)\overline{\zeta}^{-1}(t',T,W)\gamma(z')\right)^2dz' < \infty\right\} \\
&\quad \geq P\left\{\int_{[0,T]^2}\gamma^2(z)dz < \infty\right\} = 1
\end{aligned}
$$

Therefore, the stochastic integral $\int_{[0,z]}\zeta(t',s,W)\varphi(z')W(dz')$ is well defined. Let $D = \{z \in [0,T]^2 : \zeta(z) \neq 0\}$. We have

$$\zeta(z) = E\zeta(z) + \int_{[0,z]} \gamma(t_1, s)W(dz_1),$$

and by definition

$$1 + \int_{[0,z]} \zeta(t', s, W)\phi(z', W)W(dz') = 1 + \int_{[0,z]} \zeta(t', s, W)\overline{\zeta}^{-1}(t', s, W)\gamma(z')W(dz').$$

Moreover, $\zeta(z) = 0$ if $z \notin D$. Thus, for all $z \in D$

$$1 + \int_{[0,z]} \zeta(t', s, W)\varphi(z', W)W(dz') = 1 + \int_{[0,z]} \gamma(t_1, s)W(dz_1),$$

and by (2.19) we obtain the required assertion. □

Theorem 2.8 *Suppose that the following conditions hold true:*

$$P\left\{\int_{[0,T]^2} \varphi^2(u, \xi)du < \infty\right\} = 1, \tag{2.20}$$

$$P\left\{\int_{[0,T]^2} \varphi^2(u, W)du < \infty\right\} = 1. \tag{2.21}$$

Then the measures μ_ξ *and* μ_W *are equivalent, with Radon–Nikodym derivatives given by*

$$\frac{d\mu_\xi}{d\mu_W}(W) = \exp\left\{\int_{[0,T]^2} \varphi(u, W(u))W(du) - \frac{1}{2}\int_{[0,T]^2} \varphi^2(u, W(u))du\right\}, \tag{2.22}$$

$$\frac{d\mu_W}{d\mu_\xi}(\xi) = \exp\left\{-\int_{[0,T]^2} \varphi(u, \xi(u))\xi(du) - \frac{1}{2}\int_{[0,T]^2} \varphi^2(u, \xi(u))du\right\}, \tag{2.23}$$

where the equalities are satisfied with probability 1.

Proof Denote

$$\chi^{(n)}(z, x) = \chi_{\left\{x:\, x\in C,\ \int_{[0,z]} \varphi^2(z_1, x)dz_1 < n\right\}},$$

$$\varphi^{(n)}(z,x) = \varphi(z,x)\chi^{(n)}(z,x),$$

$$\xi^{(n)}(z) = \int_{[0,z]} \varphi^{(n)}(z_1,\xi)dz_1 + W(z), \quad z\in[0,T]^2, \quad n = 1,2,\ldots$$

It is easy to see that $\varphi^{(n)}(z,\xi) = \varphi^{(n)}(z,\xi^{(n)})$, $z \in [0,T]^2$, P-almost surely. Thus the field $\left(\xi^{(n)}(z),\Im_z\right)$ is a diffusion-type field, and

$$\xi^{(n)}(z) = \int_{[0,z]} \varphi^{(n)}\left(z_1,\xi^{(n)}\right)dz_1 + W(z), \quad z\in[0,T]^2, P\text{-a.s.}$$

Since

$$P\left\{\int_{[0,T]^2} \left(\varphi^{(n)}\left(z,\xi^{(n)}\right)\right)^2 dz < n\right\} = 1,$$

then, taking into account Remark 2.5, we obtain

$$E\exp\left\{-\int_{[0,T]^2} \varphi^{(n)}\left(z,\xi^{(n)}\right)W(dz) - \frac{1}{2}\int_{[0,T]^2} \left(\varphi^{(n)}\left(z,\xi^{(n)}\right)\right)^2 dz\right\} = 1,$$

which gives by Theorem 2.4 the equivalence $\mu_{\xi}{}^{(n)} \sim \mu_W$, where

$$\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right) = \exp\left\{-\int_{[0,T]^2} \varphi^{(n)}\left(z,\xi^{(n)}\right)W(dz) + \frac{1}{2}\int_{[0,T]^2} \left(\varphi^{(n)}\left(z,\xi^{(n)}\right)\right)^2 dz\right\} P\text{-a.s.}$$

The condition (2.20) guarantees that the property

$$\lim_{n\to\infty} \frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right) = \rho(\xi)$$

holds true P-a.s., where

$$\rho(\xi) = \exp\left\{-\int_{[0,T]^2} \varphi(z,\xi)W(dz) + \frac{1}{2}\int_{[0,T]^2} \varphi^2(z,\xi)dz\right\}.$$

Let us show that the set of random variables $\left\{\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right), \ n = 1,2,\ldots\right\}$ is uniformly integrable. Indeed,

$$\int_{\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right)>N}\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right)P(d\omega)=\int_{\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right)>N}\frac{d\mu_W}{d\mu_{\xi^{(n)}}}(z)\mu_{\xi^{(n)}}(dz)$$
$$=\mu_W\left\{z\ :\ \frac{d\mu_W}{d\mu_{\xi^{(n)}}}(z)>N\right\}=P\left\{z\ :\ \frac{d\mu_W}{d\mu_{\xi^{(n)}}}(W)>N\right\}$$
$$=P\left\{-\int_{[0,T]^2}\phi^{(n)}(z,W)W(dz)+\frac{1}{2}\int_{[0,T]^2}\left(\phi^{(n)}(z,W)\right)^2dz>\ln N\right\}$$
$$\leq P\left\{\left|\int_{[0,T]^2}\phi^{(n)}(z,W)W(dz)\right|>\frac{\ln N}{2}\right\}$$
$$+P\left\{\int_{[0,T]^2}\left(\phi^{(n)}(z,W)\right)^2dz>\frac{\ln N}{2}\right\}$$
$$\leq\frac{4}{\ln N}+2P\left\{\int_{[0,T]^2}\phi^2(z,W)dz>\ln N\right\},\quad N=1,2,\ldots$$

From condition (2.20) we get

$$\sup_n\int_{\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right)>N}\frac{d\mu_W}{d\mu_{\xi^{(n)}}}\left(\xi^{(n)}\right)P(d\omega)\to 0,\quad N\to\infty.$$

Thus,

$$E\exp\left\{-\int_{[0,T]^2}\varphi(z,\xi)\xi(dz)+\frac{1}{2}\int_{[0,T]^2}\varphi^2(z,\xi)dz\right\}=1$$

and conditions (2.22) and (2.23) are satisfied. Theorem is proved. □

Theorem 2.9 *Let $(\xi(z),\mathfrak{I}_z)$, z ∈ [0,T]^2 be the generalized Ito field of the form*

$$\xi(z)=\int_{[0,z]}\alpha(z')dz'+\int_{[0,z]}\beta(z')W(dz')$$

with coefficients satisfying

$$P\left\{\int_{[0,T]^2}|\alpha(z)|dz<\infty\right\}=1,\tag{2.24}$$

$$P\left\{\int_{[0,T]^2}\beta^2(z)dz<\infty\right\}=1.\tag{2.25}$$

Let $\widetilde{P}(d\omega) = \exp\{\zeta(0, T, \varphi\}P(d\omega)$, *hence* $\varphi(z)$ *satisfies*

$$P\left\{\int_{[0,T]^2} \varphi^2(z)dz < \infty\right\} = 1 \quad \text{and} \quad E\exp\{\zeta(0, T, \varphi)\} = 1.$$

Then

1. $\left(\xi(z), \Im_z, \widetilde{P}\right)$ is an Ito field with coefficients $\widetilde{\alpha}(z) = \alpha(z) + \beta(z)\varphi(z)$ and $\widetilde{\beta}(z) = E\left(\beta(z)/\Im_z^W\right)$

$$\widetilde{\xi}(z) = W(z) - \int_{[0,z]} \varphi(z')dz'.$$

2. $\mu_\xi \ll \mu\widetilde{\xi}$.

Proof It is easy to see that for the random field $\psi = (\psi(z), \Im_z)$ such that

$$P\left\{\int_{[0,T]^2} \psi^2(z)dz < \infty\right\} = 1$$

both stochastic integrals $\int_{[0,z]}\psi(z')W(dz')$ and $\int_{[0,z]} \psi(z')\widetilde{\xi}(dz')$ exist and, moreover,

$$\int_{[0,z]} \psi(z')W(dz') = \int_{[0,z]} \psi(z')\widetilde{\xi}(dz') + \int_{[0,z]} \psi(z')\varphi(z')dz',$$

implying that

$$\begin{aligned}\xi(z) &= \int_{[0,z]} \alpha(z')dz' + \int_{[0,z]} \beta(z')W(dz')\\ &= \int_{[0,z]} [\alpha(z') + \beta(z')\varphi(z')]dz' + \int_{[0,z]} \beta(z')\widetilde{\xi}(dz').\end{aligned}$$

Thus, statement (1) is proved. Statement (2) follows directly from the relation $\widetilde{P} \ll P$. □

Theorem 2.10 *Let* $(\xi(z), \Im_z)$, z ∈ *[0,T]*^{2} *be a generalized Ito field of the form*

$$\xi(z) = \int_{[0,z]} A(z')dz' + \int_{[0,z]} b(z',\xi)W(dz'),$$

and let $(\eta(z), \Im_z)$ *be a generalized diffusion-type field,*

$$\eta(z) = \int_{[0,z]} a(z',\eta)dz' + \int_{[0,z]} b(z',\eta)W(dz'). \tag{2.26}$$

Assume that the following conditions hold true:

(a) Functionals $a(z,x)$ and $b(z,x)$ are such that there exists a unique strong solution to (2.26).
(b) For any $z \in [0,T]^2$ the equation $b(z,\xi)\alpha(z) = A(z) - a(z,\xi)$ has a bounded solution with respect to $\alpha(z)$.
(c) $P\left\{\int_{[0,T]^2} \alpha^2(z)dz < \infty\right\} = 1.$
(d) $E\exp\left\{-\int_{[0,T]^2} \alpha(z)W(dz) - \frac{1}{2}\int_{[0,T]^2} \alpha^2(z)dz\right\} = 1.$

Then $\mu_\xi \sim \mu_\eta$, and

$$\frac{d\mu_\eta}{d\mu_\xi}(\xi) = E\left[\exp\left\{-\int_{[0,T]^2} \alpha(z)W(dz) - \frac{1}{2}\int_{[0,T]^2} \alpha^2(z)dz\right\}/\Im^\xi_{(T,T)}\right] P\text{-a.s.}$$

The proof can be deduced in the same way as the proof of Theorem 2.4.

Corollary 2.1 If $(\xi(z), \Im_z)$ is a generalized diffusion-type field with a drift coefficient $A(z,\xi)$, i.e.

$$\xi(z) = \int_{[0,z]} A(z',\xi)dz' + \int_{[0,z]} b(z',\xi)W(dz'), \tag{2.27}$$

then under conditions (a), (b), and (d) of Theorem 2.5 and the condition

$$P\left\{\int_{[0,T]^2} \left(\overline{b}^{-1}(z,\xi)A(z,\xi)\right)^2 < \infty\right\} = P\left\{\int_{[0,T]^2} \left(\overline{b}^{-1}(z,\xi)a(z,\xi)\right)^2 < \infty\right\} = 1,$$

we obtain $\mu_\xi \sim \mu_\eta$, and

$$\begin{aligned}\frac{d\mu_\eta}{d\mu_\xi}(\xi) = \exp\Bigg\{&-\int_{[0,T]^2} \left(\overline{b}^{-1}(z,\xi)\right)^2 (A(z,\xi) - a(z,\xi))\xi(dz) \\ &+ \frac{1}{2}\int_{[0,T]^2} \left(\overline{b}^{-1}(z,\xi)\right)^2 \left(A^2(z,\xi) - a^2(z,\xi)\right)dz\Bigg\}\end{aligned}$$

$$\frac{d\mu_\xi}{d\mu_\eta}(\eta) = \exp\left\{-\int_{[0,T]^2}\left(\overline{b}^{-1}(z,\eta)\right)^2 (A(z,\eta)-a(z,\eta))\eta(dz)\right.$$
$$\left.-\frac{1}{2}\int_{[0,T]^2}\left(\overline{b}^{-1}(z,\eta)\right)^2\left(A^2(z,\eta)-a^2(z,\eta)\right)dz\right\} \tag{2.28}$$

Theorem 2.11 *Assume that the condition (b) of Theorem 2.10 holds true, and*

(a) $P\left\{\int_{[0,T]^2}\left(\overline{b}^{-1}(z,\xi)\right)^2\left(A^2(z,\xi)-a^2(z,\xi)\right)dz<\infty\right\}$
$= P\left\{\int_{[0,T]^2}\left(\overline{b}^{-1}(z,\eta)\right)^2\left(A^2(z,\eta)-a^2(z,\eta)\right)dz<\infty\right\} = 1.$

(b) There exists a nonrandom constant K, such that

$$|a(z,x)|+|b(z,x)|\le K\left(1+\|x\|_z\right),$$

where $\|x\|_z = \max\limits_{0<z'<z}|x(z')|$.

(c) For a nonrandom constant $\widetilde{K}$

$$|a(z,x)-a(z,\widetilde{x})|+|b(z,x)-b(z,\widetilde{x})|\le \widetilde{K}\,\|x-\widetilde{x}\,\|_z.$$

Then $\mu_\xi \sim \mu_\eta$, *and the respective Radon–Nikodym derivatives are given by formulas (2.28).*

Proof Put

$$\chi^{(n)}(z,x) = \begin{cases} 1, & \int_{[0,z]}\left(\overline{b}^{-1}(z',x)[A(z',x)-a(z',x)]\right)^2 dz' < n,\\ 0, & \int_{[0,z]}\left(\overline{b}^{-1}(z',x)[A(z',x)-a(z',x)]\right)^2 dz' \ge n,\end{cases}$$

$$A^{(n)}(z) = a\left(z,x\right)+\chi^{(n)}\left(z,x\right)[A(z,x)-a(z,x)].$$

Consider the differential equations

$$\xi^{(n)}(z) = \int_{[0,z]} A^{(n)}\left(z',\xi^{(n)}\right)dz' + \int_{[0,z]} b\left(z',\xi^{(n)}\right)W(dz'). \tag{2.29}$$

Since the conditions of Theorem 2.11 are satisfied, there exists a unique strong solution to (2.29). Moreover,

$$A^{(n)}(z) - a(z,x)+\chi^{(n)}(z,x)[A(z,x)-a(z,x)],$$

which implies

$$\int_{[0,T]^2} \left(\overline{b}^{-1}(z',x) \left[A^{(n)}\left(z',\xi^{(n)}\right) - a\left(z',\xi^{(n)}\right) \right] \right)^2 dz' < nP\text{-a.s.}$$

By Theorem 2.3 and Remark 2.5 we obtain

$$E\exp\left\{ -\int_{[0,T]^2} \overline{b}^{-1}\left(z',x\right)\left[A^{(n)}\left(z',\xi^{(n)}\right) - a\left(z',\xi^{(n)}\right)\right]W\left(dz'\right) \right.$$
$$\left. -\frac{1}{2}\int_{[0,T]^2} \left(\overline{b}^{-1}\left(z',x\right)\left[A^{(n)}\left(z',\xi^{(n)}\right) - a\left(z',\xi^{(n)}\right)\right]\right)^2 dz' \right\} = 1.$$

By Theorem 2.5, $\mu_{\xi^{(n)}} \sim \mu_\eta$, and

$$\frac{d\mu_{\xi^{(n)}}}{d\mu_\eta}(\eta) = \exp\left\{ -\int_{[0,T]^2} \left(\overline{b}^{-1}(z,\eta)\right)^2 \left(A^{(n)}(z,\eta) - a(z,\eta)\right)\eta(dz) \right.$$
$$\left. -\frac{1}{2}\int_{[0,T]^2} \left(\overline{b}^{-1}(z,\eta)\right)^2 \left(\left[A^{(n)}(z,\eta)\right]^2 - a^2(z,\eta)\right)dz \right\}.$$

Since the condition (a) is fulfilled, we obtain the assertion of the theorem in the same fashion as in the case of Theorem 2.4. □

Now we are interested in the following problem: how a diffusion-type field behaves under the change of the measure, and when the new transformed field is also of diffusion type in some other space. We reconstruct the form of field if it is known that the corresponding measure is absolutely continuous with respect to the measure generated by some diffusion-type field.

Consider two probability spaces $(\Omega, \Im, P)$ and $\left(\Omega, \Im, \widetilde{P}\right)$, where $d\widetilde{P}(\omega) = \rho(T,T)$ $dP(\omega)$, and

$$\rho(z) = 1 + \int_{[0,z]} \gamma(t_1,s)W(dz_1), \quad z \in [0,T]^2.$$

Theorem 2.12 *Assume that ξ is a solution to the stochastic differential equation*

$$\xi(z) = \int_{[0,z]} a(u,\xi)du + \int_{[0,z]} b(u,\xi)W(du), \tag{2.30}$$

where functionals $a(z,f)$ and $b(z,f)$, $z \in [0,T]^2$, belong to the space $C[0,T]^2$ and do not depend on the future. Moreover, assume that the conditions below are satisfied:

(a) $P\left\{ \int_{[0,T]^2} b^2(u,\xi)du < \infty \right\} = 1.$

(b) $P\left\{\int_{[0,T]^2}\left(\overline{b}^{-1}(z',\xi)\left[a(z',\xi)+b(z',\xi)\gamma(z')\overline{\rho}^{-1}(t',s)\right]\right)^2 dz' < \infty\right\} = 1.$

(c) For almost all $z \in [0,T]^2$ $P\{b^2(z,\xi) > 0\} = 1$.

Then there exists a random field $\widetilde{\xi} = \left(\xi(z), \Im_z, \widetilde{P}\right)$ *on the probability space* $\left(\Omega, \Im, \widetilde{P}\right)$, *which is the solution to some another stochastic differential equation and, moreover, the measure* $\mu\widetilde{\xi}$ *corresponding to the field* $\widetilde{\xi}$ *is absolutely continuous with respect to the measure* μ_ξ, *generated by the field* ξ, *and the Radon–Nikodym derivative is given by*

$$\frac{d\mu\widetilde{\xi}}{d\mu_\eta} = E\left[\rho(T,T)/\Im^{\xi}_{(T,T)}\right].$$

Proof On the probability space $\left(\Omega, \Im, \widetilde{P}\right)$ the random field $\widetilde{W} = \left(\widetilde{W}(z), \Im_z\right)$, $z \in [0,T]^2$, given by

$$\widetilde{W}(z) = W(z) - \int_{[0,z]} \rho(t',s)\gamma(z')dz',$$

is a Wiener field, see Theorem 2.3. Therefore, under conditions (a) and (b) the random field $\widetilde{\xi}$ on the probability space $\left(\Omega, \Im, \widetilde{P}\right)$ is the Ito field with respect to $\widetilde{W}$ with diffusion coefficients

$$\alpha(z') = a(z',\xi) + b(z',\xi)\overline{\rho}^{-1}(t',s)\gamma(z')$$

and

$$\beta(z') = b(z',\xi).$$

In view of conditions (a)–(c) one can apply Theorem 2.1 by which there exist functionals $\widetilde{a}\,(z,f)$ and $\widetilde{b}\,(z,f)$ on the space $C[0,1]^2$, and a Wiener field $\hat{W}$ on $\left(\Omega, \Im, \widetilde{P}\right)$, such that

$$\widetilde{\xi}\,(z) = \int_{[0,z]} \widetilde{a}\,(u,\xi)du + \int_{[0,z]} \widetilde{b}\,(u,\xi)\hat{W}\,(du)\,P\text{-a.s.}$$

Let $\mu\widetilde{\xi}$ and μ_ξ be the measures corresponding, respectively, to the fields $\widetilde{\xi}$ and ξ. Then for any set $I = (x,y) \subset R$ we have

$$\mu\widetilde{\xi}\,(I) = \widetilde{P}\left\{\omega : \widetilde{\xi}\,(\omega)\in I\right\} = \int \chi_I(\xi)d\widetilde{P}\,(\omega) = \int \chi_I(\xi)\rho(1,1)dP(\omega)$$
$$= \int \chi_I(\xi)E\left[\rho(1,1)/\Im^{\xi}_{(1,1)}\right]dP(\omega).$$

The last equality can be expanded to all Borel sets from R^2. Put $\zeta(\xi) = E\left[\rho(T,T)/\Im^{\xi}_{(T,T)}\right]$. Thus, we proved

$$\mu\widetilde{\xi}(I) = \int_I \zeta(f)d\mu(f),$$

which implies the assertion of the theorem. □

Assume now that the random field ξ is a solution to (2.30), and the field ζ is such that $\mu_\zeta \ll \mu_\xi$ with

$$\frac{d\mu_\zeta}{d\mu_\xi}(z,\xi) = \rho_1(z,\xi) = 1 + \int_{[0,z]} \beta(u)W(du).$$

Put $\eta(z) = \int_{[0,z]} b(u,\xi)W(du)$. Theorem 2.3 asserts that under conditions

B1. for any $z \in [0,T]^2$ the equation $b(z,\xi)\alpha(z) = a(z,\xi)$ has (with respect to α) P-a.s. a bounded solution and

$$E\exp\left\{-\int_{[0,T]^2}\alpha(z)W(dz) - \frac{1}{2}\int_{[0,T]^2}\alpha^2(z)dz\right\} = 1,$$

B2. $P\left\{\int_{[0,T]^2}\left(\overline{b}^{-1}(u,\xi)a(u,\xi)\right)^2 du < \infty\right\} = 1$

the measures μ_ξ and μ_η are equivalent, and the Radon–Nikodym derivative is given by

$$\begin{aligned}\frac{d\mu_\xi}{d\mu_\eta}(z,\eta) &= \rho_2(z,\eta)\\ &= \exp\left\{\int_{[0,z]}\overline{b}^{-1}(u,\eta)a(u,\eta)W(du) - \int_{[0,z]}\left(\overline{b}^{-1}(u,\eta)a(u,\eta)\right)^2 du\right\}.\end{aligned}$$

Appling the Ito formula (see Theorem 1.17) we derive

$$\rho_2(z,\eta) = 1 + \int_{[0,z]}\overline{b}^{-1}(z',\eta)a(z',\eta)\rho_2(t',s,\eta)W(dz'),$$

implying $\mu_\zeta \ll \mu_\xi \ll \mu_\eta$. Let us calculate the derivative $\rho_3(z,\eta) = \frac{d\mu_\zeta}{d\mu_\xi}$. We have:

$$\begin{aligned}\rho_3(z,\eta) = 1 &+ \int_{[0,z]}\left(\beta(z')\rho_2(z',\eta) + \rho_1(z',\eta)\rho_2(t',s,\eta)\overline{b}^{-1}(z',\eta)a(z',\eta)\right)W(dz')\\ &+ \int_{[0,z]}\int_{[0,z]}\left(\beta(x,s')\rho_2\left(t,'s,\eta\right)\overline{b}^{-1}\left(t,'y,\eta\right)a(z',\eta)\right)W(dx,dy)W(dz')\\ &+ \int_{[0,z]}\int_{[0,z']}\left(\beta(t',y)\rho_2(x,s,\eta)\overline{b}^{-1}(x,s',\eta)a(z',\eta)\right)W(dx,dy)W(dz').\end{aligned}$$

Suppose that the random fields $a,\ b,\ \beta$ are such that

$$\int_{[0,z]}\int_{[0,z]}\Big(\beta(x,s')\rho_2\Big(t,'s,\eta\Big)\overline{b}^{-1}\Big(t,'y,\eta\Big)a(z',\eta)\Big)W(dx,dy)W(dz')$$
$$+\int_{[0,z]}\int_{[0,z']}\Big(\beta(t',y)\rho_2(x,s,\eta)\overline{b}^{-1}(x,s',\eta)a(z',\eta)\Big)W(dx,dy)W(dz')=0.$$

Then applying again the Ito formula we derive

$$\rho_3(z,\eta)=1+\int_{[0,z]}\alpha(u)W(du),$$

$$\alpha(z')=\beta(z')\rho_2(z',\eta)+\rho_1(z',\eta)\rho_2(t',s,\eta)\overline{b}^{-1}(z',\eta)a(z',\eta).$$

Thus, $\rho_3(z,\eta)$ is nonnegative martingale with $E\rho(T,T)=1$. For an $\mathfrak{I}^{\eta}_{z'}$-measurable random variable $\lambda=\lambda(\eta)$ we have

$$E\lambda(\eta)\rho(z,\eta)=\int\lambda(f)d\mu_{\zeta}(z,f)=\int\lambda(f)\rho_3(z',f)d\mu_{\eta}(z,f).$$

Consider the probability space $\Big(\Omega,\mathfrak{I}^{\eta},\widetilde{P}\Big)$, with $d\widetilde{P}(\omega)=\rho(T,T)dP(\omega)$. The random field

$$\widetilde{W}(z)=W(z)-\int_{[0,z]}\overline{\rho}_3^{\,-1}(t',s,\eta)b(z',\eta)\alpha(z')dz'$$

is a Wiener field on this space, hence by Theorem 2.1 there exists a functional $\widetilde{a}(z,f)$ in the space $C[0,T]^2$ such that for almost all $z'\in[0,T]^2$

$$\widetilde{a}(z',f)=\overline{\rho}_3^{\,-1}(t',s,\eta)b(z',\eta)\alpha(z')$$

and

$$\widetilde{P}\left\{\int_{[0,T]^2}\widetilde{a}^2(u,\eta)du<\infty\right\}=1.$$

Moreover,

$$P\left\{\int_{[0,T]^2}\widetilde{a}^2(\mu,\eta)du<\infty\right\}=\mu_{\zeta}\left\{\int_{[0,T]^2}\widetilde{a}^2(\mu,f)du<\infty\right\}$$
$$=\widetilde{P}\left\{\int_{[0,T]^2}\widetilde{a}^2(u,\eta)du<\infty\right\}=1,$$

and in such a way all stochastic integrals written above are well defined.

Define on the probability space $\left(\Omega, \mathfrak{I}^{\eta}, \widetilde{P}\right)$ a random field $\left(\varphi(z,\zeta), \mathfrak{I}_z^{\zeta}\right)$, $z \in [0,T]^2$, by

$$\varphi(z,f) = f(z) - \int_{[0,z]} \widetilde{a}^2(u,f)du, \quad f \in C[0,T]^2.$$

Let $\lambda(\zeta)$ be a bounded random $\mathfrak{I}_z^{\zeta *}$-measurable variable. Then for any $z_1 \in [0,T]^2$, $z_1 > z$, we have

$$\begin{aligned} E\lambda(\zeta)\,\varphi(z,z_1](\zeta) &= \int \lambda(f)\,\phi(z,z_1](f)d\mu_\zeta(f) = \int \lambda(f)\,\phi(z,z_1](f)\,\rho(T,,T)d\mu_\eta(f) \\ &= \int \lambda(\eta)\,\phi(z,z_1](\eta)\,\rho(T,T)dP = \widetilde{E}\,\lambda(\eta)\widetilde{E}\left[\phi(z,z_1](\eta)/\mathfrak{I}_z^{\zeta *}\right] = 0. \end{aligned}$$

Therefore, $\zeta(z) - \int_{[0,z]} \widetilde{a}^2(u,\zeta)du$ is a strong martingale, and there exists (see Theorems 1.12, 2.1) $\phi(z,f)$ such that for almost all $f \in C[0,1]^2$

$$\int_{[0,T]^2} \varphi^2(u,f)du < \infty$$

and

$$\zeta(z) = \int_{[0,z]} \widetilde{a}\,(u,\zeta)du - \int_{[0,z]} \phi(u,\zeta)\widetilde{W}(du), \tag{2.31}$$

where $\widetilde{W}$ is a Wiener field on $\left(\Omega, \mathfrak{I}\widetilde{P}\right)$. In such a way, we proved.

Theorem 2.13 *Assume that ξ is a strong solution to (2.30) with coefficients $a(z,f)$ and $b(z,f)$, satisfying B1 and B2. Assume that the stochastic field ζ is such that $\mu_\zeta \ll \mu_\xi$, and the Radon–Nikodym derivative is given by*

$$\frac{d\mu_\zeta}{d\mu_\xi}(z,\xi) = 1 + \int_{[0,z]} \beta(t_1,s)W(dz_1).$$

In addition, assume that $a(z,f)$, $b(z,f)$ and $\beta(z)$ satisfy

$$\begin{aligned} &\int_{[0,z]}\int_{[0,z]} \left(\beta(x,s')\rho_2\left(t,'s,\eta\right)\overline{b}^{-1}\left(t,'y,\eta\right)a(z',\eta)\right)W(dx,dy)W(dz') \\ &+ \int_{[0,z]}\int_{[0,z']} \left(\beta(t',y)\rho_2(x,s,\eta)\overline{b}^{-1}(x,s',\eta)a(z',\eta)\right)W(dx,dy)W(dz') = 0 \end{aligned}$$

Then there exist a probability space $\left(\Omega, \mathfrak{I}, \widetilde{P}\right)$, the coefficients $\widetilde{a}\,(z,f)$ and $\widetilde{b}\,(z,f)$, the field $\phi(z,f)$ and a Wiener field $\widetilde{W}$, such that random field ζ is the solution to the stochastic differential equation (2.31).

2.5 Nonparametric Estimation of a Two-Parametrical Signal from Observation with Additive Noise

In this section we study the properties of a periodic in both variables estimator, observed on enlarged part of the plane, with random errors of white noise type. Here we present results obtained in [13, 15, 17, 42–44].

Suppose that on the probability space $(\Omega, \Im, P)$ we have a real random field $(x(z),\ z \in R_+^2)$ and continuous square integrable strong martingale $(\xi(z),\ z \in R_+^2)$. We consider the problem of estimating the unknown function a from observations of the two-dimensional random field $(x(z), y(z),\ z \in R_+^2)$ in the rectangle $[0,T] \times [0,S]$ where

$$y(z) = \int_{[0,z]} a_0(u)x(u)du + \xi(z). \tag{2.32}$$

Further we assume that the random fields $x(z)$ and $\xi(z)$ are independent and the conditions below are satisfied:

C1. The function a_0 is an element of the set K of all real functions defined on the plane, 2π-periodic in both variables, whose Fourier coefficients

$$c_{kl}(a) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} a(t,s)\exp\{i(kt + ls)\}dtds, \quad k,l = 0, \pm 1, \pm 2, \ldots$$

satisfy the inequalities

$$|c_{00}(a)| \le L,\ |c_{k0}(a)||k|^{\alpha} \le L,\ |c_{0l}(a)||l|^{\beta} \le L,\ |c_{kl}(a)||k|^{\alpha}|l|^{\beta} \le L,\ kl \ne 0,$$

with some constants $L > 0$, $\alpha > 3$, $\beta > 3$.

For any function $a \in K$ put $\|a\|^2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} a(t,s)dtds$. We call the element $a \in K$ an interior point of K, if $|c_{00}(a)| \le \widetilde{L}$, $|c_{k0}(a)||k|^{\alpha} \le \widetilde{L}$, $|c_{0l}(a)||l|^{\beta} \le \widetilde{L}$, $|c_{kl}(a)||k|^{\alpha}|l|^{\beta} \le \widetilde{L}$, $kl \ne 0$, with some $\widetilde{L} < L$.

C2. $(\xi(z),\ z \in R_+^2)$, $\xi(t,0) = \xi(0,s) = 0$ is a continuous strong martingale, square integrable in any finite rectangle, with characteristic $\gamma(z) = \int_{[0,z]}\sigma^2(u)du$, where $\sigma^2(z) > 0$ P- almost surely and $E\sigma^2(z) \le C$.

C3. $(x(z),\ z \in R_+^2)$ is a real random field whose sample functions have continuous second-order derivatives with P-probability 1. The random field $(x^2(z), z \in R_+^2)$ is homogeneous in the broad sense, with $Ex^2(0) > 0$.

Denote by $r(z) = E([x^2(z) - Ex^2(0)][x^2(0) - Ex^2(0)])$ the correlation function of the field $(x^2(z),\ z \in R_+^2)$.

C4. For some $\gamma_1 > 0$, $\gamma_2 > 0$, $L_1 > 0$ and all $S \ge 1$, $T \ge 1$ the inequality $\int_0^T\int_0^S |r(z)|dz \le L_1 T^{1-\gamma_1}S^{1-\gamma_2}$ holds true.

C5. For some positive $L_2 > 0$ $|r(z)| \le \dfrac{L_2}{(1+t^2)(1+s^2)}$, $z = (t,s) \in R_+^2$.

The conditions C1–C3 guarantee that for any function from the set K one can define a stochastic integral with respect to a strong martingale.

We consider the problem of estimation a_0 from the observations $(x(z), y(z), z \in [0,T] \times [0,S])$ of the stochastic field $(y(z),\ z \in R_+^2)$, defined by (2.32) with some fixed function $a \in K$. As an estimate for a_0 we take an element $a_{TS} \in K$ which minimizes the functional

$$Q_{TS} = \frac{1}{TS}\int_0^T\int_0^S a(u)x(u)y(du) - \frac{1}{2TS}\int_0^T\int_0^S a^2(u)x^2(u)du$$

on K. In assumption C1 the maximum of Q_{TS} on K is achieved, and one can show using Theorem 1.20 that $\{a_{TS}(z), z \in R_+^2\}$ is a separable measurable field. The functional Q_{TS} can be represented as

$$\begin{aligned} Q_{TS} &= \frac{1}{TS}\int_0^T\int_0^S [a(u) - a_0(u)]x(u)\xi(du) \\ &\quad - \frac{1}{2TS}\int_0^T\int_0^S [a(u) - a_0(u)]^2 x^2(u)du + Q_{TS}(a_0). \end{aligned}$$

Further, to simplify our calculation we put $T = S$ and denote $a_{TT} := a_T$, $Q_{TT} := Q_T$.

Lemma 2.9 *Assume that conditions C1–C3 hold true. Then for any $\gamma > 0$*

$$P\left\{\lim_{T\to\infty}\max_{a\in K}\left|\frac{1}{T^{1+\gamma}}\int_{[0,T]^2} a(u)x(u)\xi(du)\right| = 0\right\} = 1.$$

Proof For $T > 0$ denote

$$\eta_T = \max_{a\in K}\left|\frac{1}{T^{1+\gamma}}\int_{[0,T]^2} a(u)x(u)\xi(du)\right|.$$

Let us estimate $E\eta_T^2$. From the Fourier decomposition of the function a and condition C1 we obtain

$$\begin{aligned} E\eta_T^2 &\le E\left\{\max_{a\in K}\left|\sum_{j,k=-\infty}^{\infty} c_{jk}(a)\frac{1}{T^{1+\gamma}}\int_{[0,T]^2}\exp\{i(jt' + ks')\}x(z')\xi(dz')\right|^2\right\} \\ &\le E\left\{\left|{\sum_{j,k=-\infty}^{\infty}}' c_{jk}(a)\frac{1}{T^{1+\gamma}}\int_{[0,T]^2}\exp\{i(jt' + ks')\}x(z')\xi(dz')\right|^2\right\}. \end{aligned}$$

Here the dash near sum sign means that we take 1 instead of $|j|^{-\alpha}$ as $j = 0$ and $|k|^{-\beta}$ as $k = 0$. By the properties of the stochastic integrals we get

$$
\begin{aligned}
E\eta_T^2 &\leq \left\{ \sum_{j,k=-\infty}^{\infty} \frac{L}{T^{1+\gamma}|j|^\alpha |k|^\beta} \left[E \left| \int_{[0,T]^2} \exp\{i(jt' + ks')\} x(z')\xi(dz') \right|^2 \right]^{1/2} \right\}^2 \\
&\leq C^* T^{-2\gamma},
\end{aligned}
\tag{2.33}
$$

where

$$
C^* = \left(\sum_{j,k=-\infty}^{\infty} \frac{L}{|j|^\alpha |k|^\beta} \right)^2 Ex^2(0)C.
$$

Let p be a fixed natural number such that $2p\gamma > 1$. From the previous estimate we obtain using the Borel-Cantelli lemma for the sequence $\{\eta_{T(n)},\ \ n \geq 1\}$, with $T(n) = n^p, p \geq 1$, that $P\left\{\lim_{n\to\infty} \eta_{T(n)} = 0\right\} = 1$. Take now $T \in [T(n), T(n + 1)]$. Then $\eta_T = \eta_{T(n)} + \zeta_n$, where $\zeta_n = \zeta_{1n} + \zeta_{2n}$, and

$$
\begin{aligned}
\zeta_{1n} &= \frac{1}{T^{1+\gamma}(n)} \max_{T\in[T(n),T(n+1)]} \max_{a\in K} \left| \int_{T(n)}^T \int_0^T a(u)x(u)\xi(du) \right|, \\
\zeta_{2n} &= \frac{1}{T^{1+\gamma}(n)} \max_{T\in[T(n),T(n+1)]} \max_{a\in K} \left| \int_0^{T(n)} \int_{T(n)}^T a(u)x(u)\xi(du) \right|.
\end{aligned}
$$

Let us estimate $E\zeta_{1n}$. We have

$$
\begin{aligned}
E\zeta_{1n} &\leq \frac{1}{T^{2(1+\gamma)}(n)} E \left[\sum_{j,k=-\infty}^{\infty} \frac{L}{|j|^\alpha |k|^\beta} \max_{T\in[T(n),T(n+1)]} \left| \int_{T(n)}^{T} \int_0^{T} \exp\{i(jt' + ks')\}x(z')\xi(dz') \right| \right]^2 \\
&\leq \frac{1}{T^{2(1+\gamma)}(n)} \left[\sum_{j,k=-\infty}^{\infty}{}' \frac{L}{|j|^\alpha |k|^\beta} \left\{ E \max_{T\in[T(n),T(n+1)]} \left| \int_{T(n)}^{T} \int_0^{T} \exp\{i(jt' + ks')\}x(z')\xi(dz') \right|^2 \right\}^{1/2} \right]^2.
\end{aligned}
$$

It is easy to see from (2.33) that

$$
\begin{aligned}
&E \left[\max_{T\in[T(n),T(n+1)]} \left| \int_{T(n)}^T \int_0^T \exp\{i(jt' + ks')\}x(z')\xi(dz') \right|^2 \right] \\
&\quad \leq 16T(n+1)[T(n+1) - T(n)]Ex^2(0)c.
\end{aligned}
$$

Thus,

$$E\zeta_{1n} \le 16c_1 \frac{T(n+1)[T(n+1)-T(n)]}{T^{2(1+\gamma)}(n)} = 16c_1 \frac{T(n+1)-T(n)}{T^{1+2\gamma}(n)} \quad \frac{T(n+1)}{T(n)}$$
$$= \frac{16c_1}{n^{2p\gamma}} \left[\left(1+\frac{1}{n}\right)^p - 1 \right] \left(\frac{n+1}{n}\right)^p.$$

Hence, $P\{\lim_{n\to\infty} \zeta_{1n} = 0\} = 1$. Similarly, one can show that $P\{\lim_{n\to\infty} \zeta_{2n} = 0\} = 1$. Lemma is proved. □

Remark 2.6 The assertions of Lemma 2.8 also hold true also if instead of the function $a \in K$ a difference of two functions from K is considered.

Lemma 2.10 *Assume that* $(\varsigma(z), z \in R^2)$ *is a real homogeneous field with zero mean and correlation function* $r(z) = E(\varsigma(z)\varsigma(0))$, $z \in R^2$, *such that for all* $T \ge 1$ *and some positive* L_3 *and* δ

$$\int_{[0,T]^2} |r(z)|dz \le L_3 T^{2-\delta}.$$

Then

$$P\left\{ \lim_{T\to\infty} \max_{a\in K} \left| \frac{1}{T^2} \int_{[0,T]^2} a(u)\varsigma(u)du \right| = 0 \right\} = 1.$$

Proof For any $T > 0$ let

$$\widetilde{\eta}_T = \sup_{a\in K} \left| \frac{1}{T^2} \int_{[0,T]^2} a(u)\varsigma(u)du \right|,$$

and for $n \ge 1$ let $\widetilde{\zeta}_n = \max_{T\in[T(n),T(n+1)]} \widetilde{\eta}_T$ where $T(n) = n^p, p \ge 1, p$ is the fixed value with $\delta p > 1$. Decompose a into the Fourier series:

$$\widetilde{\eta}_T = \sup_{a\in K} \left| \sum_{j,k=-\infty}^{\infty} c_{jk}(a) \frac{1}{T^2} \int_{[0,T]^2} \exp\{i(jt' + ks')\}\varsigma(z')dz' \right|$$
$$\le \sum_{j,k=-\infty}^{\infty} \frac{L}{|j|^\alpha |k|^\beta} \left| \frac{1}{T^2} \int_{[0,T]^2} \exp\{i(jt' + ks')\}\varsigma(z')dz' \right|.$$

The series in the right-hand side of the last formula converge with P-probability 1. For $\widetilde{\zeta}_n$ we have

$$\begin{aligned}\widetilde{\zeta}_n \le & \sum_{j,k=-\infty}^{\infty}{}' \frac{L}{T^2(n)\,|j|^\alpha|k|^\beta}\left|\int_{[0,T(n)]^2} \exp\{i(jt'+ks')\}\varsigma(z')dz'\right| \\ & + \sum_{j,k=-\infty}^{\infty}{}' \frac{L}{T^2(n)\,|j|^\alpha|k|^\beta}\int_{T(n)}^{T(n+1)}\int_0^{T(n+1)} |\varsigma(z')|dz' \\ & + \sum_{j,k=-\infty}^{\infty}{}' \frac{L}{T^2(n)\,|j|^\alpha|k|^\beta}\int_0^{T(n)}\int_{T(n)}^{T(n+1)} |\varsigma(z')|dz' = \widetilde{\zeta}_{1n}+\widetilde{\zeta}_{2n}+\widetilde{\zeta}_{3n}.\end{aligned}$$

Let us estimate $E\left(\widetilde{\zeta}_{1n}\right)^2$. Observe that

$$E\left(\widetilde{\zeta}_{1n}\right)^2 \le \left\{\sum_{j,k=-\infty}^{\infty} \frac{L}{|j|^\alpha|k|^\beta}\left[E\left|\frac{1}{T^2(n)}\int_{[0,T(n)]^2} \exp\{i(jt'+ks')\}\varsigma(z')dz'\right|^2\right]^{1/2}\right\}^2.$$

By the conditions of Lemma 2.9 we have

$$\begin{aligned} & E\left\{\frac{1}{T^4(n)}\left|\int_{[0,T(n)]^2} \exp\{i(jt'+ks')\}\varsigma(z')dz'\right|^2\right\} \\ & \le \frac{1}{T^4(n)}\int_0^{T(n)}\int_0^{T(n)}\int_0^{T(n)}\int_0^{T(n)} |r(t-t',s-s')|dzdz' \\ & \le \frac{1}{T^2(n)}\int_{[0,T]^2}|r(z)|dz \le \frac{4L_3}{T^\delta},\end{aligned}$$

which implies

$$E\left(\widetilde{\zeta}_{1n}\right)^2 \le \frac{c_2}{T^\delta(n)} = \frac{c_2}{n^{\delta p}},$$

where $c_2 = 4L_3\left(\sum_{j,k=-\infty}^{\infty}\frac{L}{|j|^\alpha|k|^\beta}\right)^2$.

Then from the Borel-Cantelli lemma we deduce that $P\left\{\lim_{n\to\infty}\widetilde{\zeta}_{1n}=0\right\}=1$. Similarly, one can show that $P\left\{\lim_{n\to\infty}\widetilde{\zeta}_{2n}=0\right\}=1$ and $P\left\{\lim_{n\to\infty}\widetilde{\zeta}_{3n}=0\right\}$ $=1$, which proves the assertion of the lemma. □

Remark 2.7 Lemmas 2.8 and 2.9 also hold true if the function $a \in K$ is substituted with $a(z)b(z)$, where b is a bounded deterministic function, or a random function with $E|b(z)|^2 \le L$, where L is some constant independent of $\xi(z)$.

Theorem 2.14 *Assume that conditions C1–C4 hold true. Then*

$$P\left\{\lim_{T\to\infty}\max_{z\in R^2}|a_T(z)-a_0(z)|=0\right\}=1.$$

Proof By the definition of a_T we have $Q_T(a_T)=\max_{a\in K}Q_T(a)$. Therefore, $Q_T(a_T)\geq Q_T(a_0)$, and thus

$$\begin{aligned}Q_T(a_T)-Q_T(a_0)&=\frac{1}{T^2}\int_{[0,T]^2}[a_T(z)-a_0(z)]x(z)\xi(dz)\\&\quad-\frac{1}{2T^2}\int_{[0,T]^2}[a_T(z)-a_0(z)]^2x^2(z)dz\geq 0.\end{aligned}$$

The last inequality implies that

$$\begin{aligned}&\max_{a\in K}\left|\frac{1}{T^2}\int_{[0,T]^2}[a_T(z)-a_0(z)]x(z)\xi(dz)\right|\\&\quad+\max_{a\in K}\left|\frac{1}{2T^2}\int_{[0,T]^2}[a_T(z)-a_0(z)]^2\left[x^2(z)-Ex^2(0)\right]dz\right|\\&\quad\geq\frac{1}{2T^2}\int_{[0,T]^2}[a_T(z)-a_0(z)]^2dz\cdot Ex^2(0).\end{aligned}$$

Both summands in the left-hand side of the last inequality converge to zero as $T\to\infty$ P-almost surely, which follows, respectively, from Lemma 2.8 and Lemma 2.9. Hence,

$$P\left\{\lim_{T\to\infty}\frac{1}{T^2}\int_{[0,T]^2}[a_T(z)-a_0(z)]^2dz=0\right\}=1,$$

and in such a way

$$P\left\{\lim_{T\to\infty}\|a_T(z)-a_0(z)\|=0\right\}=1.$$

Taking into account the Hölder inequality and the fact that the set K is compact with respect to the uniform convergence, we obtain

$$P\left\{\lim_{T\to\infty}\max_{z\in R^2}|a_T(z)-a_0(z)|=0\right\}=1. \qquad \square$$

Consider now the asymptotic distribution of some functionals that depend on $a_T(z)$ and convergence of measures generated by these estimators. Further we assume that the square integrable functional under consideration is Gaussian.

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