Preface

The study of functional equations has a long history. In 1791 and 1809, Legendre [152] and Gauss [90] attempted to provide a solution of the following functional equation:

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, which is called the *Cauchy functional equation*. A function $f : \mathbb{R} \to \mathbb{R}$ is called an *additive function* if it satisfies the Cauchy functional equation. In 1821, Cauchy [30] first found the general solution of the Cauchy functional equation, that is, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous additive function, then f is linear, that is, $f(x) = mx$, where m is a constant. Further, we can consider the biadditive function on $\mathbb{R} \times \mathbb{R}$ as follows:

A function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called an *biadditive function* if it is additive in each variable, that is,

$$f(x + y, z) = f(x, z) + f(y, z)$$

and

$$f(x, y + z) = f(x, y) + f(x, z)$$

for all $x, y, z \in \mathbb{R}$. It is well known that every continuous biadditive function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the form

$$f(x, y) = mxy$$

for all $x, y \in \mathbb{R}$, where m is a constant.

Since the time of Legendre and Gauss, several mathematicians had dealt with additive functional equations in their books [2–4, 126, 145] and a number of them have studied Lagrange's mean value theorem and related functional equations, Pompeiu's mean value theorem and associated functional equations, two-dimensional mean value theorem and functional equations as well as several kinds of functional equations. We know that the mean value theorems have been motivated to study the functional equations (see the book "Mean Value Theorems and Functional Equations" by Sahoo and Riedel, 1998 [239]).

In 1940, S.M. Ulam [250] proposed the following stability problem of functional equations:

Given a group G_1, a metric group G_2 with the metric $d(\cdot\,,\cdot)$ and a positive number ε, does there exist $\delta > 0$ such that, if a mapping $f : G_1 \to G_2$ satisfies

$$d\big(f(xy), f(x)f(y)\big) \le \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d\big(f(x), h(x)\big) \le \varepsilon$$

for all $x \in G_1$?

Since then, several mathematicians have dealt with special cases as well as generalizations of Ulam's problem.

In fact, in 1941, D.H. Hyers [107] provided a partial solution to Ulam's problem for the case of approximately additive mappings in which G_1 and G_2 are Banach spaces with $\delta = \varepsilon$ as follows:

Let X and Y be Banach spaces and let $\varepsilon > 0$. Then, for all $g : X \to Y$ with

$$\sup_{x,y\in X} \big\|g(x+y) - g(x) - g(y)\big\| \le \varepsilon,$$

there exists a unique mapping $f : X \to Y$ such that

$$\sup_{x\in X} \big\|g(x) - f(x)\big\| \le \varepsilon,$$

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

This proof remains unchanged if G_1 is an Abelian semigroup. Particularly, in 1968, it was proved by Forti (Proposition 1, [88]) that the following theorem can be proved.

Theorem F *Let $(S, +)$ be an arbitrary semigroup and E be a Banach space. Assume that $f : S \to E$ satisfies*

$$\big\|f(x+y) - f(x) - f(y)\big\| \le \varepsilon. \tag{A}$$

Then the limit

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{B}$$

exists for all $x \in S$ and $g : S \to E$ is the unique function satisfying

$$\big\|f(x) - g(x)\big\| \le \varepsilon, \quad g(2x) = 2g(x).$$

Finally, if the semigroup S is Abelian, then G is additive.

Here, the proof method which generates the solution g by the formula like (B) is called the *direct method*.

If f is a mapping of a group or a semigroup $(S, \cdot)$ into a vector space E, then we call the following expression:

$$Cf(x, y) = f(x \cdot y) - f(x) - f(y)$$

the *Cauchy difference* of f on $S \times S$. In the case that E is a topological vector space, we call the equation of homomorphism *stable* if, whenever the Cauchy difference Cf is bounded on $S \times S$, there exists a homomorphism $g : S \to E$ such that $f - g$ is bounded on S.

In 1980, Rätz [230] generalized Theorem F as follows: Let $(X, *)$ be a power-associative groupoid, that is, X is a nonempty set with a binary relation $x_1 * x_2 \in X$ such that the left powers satisfy $x^{m+n} = x^m * x^n$ for all $m, n \geq 1$ and $x \in X$. Let $(Y, |\cdot|)$ be a topological vector space over the field $\mathbb{Q}$ of rational numbers with $\mathbb{Q}$ topologized by its usual absolute value $|\cdot|$.

Theorem R *Let V be a nonempty bounded $\mathbb{Q}$-convex subset of Y containing the origin and assume that Y is sequentially complete. Let $f : X \to Y$ satisfy the following conditions: for all $x_1, x_2 \in X$, there exist $k \geq 2$ such that*

$$f\left((x_1 * x_2)^{k^n}\right) = f\left(x_1^{k^n} * x_2^{k^n}\right) \tag{C}$$

for all $n \geq 1$ and

$$f(x_1) + f(x_2) - f(x_1 * x_2) \in V. \tag{D}$$

Then there exists a function $g : X \to Y$ such that $g(x_1) + g(x_2) = g(x_1 * x_2)$ and $f(x) - g(x) \in \overline{V}$, where $\overline{V}$ is the sequential closure of V for all $x \in X$. When Y is a Hausdorff space, then g is uniquely determined.

Note that the condition (C) is satisfied when X is commutative and it takes the place of the commutativity in proving the additivity of g. However, as Rätz pointed out in his paper, the condition

$$(x_1 * x_2)^{k^n} = x_1^{k^n} * x_2^{k^n}$$

for all $x_1, x_2 \in X$, where X is a semigroup, and, for all $k \geq 1$, does not imply the commutativity.

In the proofs of Theorem F and Theorem R, the completeness of the image space E and the sequential completeness of Y, respectively, were essential in proving the existence of the limit which defined the additive function g. The question arises whether the completeness is necessary for the existence of an odd additive function g such that $f - g$ is uniformly bounded, given that the Cauchy difference is bounded.

For this problem, in 1988, Schwaiger [240] proved the following:

Theorem S *Let E be a normed space with the property that, for each function $f : \mathbb{Z} \to E$, whose Cauchy difference $Cf = f(x+y) - f(x) - f(y)$ is bounded for all $x, y \in \mathbf{Z}$ and there exists an additive mapping $g : \mathbb{Z} \to E$ such that $f(x) - g(x)$ is bounded for all $x \in \mathbb{Z}$. Then E is complete.*

Corollary 1 *The statement of theorem S remains true if $\mathbb{Z}$ is replaced by any vector space over $\mathbb{Q}$.*

In 1950, T. Aoki [14] generalized Hyers' theorem as follows:

Theorem A *Let E_1 and E_2 be two Banach spaces. If there exist $K > 0$ and $0 \leq p < 1$ such that*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq K\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in E_1$, then there exists a unique additive mapping $g : E_1 \to E_2$ such that

$$\left\| f(x) - g(x) \right\| \leq \frac{2K}{2 - 2^p} \|x\|^p$$

for all $x \in E_1$.

In 1978, Th.M. Rassias [216] formulated and proved the stability theorem for the linear mapping between Banach spaces E_1 and E_2 subject to the continuity of $f(tx)$ with respect to $t \in \mathbb{R}$ for each fixed $x \in E_1$. Thus, Rassias' theorem implies Aoki's theorem as a special case. Later, in 1990, Th.M. Rassias [218] observed that the proof of his stability theorem also holds true for $p < 0$. In 1991, Gajda [89] showed that the proof of Rassias' theorem can be proved also for the case $p > 1$ by just replacing n by $-n$ in (B). These results are stated in a generalized form as follows (see Rassias and Šemrl [228]):

Theorem RS *Let $\beta(s,t)$ be nonnegative for all nonnegative real numbers s, t and positive homogeneous of degree p, where p is real and $p \neq 1$, that is, $\beta(\lambda s, \lambda t) = \lambda^p \beta(s,t)$ for all nonnegative λ, s, t. Given a normed space E_1 and a Banach space E_2, assume that $f : E_1 \to E_2$ satisfies the inequality*

$$\left\| f(x+y) - f(x) - f(y) \right\| \leq \beta\left(\|x\|, \|y\|\right)$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $g : E_1 \to E_2$ such that

$$\left\| f(x) - g(x) \right\| \leq \delta \|x\|^p$$

for all $x \in E_1$, where

$$\delta := \begin{cases} \frac{\beta(1,1)}{2-2^p}, & p < 1, \\ \frac{\beta(1,1)}{2-2^p}, & p > 1. \end{cases}$$

The proofs for the cases $p < 1$ and $p > 1$ were provided by applying the direct methods. For $p < 1$, the additive mapping g is given by (B), while in case $p > 1$ the formula is

$$g(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

Corollary 2 *Let $f : E_1 \to E_2$ be a mapping satisfying the hypotheses of Theorem RS and suppose that f is continuous at a single point $y \in E_1$, then the additive mapping g is continuous.*

Corollary 3 *If, under the hypotheses of Theorem RS, we assume that, for each fixed $x \in E_1$, the mapping $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous, then the additive mapping g is linear.*

Remark 4 (1) For $p = 0$, Theorem RS, Corollaries 2 and 3 reduce to the results of Hyers in 1941. If we put $\beta(s,t) = \varepsilon(s^p + t^p)$, then we obtain the results of Rassias [216] in 1978 and Gajda [89] in 1991.

(2) The case $p = 1$ was excluded in Theorem RS. Simple counterexamples prove that one can not extend Rassias' Theorem when p takes the value one (see Z. Gajda [89], Rassias and Šemrl [228] and Hyers and Rassias [109] in 1992).

A further generalization of the Hyers-Ulam stability for a large class of mappings was obtained by Isac and Rassias [110] by introducing the following:

Definition 5 A mapping $f : E_1 \to E_2$ is said to be *ϕ-additive* if there exist $\Phi \geq 0$ and a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = 0$$

such that

$$\|f(x+y) - f(x) - f(y)\| \leq \Phi\left[\phi(\|x\|) + \phi(\|y\|)\right]$$

for all $x, y \in E_1$.

In [110], Isac and Rassias proved the following:

Theorem IR *Let E_1 be a real normed vector space and E_2 be a real Banach space. Let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed $x \in E_1$. If f is ϕ-additive and phi satisfies the following conditions*:

(a) $\phi(ts) \leq \phi(t)\phi(s)$ *for all* $s, t \in \mathbb{R}$;
(b) $\phi(t) < t$ *for all* $t > 1$,

then there exists a unique linear mapping $T : E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - \phi(2)} \phi(\|x\|)$$

for all $x \in E_1$.

Remark 4 (1) If $\phi(t) = t^p$ with $p < 1$, then, from Theorem IR, we obtain Rassias' theorem [216].

(2) If $p < 0$ and $\phi(t) = t^p$ with $t > 0$, then Theorem IR is implied by the result of Gajda in 1991.

Since the time the above stated results have been proven, several mathematicians (see [1, 5–13, 17, 18, 20–25, 28, 31–35, 37, 38, 42–44, 46, 47, 50–63, 67–99, 105, 108, 111–120, 124, 132–137, 144–159, 169–208, 212–215, 219–238, 243, 245–260, 262] and [263]) have extensively studied stability theorems for several kinds of functional equations in various spaces, for example, Banach spaces, 2-Banach spaces, Banach n-Lie algebras, quasi-Banach spaces, Banach ternary algebras, non-Archimedean normed and Banach spaces, metric and ultra metric spaces, Menger probabilistic normed spaces, probabilistic normed space, p-2-normed spaces, C^*-algebras, C^*-ternary algebras, Banach ternary algebras, Banach modules, inner product spaces, Heisenberg groups and others. Further, we have to pay attention to applications of the Hyers-Ulam-Rassias stability problems, for example, (partial) differential equations, Fréchet functional equations, Riccati differential equations, Volterra integral equations, group and ring theory and some kinds of equations (see [29, 114, 121–123, 128, 129, 142, 143, 153, 155, 157, 170–172, 209–211, 255, 257]). For more details on recent development in Ulam's type stability and its applications, see the papers of Brillouët-Belluot [19] and Ciepliński [41] in 2012.

The notion of random normed space goes back to Sherstnev [242] as well as the works published in [100, 101, 241] who were dulled from Menger [160], Schweizer and Sklar [241] works. After the pioneering works by several mathematicians including authors [9, 10, 148–150, 236] who focused at probabilistic functional analysis, Alsina [8] considered the stability of a functional equation in probabilistic normed spaces and, in 2008, Miheţ and Radu considered the stability of a Cauchy additive functional equation in random normed space via fixed point method [161].

The book provides a recent survey of both the latest and new results especially on the following topics:

(1) Basic theory of random normed spaces and related spaces;
(2) Stability theory for several new functional equations in random normed spaces via fixed point method, under the special t-norms as well as arbitrary t-norms;
(3) Stability theory of well known new functional equations in non-Archimedean random normed spaces;
(4) Applications in the class of fuzzy normed spaces.

We would like to express our thanks to Professors Claudi Alsina, Stefan Czerwik, Abbas Najati, Dorian Popa, Ioan A. Rus and G. Zamani Eskandani for reading the manuscript and providing valuable suggestions and comments which have helped to improve the presentation of the book.

Last but not least, it is our pleasure to acknowledge the superb assistance provided by the staff of Springer for the publication of the book.

Jinju, Republic of South Korea — Yeol Je Cho
Athens, Greece — Themistocles M. Rassias
Tehran, Iran — Reza Saadati
June 2013

http://www.springer.com/978-1-4614-8476-9

Stability of Functional Equations in Random Normed
Spaces
Cho, Y.J.; Rassias, T.M.; Saadati, R.
2013, XIX, 246 p., Hardcover
ISBN: 978-1-4614-8476-9