Chapter 2
Generalized Spaces

In this chapter, we present some generalized spaces and their properties for the main results in this chapter.

2.1 Random Normed Spaces

Random (probabilistic) normed spaces were introduced by Šerstnev in 1962 [242] by means of a definition that was closely modelled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. In the sequel, we shall adopt usual terminology, notation and conventions of the theory of random normed spaces, as in [9, 10, 148, 241].

Definition 2.1.1 A *Menger probabilistic metric space* (or *random metric spaces*) is a triple $(X, \mathcal{F}, T)$, where X is a nonempty set, T is a continuous t-norm and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at a point $(x, y) \in X \times X$, the following conditions hold: for all x, y, z in X,

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.1.2 [242] A *random normed space* (briefly, a RN-space) or *a Šerstnev (Sherstnev) probabilistic normed space* (briefly, a Šerstnev PN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$ (0 is the null vector in X);
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$ and $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$, where μ_x denotes the value of μ at a point $x \in X$.

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Note that a *triangular function* $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ is a binary operation on Δ^+ which is associative, commutative and nondecreasing in each argument and has ε_0 as the unit, that is, for all $F, G, H \in \Delta^+$,

$$\begin{aligned} \tau\big(\tau(F,G),H\big) &= \tau\big(F,\tau(G,H)\big), \\ \tau(F,G) &= \tau(G,F), \\ \tau(F,\varepsilon_0) &= F, \\ F \le G \quad &\Longrightarrow \quad \tau(F,H) \le \tau(G,H). \end{aligned}$$

The continuity of a triangular function means the continuity with respect to the topology of weak convergence in Δ^+. Triangular functions are recursively defined by $\tau^1 = \tau$ and

$$\tau^n(F_1,\dots,F_{n+1}) = \tau\big(\tau^{n-1}(F_1,\dots,F_n),F_{n+1}\big)$$

for each $n \ge 2$.

Typical continuous triangular functions are as follows:

$$\tau_T(F,G)(x) = \sup_{s+t=x} T\big(F(s),G(t)\big),$$

and

$$\tau_{T^*}(F,G) = \inf_{s+t=x} T^*\big(F(s),G(t)\big),$$

where T is a continuous t-norm, that is, a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as the identity element and T^* is a continuous t-conorm, that is, a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x,y) = 1 - T(1-x, 1-y)$.

Examples of such t-norms and t-conorms are M and M^*, respectively, defined by

$$M(x,y) = \min(x,y)$$

and

$$M^*(x,y) = \max(x,y).$$

Let τ_1 and τ_2 be two triangular functions. Then τ_1 dominates τ_2 (which is denoted by $\tau_1 \gg \tau_2$) if, for all $F_1, F_2, G_1, G_2 \in \Delta^+$,

$$\tau_1\big(\tau_2(F_1,G_1),\tau_2(F_2,G_2)\big) \ge \tau_2\big(\tau_1(F_1,F_2),\tau_1(G_1,G_2)\big).$$

In 1993, Alsina, Schweizer and Sklar gave a new definition of a probabilistic normed space [9] as follows:

A *probabilistic normed space* (briefly, PN-space) is a quadruple (V,ν,τ,τ^*), where V is a real vector space, τ, τ^* are continuous triangulares functions and ν

is a mapping from $V \to \Delta^+$ such that, for all $p, q \in V$, the following conditions hold:

(PN1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, where θ is the null vector in V;
(PN2) $\nu_{-p} = \nu_p$ for all $p \in V$;
(PN3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;
(PN4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for all $\alpha \in [0, 1]$.

If the inequality (PN4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN-space (V, ν, τ, τ^*) is called a *Šerstnev probabilistic normed space* or a *random normed space* (see Definition 2.1.2) and, as a consequence, we have the following condition stronger than (N2):

$$\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right)$$

for all $p \in V$, $\lambda \neq 0$ and $x \in \mathbb{R}$.

Example 2.1.3 Let $(X, \|\cdot\|)$ be a linear normed spaces. Define a mapping

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{t}{t+\|x\|}, & \text{if } t > 0. \end{cases}$$

Then (X, μ, T_p) is a random normed space. In fact, (RN1) and (RN2) are obvious. Now, we show (RN3).

$$\begin{aligned} T_p\big(\mu_x(t), \mu_y(s)\big) &= \frac{t}{t+\|x\|} \cdot \frac{s}{s+\|y\|} \\ &= \frac{1}{1+\frac{\|x\|}{t}} \cdot \frac{1}{1+\frac{\|y\|}{s}} \\ &\leq \frac{1}{1+\frac{\|x\|}{t+s}} \cdot \frac{1}{1+\frac{\|y\|}{t+s}} \\ &\leq \frac{1}{1+\frac{\|x\|+\|y\|}{t+s}} \\ &\leq \frac{1}{1+\frac{\|x+y\|}{t+s}} \\ &= \frac{t+s}{t+s+\|x+y\|} \\ &= \mu_{x+y}(t+s) \end{aligned}$$

for all $x, y \in X$ and $t, s \geq 0$. Also, (X, μ, T_M) is a random normed space.

Example 2.1.4 Let $(X, \|\cdot\|)$ be a linear normed spaces. Define a mapping

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \le 0, \\ e^{-(\frac{\|x\|}{t})}, & \text{if } t > 0. \end{cases}$$

Then (X, μ, T_p) is a random normed space. In fact, (RN1) and (RN2) are obvious and so, now, we show (RN3).

$$\begin{aligned} T_p\big(\mu_x(t), \mu_y(s)\big) &= e^{-(\frac{\|x\|}{t})} \cdot e^{-(\frac{\|y\|}{s})} \\ &\le e^{-(\frac{\|x\|}{t+s})} \cdot e^{-(\frac{\|y\|}{t+s})} \\ &= e^{-(\frac{\|x\|+\|y\|}{t+s})} \\ &\le e^{-(\frac{\|x+y\|}{t+s})} \\ &= \mu_{x+y}(t+s) \end{aligned}$$

for all $x, y \in X$ and $t, s \ge 0$. Also, (X, μ, T_M) is a random normed space.

Example 2.1.5 [164] Let $(X, \|\cdot\|)$ be a linear normed space. For all $x \in X$, define a mapping

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\}, & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases}$$

Then (X, μ, T_L) is a RN-space (this was essentially proved by Musthari in [179], see also [213]). Indeed, we have

$$\mu_x(t) = 1 \quad \Longrightarrow \quad \frac{\|x\|}{t} = 0 \quad \Longrightarrow \quad x = 0$$

for all $t > 0$ and, obviously,

$$\mu_{\lambda x}(t) = \mu_x\left(\frac{t}{\lambda}\right)$$

for all $x \in X$ and $t > 0$. Next, for any $x, y \in X$ and $t, s > 0$, we have

$$\begin{aligned} \mu_{x+y}(t+s) &= \max\left\{1 - \frac{\|x+y\|}{t+s}, 0\right\} \\ &= \max\left\{1 - \|\frac{x+y}{t+s}\|, 0\right\} \\ &= \max\left\{1 - \|\frac{x}{t+s} + \frac{y}{t+s}\|, 0\right\} \\ &\ge \max\left\{1 - \|\frac{x}{t}\| - \|\frac{y}{s}\|, 0\right\} \end{aligned}$$

$$= T_L\big(\mu_x(t), \mu_y(s)\big).$$

Let φ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

(a) $\varphi(-t) = \varphi(t)$ for all $t \in \mathbb{R}$;
(b) $\varphi(1) = 1$;
(c) φ is strictly increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\lim_{\alpha\to\infty} \varphi(\alpha) = \infty$.

Examples of such functions are as follows:

$$\varphi(t) = |t|, \qquad \varphi(t) = |t|^p \quad \big(p \in (0, \infty)\big), \qquad \varphi(t) = \frac{2t^{2n}}{|t| + 1}$$

for all $t \in \mathbb{R}$ and $n \geq 1$.

Definition 2.1.6 [97] A *random φ-normed space* is a triple (X, ν, T), where X is a real vector space, T is a continuous t-norm and ν is a mapping from X into D^+ such that the following conditions hold:

(φ-RN1) $\nu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(φ-RN2) $\nu_{\alpha x}(t) = \nu_x(\frac{t}{\varphi(\alpha)})$ for all x in X, $\alpha \neq 0$ and $t > 0$;
(φ-RN3) $\nu_{x+y}(t + s) \geq T(\nu_x(t), \nu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Example 2.1.7 [165] An important example is the space (X, ν, T_M), where $(X, \|\cdot\|^p)$ is a p-normed space and

$$\nu_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{t}{t+\|x\|^p}, \quad p \in (0, 1], & \text{if } t > 0. \end{cases}$$

(φ-RN1) and (φ-RN2) are obvious and so we show (φ-RN3). In fact, let $\nu_x(t) \leq \nu_y(s)$. Then we have

$$\frac{\|y\|^p}{s} \leq \frac{\|x\|^p}{t}$$

for any $x, y \in X$

Now, if $x = y$, we have $t \leq s$. Thus, otherwise, we have

$$\begin{aligned} \frac{\|x\|^p}{t} + \frac{\|x\|^p}{t} &\geq \frac{\|x\|^p}{t} + \frac{\|y\|^p}{s} \\ &\geq 2\frac{\|x\|^p}{t+s} + 2\frac{\|y\|^p}{t+s} \\ &\geq 2\frac{\|x+y\|^p}{t+s} \end{aligned}$$

and so

$$1+\frac{\|x\|^p}{t}\geq 1+\frac{\|x+y\|^p}{t+s},$$

which implies that $\nu_x(t)\leq \nu_{x+y}(t+s)$. Hence, $\nu_{x+y}(t+s)\geq T_M(\nu_x(t),\nu_y(s))$ for all $x,y\in X$ and $t,s\geq 0$.

Definition 2.1.8 Let μ and ν be measure and non-measure distribution function from $X\times(0,+\infty)$ to $[0,1]$, respectively, such that $\mu_x(t)+\nu_x(t)\leq 1$ for all $x\in X$ and $t>0$, where X is a real vector space. The triple $(X,\mathcal{P}_{\mu,\nu},\mathcal{T})$ is said to be an *intuitionistic random normed space* (briefly, IRN-space) if X is a real vector space, $\mathcal{T}$ is a continuous t-representable and $\mathcal{P}_{\mu,\nu}$ is a mapping $X\times(0,+\infty)\to L^*$ satisfying the following conditions: for all $x,y\in X$ and $t,s>0$,

(IRN1) $\mathcal{P}_{\mu,\nu}(x,0)=0_{L^*}$;
(IRN2) $\mathcal{P}_{\mu,\nu}(x,t)=1_{L^*}$ if and only if $x=0$;
(IRN3) $\mathcal{P}_{\mu,\nu}(\alpha x,t)=\mathcal{P}_{\mu,\nu}(x,\frac{t}{|\alpha|})$ for all $\alpha\neq 0$;
(IRN4) $\mathcal{P}_{\mu,\nu}(x+y,t+s)\geq_{L^*}\mathcal{T}(\mathcal{P}_{\mu,\nu}(x,t),\mathcal{P}_{\mu,\nu}(y,s))$.

In this case, $\mathcal{P}_{\mu,\nu}$ is called an *intuitionistic random norm*, where

$$\mathcal{P}_{\mu,\nu}(x,t)=\big(\mu_x(t),\nu_x(t)\big).$$

Definition 2.1.9 A *lattice random normed space* (LRN-space shortly) is a triple $(X,\mu,\mathcal{T}_\wedge)$, where X is a vector space and μ is a mapping from X into D_L^+ such that the following conditions hold:

(LRN1) $\mu_x(t)=1_{\mathcal{L}}$ for all $t>0$ if and only if $x=0$;
(LRN2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ for all x in X, $\alpha\neq 0$ and $t\geq 0$;
(LRN3) $\mu_{x+y}(t+s)\geq_L \mathcal{T}_\wedge(\mu_x(t),\mu_y(s))$ for all $x,y\in X$ and $t,s\geq 0$.

We note that, from (LRN2), $\mu_{-x}(t)=\mu_x(t)$ for all $x\in X$ and $t\geq 0$.

Example 2.1.10 Let $L=[0,1]\times[0,1]$ and the operation $\leq_L$ be defined by:

$$L=\big\{(a_1,a_2):(a_1,a_2)\in[0,1]\times[0,1]\ a_1+a_2\leq 1\big\},$$
$$(a_1,a_2)\leq_L(b_1,b_2)\iff a_1\leq b_1,\ a_2\geq b_2$$

for all $a=(a_1,a_2),b=(b_1,b_2)\in L$. Then $(L,\leq_L)$ is a complete lattice (see [49]). In this complete lattice, we denote its units by $0_L=(0,1)$ and $1_L=(1,0)$.

Let $(X,\|\cdot\|)$ be a normed linear space. Let $\mathcal{T}(a,b)=(\min\{a_1,b_1\},\max\{a_2,b_2\})$ for all $a=(a_1,a_2),b=(b_1,b_2)\in[0,1]\times[0,1]$ and μ be a mapping defined by

$$\mu_x(t)=\left(\frac{t}{t+\|x\|},\frac{\|x\|}{t+\|x\|}\right)$$

for all $t\in\mathbb{R}^+$. Then $(X,\mu,\mathcal{T})$ is a lattice random normed space.

2.2 Random Topological Structures

In this section, we give some topological structures of random normed spaces.

Definition 2.2.1 Let (X, μ, T) be an RN-space. We define the *open ball* $B_x(r, t)$ and the closed ball $B_x[r, t]$ with center $x \in X$ and radius $0 < r < 1$ for all $t > 0$ as follows:

$$B_x(r, t) = \{y \in X : \mu_{x-y}(t) > 1 - r\},$$

$$B_x[r, t] = \{y \in X \ : \ \mu_{x-y}(t) \geq 1 - r\},$$

respectively.

Theorem 2.2.2 *Let (X, μ, T) be an RN-space. Every open ball $B_x(r, t)$ is open set.*

Proof Let $B_x(r, t)$ an open ball with center x and radius r for all $t > 0$. Let $y \in B_x(r, t)$. Then $\mu_{x-y}(t) > 1 - r$. Since $\mu_{x-y}(t) > 1 - r$, there exists $t_0 \in (0, t)$ such that $\mu_{x-y}(t_0) > 1 - r$. Put $r_0 = \mu_{x,y}(t_0)$. Since $r_0 > 1 - r$, there exists $s \in (0, 1)$ such that $r_0 > 1 - s > 1 - r$. Now, for any r_0 and s such that $r_0 > 1 - s$, there exists $r_1 \in (0, 1)$ such that $T(r_0, r_1) > 1 - s$. Consider the open ball $B_y(1 - r_1, t - t_0)$.

Now, we claim that $B_y(1 - r_1, t - t_0) \subset B_x(r, t)$. In fact, let $z \in B_y(1 - r_1, t - t_0)$. Then $\mu_{y-z}(t - t_0) > r_1$ and so

$$\begin{aligned}
\mu_{x-z}(t) &\geq T\big(\mu_{x-y}(t_0), \mu_{y-z}(t - t_0)\big) \\
&\geq T(r_0, r_1) \\
&\geq 1 - s \\
&> 1 - r.
\end{aligned}$$

Thus, $z \in B_x(r, t)$ and hence $B_y(1 - r_1, t - t_0) \subset B_x(r, t)$. This completes the proof. □

Now, different kinds of topologies can be introduced in a random normed space [241]. The (r, t)*-topology* is introduced by a family of neighborhoods

$$\{B_x(r, t)\}_{x \in X, t > 0, r \in (0,1)}.$$

In fact, every random norm μ on X generates a topology ((r, t)-topology) on X which has as a base the family of open sets of the form

$$\{B_x(r, t)\}_{x \in X, t > 0, r \in (0,1)}.$$

Remark 2.2.3 Since $\{B_x(\frac{1}{n}, \frac{1}{n}) : n \geq 1\}$ is a local base at x, the (r, t)-topology is first countable.

Theorem 2.2.4 *Every RN-space* (X, μ, T) *is a* Hausdorff space.

Proof Let (X, μ, T) be an RN-space. Let x and y be two distinct points in X and $t > 0$. Then $0 < \mu_{x-y}(t) < 1$. Put $r = \mu_{x-y}(t)$. For each $r_0 \in (r, 1)$, there exists r_1 such that $T(r_1, r_1) \geq r_0$. Consider the open balls $B_x(1 - r_1, \frac{t}{2})$ and $B_y(1 - r_1, \frac{t}{2})$. Then, clearly, $B_x(1 - r_1, \frac{t}{2}) \cap B_y(1 - r_1, \frac{t}{2}) = \emptyset$. In fact, if there exists

$$z \in B_x\left(1 - r_1, \frac{t}{2}\right) \cap B_y\left(1 - r_1, \frac{t}{2}\right),$$

then we have

$$\begin{aligned} r &= \mu_{x-y}(t) \\ &\geq T\left(\mu_{x-z}\left(\frac{t}{2}\right), \mu_{y-z}\left(\frac{t}{2}\right)\right) \\ &\geq T(r_1, r_1) \\ &\geq r_0 \\ &> r, \end{aligned}$$

which is a contradiction. Hence (X, μ, T) is a Hausdorff space. This completes the proof. □

Definition 2.2.5 Let (X, μ, T) be an RN-space. A subset A of X is said to be *R-bounded* if there exist $t > 0$ and $r \in (0, 1)$ such that $\mu_{x-y}(t) > 1 - r$ for all $x, y \in A$.

Theorem 2.2.6 *Every compact subset* A *of an RN-space* (X, μ, T) *is* R*-bounded.*

Proof Let A be a *compact* subset of an RN-space (X, μ, T). Fix $t > 0$, $0 < r < 1$ and consider an open cover $\{B_x(r, t) : x \in A\}$. Since A is compact, there exist $x_1, x_2, \dots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_{x_i}(r, t).$$

Let $x, y \in A$. Then $x \in B_{x_i}(r, t)$ and $y \in B_{x_j}(r, t)$ for some $i, j \geq 1$. Thus we have $\mu_{x-x_i}(t) > 1 - r$ and $\mu_{y-x_j}(t) > 1 - r$. Now, let

$$\alpha = \min\bigl\{\mu_{x_i, x_j}(t) : 1 \leq i, j \leq n\bigr\}.$$

Then we have $\alpha > 0$ and

$$\begin{aligned} \mu_{x-y}(3t) &\geq T^2\bigl(\mu_{x-x_i}(t), \mu_{x_i, x_j}(t), \mu_{y-x_j}(t)\bigr) \\ &\geq T^2(1 - r, 1 - r, \alpha) \\ &> 1 - s. \end{aligned}$$

Taking $t' = 3t$, it follows that $\mu_{x-y}(t') > 1 - s$ for all $x, y \in A$. Hence A is R-bounded. This completes the proof. □

Remark 2.2.7 In an RN-space (X, μ, T), every compact set is closed and R-bounded.

Definition 2.2.8 Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x}(\epsilon) > 1 - \lambda$$

whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that

$$\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$$

whenever $n \geq m \geq N$.

(3) An RN-space (X, μ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.2.9 [241] *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Theorem 2.2.10 *Let (X, μ, T) be an RN-space such that every Cauchy sequence in X has a convergent subsequence. Then (X, μ, T) is complete.*

Proof Let $\{x_n\}$ be a Cauchy sequence in X and $\{x_{i_n}\}$ be a subsequence of $\{x_n\}$ which converges to a point $x \in X$.

Now, we prove that $x_n \to x$. Let $t > 0$ and $\epsilon \in (0, 1)$ such that

$$T(1 - r, 1 - r) \geq 1 - \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence, there exists $n_0 \geq 1$ such that

$$\mu_{x_m - x_n}(t) > 1 - r$$

for all $m, n \geq n_0$. Since $x_{i_n} \to x$, there exists a positive integer i_p such that $i_p > n_0$ and

$$\mu_{x_{i_p} - x}\left(\frac{t}{2}\right) > 1 - r.$$

Then, if $n \geq n_0$, we have

$$\mu_{x_n - x}(t) \geq T\left(\mu_{x_n - x_{i_p}}\left(\frac{t}{2}\right), \mu_{x_{i_p} - x}\left(\frac{t}{2}\right)\right)$$

$$> T(1-r, 1-r)$$
$$\geq 1-\epsilon.$$

Therefore, $x_n \to x$ and hence (X, μ, T) is complete. This completes the proof. □

Lemma 2.2.11 *Let (X, μ, T) be an RN-space. If we define*

$$F_{x,y}(t) = \mu_{x-y}(t)$$

for all $x, y \in X$ and $t > 0$, then F is a random (probabilistic) metric on X, which is called the random (probabilistic) metric induced by the random norm μ.

Lemma 2.2.12 *A random (probabilistic) metric F which is induced by a random norm on a RN-space (X, μ, T) has the following properties: for all $x, y, z \in X$ and scalar $\alpha \neq 0$,*

(1) $F_{x+z,y+z}(t) = F_{x,y}(t)$;
(2) $F_{\alpha x,\alpha y}(t) = F_{x,y}(\frac{t}{|\alpha|})$.

Proof We have the following:

$$F_{x+z,y+z}(t) = \mu_{(x+z)-(y+z)}(t) = \mu_{x-y}(t) = F_{x,y}(t)$$

and, also,

$$F_{\alpha x,\alpha y}(t) = \mu_{\alpha x-\alpha y}(t) = \mu_{x-y}\left(\frac{t}{|\alpha|}\right) = F_{x,y}\left(\frac{t}{|\alpha|}\right).$$

Therefore, we have (1) and (2). This completes the proof. □

Lemma 2.2.13 *If (X, μ, T) is an RN-space, then we have*

(1) *The function $(x, y) \to x + y$ is continuous*;
(2) *The function $(\alpha, x) \to \alpha x$ is continuous.*

Proof If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then we have

$$\mu_{(x_n+y_n)-(x+y)}(t) \geq T\left(\mu_{x_n-x}\left(\frac{t}{2}\right), \mu_{y_n-y}\left(\frac{t}{2}\right)\right) \to 1$$

as $n \to \infty$. This proves (1).

Now, if $x_n \to x$ and $\alpha_n \to \alpha$ as $n \to \infty$, where $\alpha_n \neq 0$, then we have

$$\mu_{\alpha_n x_n-\alpha x}(t) = \mu_{\alpha_n(x_n-x)+x(\alpha_n-\alpha)}(t)$$
$$\geq T\left(\mu_{\alpha_n(x_n-x)}\left(\frac{t}{2}\right)\mu_{x(\alpha_n-\alpha)}\left(\frac{t}{2}\right)\right)$$

$$= T\left(\mu_{x_n - x}\left(\frac{t}{2\alpha_n}\right), \mu_x\left(\frac{t}{2(\alpha_n - \alpha)}\right)\right) \to 1$$

as $n \to \infty$. This proves (2). This completes the proof. □

Definition 2.2.14 An RN-space (X, μ, T) is called a *random Banach space* whenever X is complete with respect to the random metric induced by random norm.

Lemma 2.2.15 *Let (X, μ, T) be an RN-space and define*

$$E_{\lambda,\mu} : X \to \mathbb{R}^+ \cup \{0\}$$

by

$$E_{\lambda,\mu}(x) = \inf\{t > 0 : \mu_x(t) > 1 - \lambda\}$$

for all $\lambda \in (0, 1)$ and $x \in X$. Then we have

(1) $E_{\lambda,\mu}(\alpha x) = |\alpha| E_{\lambda,\mu}(x)$ *for all $x \in X$ and $\alpha \in \mathbb{R}$;*
(2) *If T satisfies* (1.1.2), *then, for any $\alpha \in (0, 1)$, there exists $\beta \in (0, 1)$ such that*

$$E_{\gamma,\mu}(x_1 + \cdots + x_n) \le E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n)$$

for all $x, y \in X$;
(3) *A sequence $\{x_n\}$ is convergent with respect to the random norm μ if and only if $E_{\lambda,\mu}(x_n - x) \to 0$. Also, the sequence $\{x_n\}$ is a Cauchy sequence with respect to the random norm μ if and only if it is a Cauchy sequence with $E_{\lambda,\mu}$.*

Proof For (1), we have

$$\begin{aligned} E_{\lambda,\mu}(\alpha x) &= \inf\{t > 0 : \mu_{\alpha x}(t) > 1 - \lambda\} \\ &= \inf\left\{t > 0 : \mu_x\left(\frac{t}{|\alpha|}\right) > 1 - \lambda\right\} \\ &= |\alpha| \inf\{t > 0 : \mu_x(t) > 1 - \lambda\} \\ &= |\alpha| E_{\lambda,\mu}(x). \end{aligned}$$

For (2), by (1.1.2), for all $\alpha \in (0, 1)$, we can find $\lambda \in (0, 1)$ such that

$$T^{n-1}(1 - \lambda, \ldots, 1 - \lambda) \ge 1 - \alpha.$$

Thus, we have

$$\begin{aligned} &\mu_{x_1 + \cdots + x_n}\big(E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n) + n\delta\big) \\ &\quad \ge_L T^{n-1}\big(\mu_{x_1}\big(E_{\lambda,\mathcal{M}}(x_1) + \delta\big), \ldots, \mu_{x_n}\big(E_{\lambda,\mathcal{P}}(x_n) + \delta\big)\big) \end{aligned}$$

$$\geq T(1-\lambda, \ldots, 1-\lambda)$$
$$\geq 1-\alpha$$

for all $\delta > 0$, which implies that

$$E_{\alpha,\mu}(x_1 + \cdots + x_n) \leq E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n) + n\delta.$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\alpha,\mu}(x_1 + \cdots + x_n) \leq E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n).$$

For (3), since μ is continuous, $E_{\lambda,\mu}(x)$ is not an element of the set $\{t > 0 : \mu_x(t) > 1-\lambda\}$ for all $x \in X$ with $x \neq 0$. Hence, we have

$$\mu_{x_n - x}(\eta) > 1-\lambda \quad \Longleftrightarrow \quad E_{\lambda,\mu}(x_n - x) < \eta$$

for all $\eta > 0$. This completes the proof. □

Definition 2.2.16 A function f from a RN-space (X, μ, T) to a RN-space (Y, ν, T') is said to be *uniformly continuous* if, for all $r \in (0, 1)$ and $t > 0$, there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that

$$\mu_{x-y}(t_0) > 1 - r_0 \quad \Longrightarrow \quad \nu_{f(x),f(y)}(t) > 1 - r.$$

Theorem 2.2.17 (Uniform Continuity Theorem) *If f is continuous function from a compact RN-space (X, μ, T) to an RN-space (Y, ν, T'), then f is uniformly continuous.*

Proof Let $s \in (0, 1)$ and $t > 0$ be given. Then we can find $r \in (0, 1)$ such that

$$T'(1-r, 1-r) > 1-s.$$

Since $f : X \to Y$ is continuous, for any $x \in X$, we can find $r_x \in (0, 1)$ and $t_x > 0$ such that

$$\mu_{x-y}(t_x) > 1 - r_x \quad \Longrightarrow \quad \nu_{f(x)-f(y)}\left(\frac{t}{2}\right) > 1 - r.$$

But $r_x \in (0, 1)$ and then we can find $s_x < r_x$ such that

$$T(1 - s_x, 1 - s_x) > 1 - r_x.$$

Since X is compact and

$$\left\{B_x\left(s_x, \frac{t_x}{2}\right) : x \in X\right\}$$

is an open covering of X, there exist $x_1, x_2, \ldots, x_k$ in X such that

$$X = \bigcup_{i=1}^{k} B_{x_i}\left(s_{x_i}, \frac{t_{x_i}}{2}\right).$$

Put $s_0 = \min s_{x_i}$ and $t_0 = \min \frac{t_{x_i}}{2}$, $i = 1, 2, \ldots, k$. For any $x, y \in X$, if $\mu_{x-y}(t_0) > 1 - s_0$, then $\mu_{x-y}(\frac{t_{x_i}}{2}) > 1 - s_{x_i}$. Since $x \in X$, there exists $x_i \in X$ such that

$$\mu_{x-x_i}\left(\frac{t_{x_i}}{2}\right) > 1 - s_{x_i}.$$

Hence, we have

$$\nu_{f(x),f(x_i)}\left(\frac{t}{2}\right) > 1 - r.$$

Now, note that

$$\begin{aligned}
\mu_{y-x_i}(t_{x_i}) &\geq T\left(\mu_{x-y}\left(\frac{t_{x_i}}{2}\right), \mu_{x-x_i}\left(\frac{t_{x_i}}{2}\right)\right) \\
&\geq T(1 - s_{x_i}, 1 - s_{x_i}) \\
&> 1 - r_{x_i}.
\end{aligned}$$

Therefore, we have

$$\nu_{f(y)-f(x_i)}\left(\frac{t}{2}\right) > 1 - r$$

and so

$$\begin{aligned}
\nu_{f(x)-f(y)}(t) &\geq T\left(\nu_{f(x)-f(x_i)}\left(\frac{t}{2}\right), \nu_{f(y)-f(x_i)}\left(\frac{t}{2}\right)\right) \\
&\geq T(1 - r, 1 - r) \\
&> 1 - s.
\end{aligned}$$

Therefore, f is uniformly continuous. This completes the proof. □

Remark 2.2.18 Let f be an uniformly continuous function from an RN-space (X, μ, T) to an RN-space (Y, ν, T'). If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.

Theorem 2.2.19 *Every compact RN-space is separable.*

Proof Let (X, μ, T) be a compact RN-space. Let $r \in (0, 1)$ and $t > 0$. Since X is compact, there exist $x_1, x_2, \dots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} B_{x_i}(r, t).$$

In particular, for each $n \geq 1$, we can choose a finite subset A_n of X such that

$$X = \bigcup_{a \in A_n} B_a\left(r_n, \frac{1}{n}\right)$$

in which $r_n \in (0, 1)$. Let

$$A = \bigcup_{n \geq 1} A_n.$$

Then A is countable.

Now, we claim that $X \subset \overline{A}$. Let $x \in X$. Then, for each $n \geq 1$, there exists $a_n \in A_n$ such that $x \in B_{a_n}(r_n, \frac{1}{n})$. Thus, $\{a_n\}$ converges to the point $x \in X$. But, since $a_n \in A$ for all $n \geq 1$, $x \in \overline{A}$ and so A is dense in X. Therefore, X is separable. This completes the proof. □

Definition 2.2.20 Let X be a nonempty set and (Y, ν, T') be an RN-space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be *converge uniformly* to a function f from X to Y if, for any $r \in (0, 1)$ and $t > 0$, there exists $n_0 \geq 1$ such that

$$\nu_{f_n(x)-f(x)}(t) > 1 - r$$

for all $n \geq n_0$ and $x \in X$.

Definition 2.2.21 A family $\mathcal{F}$ of functions from an RN-space (X, μ, T) to a complete RN-space (Y, ν, T') is said to be *equicontinuous* if, for any $r \in (0, 1)$ and $t > 0$, there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that

$$\mu_{x-y}(t_0) > 1 - r_0 \quad \Longrightarrow \quad \nu_{f(x)-f(y)}(t) > 1 - r$$

for all $f \in \mathcal{F}$.

Lemma 2.2.22 *Let $\{f_n\}$ be an equicontinuous sequence of functions from an RN-space (X, μ, T) to a complete RN-space (Y, ν, T'). If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges for each point of X and the limit function is continuous.*

Proof Let $s \in (0, 1)$ and $t > 0$ be given. Then we can find $r \in (0, 1)$ such that

$$T'^2(1 - r, 1 - r, 1 - r) > 1 - s.$$

Since $\mathcal{F} = \{f_n\}$ is an equicontinuous family, for any $r \in (0, 1)$ and $t > 0$, there exist $r_1 \in (0, 1)$ and $t_1 > 1$ such that, for each $x, y \in X$,

$$\mu_{x-y}(t_1) > 1 - r_1 \quad \Longrightarrow \quad \nu_{f_n(x)-f_n(y)}\left(\frac{t}{3}\right) > 1 - r$$

for all $f_n \in \mathcal{F}$. Since D is dense in X, there exists

$$y \in B_x(r_1, t_1) \cap D$$

and $\{f_n(y)\}$ converges for the point y. Since $\{f_n(y)\}$ is a Cauchy sequence, for any $r \in (0, 1)$ and $t > 0$, there exists $n_0 \geq 1$ such that

$$\nu_{f_n(y)-f_m(y)}\left(\frac{t}{3}\right) > 1 - r$$

for all $m, n \geq n_0$. Now, for any $x \in X$, we have

$$\begin{aligned}
&\nu_{f_n(x)-f_m(x)}(t) \\
&\quad \geq T'^2\left(\nu_{f_n(x)-f_n(y)}\left(\frac{t}{3}\right), \nu_{f_n(y)-f_m(y)}\left(\frac{t}{3}\right), \nu_{f_m(x)-f_m(y)}\left(\frac{t}{3}\right)\right) \\
&\quad \geq T'^2(1-r, 1-r, 1-r) \\
&\quad > 1 - s.
\end{aligned}$$

Hence, $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $f_n(x)$ converges and so let $f(x) = \lim f_n(x)$.

Now, we claim that f is continuous. Let $s_o \in 1 - r$ and $t_0 > 0$ be given. Then we can find $r_0 \in 1 - r$ such that

$$T'^2(1 - r_0, 1 - r_0, 1 - r_0) > 1 - s_0.$$

Since $\mathcal{F}$ is equicontinuous, for any $r_0 \in (0, 1)$ and $t_0 > 0$, there exist $r_2 \in (0, 1)$ and $t_2 > 0$ such that

$$\mu_{x-y}(t_2) > 1 - r_2 \quad \Longrightarrow \quad \nu_{f_n(x)-f_n(y)}\left(\frac{t_0}{3}\right) > 1 - r_0$$

for all $f_n \in \mathcal{F}$. Since $f_n(x)$ converges to $f(x)$, for any $r_0 \in (0, 1)$ and $t_0 > 0$, there exists $n_1 \geq 1$ such that

$$\nu_{f_n(x)-f(x)}\left(\frac{t_0}{3}\right) > 1 - r_0.$$

Also, since $f_n(y)$ converges to $f(y)$, for any $r_0 \in (0, 1)$ and $t_0 > 0$, there exists $n_2 \geq 1$ such that

$$\nu_{f_n(y)-f(y)}\left(\frac{t_0}{3}\right) > 1 - r_0$$

for all $n \geq n_2$. Now, for all $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned}
&\nu_{f(x)-f(y)}(t_0) \\
&\geq T'^2\left(\nu_{f(x)-f_n(x)}\left(\frac{t_0}{3}\right), \nu_{f_n(x)-f_n(y)}\left(\frac{t_0}{3}\right), \nu_{f_n(y)-f(y)}\left(\frac{t_0}{3}\right)\right) \\
&\geq T'^2(1-r_0, 1-r_0, 1-r_0) \\
&> 1-s_0.
\end{aligned}$$

Therefore, f is continuous. This completes the proof. □

Theorem 2.2.23 (Ascoli–Arzela Theorem) *Let (X, μ, T) be a compact RN-space and (Y, ν, T') be a complete RN-space. Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y. If $\{f_n\}$ is a sequence in $\mathcal{F}$ such that*

$$\overline{\{f_n(x) : n \in \mathbb{N}\}}$$

is a compact subset of Y for any $x \in X$, then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n\}$ converges uniformly to f on X.

Proof Since (X, μ, T) be a compact RN-space, by Theorem 2.2.19, X is separable. Let

$$D = \{x_i : i = 1, 2, \ldots\}$$

be a countable dense subset of X. By hypothesis, for each $i \geq 1$,

$$\overline{\{f_n(x_i) : n \geq 1\}}$$

is compact subset of Y. Since every $\mathcal{L}$-fuzzy metric space is first countable space, every compact subset of Y is sequentially compact. Thus, by standard argument, we have a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each $i \geq 1$. Thus, by Lemma 2.2.22, there exists a continuous function f from X to Y such that $\{g_n(x)\}$ converges to $f(x)$ for all $x \in X$.

Now, we claim that $\{g_n\}$ converges uniformly to a functions f on X. Let $s \in (0, 1)$ and $t > 0$ be given. Then we can find $r \in (0, 1)$ such that

$$T'^2(1-r, 1-r, 1-r) > 1-s.$$

Since $\mathcal{F}$ is equicontinuous, there exist $r_1 \in (0, 1)$ and $t_1 > 0$ such that

$$\mu_{x-y}(t_1) > 1-r_1 \quad \Longrightarrow \quad \nu_{g_n(x), g_n(y)}\left(\frac{t}{3}\right) > 1-r$$

for all $n \geq 1$. Since X is compact, by Theorem 2.2.17, f is uniformly continuous. Hence, for any $r \in (0,1)$ and $t > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that

$$\mu_{x-y}(t_2) > 1 - r_2 \quad \Longrightarrow \quad \nu_{f(x)-f(y)}\left(\frac{t}{3}\right) > 1 - r$$

for all $x, y \in X$. Let $r_0 = \min\{r_1, r_2\}$ and $t_0 = \min\{t_1, t_2\}$. Since X is compact and D is dense in X, we have

$$X = \bigcup_{i=1}^{k} B_{x_i}(r_0, t_0)$$

for some $k \geq 1$. Thus, for any $x \in X$, there exists i, $i \leq i \leq k$, such that

$$\mu_{x-x_i}(t_0) > 1 - r_0.$$

But, since $r_0 = \min\{r_1, r_2\}$ and $t_0 = \min\{t_1, t_2\}$, we have, by the equicontinuity of $\mathcal{F}$,

$$\nu_{g_n(x)-g_n(x_i)}\left(\frac{t}{3}\right) > 1 - r$$

and we also have, by the uniform continuity of f,

$$\nu_{f(x)-f(x_i)}\left(\frac{t}{3}\right) > 1 - r.$$

Since $\{g_n(x_j)\}$ converges to $f(x_j)$, for any $r \in (0,1)$ and $t > 0$, there exists $n_0 \geq 1$ such that

$$\nu_{g_n(x_j)-f(x_j)}\left(\frac{t}{3}\right) > 1 - r$$

for all $n \geq n_0$. Now, for all $x \in X$, we have

$$\begin{aligned}
&\nu_{g_n(x)-f(x)}(t) \\
&\quad \geq T'^2\left(\nu_{g_n(x)-g_n(x_i)}\left(\frac{t}{3}\right), \nu_{g_n(x_i)-f(x_i)}\left(\frac{t}{3}\right), \nu_{f(x_i)-f(x)}\left(\frac{t}{3}\right)\right) \\
&\quad \geq T'^2(1-r, 1-r, 1-r) \\
&\quad > 1 - s.
\end{aligned}$$

Therefore, $\{g_n\}$ converges uniformly to a function f on X. This completes the proof. □

We recall that a subset A is said R-bounded in (X, μ, T), if there exist $t_0 > 0$ and $r_0 \in (0,1)$ such that $\mu_x(t_0) > 1 - r_0$ for all $x \in A$.

Lemma 2.2.24 *A subset A of $\mathbb{R}$ is R-bounded in $(\mathbb{R}, \mu, T)$ if and only if it is bounded in $\mathbb{R}$.*

Proof Let A be a subset in $\mathbb{R}$ which is R-bounded in $(\mathbb{R}, \mu, T)$. Then there exist $t_0 > 0$ and $r_0 \in (0, 1)$ such that $\mu_x(t_0) > 1 - r_0$ for all $x \in A$. Thus, we have

$$t_0 \geq E_{r_0,\mu}(x) = |x| E_{r_0,\mu}(1).$$

Now, $E_{r_0,\mu}(1) \neq 0$. If we put $k = \frac{t_0}{E_{r_0,\mu}(1)}$, then we have $|x| \leq k$ for all $x \in A$, that is, A is bounded in $\mathbb{R}$.

The converse is easy to see. This completes the proof. □

Lemma 2.2.25 *A sequence* $\{\beta_n\}$ *is convergent in an RN-space* $(\mathbb{R}, \mu, T)$ *if and only if it is convergent in* $(\mathbb{R}, |\cdot|)$.

Proof Let $\beta_n \to \beta$ in $\mathbb{R}$. Then, by Lemma 2.2.15(1), we have

$$E_{\lambda,\mu}(\beta_n - \beta) = |\beta_n - \beta| E_{\lambda,\mu}(1) \to 0.$$

Thus, by Lemma 2.2.15 (3), $\beta_n \stackrel{\mu}{\to} \beta$.

Conversely, let $\beta_n \stackrel{\mu}{\to} \beta$. Then, by Lemma 2.2.15,

$$\lim_{n\to+\infty} |\beta_n - \beta| E_{\lambda,\mu}(1) = \lim_{n\to+\infty} E_{\lambda,\mu}(\beta_n - \beta) = 0.$$

Now, $E_{\lambda,\mu}(1) \neq 0$ and so $\beta_n \to \beta$ in $\mathbb{R}$. This completes the proof. □

Corollary 2.2.26 *If a real sequence* $\{\beta_n\}$ *is* R*-bounded, then it has at least one limit point.*

Lemma 2.2.27 *A subset* A *of* $\mathbb{R}$ *is* R*-bounded in* $(\mathbb{R}, \mu, T)$ *if and only if it is bounded in* $\mathbb{R}$.

Proof Let the subset A is R-bounded in $(\mathbb{R}, \mu, T)$. Then there exist $t_0 > 0$ and $r_0 \in (0, 1)$ such that

$$\mu_x(t_0) > 1 - r_0$$

for all $x \in A$ and so

$$t_0 \geq E_{r_0,\mu}(x) = |x| E_{r_0,\mu}(1).$$

Now, $E_{r_0,\mu}(1) \neq 0$. If we put $k = \frac{t_0}{E_{r_0,\mu}(1)}$, then we have $|x| \leq k$ for all $x \in A$, i.e., A is bounded in $\mathbb{R}$.

The converse is easy. This completes the proof. □

Definition 2.2.28 A triple $(\mathbb{R}^n, \Phi, T)$ is called an *random Euclidean normed space* if T is a continuous t-norm and $\Phi_x(t)$ is a random Euclidean norm defined by

$$\Phi_x(t) = \prod_{j=1}^{n} \mu_{x_j}(t),$$

where $\prod_{j=1}^{n} a_j = T'^{n-1}(a_1, \ldots, a_n)$, $T' \gg T$, $x = (x_1, \ldots, x_n)$, $t > 0$ and μ is a random norm.

For example, let $\Phi_x(t) = \exp(\frac{\|x\|}{t})^{-1}$, $\mu_{x_j}(t) = \exp(\frac{|x_j|}{t})^{-1}$ and $T = \min$. Then we have $\Phi_x(t) = \min_j \mu_{x_j}(t)$ or, equivalently, $\|x\| = \max_j |x_j|$.

Lemma 2.2.29 *Suppose that the hypotheses of Definition 2.2.28 are satisfied. Then $(\mathbb{R}^n, \Phi, T)$ is an RN-space.*

Proof The properties of (RN1) and (RN2) follow immediately from the definition. For the triangle inequality (RN3) suppose that $x, y \in X$ and $t, s > 0$. Then we have

$$
\begin{aligned}
T\big(\Phi_x(t), \Phi_y(s)\big) &= T\left(\prod_{j=1}^{n} \mathcal{P}_{x_j}(t), \prod_{j=1}^{n} \mathcal{P}_{y_j}(s)\right) \\
&= T\big(T'^{n-1}\big(\mathcal{P}_{x_1}(t), \ldots, \mathcal{P}_{x_n}(t)\big), T'^{n-1}\big(\mathcal{P}_{y_1}(t), \ldots, \mathcal{P}_{y_n}(t)\big)\big) \\
&\le T'^{n-1}\big(T\big(\mathcal{P}_{x_1}(t), \mathcal{P}_{y_1}(t)\big), \ldots, T\big(\mathcal{P}_{x_n}(t), \mathcal{P}_{y_n}(t)\big)\big) \\
&\le T'^{n-1}\big(\mathcal{P}_{x_1+y_1}(t+s), \ldots, \mathcal{P}_{x_n+y_n}(t+s)\big) \\
&= \prod_{j=1}^{n} \mathcal{P}_{x_j+y_j}(t+s) \\
&= \Phi_{x+y}(t+s).
\end{aligned}
$$

This completes the proof. $\square$

Lemma 2.2.30 *Suppose that $(\mathbb{R}^n, \Phi, T)$ is a random Euclidean normed space and A is an infinite and R-bounded subset of $\mathbb{R}^n$. Then A has at least one limit point.*

Proof Let $\{x^{(m)}\}$ be an infinite sequence in A. Since A is R-bounded, so is $\{x^{(m)}\}_{m\ge 1}$. Therefore, there exist $t_0 > 0$ and $r_0 \in (0, 1)$ such that

$$1 - r_0 < \Phi_x(t_0)$$

for all $x \in A$, which implies that $E_{r_0,\Phi}(x) \le t_0$. However, we have

$$
\begin{aligned}
E_{r_0,\Phi}(x) &= \inf\big\{t > 0 : 1 - r_0 < \Phi_x(t)\big\} \\
&= \inf\left\{t > 0 : 1 - r_0 < \prod_{j=1}^{n} \mu_{x_j}(t)\right\} \\
&\ge \inf\big\{t > 0 : 1 - r_0 < \mu_{x_j}(t)\big\} \\
&= E_{r_0,\mu}(x_j)
\end{aligned}
$$

for each $1 \leq j \leq n$. Therefore, $|x_j| \leq k$ in which $k = \frac{t_0}{E_{r_0,\mu}(1)}$, that is, the real sequences $\{x_j^{(m)}\}$ for each $j \in \{1, \dots, n\}$ are bounded. Hence, there exists a subsequence $\{x_1^{(m_{k_1})}\}$ which converges to x_1 in A with respect to the random norm μ. The corresponding sequence $\{x_2^{(m_{k_1})}\}$ is bounded and so there exists a subsequence $\{x_2^{(m_{k_2})}\}$ of $\{x_2^{(m_{k_1})}\}$ which converges to x_2 with respect to the random norm μ.

Continuing like this, we find a subsequence $\{x^{(m_k)}\}$ converging to $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. This completes the proof. □

Lemma 2.2.31 *Let* $(\mathbb{R}^n, \Phi, T)$ *be a random Euclidean normed space. Let* $\{Q_1, Q_2, \dots\}$ *be a countable collection of nonempty subsets in* $\mathbb{R}^n$ *such that* $Q_{k+1} \subseteq Q_k$, *each* Q_k *is closed and* Q_1 *is* R*-bounded. Then* $\bigcap_{k=1}^{\infty} Q_k$ *is nonempty and closed.*

Proof Using the above lemma, the proof proceeds as in the classical case (see Theorem 3.25 in [15]). □

We call an n-dimensional ball $B_x(r, t)$ a *rational ball* if $x \in \mathbb{Q}^n$, $r_0 \in (0, 1)$ and $t \in \mathbb{Q}^+$.

Theorem 2.2.32 *Let* $(\mathbb{R}^n, \Phi, T)$ *be a random Euclidean normed space in which* T *satisfies* (1.1.2). *Let* $G = \{A_1, A_2, \dots\}$ *be a countable collection of* n*-dimensional rational open balls. If* $x \in \mathbb{R}^n$ *and* S *is an open subset of* $\mathbb{R}^n$ *containing* x, *then there exists* $A_k \in G$ *such that* $x \in A_k \subseteq S$ *for some* $k \geq 1$.

Proof Since $x \in S$ and S is open, there exist $r \in (0, 1)$ and $t > 0$ such that $B_x(r, t) \subseteq S$. By (1.1.2), we can find $\eta \in\in (0, 1)$ such that $1 - r < T(1 - \eta, 1 - \eta)$. Let $\{\xi_k\}_{k=1}^n$ be a finite sequence such that $1 - \eta < \prod_{k=1}^n (1 - \xi_k)$ and $x = (x_1, \dots, x_n)$. Then we can find $y = (y_1, \dots, y_n) \in \mathbb{Q}^n$ such that $(1 - \xi_k) < \mu_{x_k - y_k}(\frac{t}{2})$. Therefore, we have

$$1 - \eta < \prod_{k=1}^{n} (1 - \xi_k) \leq \Phi_{x-y}\left(\frac{t}{2}\right) = \prod_{k=1}^{n} \mu_{x_k - y_k}\left(\frac{t}{2}\right)$$

and so $x \in B_y(\eta, \frac{t}{2})$.

Now, we prove that $B_y(\eta, \frac{t}{2}) \subseteq B_x(r, t)$. Let $z \in B_y(\eta, \frac{t}{2})$. Then $\Phi_{y-z}(\frac{t}{2}) > 1 - \eta$ and hence

$$1 - r < T(1 - \eta, 1 - \eta) \leq T\left(\Phi_{x-y}\left(\frac{t}{2}\right), \Phi_{y-z}\left(\frac{t}{2}\right)\right) \leq \Phi_{x-z}(t).$$

On the other hand, there exists $t_0 \in \mathbb{Q}$ such that $t_0 < \frac{t}{2}$ and $x \in B_y(\eta, t_0) \subseteq B_y(\eta, \frac{t}{2}) \subseteq B_x(r, t) \subseteq S$. Now, $B_y(\eta, t_0) \in G$. This completes the proof. □

Corollary 2.2.33 *In a random Euclidean normed space* $(\mathbb{R}^n, \Phi, T)$ *in which* T *satisfies* (1.1.2), *every closed and* R*-bounded set is compact.*

Proof The proof is similar to the proof of Theorem 3.29 in [15]. □

Corollary 2.2.34 *Let* $(\mathbb{R}^n, \Phi, T)$ *be a random Euclidean normed space in which* T *satisfies* (1.1.2) *and* $S \subseteq \mathbb{R}^n$. *Then* S *is compact set if and only if it is* R*-bounded and closed.*

Corollary 2.2.35 *The random Euclidean normed space* $(\mathbb{R}^n, \Phi, T)$ *is complete.*

Proof Let $\{x_m\}$ be a Cauchy sequence in the random Euclidean normed space $(\mathbb{R}^n, \Phi, T)$. Since

$$\begin{aligned} E_{\lambda,\Phi}(x_n - x_m) &= \inf\{t > 0 : \Phi_{x_n - x_m}(t) > 1 - \lambda\} \\ &= \inf\left\{t > 0 : \prod_{j=1}^{n} \mathcal{P}_{x_{m,j} - x_{n,j}}(t) > 1 - \lambda\right\} \\ &\geq \inf\{t > 0 : \mathcal{P}_{x_{m,j} - x_{n,j}}(t) > 1 - \lambda\} \\ &= E_{\lambda,\mathcal{P}}(x_{m,j} - x_{n,j}) = |x_{m,j} - x_{n,j}| E_{\lambda,\mathcal{P}}(1), \end{aligned}$$

the sequence $\{x_{m,j}\}$ for each $j = 1, \dots, n$ is a Cauchy sequence in $\mathbb{R}$ and so it convergent to $x_j \in \mathbb{R}$. Then, by Lemma 2.2.15, the sequence $\{x_{m,j}\}$ is convergent in RN-space $(\mathbb{R}, \mu, T)$.

Now, we prove that $\{x_m\}$ convergent to $x = (x_1, \dots, x_n)$. In fact, we have

$$\lim_m \Phi_{x_m - x}(t) = \lim_m \prod_{j=1}^{n} \mathcal{P}_{x_{m,j} - x_j}(t) = T'^{n-1}(1, \dots, 1) = 1.$$

This completes the proof. □

2.3 Random Functional Analysis

In this section, we discuss some important results dealing with topological isomorphisms and also give the proofs of Open Mapping Theorem, Closed Graph Theorem and some other fundamental theorems in the framework of Random Functional Analysis.

Theorem 2.3.1 *Let* $\{x_1, \dots, x_n\}$ *be a linearly independent set of vectors in vector space* X *and* (X, μ, T) *be an RN-space. Then there exist* $c \neq 0$ *and an RN-space* $(\mathbb{R}, \mu', T)$ *such that, for every choice of the* n *real scalars* $\alpha_1, \dots, \alpha_n$,

$$\mu_{\alpha_1 x_1 + \cdots + \alpha_n x_n}(t) \leq \mu'_{c \sum_{j=1}^{n} |\alpha_j|}(t). \tag{2.3.1}$$

Proof Put $s = |\alpha_1| + \cdots + |\alpha_n|$. If $s = 0$, all α_j's must be zero and so (2.3.1) holds for any c. Let $s > 0$. Then (2.3.1) is equivalent to the inequality that we obtain from (2.3.1) by dividing by s and putting $\beta_j = \frac{\alpha_j}{s}$, that is,

$$\mu_{\beta_1 x_1 + \cdots + \beta_n x_n}\big(t'\big) \le \mu'_c\big(t'\big), \tag{2.3.2}$$

where $t' = \frac{t}{s}$ and $\sum_{j=1}^n |\beta_j| = 1$. Hence, it suffices to prove the existence of $c \neq 0$ and the random norm μ' such that (2.3.2) holds. Suppose that this is not true. Then there exists a sequence $\{y_m\}$ of vectors

$$y_m = \beta_{1,m}x_1 + \cdots + \beta_{n,m}x_n, \qquad \sum_{j=1}^n |\beta_{j,m}| = 1,$$

such that

$$\mu_{y_m}(t) \to 1$$

as $m \to \infty$ for any $t > 0$. Since $\sum_{j=1}^n |\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \le 1$ and so, by the Lemma 2.2.24, the sequence of $\{\beta_{j,m}\}$ is R-bounded. According to Corollary 2.2.26, $\{\beta_{1,m}\}$ has a convergent subsequence. Let β_1 denote the limit of the subsequence and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument, $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding of real scalars $\beta_2^{(m)}$ convergence. Let β_2 denote the limit. Continuing this process, after n steps, we obtain a subsequence $\{y_{n,m}\}_{m\ge1}$ of $\{y_m\}$ such that

$$y_{n,m} = \sum_{j=1}^n \gamma_{j,m}x_j,$$

where $\sum_{j=1}^n |\gamma_{j,m}| = 1$, and $\gamma_{j,m} \to \beta_j$ as $m \to \infty$. By the Lemma 2.2.15 (2), for any $\alpha \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$\begin{aligned} E_{\alpha,\mu}\Bigg(y_{n,m} - \sum_{j=1}^n \beta_j x_j\Bigg) &= E_{\alpha,\mu}\Bigg(\sum_{j=1}^n (\gamma_{j,m} - \beta_j)x_j\Bigg) \\ &\le \sum_{j=1}^n |\gamma_{j,m} - \beta_j| E_{\lambda,\mu}(x_j) \to 0 \end{aligned}$$

as $m \to \infty$. By Lemma 2.2.15 (3), we conclude

$$\lim_{m\to\infty} y_{n,m} = \sum_{j=1}^n \beta_j x_j,$$

where $\sum_{j=1}^n |\beta_j| = 1$, and so all β_j cannot be zero. Put $y = \sum_{j=1}^n \beta_j x_j$. Since $\{x_1, \ldots, x_n\}$ is a linearly independent set, we have $y \neq 0$. Since $\mu_{y_m}(t) \to 1$, by the assumption, we have $\mu_{y_{n,m}}(t) \to 1$. Hence, we have

$$\begin{aligned}\mu_y(t) &= \mu_{(y-y_{n,m})+y_{n,m}}(t)\\ &\geq T(\mu_{y-y_{n,m}}t/2), \mu_{y_{n,m}}(t/2)) \to 1\end{aligned}$$

and so $y = 0$, which is a contradiction. This completes the proof. □

Definition 2.3.2 Let (X, μ, T) and (X, ν, T') be two RN-spaces. Then two random norms μ and ν are said to be *equivalent* whenever $x_n \xrightarrow{\mu} x$ in (X, μ, T) If and only if $x_n \xrightarrow{\nu} x$ in (X, ν, T').

Theorem 2.3.3 *In a finite dimensional vector space X, every two random norms μ and ν are equivalent.*

Proof Let $\dim X = n$ and $\{v_1, \dots, v_n\}$ be a basis for X. Then every $x \in X$ has a unique representation $x = \sum_{j=1}^n \alpha_j v_j$. Let $x_m \xrightarrow{\mu} x$ in (X, μ, T), but, for each $m \geq 1$, suppose that x_m has a unique representation, that is,

$$x_m = \alpha_{1,m} v_1 + \cdots + \alpha_{n,m} v_n.$$

By Theorem 2.3.1, there exist $c \neq 0$ and the random norm μ' such that (2.3.1) holds. thus we have

$$\mu_{x_m - x}(t) \leq \mu'_{c \sum_{j=1}^n |\alpha_{j,m} - \alpha_j|}(t) \leq \mu'_{c|\alpha_{j,m} - \alpha_j|}(t).$$

Now, if $m \to \infty$, then we have

$$\mu_{x_m - x}(t) \to 1$$

for all $t > 0$ and hence $|\alpha_{j,m} - \alpha_j| \to 0$ in $\mathbb{R}$.

On the other hand, by the Lemma 2.2.15 (2), for any $\alpha \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$E_{\alpha,\nu}(x_m - x) \leq \sum_{j=1}^n |\alpha_{j,m} - \alpha_j| E_{\lambda,\nu}(v_j).$$

Since $|\alpha_{j,m} - \alpha_j| \to 0$, we have $x_m \xrightarrow{\nu} x$ in (X, ν, T'). Therefore, with the same argument, $x_m \to x$ in (X, ν, T') imply $x_m \to x$ in (X, μ, T). This completes the proof. □

Definition 2.3.4 A linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is said to be *random bounded* if there exists a constant $h \in \mathbb{R} - \{0\}$ such that, for all $x \in X$ and $t > 0$,

$$\nu_{\Lambda x}(t) \geq \mu_{hx}(t). \tag{2.3.3}$$

Note that, by Lemma 2.2.15 and the last definition, we have

$$\begin{aligned} E_{\lambda,\nu}(\Lambda x) &= \inf\{t > 0 : \nu_{\Lambda x}(t) > 1-\lambda\} \\ &\le \inf\{t > 0 : \mu_x(t/|h|) > 1-\lambda\} \\ &= |h| \inf\{t > 0 : \mu_x(t) > 1-\lambda\} \\ &= |h| E_{\lambda,\mu}(x). \end{aligned}$$

Theorem 2.3.5 *Every linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is random bounded if and only if it is continuous.*

Proof By (2.3.3), every random bounded linear operator is continuous.

Now, we prove the converse. Let the linear operator Λ be continuous, but is not random bounded. Then, for each $n \ge 1$, there exists $x_n \in X$ such that $E_{\lambda,\nu}(\Lambda x_n) \ge nE_{\lambda,\mu}(p_n)$.

If we let

$$y_n = \frac{x_n}{nE_{\lambda,\mu}(x_n)},$$

then it is easy to see $y_n \to 0$, but $\{\Lambda y_n\}$ do not tend to 0. This completes the proof. □

Definition 2.3.6 A linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is an *random topological isomorphism* if Λ is one-to-one, onto and both Λ, Λ^{-1} are continuous. The RN-spaces (X, μ, T) and (Y, ν, T') for which such a Λ exists are said to be *random topologically isomorphic*.

Lemma 2.3.7 *A linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is random topological isomorphism if Λ is onto and there exist constants $a, b \ne 0$ such that*

$$\mu_{ax}(t) \le \nu_{\Lambda x}(t) \le \mu_{bx}(t).$$

Proof By the hypothesis, Λ is random bounded and, by last theorem, is continuous. Since $\Lambda x = 0$ implies that

$$1 = \nu_{\Lambda x}(t) \le \mu_x\left(\frac{t}{|b|}\right)$$

and so $x = 0$, it follows that Λ is one-to-one. Thus Λ^{-1} exists and, since

$$\nu_{\Lambda x}(t) \le \mu_{bx}(t)$$

is equivalent to

$$\nu_y(t) \le \mu_{b\Lambda^{-1}y}(t) = \mu_{\Lambda^{-1}y}\left(\frac{t}{|b|}\right)$$

or

$$\nu_{\frac{1}{b}y}(t) \leq \mu_{\Lambda^{-1}y}(t),$$

where $y = \Lambda x$, we see that Λ^{-1} is random bounded and, by last theorem, is continuous. Therefore, Λ is an random topological isomorphism. This completes the proof. □

Corollary 2.3.8 *Ever random topologically isomorphism preserves completeness.*

Theorem 2.3.9 *Every linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$, where $\dim X < \infty$, but other is not necessarily finite dimensional, is continuous.*

Proof If we define

$$\eta_x(t) = T'\big(\mu_x(t), \nu_{\Lambda x}(t)\big), \tag{2.3.4}$$

where $T' \gg T$. Then (X, η, T) is an RN-space since (RN1) and (RN2) are immediate from the definition and, for the triangle inequality (RN3),

$$\begin{aligned}
T\big(\eta_x(t), \eta_z(s)\big) &= T\big[T'\big(\mu_x(t), \nu_{\Lambda x}(t)\big), T'\big(\mu_z(s), \nu_{\Lambda z}(s)\big)\big] \\
&\leq T'\big[T\big(\mu_x(t), \mu_z(s)\big)T\big(\nu_{\Lambda x}(t), \nu_{\Lambda z}(s)\big)\big] \\
&\leq T'\big(\mu_{x+z}(t+s), \nu_{\Lambda(x+z)}(t+s)\big) \\
&= \eta_{x+z}(t+s).
\end{aligned}$$

Now, let $x_n \xrightarrow{\mu} x$. Then, by Theorem 2.3.3, $x_n \xrightarrow{\eta} x$, but, by (2.3.3), since

$$\nu_{\Lambda x}(t) \geq \eta_x(t),$$

we have $\Lambda x_n \xrightarrow{\nu} \Lambda x$. Hence, Λ is continuous. This completes the proof. □

Corollary 2.3.10 *Every linear isomorphism between finite dimensional RN-spaces is a topological isomorphism.*

Corollary 2.3.11 *Every finite dimensional RN-space (X, μ, T) is complete.*

Proof By Corollary 2.3.10, (X, μ, T) and $(\mathbb{R}^n, \Phi, T)$ are random topologically isomorph. Since $(\mathbb{R}^n, \Phi, T)$ is complete and every random topological isomorphism preserves completeness, (X, μ, T) is complete. □

Definition 2.3.12 Let (V, μ, T) be an RN-space, W be a linear manifold in V and $Q : V \to V/W$ be the natural mapping with $Qx = x + W$. For any $t > 0$, we define

$$\bar{\mu}(x + W, t) = \sup\big\{\mu_{x+y}(t) : y \in W\big\}.$$

Theorem 2.3.13 *Let W be a closed subspace of an RN-space (V, μ, T). If $x \in V$ and $\epsilon > 0$, then there exists $x' \in V$ such that*

$$x' + W = x + W, \qquad E_{\lambda,\mu}\bigl(x'\bigr) < E_{\lambda,\bar{\mu}}(x + W) + \epsilon.$$

Proof By the properties of sup, there always exists $y \in W$ such that

$$E_{\lambda,\mathcal{P}}(x + y) < E_{\lambda,\bar{\mu}}(x + W) + \epsilon.$$

Now, it is enough to put $x' = x + y$. □

Theorem 2.3.14 *Let W be a closed subspace of an RN-space (V, μ, T) and $\bar{\mu}$ be given in the above definition. Then we have*

(1) *$\bar{\mu}$ is an RN-space on V/W;*
(2) *$\bar{\mu}_{Qx}(t) \geq \mu_x(t)$;*
(3) *If (V, μ, T) is an random Banach space, then so is $(V/W, \bar{\mu}, T)$.*

Proof (1) It is clear that $\bar{\mu}_{x+W}(t) > 0$. Let $\bar{\mu}_{x+W}(t) = 1$. By the definition, there exists a sequence $\{x_n\}$ in W such that $\mu_{x+x_n}(t) \to 1$. Thus, $x + x_n \to 0$ or, equivalently, $x_n \to (-x)$ and since W is closed, $x \in W$ and $x + W = W$, the zero element of V/W. Now, we have

$$\begin{aligned}\bar{\mu}_{(x+W)+(y+W)}(t) &= \bar{\mu}_{(x+y)+W}(t)\\ &\geq \mu_{(x+m)+(y+n)}(t)\\ &\geq T\bigl(\mu_{x+m}(t_1), \mu_{y+n}(t_2)\bigr)\end{aligned}$$

for all $m, n \in W$, $x, y \in V$ and $t_1 + t_2 = t$. Now, if we take the sup, then we have

$$\bar{\mu}_{(x+W)+(y+W)}(t) \geq T\bigl(\bar{\mu}_{x+W}(t_1), \bar{\mu}_{y+W}(t_2)\bigr).$$

Therefore, $\bar{\mu}$ is random norm on V/W.

(2) By Definition 2.3.12, we have

$$\bar{\mu}_{Qx}(t) = \bar{\mu}_{x+W}(t) = \sup\bigl\{\mu_{x+y}(t) : y \in W\bigr\} \geq \mu_x(t).$$

Note that, by Lemma 2.2.15,

$$\begin{aligned}E_{\lambda,\bar{\mu}}(Qx) &= \inf\bigl\{t > 0 : \bar{\mu}_{Qx}(t) > 1 - \lambda\bigr\}\\ &\leq \inf\bigl\{t > 0 : \mu_x(t) > 1 - \lambda\bigr\}\\ &= E_{\lambda,\mu}(x).\end{aligned} \tag{2.3.5}$$

(3) Let $\{x_n + W\}$ be a Cauchy sequence in V/W. Then there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$,

$$E_{\lambda,\bar{\mu}}\bigl((x_n + W) - (x_{n+1} + W)\bigr) \leq 2^{-n}.$$

Let $y_1 = 0$ and choose $y_2 \in W$ such that

$$E_{\lambda,\mu}\big(x_1 - (x_2 - y_2), t\big) \le E_{\lambda,\bar{\mu}}\big((x_1 - x_2) + W\big) + \frac{1}{2}.$$

However, $E_{\lambda,\bar{\mu}}((x_1 - x_2) + W) \le \frac{1}{2}$ and so $E_{\lambda,\mu}(x_1 - (x_2 - y_2)) \le (\frac{1}{2})^2$.

Now, suppose that y_{n-1} has been chosen. Then choose $y_n \in W$ such that

$$E_{\lambda,\mu}\big((x_{n-1} + y_{n-1}) - (x_n + y_n)\big) \le E_{\lambda,\bar{\mu}}\big((x_{n-1} - x_n) + W\big) + 2^{-n+1}.$$

Hence, we have

$$E_{\lambda,\mu}\big((x_{n-1} + y_{n-1}) - (x_n + y_n)\big) \le 2^{-n+2}.$$

However, by Lemma 2.2.15, for each positive integer $m > n$ and $\lambda \in (0, 1)$, there exists $\gamma \in (0, 1)$ such that

$$\begin{aligned} E_{\lambda,\mu}\big((x_m + y_m) - (x_n + y_n)\big) &\le E_{\gamma,\mu}\big((x_{n+1} + y_{n+1}) - (x_n + y_n)\big) + \cdots \\ &\quad + E_{\gamma,\mu}\big((x_m + y_m) - (x_{m-1} + y_{m-1})\big) \\ &\le \sum_{i=n}^{m} 2^{-i}. \end{aligned}$$

By Lemma 2.2.15, $\{x_n + y_n\}$ is a Cauchy sequence in V. Since V is complete, there exists x_0 in V such that $x_n + y_n \to x_0$ in V.

On the other hand, we have

$$x_n + W = Q(x_n + y_n) \to Q(x_0) = x_0 + W.$$

Therefore, every Cauchy sequence $\{x_n + W\}$ is convergent in V/W and so V/W is complete. Thus $(V/W, \bar{\mu}, T)$ is a random Banach space. This completes the proof. □

Theorem 2.3.15 *Let W be a closed subspace of an RN-space (V, μ, T). If two of the spaces V, W and V/W are complete, then so is the third one.*

Proof If V is a random Banach space, then so are V/W and W. Hence, the fact that needs to be checked is that V is complete whenever both W and V/W are complete. Suppose that W, V/W are random Banach spaces and $\{x_n\}$ is a Cauchy sequence in V. Since

$$E_{\lambda,\bar{\mu}}\big((x_n - x_m) + W\big) \le E_{\lambda,\mu}(x_n - x_m)$$

for each $m, n \ge 1$, the sequence $\{x_n + W\}$ is a Cauchy sequence in V/W and so converges to $y + W$ for some $y \in W$. Thus, there exists $n_0 \ge 1$ such that, for each $n \ge n_0$,

$$E_{\lambda,\bar{\mu}}\big((x_n - y) + W\big) < 2^{-n}.$$

Now, by the last theorem, there exist a sequence $\{y_n\}$ in V such that

$$y_n + W = (x_n - y) + W, \qquad E_{\lambda,\mu}(y_n) < E_{\lambda,\bar{\mu}}\big((x_n - y) + W\big) + 2^{-n}.$$

Thus, we have

$$\lim_{n\to\infty} E_{\lambda,\mu}(y_n) \le 0$$

and so, by Lemma 2.2.15, $\mu_{y_n}(t) \to 1$ for any $t > 0$, that is, $\lim_{n\to\infty} y_n = 0$. Therefore, $\{x_n - y_n - y\}$ is a Cauchy sequence in W and so it is convergent to a point $z \in W$. This implies that $\{x_n\}$ converges to $z + y$ and hence V is complete. This completes the proof. □

Theorem 2.3.16 (Open Mapping Theorem) *If T is a random bounded linear operator from a RN-space (V, μ, T) onto an RN-space (V', ν, T), then T is an open mapping.*

Proof The theorem will be proved by the following steps:

Step 1: Let E be a neighborhood of the 0 in V. We show that $0 \in (\overline{T(E)})^o$. Let W be a balanced neighborhood of 0 such that $W + W \subset E$. Since $T(V) = V'$ and W is absorbing, it follows that $V' = \bigcap_n T(nW)$ and so there exists $n_0 \ge 1$ such that $\overline{T(n_0 W)}$ has a nonempty interior. Therefore, we have

$$0 \in \big(\overline{T(W)}\big)^o - \big(\overline{T(W)}\big)^o.$$

On the other hand, we have

$$\begin{aligned}\big(\overline{T(W)}\big)^o - \big(\overline{T(W)}\big)^o &\subset \overline{T(W)} - \overline{T(W)} = \overline{T(W) + T(W)}\\ &\subset \overline{T(E)}.\end{aligned}$$

Thus, the set $\overline{T(E)}$ includes the neighborhood $(\overline{T(W)})^o - (\overline{T(W)})^o$ of 0.

Step 2: We show $0 \in (T(E))^o$. Since $0 \in E$ and E is an open set, there exist $0 < \alpha < 1$ and $t_0 \in (0, \infty)$ such that $B_0(\alpha, t_0) \subset E$. However, $0 < \alpha < 1$ and so a sequence $\{\epsilon_n\}$ can be found such that

$$T^{m-n}(1 - \epsilon_{n+1}, \ldots, 1 - \epsilon_m) \to 1$$

and

$$1 - \alpha < \lim_n T^{n-1}(1 - \epsilon_1, 1 - \epsilon_n),$$

in which $m > n$.

On the other hand, $0 \in \overline{T(B_0(\epsilon_n, t_n'))}$, where $t_n' = \frac{1}{2^n} t_0$, and so, by Step 1, there exist $0 < \sigma_n < 1$ and $t_n > 0$ such that

$$B_0(\sigma_n, t_n) \subset \overline{T\big(B_0\big(\epsilon_n, t_n'\big)\big)}.$$

Since the set $\{B_0(r, 1/n)\}$ is a countable local base at zero and $t'_n \to 0$ as $n \to \infty$, t_n and σ_n can be chosen such that $t_n \to 0$ and $\sigma_n \to 0$ as $n \to \infty$.

Now, we show that

$$B_0(\sigma_1, t_1) \subset \big(T(E)\big)^o.$$

Suppose that $y_0 \in B_0(, \sigma_1, t_1)$. Then $y_0 \in \overline{T(B_0(\epsilon_1, t'_1))}$ and so for any $0 < \sigma_2$ and $t_2 > 0$, the ball $B_{y_0}(\sigma_2, t_2)$ intersects $T(B_0(\epsilon_1, t'_1))$. Therefore, there exists $x_1 \in B_0(\epsilon_1, t'_1)$ such that $Tx_1 \in B_{y_0}(\sigma_2, t_2)$, that is,

$$\nu_{y_0 - Tx_1}(t_2) > 1 - \sigma_2$$

or, equivalently,

$$y_0 - Tx_1 \in B_0(\sigma_2, t_2) \subset \overline{T\big(B_0\big(\epsilon_1, t'_1\big)\big)}.$$

By the similar argument, there exist $x_2 \in B_0(\epsilon_2, t'_2)$ such that

$$\nu_{y_0 - (Tx_1 + Tx_2)}(t_3) = \nu_{(y_0 - Tx_1) - Tx_2}(t_3) > 1 - \sigma_3.$$

If this process is continued, it leads to a sequence $\{x_n\}$ such that

$$x_n \in B_0\big(\epsilon_n, t'_n\big), \quad \nu_{y_0 - \sum_{j=1}^{n-1} Tx_j}(t_n) > 1 - \sigma_n.$$

Now, if $n, m \geq 1$ and $m > n$, then we have

$$\begin{aligned}\mu_{\sum_{j=1}^{n} x_j - \sum_{j=n+1}^{m} x_j}(t) &= \mu_{\sum_{j=n+1}^{m} x_j}(t) \\ &\geq T^{m-n}\big(\mu_{x_{n+1}}(t_{n+1}), \mu_{x_m}(t_m)\big),\end{aligned}$$

where $t_{n+1} + t_{n+2} + \cdots + t_m = t$. Put $t'_0 = \min\{t_{n+1}, t_{n+2}, \ldots, t_m\}$. Since $t'_n \to 0$, there exists $n_0 \geq 1$ such that $0 < t'_n \leq t'_0$ for all $n > n_0$. Therefore, for all $m > n$, we have

$$\begin{aligned}T^{m-n}\big(\mu_{x_{n+1}}\big(t'_0\big), \mu_{x_m}\big(t'_0\big)\big) &\geq T^{m-n}\big(\mu_{x_{n+1}}\big(t'_{n+1}\big), \mu_{x_m}\big(t'_m\big)\big) \\ &\geq T^{m-n}(1 - \epsilon_{n+1}, 1 - \epsilon_m)\end{aligned}$$

and so

$$\lim_{n\to\infty} \mu_{\sum_{j=n+1}^{m} x_j}(t) \geq \lim_{n\to\infty} T^{m-n}(1 - \epsilon_{n+1}, 1 - \epsilon_m) = 1,$$

that is,

$$\mu_{\sum_{j=n+1}^{m} x_j}(t) \to 1$$

for all $t > 0$. Thus, the sequence $\{\sum_{j=1}^{n} x_j\}$ is a Cauchy sequence and so the series $\{\sum_{j=1}^{\infty} x_j\}$ converges to a point $x_0 \in V$ since V is a complete space. For any fixed

$t > 0$, there exists $n_0 \geq 1$ such that $t > t_n$ for all $n > n_0$ since $t_n \to 0$. Thus, we have

$$\begin{aligned} \nu_{y_0 - T(\sum_{j=1}^{n-1} x_j)}(t) &\geq \nu_{y_0 - T(\sum_{j=1}^{n-1} x_j)}(t_n) \\ &\geq 1 - \sigma_n \end{aligned}$$

and so

$$\nu_{y_0 - T(\sum_{j=1}^{n-1} x_j)}(t) \to 1.$$

Therefore, we have

$$y_0 = \lim_{n\to\infty} T\left(\sum_{j=1}^{n-1} x_j\right) = T\left(\lim_{n\to\infty} \sum_{j=1}^{n-1} x_j\right) = Tx_0.$$

But, we have

$$\begin{aligned} \mu_{x_0}(t_0) &= \lim_{n\to\infty} \mu_{\sum_{j=1}^{n} x_j}(t_0) \\ &\geq T^n\Big(\lim_{n\to\infty} \big(\mu_{x_1}(t_1'), \mu_{x_n}(t_n')\big)\Big) \\ &\geq \lim_{n\to\infty} T^{n-1}(1-\epsilon_1, \ldots, 1-\epsilon_n) \\ &> 1 - \alpha. \end{aligned}$$

Therefore, $x_0 \in B_0(\alpha, t_0)$.

Step 3: Let G be an open subset of V and $x \in G$. Then we have

$$T(G) = Tx + T(-x+G) \supset Tx + \big(T(-x+G)\big)^o.$$

Hence, $T(G)$ is open since it includes a neighborhood of each of its point. This completes the proof. $\square$

Corollary 2.3.17 *Every one-to-one random bounded linear operator from a random Banach space onto a random Banach space has a random bounded converse.*

Theorem 2.3.18 (Closed Graph Theorem) *Let T be a linear operator from a random Banach space (V, μ, T) into a random Banach space (V', ν, T). Suppose that, for every sequence $\{x_n\}$ in V such that $x_n \to x$ and $Tx_n \to y$ for some elements $x \in V$ and $y \in V'$, it follows that $Tx = y$. Then T is random bounded.*

Proof For any $t > 0$, $x \in X$ and $y \in V'$, define

$$\Phi_{(x,y)}(t) = T'\big(\mu_x(t), \nu_y(t)\big),$$

where $T' \gg T$.

First, we show that $(V \times V', \Phi, T)$ is a complete RN-space. The properties of (RN1) and (RN2) are immediate from the definition. For the triangle inequality (RN3), suppose that $x, z \in V$, $y, u \in V'$ and $t, s > 0$. Then we have

$$\begin{aligned} T\big(\Phi_{(x,y)}(t), \Phi_{(z,u)}(s)\big) &= T\big[T'\big(\mu_x(t), \nu_y(t)\big), T'\big(\mu_z(s), \nu_u(s)\big)\big] \\ &\le T'\big[T\big(\mu_x(t), \mu_z(s)\big), T\big(\nu_y(t), \nu_u(s)\big)\big] \\ &\le T'\big(\mu_{x+z}(t+s), \nu_{y+u}(t+s)\big) \\ &= \Phi_{(x+z,y+u)}(t+s). \end{aligned}$$

Now, if $\{(x_n, y_n)\}$ is a Cauchy sequence in $V \times V'$, then, for any $\epsilon > 0$ and $t > 0$, there exists $n_0 \ge 1$ such that

$$\Phi_{(x_n,y_n)-(x_m,y_m)}(t) > 1 - \epsilon$$

for all $m, n > n_0$. Thus, for all $m, n > n_0$, we have

$$\begin{aligned} T'\big(\mu_{x_n-x_m}(t), \nu_{y_n-y_m}(t)\big) &= \Phi_{(x_n-x_m,y_n-y_m)}(t) \\ &= \Phi_{(x_n,y_n)-(x_m,y_m)}(t) \\ &> 1-\epsilon. \end{aligned}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V and V', respectively, and there exist $x \in V$ and $y \in V'$ such that $x_n \to x$ and $y_n \to y$ and so $(x_n, y_n) \to (x, y)$. Hence, $(V \times V', \Phi, T)$ is a complete RN-space. The remainder of the proof is the same as the classical case. This completes the proof. □

2.4 Non-Archimedean Random Normed Spaces

By a *non-Archimedean field* we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathcal{K}$ into $[0, \infty)$ such that

(1) $|r| = 0$ if and only if $r = 0$;
(2) $|rs| = |r||s|$;
(3) $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$.

Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \ge 1$. By the *trivial valuation,* we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$.

Let X be a vector space over a field $\mathcal{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$, that is, there exists $a_0 \in \mathcal{K}$ such that $|a_0|$ is not in $\{0, 1\}$.

The most important examples of non-Archimedean spaces are p-adic numbers. In 1897, Hensel [106] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b} p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the *p-adic number field*.

A function $\|\cdot\| : X \to [0, \infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(NAN1) $\|x\| = 0$ if and only if $x = 0$;
(NAN2) for any $r \in \mathcal{K}, x \in X$, $\|rx\| = |r|\|x\|$;
(NAN3) the strong triangle inequality (ultrametric), namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\}$$

for all $n, m \geq 1$ with $n > m$, a sequence $\{x_n\}$ is a Cauchy sequence in X if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space.

By a *complete non-Archimedean normed space*, we mean one in which every Cauchy sequence is convergent.

Definition 2.4.1 A *non-Archimedean random normed space* (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(NA-RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(NA-RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$ and $\alpha \neq 0$;
(NA-RN3) $\mu_{x+y}(\max\{t, s\}) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

It is easy to see that, if (NA-RN3) holds, then so is

(RN3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$.

Example 2.4.2 As a classical example, if $(X, \|\cdot\|)$ is a non-Archimedean normed linear space, then the triple (X, μ, T_M), where

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq \|x\|, \\ 1, & \text{if } t > \|x\|, \end{cases}$$

is a non-Archimedean RN-space.

Example 2.4.3 Let $(X, \|\cdot\|)$ be a non-Archimedean normed linear space. Define

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $x \in X$ and $t > 0$. Then (X, μ, T_M) is a non-Archimedean RN-space.

Definition 2.4.4 Let (X, μ, T) be a non-Archimedean RN-space. Let $\{x_n\}$ be a sequence in X.

(1) The sequence $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$$

for all $t > 0$. In this case, the point x is called the *limit* of the sequence $\{x_n\}$.

(2) The sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$ and $p > 0$,

$$\mu_{x_{n+p}-x_n}(t) > 1 - \varepsilon.$$

(3) If each Cauchy sequence in X is convergent, then the random normed space is said to be *complete* and the non-Archimedean RN-space (X, μ, T) is called a *non-Archimedean random Banach space*.

Remark 2.4.5 [168] Let (X, μ, T_M) be a non-Archimedean RN-space. Then we have

$$\mu_{x_{n+p}-x_n}(t) \geq \min\bigl\{\mu_{x_{n+j+1}-x_{n+j}}(t) : j = 0, 1, 2, \ldots, p-1\bigr\}.$$

Thus, the sequence $\{x_n\}$ is a Cauchy sequence in X if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$,

$$\mu_{x_{n+1}-x_n}(t) > 1 - \varepsilon.$$

2.5 Fuzzy Normed Spaces

Now, we define the concept of fuzzy normed spaces and give some examples of these spaces. Here the t-norms notation is denoted by $*$.

Definition 2.5.1 The triple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(FM1) $M(x, y, 0) > 0$;
(FM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(FM3) $M(x, y, t) = M(y, x, t)$;
(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$;
(FM5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Definition 2.5.2 The triple $(X, N, *)$ is called a *fuzzy normed space* if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(FN1) $N(x, t) > 0$;

(FN2) $N(x,t)=1$ if and only if $x=0$;
(FN3) $N(\alpha x,t)=N(x,t/|\alpha|)$ for all $\alpha\neq 0$;
(FN4) $N(x,t)*N(y,s)\leq N(x+y,t+s)$;
(FN5) $N(x,\cdot):(0,\infty)\to[0,1]$ is continuous;
(FN6) $\lim_{t\to\infty}N(x,t)=1$.

Lemma 2.5.3 *Let N be a fuzzy norm. Then we have*

(1) *$N(x,t)$ is nondecreasing with respect to t for all $x\in X$;*
(2) *$N(x-y,t)=N(y-x,t)$.*

Proof Let $t<s$. Then $k=s-t>0$ and we have

$$N(x,t)=N(x,t)*1=N(x,t)*N(0,k)\leq N(x,s),$$

which proves (1).

To prove (2), we have

$$N(x-y,t)=N\big((-1)(y-x),t\big)=N\left(y-x,\frac{t}{|-1|}\right)=N(y-x,t).$$

This completes the proof. □

Example 2.5.4 Let $(X,\|\cdot\|)$ be a normed linear space. Define $a*b=ab$ or $a*b=\min(a,b)$ and

$$N(x,t)=\frac{kt^n}{kt^n+m\|x\|}$$

for all $k,m,n\in\mathbb{R}^+$. Then $(X,N,*)$ is a fuzzy normed space. In particular, if $k=n=m=1$, then we have

$$N(x,t)=\frac{t}{t+\|x\|},$$

which is called the *standard fuzzy norm* induced by the norm $\|\cdot\|$.

Lemma 2.5.5 *Let $(X,N,*)$ be a fuzzy normed space. If we define*

$$M(x,y,t)=N(x-y,t),$$

then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

We can see that both definition and properties on fuzzy normed spaces are very similar to those of random normed spaces. Then X equipped with $\mu_x(t)=N(x,t)$ and $T=*$ can be regarded as a RN-space.

Now, we extend the definition of fuzzy metric space. In fact, we extend the range of fuzzy sets to arbitrary lattice.

Definition 2.5.6 The triple $(X, \mathcal{P}, \mathcal{T})$ is called an $\mathcal{L}$*-fuzzy normed space* (briefly, $\mathcal{L}F$-normed space) if X is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions: for all $x, y \in X$ and $t, s \in (0, +\infty)$,

(LFN1) $\mathcal{P}(x, t) >_L 0_{\mathcal{L}}$;
(LFN2) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(LFN3) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for any $\alpha \neq 0$;
(LFN4) $\mathcal{T}(\mathcal{P}(x, t), \mathcal{P}(y, s)) \leq_L \mathcal{P}(x + y, t + s)$;
(LFN5) $\mathcal{P}(x, \cdot) : (0, \infty) \to L$ is continuous;
(LFN6) $\lim_{t\to\infty} \mathcal{P}(x, t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called an $\mathcal{L}$*-fuzzy norm* (briefly, $\mathcal{L}F$-norm). If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set and the t-norm $\mathcal{T}$ is t-representable, then the triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space* (briefly, *IF*-normed space).

Example 2.5.7 Let $(X, \|\cdot\|)$ be a normed linear space. Denote $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and let M, N be the fuzzy sets on $X \times (0, \infty)$ defined as follows:

$$\mathcal{P}_{M,N}(x, t) = \left(\frac{ht^n}{ht^n + m\|x\|}, \frac{m\|x\|}{ht^n + m\|x\|}\right)$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{M,N}, \mathcal{T})$ is an *IF*-normed space.

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