2

Return and Risk

This chapter is about the fundamentals of investment growth. It introduces important calculations with interest rates, returns, and discounting. These ideas will be needed in later chapters. It is also about investment risk, how it can be measured and how it can be minimized in the formation and maintenance of an investment portfolio. This is possible through one of the great financial breakthroughs, the *mean-variance theory* and CAPM, the capital asset and pricing model, due to Markowitz and his followers. In the 50 years since its introduction and subsequent development shortcomings of the theory have surfaced and improvements offered. Yet it is a starting point for these advanced theories and basic to a financial course of study.

Since most of the content of this chapter is deterministic, opportunities for Monte Carlo analysis are limited.

Market risk refers to the possibility of suffering a loss or even a less-than-expected return as a result of unexpected movements in some market, for example the currency, or real estate, or commodities market. However in this book our primary focus is on the stock market.

With few exceptions risk is ever present. A blue chip equity can go along smoothly for years issuing dividends on a regular basis only to succumb to unforeseen events.[1] Market prices for stocks, real estate, currencies, and precious metals rise and fall, for the most part, according to models we investigated in the last chapter. At the point of sale, which may not be at a propitious moment for the investor, a position is subject to the market price prevailing at that time. This very concrete "mark-to-market" valuation often results in a loss.

Over time risk has become better understood. Especially so upon the advent of the science of probability (see [Ber96]). Quantifying risk entails two components: fixing the amount of loss and second, its probability. Once risk became quantifiable, the investment community invented instruments for managing risk. This includes portfolio diversification, which we take up in this chapter, but also futures and option contracts which we take up in the sequel.

[1] GM was declared bankrupt in 2009.

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2.1 The Risk-Free Rate

As already noted, in general all investment entails some risk. One exception to the rule, as near as possible, is an investment in government bonds. So much so that certain government securities are referred to as a *risk-free investment*.[2] U.S. bonds are one form of treasury securities; treasury bills and notes are two others.

Unlike a hard asset, a *financial instrument* is a item that derives its value from a promise to pay. If there is a well-developed market for them, then it is called a *security*. A *fixed-income security* is a security whose promise to pay is a definite amount to the holder over a given span of time. An *equity* is a security in which the investor has an ownership share, for example as in stock.

A *bond* is a fixed-income security representing the debt of the issuer, a company or government, to the holder of the bond. A bond has a stated *face* or *par* value which the issuer promises to pay the holder on the stated *maturity date*. In addition to that, a bond can have *coupons*, stated as a percentage of the face value, which the issuer pays the holder on an annual basis. (In some cases half the coupon payment is made semi-annually.) The final payment at maturity includes both the face value and the last coupon payment. A *zero-coupon bond* is one that has no coupons and just pays the face value at maturity. Originally the issuer sold the bond to borrow money. In short, a bond is an IOU for a loan with explicit payback terms.

U.S. Treasury *bills* are zero-coupon bonds. They pay no interest but sell at a discount of par value. U.S. Treasury *notes* have maturities between 2 and 10 years and have a coupon payment every 6 months. U.S. Treasury *bonds* have maturities between 20 and 30 years and a coupon payment every 6 months. These, and other, risk-free investments are centrally important throughout finance. The rate of return on a risk-free investment is called the *risk-free rate*, r_f. For investments in U.S. dollars, this is often taken as the yield rate on short-term treasury bills. These rates can be found at www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml.

The risk-free rate is a very important tool in use throughout finance. As we will see, it serves as a basis of comparison for all other investments. One of these is the determination of "fair" prices for many financial instruments such as futures and options. We will take up this dependence in a later chapter. It is important to understand that any rate of return exceeding the risk-free rate is considered to have risk, for example, dividend yields exceeding the risk-free rate.

The risk-free rate impacts many other investment rates throughout the financial system, for example interest rates on insured bank deposits, home mortgage rates, and corporate bond rates.

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[2] This is so for US government bonds. A financial crisis was precipitated in 1998 when the Russian government defaulted on its debt.

2.2 Fixed-Income Securities Calculations

When return rates are predictable future payments can be calculated exactly. A *zero-coupon bond* is an example. It has a face value F and a maturity date T. The holder of the bond may exchange the bond for its face value on the maturity date. The original cost of the bond, P, is the investment. This fixes the rate of return.

Conversely, at the end of a financial transaction, when all the payments are known after the fact, then an analysis can be made as to the true return of the investment.

2.2.1 Simple Interest

Let P dollars be the value of an original investment, or bank deposit, and let ΔP be the gain (or loss if negative) after a period of time t in years. The value of the investment at that time is

$$A_t = P + \Delta P. \tag{2.1}$$

The *return* is the relative gain,

$$\frac{\Delta P}{P}. \tag{2.2}$$

It is usually expressed in percent. By *logarithmic return*, in brief *log-return*, we mean

$$\log(\frac{A_t}{P}) = \log(1 + \frac{\Delta P}{P}). \tag{2.3}$$

From the Taylor series for the logarithm, see (A.4), we have

$$\log(1 + \frac{\Delta P}{P}) = \frac{\Delta P}{P} - \frac{(\frac{\Delta P}{P})^2}{2} + \dots .$$

Thus, to first order, logarithmic returns and returns are the same.

The *rate of return*, or interest rate in the case of bank deposits, is

$$r = \frac{\Delta P}{Pt} \tag{2.4}$$

expressed in percent per year.[3] If the time period is 1 year, then the return and the rate of return are numerically the same.

Turning this equation around, after t years, an investment of P dollars earns the gain

$$\Delta P = Prt, \tag{2.5}$$

and the value at that time is

$$A_t = P(1 + rt). \tag{2.6}$$

[3] Some authors define the rate of return as $\frac{\Delta P}{P}$, [CZ03, Lun98]; others as we have done here, [Sch03]. As in most science and engineering applications, we prefer to reserve the term rate for changes per unit time.

The time t need not be a multiple of a year. If it is not, the amount earned is still proportional to the time invested. For example, if t is 3/2 years, then the investment returns one full years' amount and half of another.

Example 2.1. Two hundred dollars are deposited in an account paying 2 % per quarter. After 3 and 1/4th quarters the account is closed. The interest earned is $\$200(0.02)(3+1/4) = \$13$. This calculation could be put on an annual basis: 2 % per quarter is 8 % per year and 3 and 1/4 quarters is 13/16th years. Thus we have $0.02(13/4) = 0.02(4)(13/4)(1/4) = 13$. The amount of the investment when closed is \$213. □

2.2.2 Compounding

The rate r above is a *simple* earnings rate meaning the amounts accrued were not reinvested over the period; there is no interest on interest. But if accrued amounts are reinvested, the situation becomes quite different, it is called *compounding.*

Let the earnings be calculated annually, $t = 1$ year. The value of the investment after 1 year is

$$A_1 = P(1+r).$$

If that money is reinvested, then $P(1+r)$ plays the role of the original investment and after 2 years it grows to

$$A_2 = P(1+r)(1+r) = P(1+r)^2.$$

Continuing, after t years it becomes

$$A_t = P(1+r)^t.$$

Compare this with (2.6).

Suppose an investment offers more frequent compounding periods. If r is the annual rate of return but compounding is quarterly, then the quarterly rate is $r/4$. Now quarterly periods play the role of years in the previous equations, hence after n quarters, or $t = n/4$ years, we have

$$A_t = P(1+\frac{r}{4})^n = P(1+\frac{r}{4})^{4t}.$$

In particular, at the end of 1 year, the amount becomes

$$A_1 = P(1+\frac{r}{4})^4.$$

The earnings are $\Delta P = A_1 - P$ and the return rate is (since $t = 1$)

$$\begin{aligned} R &= \frac{A_1 - P}{P} = (1+\frac{r}{4})^4 - 1 \\ &= 4(\frac{r}{4}) + 6(\frac{r}{4})^2 + 4(\frac{r}{4})^3 + (\frac{r}{4})^4. \end{aligned}$$

It is evident that $R > r$. For example, if r is 10 %, then R is 10.38 %. In other words, an annual rate of 10 % compounded quarterly earns the same as 10.38 % compounded annually. To distinguish between them, r is called the *nominal rate* and R is the *effective rate.*

Example 2.2. Two hundred dollars are deposited in an account paying 8 % per year compounded quarterly (2 % per quarter). After 3 and 1/4th quarters the account is closed. The interest earned in the first quarter is $200 * 0.02$ and this is added to the account making the value equal to $200(1 + 0.02)$. This repeats for the second quarter and third. The account value at that time is $200(1 + 0.02)^3$. Over the next 1/4 of a quarter the interest earned is this amount as principal times the interest rate for that fractional time, $200(1 + 0.02)^3(1/4)0.02$. This is 3 periods of compounding plus one-fourth period of simple interest. Hence the closing value is

$$200(1 + 0.02)^3(1 + (1/4)0.02) = 212.24(1 + 0.005) = 213.30.$$

Since $\left(1 + (\frac{1}{4})0.02\right) \neq (1 + 0.02)^{1/4}$, the exponent 3.25 does not give the exact answer.[4] This problem disappears under continuous compounding as we discuss next. □

As the number of compounding periods increases, so does the effective rate. If there are m compounding periods per year, the effective rate is given by

$$R = \left(1 + \frac{r}{m}\right)^m - 1. \tag{2.7}$$

The right-hand side is an increasing function of m, see Table 2.1.

Table 2.1 Effective rates at 10 % nominal for various compounding periods

Times/year	1	2	4	6	12	365	730
Rate	10.000	10.250	10.381	10.463	10.471	10.515	10.516

With m compounding periods per year, in t years there are $n = tm$ compounding periods. It follows that the value of the investment after t years is

$$A_t = P\left(1 + \frac{r}{m}\right)^{mt}. \tag{2.8}$$

As the number of compounding periods tends to infinity the expression on the right-hand side tends to a limit,

$$A_t = \lim_{m\to\infty} P\left(1 + \frac{r}{m}\right)^{mt} = Pe^{rt} \tag{2.9}$$

where $\boldsymbol{e}$ is a mathematical constant equal to 2.71828 accurate to 5 decimal places. This then is the accrued value for *continuous compounding* at a nominal annual rate of r. In this equation t does not have to be an exact number of years. Since compounding occurs continuously, t can be any non-negative real number.

From (2.6) the effective rate R satisfies the equation $1+R = e^r$, and therefore

$$R = e^r - 1 = r + \frac{r^2}{2!} + \frac{r^3}{3!} + \ldots;$$

[4] The exponent $3.25186\ldots = 3 + \log(1 + 0.02/4)/\log(1 + 0.02)$ is required.

the second equality follows from the power series expansion of the exponential function (A.3). For example, for a nominal rate of 10 %, the effective rate is 10.517 % under continuous compounding.

Fig. 2.1 shows in a dramatic way the effect of compounding. One unit of currency on deposit for 40 years at 8% earns about 4 times the original value under simple interest, but compounded continuous earns about 25 times its original value.

Doubling Time

An alternate way of characterizing the return rate is specifying the time required for an investment to double in value. Under continuous compounding we seek the time t_2 for which $Pe^{rt_2} = 2P$. Solving gives,

$$t_2 = \frac{\log 2}{r} \approx \frac{.7}{r}. \tag{2.10}$$

For example when r is 10 % it takes about 7 years for an investment to double.

This equation is the origin of the *Seven-Ten Rule*: Money invested at 7 % doubles in approximately 10 years and money invested at 10 % doubles in approximately 7 years.

For discrete compounding we may use (2.8),

$$t_2 = \frac{1}{m}\frac{\log(2)}{\log(1+\frac{r}{m})}.$$

Recall m is the number of compounding periods per year. By using the effective rate R, m can be taken as 1, compare (2.7),

$$t_2 = \frac{\log(2)}{\log(1+R)}. \tag{2.11}$$

Average Rates

As a rule return rates vary from time to time. Then the average rate becomes important. Suppose r_1 is the (simple) rate over the first compounding period, r_2 the rate over the second and so on through r_n, the nth. The amount of an investment P after this time is

$$A = P(1+r_1)(1+r_2)\ldots(1+r_n).$$

Therefore the average rate $\bar{r}$ over these n periods satisfies the equation $A = P(1+\bar{r})^n$; it follows that the average is given by the nth root,

$$\bar{r} = ((1+r_1)(1+r_2)\ldots(1+r_n))^{1/n} - 1. \tag{2.12}$$

This is the *geometric average* of the several rates.

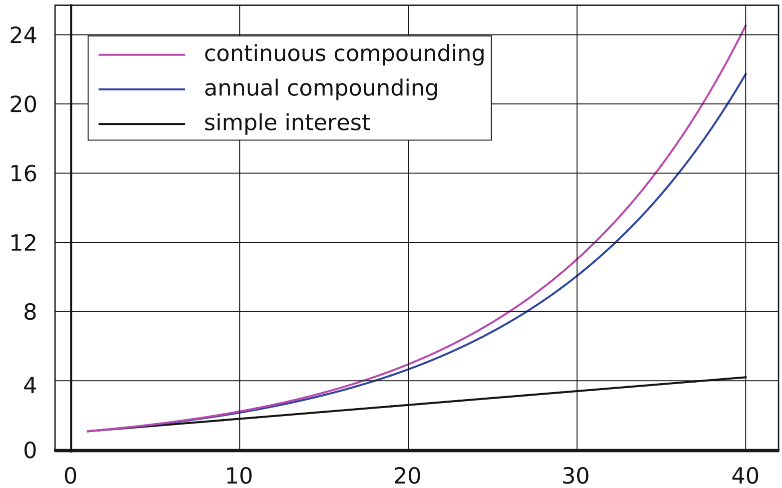


Fig. 2.1. A comparison of the growth of \$1 over 40 years at the annual rate of 8 %. Under simple interest the principal grows by about 4 times, under annual compounding, by about 22 times, and under continuous compounding by about 25 times

If compounding is continuous, the calculation is even simpler. Let r_1 be the rate for an arbitrary period of time t_1. That amount is then reinvested at the rate of r_2 over time t_2 and so on for n periods of time. Then the accrued amount is

$$A = Pe^{r_1 t_1} e^{r_2 t_2} \dots e^{r_n t_n} = Pe^{r_1 t_1 + r_2 t_2 + \dots + r_n t_n}.$$

Let $T = t_1 + t_2 + \dots + t_n$ be the total time; since the average rate satisfies $A = Pe^{\bar{r}T}$, we get

$$\bar{r} = \frac{1}{T}\left(r_1 t_1 + r_2 t_2 + \dots + r_n t_n\right), \tag{2.13}$$

the *arithmetic average* rate.

Example 2.3. What is the average rate of compounded return over four quarters if the first quarter rate is 0.01, the second is 0.02, the third is 0.03, and the fourth is 0.04?

Under discrete compounding the average quarterly rate is

$$\bar{r} = \sqrt[4]{(1+0.01)(1+0.02)(1+0.03)(1+0.04)} - 1 = 0.0249\dots.$$

Under continuous compounding it is

$$\bar{r} = \frac{0.01 + 0.02 + 0.03 + 0.04}{4} = 0.025.$$

Note that the geometric average is less than the arithmetic average (slightly so here). This is always the case.[5] □

[5] Because the log function is concave down, for positive arguments, $\frac{1}{2}(\log x_1 + \log x_2) < \log(\frac{1}{2}(x_1 + x_2))$ unless $x_1 = x_2$. So $((1+x_1)(1+x_2))^{1/2} - 1 < \exp(\frac{1}{2}(\log(1+x_1) + \log(1+x_2))) - 1 < \exp(\log(\frac{1}{2}((1+x_1)+(1+x_2)))) - 1 = \frac{1}{2}(x_1 + x_2)$; same argument for n terms.

Example 2.4. An amount P is put into savings certificates. The first year it earned 6 % interest and in the second 5 %. In the third year it was to have earned 8 % but the account was closed at mid-year (at no penalty). What was the average annual rate earned?

We want to find $\bar{r}$ solving the following equation

$$P(1.06)(1.05)(1+\frac{1}{2}0.08) = P(1+\bar{r})^2(1+\frac{1}{2}\bar{r}).$$

Numerical methods are required to discover that the answer is 6.0077 %. The approximate answer of 6.025 % can be gotten by approximating $(1+\bar{r}/2)$ by $(1+\bar{r})^{1/2}$.

If compounding were continuous the problem is much easier. The total time is $T = 1+1+0.5 = 2.5$, thus solving

$$Pe^{0.05+0.06+\frac{1}{2}0.08} = Pe^{\bar{r}2.5}$$

gives $\bar{r} = 6\,\%$.

This example shows why continuous compounding is often used in financial transactions. □

2.2.3 Discounting

Having P dollars today is worth more than having P dollars next week, or next month, or next year. For one thing, one could invest it at the risk-free rate r_f. Then in time t, those dollars will become

$$V = Pe^{r_f t}$$

in the case of continuous compounding. This demonstrates the time value of money.

Turning the argument around, it shows that P is today's value of a payment of V dollars at time t,

$$P = Ve^{-r_f t}. \tag{2.14}$$

This is called *discounting* future money to the present time. One must discount in this way when dealing with future payments. It then becomes possible to place transactions occurring at different times on an equal basis to compare them. We refer to P in (2.14) as the *present value* of V.

It is easier to discount under the assumption of continuous compounding as there is no need to interpolate but sometimes discrete compounding is called for. From (2.8) we get that

$$P = \frac{V}{\left(1+\frac{r}{m}\right)^{tm}} = V\left(1+\frac{r}{m}\right)^{-tm} \tag{2.15}$$

when t is an exact multiple of the compounding period $1/m$. It has the correct value when tm is an integer and interpolates for the other values of t,

Mortgages

As an example of a more intricate calculation with interest rates let us work through the problem of periodic mortgage payments. A loan of A dollars is to be paid back in n equal installments of Y dollars each. Assume the rate of interest is r per payment period (for example, if payments are monthly, then r is the annual rate divided by 12).

Let A_i be the remaining balance on the loan just after the ith payment, $A_0 = A$. At the end of the first period the balance has grown to $A(1+r)$; the payment reduces that by Y, hence the balance remaining is

$$A_1 = A(1+r) - Y.$$

For the second period, A_1 acts as the loan amount, and so

$$A_2 = A_1(1+r) - Y = A(1+r)^2 - Y(1+r) - Y.$$

Continuing, after n payments the balance is

$$\begin{aligned} A_n &= A(1+r)^n - Y\left((1+r)^{n-1} + \ldots + (1-r) + 1\right) \\ &= A(1+r)^n - Y\left(\frac{(1+r)^n - 1}{r}\right). \end{aligned} \tag{2.16}$$

But this final amount is zero, $A_n = 0$. Solving for Y we get

$$Y = \frac{Ar}{1 - \frac{1}{(1+r)^n}}$$

per period. The term

$$a = \frac{1 - (1+r)^{-n}}{r}$$

is called the *annuity-immediate factor*; in this notation, $Y = A/a$.

The equation derived in (2.16) can be used to construct a table of remaining balances, see Table 2.2. These are of considerable interest to the homeowner.

Table 2.2 Remaining mortgage balance
\$200,000 at 6 % for 15 years, monthly payment: \$1,687.71

Month	Interest	Towards principal	Principal remaining
1	1,000	687.71	199,312.29
2	996.56	691.15	198,621.13
3	993.11	694.60	197,926.53
4	989.63	698.08	197,228.45
5	986.14	701.57	196,526.87
6	982.63	705.08	195,821.79

Example 2.5. Consider financing $A = \$200,000$ over 15 years ($n = 180$ months) at a rate of 6 % annually ($r = 1/2\,\%$ monthly). From above, this requires a monthly payment of

$$Y = \frac{200{,}000 * .005}{1 - \frac{1}{1.005^{180}}} = \$1,687.71.$$

□

Example 2.6. As a second example we will solve the same problem in an entirely different way using the principle of discounting. The lender receives a stream of payments each in the amount of Y. The present value of the first is $Y/(1 + r)$. The present value of the second is, following (2.15), $Y/(1 + r)^2$. Continuing in this fashion, the present value of all the payments equals the amount of the loan, and so

$$\begin{aligned} A &= \frac{Y}{1+r} + \frac{Y}{(1+r)^2} + \ldots + \frac{Y}{(1+r)^n} \\ &= \frac{Y}{(1+r)^n}\left((1+r)^{n-1} + \ldots + (1+r) + 1\right) \\ &= \frac{Y}{r}\left(1 - \frac{1}{(1+r)^n}\right). \end{aligned}$$

This gives the same solution as above. □

Annuities

An *annuity* is a series of payments to a beneficiary made at fixed intervals of time. If the number of payments is known in advanced, it is an *ordinary annuity*. A *perpetuity* is an annuity in which the payments continue forever.

The cost of an annuity can be derived by calculating its present value. Let Y denote the payments and r the per period discount rate. Upon reflection one sees that this problem is exactly like the mortgage calculation above. Indeed, from the lender's position, it is an annuity. The present value of the first payment made at the end of the first period is $Y/(1 + r)$, the present value of the second payment is $Y/(1 + r)^2$ and so on. Therefore for an ordinary annuity having n payments

$$PV = \sum_{k=1}^{n} \frac{Y}{(1+r)^k} = \frac{Y}{r}\left(1 - \frac{1}{(1+r)^n}\right). \tag{2.17}$$

And for a perpetuity

$$\begin{aligned} PV &= \sum_{k=1}^{n} \frac{Y}{(1+r)^k} = \frac{Y}{r}\left(1 - \frac{1}{(1+r)^n}\right) \\ &\to_{n\to\infty} \frac{Y}{r}. \end{aligned} \tag{2.18}$$

Yield

The annual rate of return of a bond over its lifetime is called its *yield to maturity* (YTM). Suppose a bond with a face value of F makes m coupon payments per year in the amount C/m and there are n payments remaining until its maturity. If the bond costs P, what is its YTM?

This can be calculated by discounting to the present time all the future payments of the bond at the yield rate. Denote this annual rate by r. Again invoking (2.15), the discount factor for the kth coupon payment is $(1+(r/m))^{-k}$. The discounted amount for each is C/m; over the lifetime there will be n such payments. It remains to discount the face value F. If there were an exact number of years y remaining it would be appropriate to use $(1+r)^{-y}$ as if compounding were annually. Or, since there are exactly n coupon periods remaining, one could use $(1+(r/m))^{-n}$ and thereby invoke per period compounding. The two are not the same, $(1+r)^{-n/m} \neq (1+(r/m))^{-n}$. The custom is to use the latter. And so we have

$$P = \frac{F}{(1+(r/m))^n} + \sum_{k=1}^{n} \frac{C/m}{(1+(r/m))^k};$$

upon summing the series we obtain

$$P = \frac{F}{(1+(r/m))^n} + \frac{C}{r}\left(1 - \frac{1}{(1+(r/m))^n}\right). \tag{2.19}$$

For given values of P, F, C, m, and n, this must be solved for r. Since r cannot be solved in closed form in (2.19), numerical methods have to be used. For example the bisection method discussed in Section A.14.

Example 2.7. Find the YTM of a 10 year \$10,000 bond with coupon payments of \$400 annually. The bond costs \$9,500.

We must solve

$$9{,}500 = \frac{10{,}000}{(1+r)^{10}} + \frac{400}{r}\left(1 - \frac{1}{(1+r)^{10}}\right).$$

The value $r = 0.01$ is too low and $r = 0.06$ is too high so they can be starting points for the bisection method. The method quickly gives $r = 0.04636$. □

2.3 Risk for a Single Investment

Let us suppose an investment of S_0 dollars is initiated by the purchase of stock in a particular company. To avoid certain complications, assume the company does not pay dividends. Let us also suppose that at some time in the future, $t = T$, we sell the stock (or at least assess whether we have made a profit or a loss to date).

If we assume the dynamics of the last chapter, then at the end of the investment period the value of the asset, S_T, will be lognormally distributed. We can calculate the investment's probability that $S_T < S_0$, the probability of losing

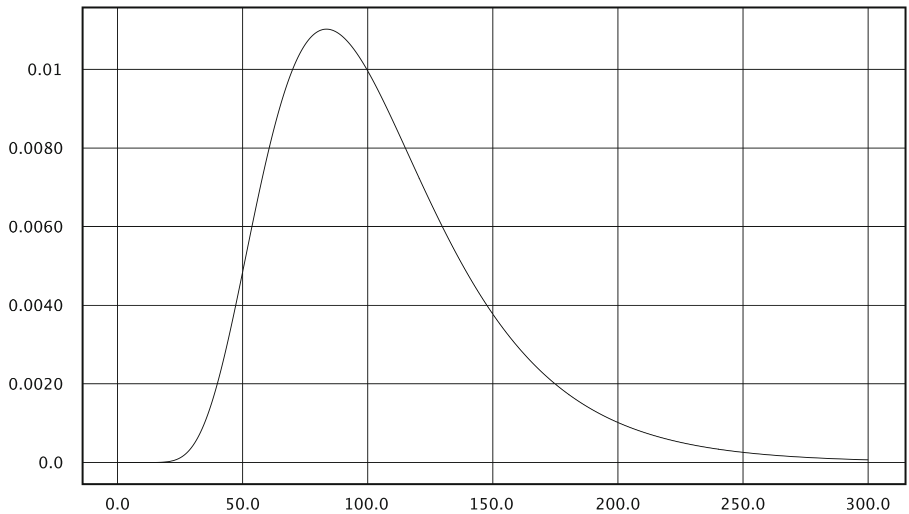


Fig. 2.2. Lognormal ending prices for: $S_0 = 100$, $\mu = 0.06$, $\sigma = 0.4$, $T = 1$ year

money, by integrating over the lognormal distribution from zero up to S_0, see Fig. 2.2.

But an easier method is available. Since S_T is lognormally distributed, $Y = \log(S_T)$ is normally distributed. From (1.23),

$$Z = \frac{1}{\sigma\sqrt{t}}\left(Y - \log(S_0) - (\mu - \frac{1}{2}\sigma^2)T\right)$$

is standard normal. It follows that for any $x > 0$,

$$\begin{aligned}\Pr(S_T < x) &= \Pr(\log(S_T) < \log(x)) \\ &= \Pr\left(\frac{\log(S_T) - \log(S_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) < d\right) \\ &= \Pr(Z < d) \end{aligned} \tag{2.20}$$

where

$$d = \frac{\log(x) - \log(S_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \tag{2.21}$$

The probability can now be looked up in the cumulative normal table or, more conveniently, calculated from the cumulative normal rational approximation given in Section A.5 (or looked up online).

Example 2.8. Using the parameters as in Fig. 2.2: $S_0 = 100$, drift $\mu = 0.06$, volatility $\sigma = 0.4$, and $T = 1$ year and taking $x = S_0$ in (2.21), we get

$$d = \frac{\log(100) - \log(100) - (0.06 - \frac{1}{2}0.4^2)1}{0.4\sqrt{1}} = 0.05.$$

For this argument a cumulative normal table puts the loss probability at 52 %. □

In the example it is perhaps surprising that the probability exceeds 50 % even though the drift is positive. The explanation lies in the asymmetry of the lognormal distribution; recall equation (1.32) for the median of a lognormal. While the lognormal mean is calculated to be 106.18 according to (1.28), the probability mass extending far upscale must be compensated for by greater probabilities on the downside.

An added complication in figuring the probability of loss is accounting for the possibility that the company goes bankrupt and our investment devaluates to zero. Suppose this probability is B. From www.bloomberg.com[6] over the years 2000–2008 an average of 2.1 % of NYSE companies are *delisted* per year, so a value of $B = 0.02$ is a natural guess. The above calculation is now effected by scaling down the curve by $1 - B$, and then adding B to the result,

$$\begin{aligned} \Pr(S_T < S_0) &= \Pr(S_t < S_0 \,|\, \text{bankrupt})\Pr(\text{bankrupt}) \\ &\qquad + \Pr(S_T < S_0 \,|\, \text{not bankrupt})\Pr(\text{not bankrupt}) \\ &= B + (1 - B)\int_0^{S_0} f(s)\, ds. \end{aligned} \tag{2.22}$$

Here f is the lognormal density function with the appropriate parameter values. It is as though a histogram bar of probability B is placed at $S_T = 0$ and the rest of the histogram is scaled by $(1 - B)$.

Example 2.9. To include the probability of bankruptcy in our previous example make use of (2.22); we get

$$\Pr(\text{loss}) = 0.02 + 0.98 * 0.52 = 0.523. \tag{2.23}$$

□

2.3.1 The Solution by Simulation

We may also calculate the result through simulation. The algorithm is quite simple: run the geometric random walk algorithm on page 12 and count a hit if the end point $S_T < S_0$. Repeat this for a large number of times N (the results below are for N = 90,000) and use the number of hits divided by N as the estimate.

Example 2.10. Carrying out such a simulation with the parameters in Fig. 2.2 we get the risk of loss to be 52.2 % agreeing with the analytical calculation of Example 2.8. □

With such an algorithm in hand, we can pose and answer many relevant questions about our investment. How does the risk vary with volatility? with drift? with the time horizon? These involve making simple parameter changes in the algorithm and re-running the simulation.

[6] Search "NYSE Companies Delisted for Noncompliance" for this lengthy reference. See also www.moneycontrol.com/stocks/marketinfo/delisting/index.php

2.3.2 The Effect of Dividends

An important advantage of simulation is that more complicated situations can be easily accommodated. Such a complication is gaging the effect that issuing dividends has on the ending price distribution. The dividend payments can be fixed amounts at fixed times or amounts tied to the current stock price, or virtually any scheme.

When dividend payments are made, usually on a per share basis, the price of the stock immediately drops by the same amount. This is the result of the reduction in value of the company.

The *book value* of a company is the net worth of its tangible assets, for example the property it owns and its cash. (In particular, the talents and ideas of its employees are excluded.) Book per share is, theoretically, what a shareholder would get if the company liquidated and its proceeds were distributed to the shareholders.

While a stock's price can trade at several times its book per share value, often there is a close relationship between the two. When a company issues a dividend, its book value drops by the total amount dispersed. And so the book per share drops by that amount divided by the number of shares outstanding. This is exactly equal to a drop in price by the dividend per share.

There is a second reason why the share price must drop by the amount of the dividend. Suppose an investor buys the stock just prior to *ex-dividend day*[7] and thereby joins the rolls of those receiving a dividend. Later, maybe even the next day, the investor now sells the stock. Assuming the price does not fall by the dividend amount, this sale price will be about the same as the purchase price (possibly higher) thereby earning the investor the dividend as free money. Such a practice cannot last long. Many others will want to get in on it. The result will be to cause the stock price prior to going ex-dividend to escalate and then fall back afterwards.

To make a simulation of the problem treated above, but now with dividends, assume dividend payments are made quarterly (so the number of days between dividends is 91) with a yield of 8 % per year or 2 % per quarterly period. Assume dividends are not reinvested (otherwise there is no dividend essentially).

The modification to the algorithm consists in calculating the dividend amount on the last day of each quarter, adding this to the accumulated dividend payout, reducing the stock by the same amount and continuing. The accumulated dividends are assumed to be reinvested at the risk-free rate. At the end of the time frame, record a "hit" if the stock price plus accumulated dividends is less than S_0.

Algorithm 6. Ending Value with Dividends

```
inputs: S_0, nDays (T in days), μ, σ, daysBtwnDiv
        accumDiv, periodYield, periodRFR
dt=1/365  ▷1-day walk resolution
```

[7] The ex-dividend day and afterward is when a stock purchase does not qualify for the current dividend. Buying prior to ex-dividend day is required.

```
S = S_0  ▷initialize stock price
accumDiv = 0  ▷initialize accumulated dividends
j = 0   ▷initialize days since last dividend
for i = 1,..., nDays
  Z ~ N(0,1)    ▷N(0,1) sample
  S = S ∗ (1 + μΔt + σ√(Δt)Z)
  j = j + 1
  if( j == daysBtwnDiv )
      ▷grow accumulated dividends
    accumDiv = accumDiv∗(1+periodRFR)
    divAmt = S∗periodYield   ▷dividend this period
    accumDiv = accumDiv + divAmt
    S = S − divAmt  ▷decrease by amount of the dividend
    j = 0  ▷reset j
  endif
endfor
  ▷dividend growth over partial period
accumDiv = accumDiv∗(1+periodRFR∗j/daysBtwnDiv)
endValue = S + accumDiv
```

Example 2.11. Such a simulation applied to the problem of Example 2.8 with quarterly dividends at 8 % yield gives the result that the risk of losing money is the slightly reduced value 51.8 %. □

2.3.3 Stocks Follow the Market

In Chapter 1 we mentioned that a stock's price is subject to general market influences as well as the random walk fine structure. Let us now take that into account. Our goal is to calculate the risk for a particular stock under different market scenarios given the degree to which the stock follows the market. For this purpose we must learn how to generate market prices, possibly engineered to have specified characteristics, and how to generate individual stock prices in relation to the market.

Generating Market Prices

Start with the former. Of course we could generate market prices in the usual way using the algorithm on page 12. We could obtain markets with specified volatilities and drifts as desired in this way. But suppose we want more; for example we might want to specify trends, an improving market or a declining one or an oscillating one. In fact this is possible as we demonstrate by example.

Assume we want to simulate daily prices over 1 year but we want the market to have a given price each quarter, $(0, m_0)$, $(91, m_1)$, $(182, m_2)$, $(173, m_3)$ and $(364, m_4)$. We will see that in generating the stock prices, only the day to day market increments matter and therefore we take m_0 to be some convenient value such as $m_0 = 100$.

Now generate a GRW, $P[i]$, $i = 0, \ldots, 364$ with $P[0] = m_0$ and having the desired volatility, σ_M (and 0 drift for convenience).

Next let $\ell(\cdot)$ be the piecewise linear curve passing through the points $(0, m_0 - P[0])$, $(91, m_1 - P[91])$, $(182, m_2 - P[182])$, $(273, m_3 - P[273])$, and $(364, m_4 - P[364])$. The array of t-coordinates is

$$\text{tPts:} \quad 0, 91, 182, 273, 364$$

and the array of y-coordinates is

$$\text{yPts:} \quad m_0 - P[0], m_1 - P[91], m_2 - P[182], m_3 - P[273], m_4 - P[364].$$

Generically, the straight line through two points (a, A) and (b, B) is given by

$$y = \frac{1}{b-a}\Big(B(t-a) - A(t-b)\Big). \tag{2.24}$$

Letting $\mathbb{1}_{[a,b]}(t)$ denote the indicator function of the interval $[a, b]$ which is 1 for $a \le t \le b$ and 0 otherwise, put

$$\ell(t; a, b) = \frac{1}{b-a}\Big(B(t-a) - A(t-b)\Big)\mathbb{1}_{[a,b]}(t). \tag{2.25}$$

This is the line segment we want for t between a and b. Then $\ell(\cdot)$ is their sum

$$\ell(t) = \ell(t; 0, 91) + \ell(t; 91, 182) + \ell(t; 182, 273) + \ell(t; 273, 364). \tag{2.26}$$

Finally the array of market prices having the desired properties is just the sum

$$M[t] = P[t] + \ell(t), \qquad t = 0, 1, \ldots, 364.$$

Figure 2.3 portrays an example market scenario.

The function $\ell(\cdot)$ above can be implemented via the following code.

Algorithm 7. Generating piecewise linear ordinates

```
inputs: tPts, yPts
        t  ▷tPts[0] ≤ t ≤ tPts[4]
output: y = ℓ(t)
a = tPts[0]; A= yPts[0]
for i = 1,...,4
  b=tPts[i]; B=yPts[i]
  if( a ≤ t and t ≤ b )
    y=(B(t-a)-A(t-b))/(b-a)
    return y
  endif
  a=b; A=B
endfor
```

In the next section we will want to generate stock prices that correlate with the market. For this purpose we need the equivalent random increments W_i,

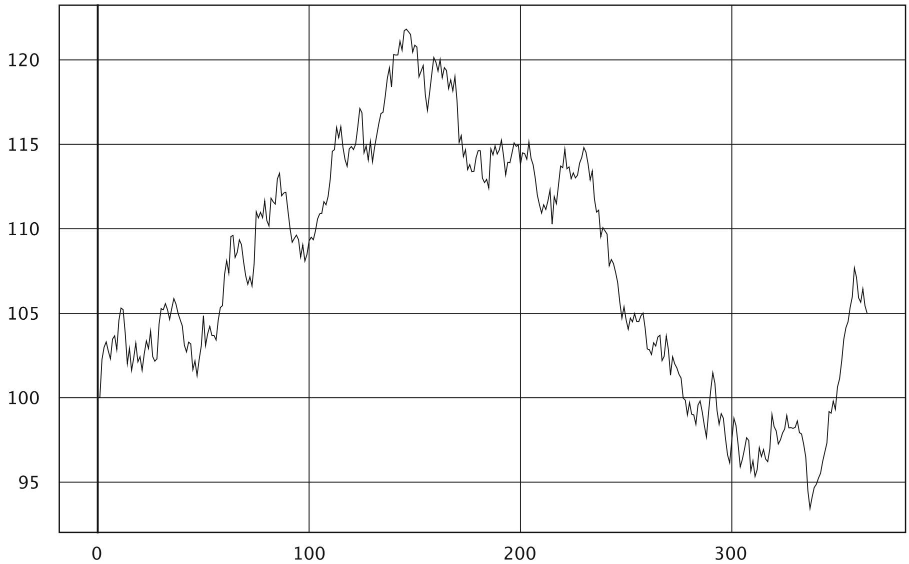


Fig. 2.3. Market prices passing through (0,100),(91,110), (182,113), (273,102) and (364,105). The volatility is 10 %

$i = 1, 2, \ldots,$ from which the market could be regenerated.[8] Since $M_i = M_{i-1} + M_{i-1}\sigma_M\sqrt{dt}W_i$,

$$W_i = \frac{M_i - M_{i-1}}{M_{i-1}\sigma_M\sqrt{dt}}. \tag{2.27}$$

2.3.4 Correlated Stock Prices

Given market behavior, now we want to generate the prices of individual stocks that are influenced by the market. We consider stocks whose prices follow the market to an extent but not completely and not all the time. Specifically, let ρ, a number between -1 and 1, quantify the degree to which the stock's movement tracks the market's movement. If $\rho = 1$ then it tracks exactly in the sense of rising when the market rises and falls when the market falls. If $\rho = 1/2$ then it follows the general market about one half the time otherwise it moves independently of the market. If $\rho = 0$ then the stock moves independently all the time. A stock might even move contrary to the market, in this case ρ is negative.

Such a parameter is exemplified by the correlation coefficient defined next. First the *covariance* between two random variables X and Y is defined to be

$$\mathrm{covar}(X, Y) = \mathbb{E}((X - \mu_X)(Y - \mu_Y)), \tag{2.28}$$

where $\mu_X = \mathbb{E}(X)$ and $\mu_Y = \mathbb{E}(Y)$. The *correlation coefficient* is its normalization,

$$\rho_{XY} = \frac{\mathrm{covar}(X, Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}} = \frac{\mathrm{covar}(X, Y)}{\sigma_X\sigma_Y}. \tag{2.29}$$

[8] The calculated market path is a possible realization of a geometric random walk.

Notice from (2.28) that if X and Y both tend to be greater than their means at the same time and likewise lesser than their means at the same time, then their covariance will be a large positive value. It follows that ρ_{XY} will be near 1.[9] Conversely if Y tends to be below its mean when X is above and vice-versa, then their covariance will be a large negative value, and ρ_{XY} will be near -1. If X and Y are independent, then their correlation is 0, $\rho_{XY} = 0$.

The *covariance matrix* C for two random variables X and Y is defined as

$$\begin{aligned} C &= \begin{bmatrix} \mathrm{var}(X) & \mathrm{covar}(X,Y) \\ \mathrm{covar}(Y,X) & \mathrm{var}(Y) \end{bmatrix} \\ &= \begin{bmatrix} \sigma_X^2 & \rho_{XY}\sigma_X\sigma_Y \\ \rho_{YX}\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}. \end{aligned} \tag{2.30}$$

We have used (2.29) to obtain the off-diagonal elements. Since $\rho_{YX} = \rho_{XY}$, the covariance matrix is symmetric meaning $C^T = C$ where superscript T designates matrix transpose.

Let $\mathbf{V}$ denote the 2×1 matrix, that is column vector, consisting of X and Y,

$$\mathbf{V} = \begin{bmatrix} X \\ Y \end{bmatrix}.$$

If $\mu_{\mathbf{V}}$ is the column vector of their means μ_X and μ_Y, and letting the expectation of a matrix mean the expectation of each of its elements, then the covariance matrix is given by

$$\begin{aligned} C &= \begin{bmatrix} \mathbb{E}((X-\mu_X)(X-\mu_X)) & \mathbb{E}((X-\mu_X)(Y-\mu_Y)) \\ \mathbb{E}((Y-\mu_Y)(X-\mu_X)) & \mathbb{E}((Y-\mu_Y)(Y-\mu_Y)) \end{bmatrix} \\ &= \mathbb{E}((\mathbf{V}-\mu_{\mathbf{V}})(\mathbf{V}-\mu_{\mathbf{V}})^T). \end{aligned} \tag{2.31}$$

Since a covariance matrix is symmetric[10] it has a *Cholesky decomposition*,

$$C = HH^T \tag{2.32}$$

where H is lower triangular. For example a 2×2 covariance matrix has the decomposition

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix}.$$

[9] By its definition, $-1 \le \rho_{XY} \le 1$. This follows from the well-known Cauchy-Schwarz inequality as indicated by the following. If x_i and y_i for $i = 1, \ldots, n$ are empirical values of X and Y, then statistically

$$\begin{aligned} \mathrm{covar}(X,Y) &= \frac{1}{n}\sum_i (x_i-\mu_X)(y_i-\mu_Y) \\ &\le \sqrt{\frac{1}{n}\sum_i (x_i-\mu_X)^2}\sqrt{\frac{1}{n}\sum_i (y_i-\mu_Y)^2} = \sqrt{\mathrm{var}_X}\sqrt{\mathrm{var}_Y}. \end{aligned}$$

[10] It is also positive semi-definite but that is not needed for a Cholesky decomposition.

Now let Z and Z' be two independent, mean 0, unit variance random variables; their covariance matrix is therefore the identity matrix I. Put

$$\begin{bmatrix} X \\ Y \end{bmatrix} = H \begin{bmatrix} Z \\ Z' \end{bmatrix}, \tag{2.33}$$

then X and Y have covariance matrix C and therefore are correlated with coefficient ρ. The reason is that X and Y are mean 0 and, from (2.31),

$$\begin{aligned} \mathbb{E}(\begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}^T) &= \mathbb{E}(H \begin{bmatrix} Z \\ Z' \end{bmatrix} (H \begin{bmatrix} Z \\ Z' \end{bmatrix})^T) \\ &= \mathbb{E}(H \begin{bmatrix} Z \\ Z' \end{bmatrix} \begin{bmatrix} Z \\ Z' \end{bmatrix}^T H^T) = H\mathbb{E}(\begin{bmatrix} Z \\ Z' \end{bmatrix} \begin{bmatrix} Z \\ Z' \end{bmatrix}^T) H^T \\ &= HIH^T = C. \end{aligned} \tag{2.34}$$

Since H is constant it can be moved outside the expectation operation.

Finally (2.33) shows how to construct correlated Gaussian random variables. Let Z and Z' be independent $N(0,1)$ random variables and put

$$\begin{aligned} X &= \sigma_1 Z \\ Y &= \rho\sigma_2 Z + \sqrt{1-\rho^2}\sigma_2 Z', \end{aligned} \tag{2.35}$$

then X and Y are correlated and normally distributed with variances σ_1 and σ_2 respectively and correlation coefficient ρ. Note that Z and Y are also correlated with coefficient ρ.

With correlated $N(0,1)$ samples in hand, to obtain correlated random walks, we simply generate them in the usual way using these samples. Let X_i and X_i', $i = 1, 2, \ldots$, be correlated $N(0,1)$ samples and set

$$\begin{aligned} S_i &= S_{i-1}(1 + \mu dt + \sigma\sqrt{dt}X_i) \\ S_i' &= S_{i-1}'(1 + \mu' dt + \sigma'\sqrt{dt}X_i'). \end{aligned}$$

Note that it is not the prices themselves that are correlated but rather the price increments. Correlating the increments is preferable because, for one thing, it is the day-to-day increments that are market correlated and, for a another, price increments are stationary in the sense of having constant means and variances, see [Mar78].

To obtain equity prices that follow the market we put the two constructions together. First construct a market scenario, M_i, engineered as desired using the techniques earlier in this section. Back out the equivalent price increments W_i, $i = 1, 2, \ldots$, given by (2.27). Finally, using a sequence of independent $N(0,1)$ samples Z_i, $i = 1, 2, \ldots$, put

$$Y_i = \rho W_i + \sqrt{1-\rho^2} Z_i$$

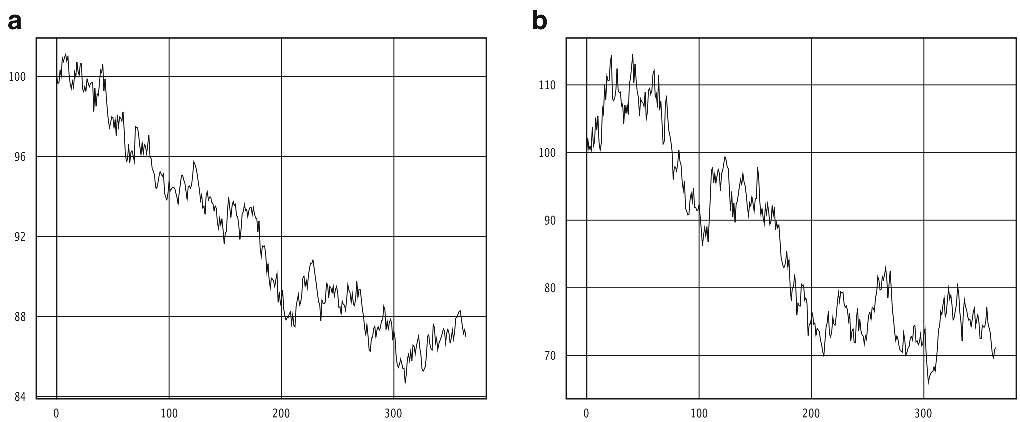
and then generate the prices as usual using the Y_i,

$$S_i = S_{i-1}(1 + \mu dt + \sigma\sqrt{dt}Y_i).$$

The technique is outlined in the following algorithm. Figure 2.4 shows an example run of the algorithm.

Algorithm 8. Stock prices correlated with the market

```
inputs: ρ, σ_m (market volatility)
        μ_s, σ_s (equity parameters)
• generate a market scenario:
m_0, m_1, ...   ▷e.g. quarterly prices manually assigned
  ▷generate preliminary market prices
P_i = P_{i-1}(1 + σ_m √dt Z_i)   ▷Z_i ~ N(0,1)
• calculate the piecewise linear correction ℓ(t) (pp 48)
• calculate the market prices M_i = P_i + ℓ(i)
  ▷back out the market increments
W_i = (M_i − M_{i-1})/(σ_m √dt M_{i-1})
label AA:
  ▷generate correlated N(0,1) increments
Y_i = ρW_i + √(1 − ρ²) Z_i   ▷Z_i ~ N(0,1)
  ▷generate the correlated stock prices using the Y_i
S_i = S_{i-1}(1 + μ_s dt + σ_s √dt Y_i)
```



A falling market trend

Correlated price changes, ρ =0.7

Fig. 2.4. Example prices of a stock whose price movements are correlated with a market trend

It may become necessary to generate a large number of stock price histories all correlated with the same market; this is possible. In the algorithm, simply repeat starting from label AA, as many times as desired, to generate a new history.

Extension to More Variables

The technique for correlated samples given here extends to any number of random variables. For example to generate three pairwise correlated Gaussian random variables, let C be the desired covariance matrix and let HH^T be its Cholesky decomposition. If

$$C = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{13}\sigma_1\sigma_3 & \rho_{23}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix}$$

Then

$$H = \begin{bmatrix} \sigma_1 & 0 & 0 \\ \rho_{12}\sigma_2 & \sqrt{1-\rho_{12}^2}\,\sigma_2 & 0 \\ \rho_{13}\sigma_3 & \sigma_3 \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} & \sqrt{\sigma_3^2 - h_{31}^2 - h_{32}^2} \end{bmatrix} \tag{2.36}$$

where h_{31} and h_{32} are the 31 and 32 elements of H,

$$h_{31} = \rho_{13}\sigma_3 \qquad h_{32} = \sigma_3 \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}.$$

If the h_{33} square root should result in an imaginary number, it means C is not positive semi-definite and therefore not a valid covariance matrix.[11]

Put

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = H \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \tag{2.37}$$

where Z_1, Z_2, and Z_3 are independent $N(0,1)$ Gaussians. Then X_1 and X_2 have correlation coefficient ρ_{12}, X_1 and X_3 have correlation coefficient ρ_{13} and X_2 and X_3 have correlation coefficient ρ_{23}.

2.4 Risk for Two Investments

Most portfolios consist of more than one investment. The interplay between several investments has a profound effect on risk. If the components of the portfolio are oppositely correlated, then the portfolio's prices tend to be more constant, avoiding large swings either way.

To see this, suppose an investment consists of equal positions in two stocks trading at about the same price. Suppose one is positively correlated and the other is negatively correlated with the market, see Fig. 2.5a and b. The day-to-day value of a 50–50 mix of these two stocks is shown in (c). Since the portfolio is the day-to-day average of the two, its value must necessarily lie halfway between them.

[11] An arbitrarily constructed real symmetric matrix is not necessarily positive-semi-definite. A little algebra shows that

$$h_{33}^2 = \frac{\sigma_3^2}{(1-\rho_{12}^2)}\left(1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + \rho_{12}\rho_{13}\rho_{23}\right).$$

This can be negative for the choices $\rho_{12} = \rho_{13} = -\rho_{23} = \frac{3}{4}$. Of course if 1 and 2 are highly correlated then 1 and 3 can't be highly correlated while 2 and 3 highly uncorrelated.

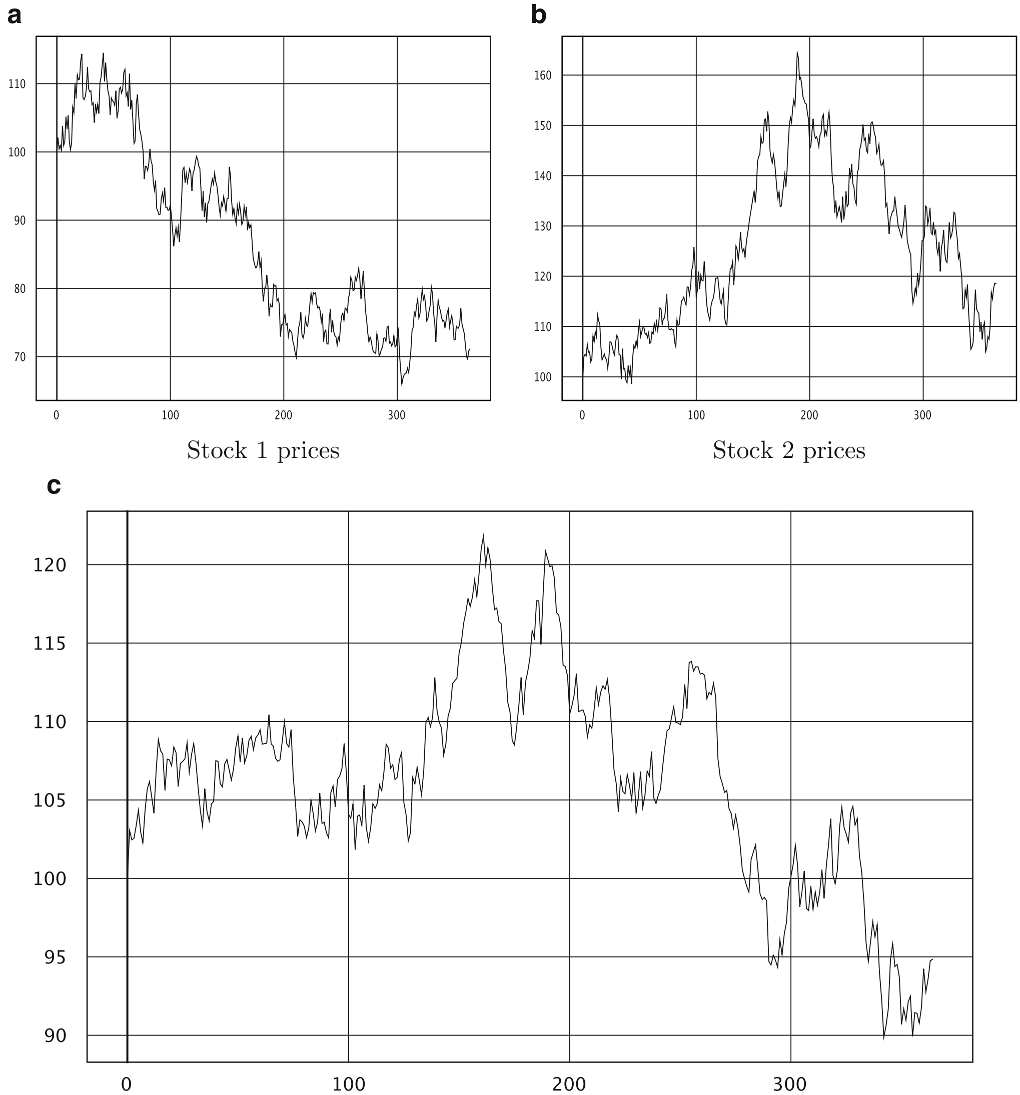


Fig. 2.5. The market for this example is that of Fig. 2.4a. The equity in (**a**) is positively correlated with the market while (**b**) is negatively correlated with the market. The individual stocks are subject to large swings but the portfolio remains between 90 and 125

The figure shows what can happen in one particular scenario of market and stock prices. To gauge the effect on the risk of such a portfolio, we must run a simulation as before over many such scenarios and count how often the result is a loss. Effectively we are integrating over the distribution of portfolio prices under the constraint of the assigned correlations.

Algorithmically the new simulation goes as follows.

Algorithm 9. Correlated Portfolio Risk

`for` $i = 1, 2, \ldots, N$
- `generate a random market scenario`

```
    • generate stock 1 prices with ρ = ρ1 (pp. 52)
    • generate stock 2 prices with ρ = ρ2
    • average to generate the portfolio prices
    • record a 'hit' if the S_T < S_0
endfor
• output (number of hits)/N
```

In this algorithm the market trend is random, therefore any difference between the risk predicted by simulations of this algorithm and that of the one-stock portfolio is due to the attributes of the portfolio.

Example 2.12. Consider the problem of Example 2.8 on page 44. Let a portfolio consist of two stocks having exactly the same financial parameters as in that example, $S_0 = 100$, $\mu = 0.06$, $\sigma = 0.4$, and $T = 1$. But let the first have correlation $\rho_1 = 0.7$ and the second have correlation $\rho_2 = -0.5$ with respect to the market. Let the market itself have drift 8 % and volatility 20 %. Then the risk of loss predicted by simulation is 41.9 %, an improvement of about 10 % from that of a stock by itself. Even if the two stocks are uncorrelated with the market or to each other there is still an improvement of about 5 %. This is due to the fact that the average of two or more values is less extreme than any of the values individually. □

Referring to Algorithm 9, shifting the market scenario generation outside the loop allows for testing portfolios under specific types of markets.

2.5 Value at Risk

The value at risk (VaR) is a measure that attempts to capture in a single number the total risk of a portfolio. The value at risk V is the maximum loss that can be expected with a given confidence over a specified period of time. For example one might assert "We are 99 % sure that over the next 30 days the portfolio will not lose more than \$10,000."

Of course the prediction is made on the basis of a model for stock prices, for example the GBM model. The prediction can also be made drawing on the pattern of historical prices for the portfolio. In this case, the model is that the economic conditions of the past and the underlying basis for stock price movement are projected to hold in the future.

For a portfolio consisting of a single stock, the GBM model predicts the future price will be lognormally distributed as in Fig. 2.2 which shows the ending price distribution after 1 year. As worked out in Section 2.3, the probability δ that the ending price will be less than $S_0 - V$ is given by the integral

$$\delta = \int_0^{S_0 - V} f(s)\, ds \tag{2.38}$$

where f is the lognormal density function with the appropriate parameter values. According to the model then, with probability $1 - \delta$ the stock price will not be less than $S_0 - V$.

Example 2.13. For a VaR confidence level of 99 %, take $\delta = 0.01$ and solve for $x = S_0 - V$. Using the parameters as shown in Fig. 2.2, from (2.20) we have

$$\begin{aligned} 0.01 = \Pr(S_T < x) &= \Pr(\log(S_T) < \log(x)) \\ &= \Pr\left(\frac{\log(S_T) - \log(S_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} < d_T\right) \\ &= \Pr(Z < d_T) \end{aligned} \tag{2.39}$$

where

$$d_T = \frac{\log(x) - \log(S_0) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \tag{2.40}$$

Hence we want to find $d_T = \Phi^{-1}(0.01)$. From the cumulative normal table, or the rational interpolation of it, (A.9), find that $d_T = -2.3263$ and hence, from (2.40),

$$\log(x/S0) = -2.3263 * 0.4 + 0.06 - \frac{1}{2}(.4^2) = -0.95052;$$

consequently $x = 38.65$. This gives $V = 100 - 38.65 = 61.35$. Therefore with probability 99 %, the single stock portfolio is predicted to lose at most \$61.35 over the course of 1 year. (Keep in mind this is worst case (at the 99 % level); the portfolio might in fact gain in value over the year.) □

In more complex situations one can compute the VaR by Monte Carlo. We can see how it works by applying the technique to the single stock portfolio above. One simulates the price history a large number of times and notes the final price. A histogram of these produces the approximate price density but this is not what we want here.

Instead we want the cumulative price distribution. Recall that the cumulative distribution for an argument x is the integral of the density up to x. Statistically this means the sum of the number of prices that come in less than x (divided by the size of the sample). By sorting the ending prices low to high and plotting the sum of the number of sorted prices against price, the cumulative distribution is approximated, see Fig. 2.6.[12]

Having the sorted prices makes it easy to solve $\mathrm{cdf}(x) = \delta$ for x given any δ. For example, for $\delta = 0.2 = 1/5$, the solution x is the sorted price one-fifth the way up the list, see Fig. 2.6. Let $S_T[i]$, $i = 1, \ldots, n$, be the sorted ending prices. Then the kth sorted price, $S_T[k]$, for

$$k = (\text{integer part})\delta n$$

gives the simulated solution $x = S_T[k]$, and in turn $V = S_0 - x$.

Example 2.14. For the problem in Example 2.13, a simulation gives $x = 38.87$ closely agreeing with the analytical value obtained there. □

The advantage of the Monte Carlo method is that it is easily implemented against any portfolio so long as there is a joint price model accounting for each constituent.

[12] A sorting subroutine is given in Appendix E.

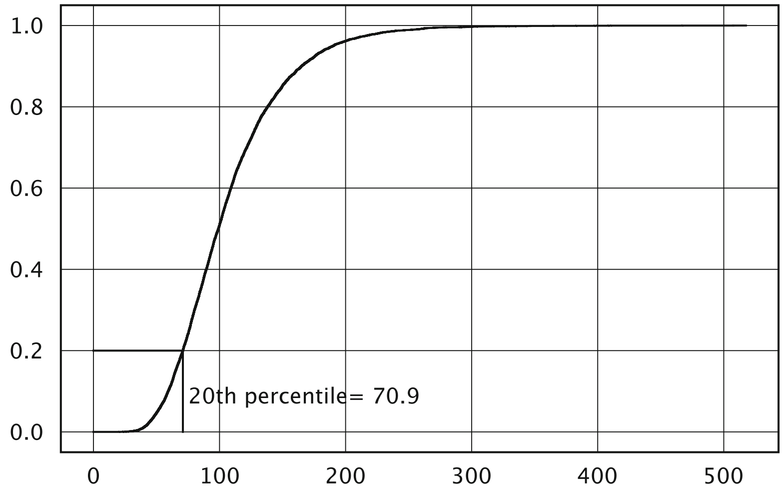


Fig. 2.6. Cumulative price distribution, $S_0 = 100$, drift$= 0.06$, $\sigma = 0.4$, $T = 1$ year. The calculation for the 20th percentile point is shown

Example 2.15. To apply the method to the two-stock portfolio in Example 2.12 of page 55 it is just a matter of sorting the price outcomes instead of recording hits. The sorted price for $\delta = 0.01$ is $x = 60.42$; hence the value at risk at the 99 % level is $V =$\$39.58; significantly less than the \$61.35 obtained above. This is due to the negative correlation among the constituents of the portfolio. □

2.5.1 Historical Simulation Method

In the technique of *historical simulation* one needs historical data for every market variable that affects the portfolio. For our simple portfolio of two equities, this means their price histories. For more complicated portfolios it could include interest rates, real estate values, exchange rates and so on.

The period of time in days, n, over which the VaR applies is called the *time horizon*. Ideally, we would like to have historical data covering several n-day periods. Usually there is not enough data for this purpose. Furthermore, the more time between the data observations and the present means the less likely it is that the economic conditions are the same. As a result, the n-day time horizon is estimated using 1-day data and the assumption that

$$n\text{-day VaR} = 1\text{-day Var } \times\sqrt{n}. \tag{2.41}$$

This assumption is correct for an arithmetical random walk (a Wiener process, see page 7) implying the data over disjoint time periods of equal length are independent and have identical normal distributions.[13]

[13] An extension formula for a portfolio consisting of a single GBM constituent can be derived from (2.40). But the day-to-day value of a portfolio consisting of several GBM constituents does not itself follow a GBM. Thus an accurate n-day VaR for such a portfolio requires a direct n-day simulation as above.

To see why, let the random variable Y_n be the portfolio value after n days starting from an initial value of Y_0. The assumption is that $Z_n = Y_n - Y_0$ is distributed as $N(0, \sigma^2 n)$. Further the 1-day VaR, V_1, is defined as $\Pr(Y_1 - Y_0 < -V_1) = \delta$ and the n-day VaR, V_n, is $\Pr(Y_n - Y_0 < -V_n) = \delta$. Since $Z_n = \sqrt{n} Z_1$,

$$\begin{aligned}\delta &= \Pr(Z_n < -V_n) = \Pr(\sqrt{n} Z_1 < -V_n) = \Pr(Z_1 < -\frac{V_n}{\sqrt{n}}) \\ &= \Pr(Z_1 < -V_1),\end{aligned}$$

provided $V_n = \sqrt{n} V_1$.

To illustrate the historical simulation method for the 360-day VaR, assume that Table 2.3 is the record of the stock prices for the two-equity portfolio over the last 360 days.[14] The historical record should include about the same number of days as the VaR to be predicted in order that the normally distributed data have approximately the same variation as expected over the VaR period.

Day 0 in the first column of the table is 360 days ago, day 360 of the table gives today's values. The relative change in the value of each market constituent is computed between each successive day; these values are calculated in columns 4 and 7 of the table. This provides 360 "experimental" observations for the change in value of each constituent.

Then, one-by-one, each observed percentage change is applied to today's constituent value to produce a possible future value for that constituent. This is shown in columns three and six of Table 2.4. Each such predicted constituent value is a possible future scenario and gives rise to a corresponding value of the portfolio, this is shown in column eight.

Table 2.3. Historical prices for the two-stock portfolio and their relative day-to-day changes

Day	c1	$\Delta c1$	$\Delta c1/c1$	c2	$\Delta c2$	$\Delta c2/c2$
364	116.80	2.225	0.01905	80.25	−1.827	−0.02276
363	114.57	−0.828	−0.00723	82.08	3.051	0.03717
362	115.40	−2.187	−0.01895	79.03	−0.259	−0.00328
361	117.59	0.944	0.00803	79.29	−4.031	−0.05084
360	116.64	−0.459	−0.00393	83.32	1.965	0.02358
⋮	⋮	⋮	⋮	⋮	⋮	⋮
4	103.96	1.948	0.01873	100.18	−0.340	−0.00339
3	102.02	−0.150	−0.00147	100.52	1.621	0.01612
2	102.17	−0.481	−0.00471	98.90	−0.412	−0.00416
1	102.65	2.648	0.02579	99.31	−0.687	−0.00692
0	100.00			100.00		

[14] We use 360 days here for illustrative and comparison purposes. In actual practice 252 "trading day" years is more likely to be used by company management. Further, the international Basel regulations specify the following VaR parameters: 10 day horizon, 99 % confidence level, and at least 1 year of historical data.

Then, as above, the 360 possible portfolio values are sorted low to high and the δ-th percentile noted. The 1-day VaR is calculated from this value and the 360-day VaR is calculated from (2.41).

Example 2.16. These particular tables were generated based upon the problem described in Example 2.15 above. From the tables we calculate that the 1-day VaR is \$3.05 and the 360-day VaR is \$58.16 by the historical method. The actual 1-day VaR from the simulation is \$2.65 which extends to a 360-day VaR of \$50.61.

Contrast these numbers with the direct 360 day simulation value of \$39.58 calculated above. □

2.6 Mean-Variance Portfolio Theory

The breakthrough that enabled mean-variance theory was the mathematical definition of risk and the attempt to deal with it through portfolio diversification. The treatment of risk is sufficiently precise that a rich theory may be developed, a theory that has proved to be useful in practice and remains the workhorse of analytical portfolio management.

Table 2.4. Three hundred and sixty 1-day price change scenarios; the third smallest portfolio value is the 1-percentile point

Scenario	Base c1	% change	c1	Base c2	% change	c2	Portfolio
1	116.80	1.905	119.02	80.25	−2.276	78.43	98.72
2	116.80	−0.723	115.95	80.25	3.717	83.24	99.59
3	116.80	−1.895	114.58	80.25	−0.328	79.99	97.29
4	116.80	0.803	117.73	80.25	−5.084	76.17	96.95
5	116.80	−0.393	116.34	80.25	2.358	82.15	99.24
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
361	116.80	1.873	118.98	80.25	−0.339	79.98	99.48
362	116.80	−0.147	116.62	80.25	1.612	81.55	99.09
363	116.80	−0.471	116.25	80.25	−0.416	79.92	98.08
364	116.80	2.579	119.81	80.25	−0.692	79.70	99.75

The assumptions of the mean-variance analysis are:

- A single period model
- At a given risk, investors prefer higher returns
- At a given return, investors prefer lower risk
- Markets are frictionless, meaning
  - Assets trade at any price and quantity
  - There are no transaction costs
  - There are no taxes

The single period model assumption means the investment is not dynamic, it does not adjust over time. All parameters of the model are fixed in advance (via estimates) and are applied as constants over the investment period. (The parameters being means, variances, and co-variances.) For example, dividends that occur over the investment period are only taken into account if incorporated into the parameters in advance. It also means that the investors preferences remain fixed over the investment period.

As mentioned, risk is central to the mean-variance analysis. The risk of the previous section is that of the actual loss of money. In his Ph.D. thesis of 1952, H. Markowitz introduced a definition of risk applicable to the *potential* for losing money. It also has the virtue of being mathematically quantifiable. For Markowitz, the risk of an investment is the variability of its returns; precisely, the standard deviation of its sequence of returns through time. (Often variance is used interchangeably with standard deviation in this context.)

A rationale for this definition stems from the fact that the greater the variance of an investment's return, the greater the uncertainty about future returns.

Using the techniques of the previous section, we can show that greater price variance aggravates the risk of actual loss as well. With parameters as in Fig. 2.2, we calculated the probability of loss to be 52 %, see page 45. If we now increase the volatility to 60 %, the probability of loss grows to 57.8 %.

2.6.1 A Two Scenario Example

The following simple example shows how effective reducing variability in a portfolio works to improve returns.

Consider a hypothetical situation in which the future value of an investment has two possible outcomes depending on which of two scenarios occur. The first scenario, ω_1, has probability 1/4 of occurring, and in this case the return on the investment will be 20 %. In the second scenario, ω_2, with probability 3/4, the return will be 5 %.[15] Under these conditions what is the risk of the investment and is it a good one?

To answer the first question we calculate the *mathematically expected* return. This is the sum of the possible outcomes each weighted by its probability. We get

$$\mu_A = \mathbb{E}(\text{return}) = \frac{1}{4}20 + \frac{3}{4}5 = 8.75\,\%.$$

The answer to the second question could depend on the other investment opportunities available. Suppose the money could be deposited in a bank account instead for a return of 8 %. The bank account is assumed safe and therefore has an expectation of 8 % as well.

From the standpoint of expected payoff the risky investment is better.

But what is the risk? As remarked above, we use the standard deviation (or variance) of the return to quantify it. For the bank account the variance of the return is 0. In the other it is

[15] Throughout this section we will measure returns in percent.

$$\text{var} = \frac{1}{4}(20 - 8.75)^2 + \frac{3}{4}(5 - 8.75)^2 = 42.1875$$

and the standard deviation is 6.5 approximately. (The standard deviation puts the value on the same numerical footing as the returns themselves.)

Risk Aversion

We have encountered a key element of investment science, *risk aversion*. Is it better to accept 8 % with certainty or take a chance on earning 20 % but with the prospect of having to settle for 5 % instead, even given that the expectation is favorable? The choice is personal and depends on one's level of willingness to gamble or not. The question of risk aversion will occur often in the sequel. In particular, if two investment returns have the same expectation, an investor is said to be *risk-neutral* upon being completely indifferent about the choice.

Aside from the question of risk aversion, a point to be made here is that the standard deviation of a return has merit as a measure of risk.

Example 2.17. It may seem that one should always choose the investment that has the biggest expected payoff. And this is a good choice if that opportunity presents itself over and over a large number of times. We will take up this topic in some detail in Chapter 7. But what if it presents itself just once?

Offered a one-time chance to win an amount of money equal in value to one's house or to lose the house altogether equally likely is not a bet most people would take.

As previously mentioned, risk-neutral means making decisions based on the best expected outcome and, if both have the same expectation, choose equally likely. For the home owner behaving in a risk neutral manner, either choice is just as good. □

Now consider another situation. Let the original investment choice, 20 % return with probability 1/4 and 5 % return with probability 3/4, be designated investment A. Suppose there is a second choice, investment B, with particulars: 2 % return under scenario 1 and 10 % return under scenario 2. These are spelled out in the following table.

Table 2.5 Two scenario risks and returns

	ω_1 1/4	ω_2 3/4	μ	Var	σ
Bank	8	8	8	0	0
A	20	5	8.75	42.18	6.50
B	2	10	8	12	3.46
I_{50-50}	11	7.5	8.375	2.3	1.52

Investment B has the same expectation, $\mu_B = 8$, as the bank account but its risk is larger and hence is inferior (as judged by the risk averse investor who measures risk by variance). Also, if an investor chooses A over the bank account, then B is likewise unattractive since B is already inferior to the bank account.

But what about a 50–50 mix of A and B? Under scenario 1 the return on such an investment is 11 % and in scenario 2 it is 7.5 %. The expectation under 50–50 can be computed as the scenario weighting of these 50–50 averaged per scenario returns,

$$\mu_{50-50} = \frac{1}{4}11 + \frac{3}{4}7.5 = 8.375.$$

For future reference we note here that it can also be calculated as the 50–50 weighted per individual returns

$$\mu_{50-50} = \frac{1}{2}8.75 + \frac{1}{2}8 = 8.375.$$

Likewise the variance can be computed as the scenario weighting of the 50–50 variances

$$\mathrm{var}_{50-50} = \frac{1}{4}(11 - 8.375)^2 + \frac{3}{4}(7.5 - 8.375)^2 = 2.2968.$$

We see that the 50–50 variance is unexpectedly low, only 2.3; much lower than either A or B alone

To see why, we calculate the variance by another method. In general, for random variables X and Y with means μ_X and μ_Y respectively, and weights α and β, we have

$$\begin{aligned}\mathrm{var}(\alpha X + \beta Y) &= \mathbb{E}\left([(\alpha X + \beta Y) - (\alpha\mu_X + \beta\mu_Y)]^2\right)\\ &= \mathbb{E}\left([\alpha(X - \mu_X) + \beta(Y - \mu_Y)]^2\right)\\ &= \mathbb{E}\left(\alpha^2(X - \mu_X)^2 + 2\alpha\beta(X - \mu_X)(Y - \mu_Y) + \beta^2(Y - y)^2\right)\\ &= \alpha^2\mathrm{var}_X + 2\alpha\beta\mathbb{E}\left((X - \mu_X)(Y - \mu_Y)\right) + \beta^2\mathrm{var}_Y. \end{aligned} \tag{2.42}$$

The middle term (of the last line) is the *covariance* of X and Y we encountered in the previous section, see equation (2.28).

For these two investments the covariance is negative,

$$\mathrm{covar} = \frac{1}{4}(20 - 8.75)(2 - 8) + \frac{3}{4}(5 - 8.75)(10 - 8) = -22.5$$

because when one is greater than its mean, the other is less. These investments are *negatively correlated*. The negative covariance subtracts from the positive variances. From (2.42) with $\alpha = \beta = .5$, $X = A$, and $Y = B$, we have

$$\sigma^2_{50\text{-}50} = \left(\frac{1}{2}\right)^2(42.18) + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(-22.5) + \left(\frac{1}{2}\right)^2 12 = 2.30. \tag{2.43}$$

The 50–50 investment is attractive because it has better expected return than the bank account but at the same time it has nearly zero risk (in terms of variance).

Moreover, a 50–50 split might not be the optimal split from the stand point of variance. Let w_A be the fraction of resources allocated to investment A and hence $w_B = 1 - w_A$ is allocated to B. Then as a function of w_A, return is given by

$$\mu_{w_A} = w_A\mu_A + (1 - w_A)\mu_B = 8.75w_A + 8(1 - w_A),$$

and variance by

$$\begin{aligned} \sigma^2_{w_A} &= w_A^2\sigma_A^2 + 2w_A(1 - w_A)\text{covar} + (1 - w_A)^2\sigma_B^2 \\ &= 42.18w_A^2 + 2w_A(1 - w_A)(-22.5) + 12(1 - w_A)^2. \end{aligned} \tag{2.44}$$

In Fig. 2.7 we plot return vs risk as a function of w_A. The figure encompasses all the (return, risk) pairs calculated above ($w_A = 0$ for B only, $w_A = 1$ for A only, and $w_A = 1/2$ for I_{50-50}). It also shows that for a certain value of w_A the variance can actually be brought to zero. We can find the minimum variance, be it zero or not, by differentiating the risk function with respect to w_A, setting the derivative to 0 and solving.

Example 2.18. Differentiating (2.44) gives

$$\begin{aligned} \frac{d\sigma^2_{w_A}}{dw_A} &= 2w_A\sigma_A^2 + 2\text{covar} - 4w_A\text{covar} - 2(1 - w_A)\sigma_B^2 \\ 0 &= (2\sigma_A^2 - 4\text{covar} + 2\sigma_B^2)w_A + 2\text{covar} - 2\sigma_B^2 \\ w_A &= \frac{\sigma_B^2 - \text{covar}}{\sigma_A^2 - 2\text{covar} + \sigma_B^2}. \end{aligned} \tag{2.45}$$

For $\sigma_A^2 = 42.18$, covar $= -22.5$, and $\sigma_B^2 = 12$, the minimum risk occurs for $w_A = 0.348$. As shown in the figure, the risk then is 0 for a return of 8.26 %. This is a strategy that is superior to the bank account. □

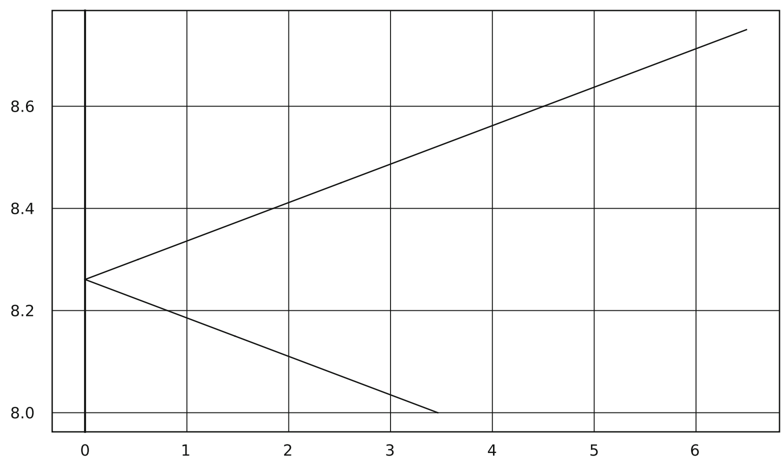


Fig. 2.7. Return vs. risk pairs (μ,σ) for two investments plotted as a function of w_A in the range $0 \leq w_A \leq 1$

Clearly investment B added an element of beneficial possibilities. Using Fig. 2.7, an investor can pick the allocation split satisfying his or her personal comfort of return versus risk.

In the next section we extend these ideas to portfolios of arbitrary size. But first we make the observation that two-investment portfolios generated as above are degenerate in a certain way: the product of the variances equals the covariance squared,

$$\sigma_A^2 \sigma_B^2 = \text{covar}^2. \tag{2.46}$$

Here

$$42.1875 * 12 = 22.5^2.$$

Since covar $= \sigma_A \sigma_B \rho$, it means that either $\rho = 1$ or $\rho = -1$. In this case it is the latter since the covariance is negative.

Substituting (2.46) into (2.44) yields

$$\begin{aligned} \sigma_{w_A}^2 &= w_A^2 \sigma_A^2 \pm 2 w_A (1 - w_A) \sigma_A \sigma_B + (1 - w_A)^2 \sigma_B^2 \\ &= \Big(w_A \sigma_A \pm (1 - w_A) \sigma_B \Big)^2. \end{aligned} \tag{2.47}$$

This explains why the (return, risk) plot is a straight line (broken at 0 if $\rho = -1$).

It is straightforward to show that (2.46) holds for any any assignment of returns in this two investment, two scenario example.

2.6.2 Portfolio Risk

Let V be the value of a portfolio B consisting of two equities with initial prices $S_1(0)$ and $S_2(0)$. The initial capital invested in the portfolio is $V(0)$. Let weights w_1 and w_2 be the allocation of capital to these equities respectively. The amount allocated to the first is $w_1 V(0)$ and the number of shares of this equity is

$$x_1 = \frac{w_1 V(0)}{S_1(0)}.$$

Similarly the number of shares of the second is

$$x_2 = \frac{w_2 V(0)}{S_2(0)}.$$

At any time t, the value of the portfolio depends on the prices of the equities at that time and is given by

$$V(t) = x_1 S_1(t) + x_2 S_2(t).$$

Note that the weights change as the equity prices change but the number of shares do not.

Next let K_1 and K_2 be the returns of the two securities at $t = 1$,

$$K_1 = \frac{S_1(1) - S_1(0)}{S_1(0)} \quad \text{and} \quad K_2 = \frac{S_2(1) - S_2(0)}{S_2(0)}.$$

The portfolio's return is

$$\begin{aligned}
K_B &= \frac{V(1)-V(0)}{V(0)} = \frac{x_1(S_1(1)-S_1(0)) + x_2(S_2(1)-S_2(0))}{V(0)} \\
&= \frac{\frac{w_1 V(0)}{S_1(0)}(S_1(1)-S_1(0) + \frac{w_2 V(0)}{S_2(0)}(S_2(1)-S_2(0))}{V(0)} \\
&= w_1 K_1 + w_2 K_2. \qquad (2.48)
\end{aligned}$$

Thus the portfolio return is linear with respect to the weights.

To calculate portfolio variance, let σ_1^2 and σ_2^2 be the variances of K_1 and K_2 respectively and let ρ_{12} be the correlation coefficient between them, see (2.29); then from (2.42),

$$\sigma_B^2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} + w_2^2\sigma_2^2. \qquad (2.49)$$

since $\sigma_1\sigma_2\rho_{12} = \mathrm{covar}(K_1, K_2)$.

As above, the minimum risk for the portfolio is found by minimizing this equation under the constraint $w_1 + w_2 = 1$. Put $s = w_2$, then $w_1 = 1 - s$ and (2.49) becomes

$$\sigma_B^2 = (1-s)^2\sigma_1^2 + 2s(1-s)\sigma_1\sigma_2\rho_{12} + s^2\sigma_2^2. \qquad (2.50)$$

Differentiating with respect to s gives

$$\frac{d\sigma_B^2}{ds} = -2(1-s)\sigma_1^2 + 2(1-2s)\sigma_1\sigma_2\rho_{12} + 2s\sigma_2^2.$$

Setting the derivative to zero and solving for s we get

$$s_0 = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 - 2\sigma_1\sigma_2\rho_{12} + \sigma_2^2} \qquad (2.51)$$

provided the denominator is not zero.

In fact, the minimum value the denominator can have is when $\rho_{12} = 1$, then

$$\sigma_1^2 - 2\sigma_1\sigma_2\rho_{12} + \sigma_2^2 \geq \sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 = (\sigma_1 - \sigma_2)^2.$$

Hence the denominator is either positive or zero. The latter occurring when $\rho_{12} = 1$ and $\sigma_2 = \sigma_1$. In this case, from (2.49), we have $\sigma_B = \sigma_1$. This case is not essentially different from that in which the two stocks are the same.

The zero derivative value of s given by (2.51) can be bigger than 1 or less than 0. To see this, divide the equation, numerator and denominator by $\sigma_1\sigma_2$ and let $r = \sigma_1/\sigma_2$, we get

$$s_0 = \frac{r - \rho_{12}}{(r + \frac{1}{r}) - 2\rho_{12}} = \frac{r - \rho_{12}}{r - \rho_{12} + (\frac{1}{r} - \rho_{12})}. \qquad (2.52)$$

Here we see that the denominator can be made arbitrarily small by choosing both r and ρ_{12} near 1. But the numerator is positive or negative depending on whether $r > \rho_{12}$ or $r < \rho_{12}$.

Example 2.19. Let $\sigma_1 = 1.04$, $\sigma_2 = 1$, and $\rho_{12} = 0.98$. Then $r = 1.04$ and, from (2.52),

$$s_0 = \frac{1.04 - 0.98}{1.04 - 0.98 + (0.9615 - 0.98)} = 1.44.$$

$\square$

Recall that s is the weight w_2. Solutions for which s_0 is not between 0 and 1 correspond to going short, either in stock 2 if $s_0 < 0$ or stock 1 if $s_0 > 1$.

2.6.3 Efficient Frontier

The considerations of the previous section are a prototype of the general situation in which a portfolio consists of several stocks and the scenarios are the returns resulting from the infinity of possible price histories over the time horizon. While actual expected returns and variances can only be estimated, nevertheless the following theory, with its heavy emphasis on *diversification*, forms the bedrock of guiding principles for managing a portfolio.

Assume then a portfolio B of several equities, $i = 1, \ldots, n$, whose expected returns $\mu_i = \mathbb{E}(K_i)$ and return variances σ_i^2 and covariances covar_{ij} are known. Each point (σ_i, μ_i) may be plotted in the risk-return plane introduced in the previous section.[16]

For a given set of weights, w_i, $i = 1, \ldots, n$, a portfolio is constructed with w_i fraction of the total investment allocated to equity i. The portfolio return K_B is given by

$$K_B = w_1 K_1 + \ldots + w_n K_n = \sum_i w_i K_i.$$

It follows that the expected return is

$$\mu_B = \sum_i w_i \mu_i. \tag{2.53}$$

And the risk is given by

$$\begin{aligned}
\sigma_B^2 &= \mathbb{E}\left((\sum_i w_i (K_i - \mu_i))^2 \right) = \mathbb{E}\left(\sum_i \sum_j w_i w_j (K_i - \mu_i)(K_j - \mu_j) \right) \\
&= \sum_i w_i^2 \mathbb{E}((K_i - \mu_i)^2) + \sum_i \sum_{j \neq i} w_i w_j \mathbb{E}((K_i - \mu_i)(K_j - \mu_j)) \\
&= \sum_i w_i^2 \sigma_i^2 + 2 \sum_i \sum_{j > i} w_i w_j \mathrm{covar}_{ij}. \qquad (2.54)
\end{aligned}$$

By letting C be the covariance matrix,

[16] In this section and the next we are, more exactly, plotting the rate of return versus risk. Since the time period is fixed, the two only differ by a constant factor. In the next section we will add the risk-free rate to the diagram.

$$C = \begin{bmatrix} \sigma_1^2 & \mathrm{covar}_{12} & \dots & \mathrm{covar}_{1n} \\ \mathrm{covar}_{21} & \sigma_2^2 & \dots & \mathrm{covar}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{covar}_{n1} & \mathrm{covar}_{n2} & \dots & \sigma_n^2 \end{bmatrix}$$

we can write

$$\sigma_B^2 = \mathbf{w} \cdot C\mathbf{w} \tag{2.55}$$

where $\mathbf{w}$ is the vector of weights and the dot means the dot product of the two vectors.[17]

When there are only two investments, the subset of the risk-return plane spanned by (σ_B, μ_B) over the set of all possible weights is a curve as in Fig. 2.7. (A parabola except in degenerate cases.) But here the subset spanned is an entire region. In Fig. 2.8 we show the risk-return region spanned by three investments. The three two-investment curves are visible as parabolas within or marking the edge of the region. It can be seen that the optimal mix, for instance point MP, is a mix of all three investments.

The risk-return region in this figure was obtained by a very simple Monte Carlo calculation as follows:

Algorithm 10. Calculating Risk-Return Points

```
inputs: N, μ1, μ2, μ3, σ1², σ2², σ3²
        covar12, covar13, covar23
for i = 1, ..., N
  sum=0
  for j = 1,2,3
    wj ~ U(0,1)   ▷wj is a uniform [0,1) sample
    sum = sum + wj
  endfor
  for j = 1,2,3
    wj = wj/sum   ▷the weights are now normalized
  endfor
  • compute μB by (2.53)
  • compute σB using (2.54)
  • plot
endfor
```

The risk-return calculation induces a partial order on the set of investment mixtures. An investment that has the same risk as another but greater return *dominates* the latter. That is, up is better.

Likewise, an investment that has the same return as another but less risk also dominates the other. So leftwards is better. More generally, if $\mu_A \geq \mu_B$ and $\sigma_A \leq \sigma_B$, then investment A dominates investment B. Therefore the boundary of the risk-return region from the vertex of the abc-mno parabola running up to xyz consists of undominated portfolios; all others are dominated by these.

[17] $\mathbf{x} \cdot \mathbf{y} = (\,x_1 \quad x_2 \quad \dots \quad x_n\,) \cdot (\,y_1 \quad y_2 \quad \dots \quad y_n\,) = \sum_1^n x_i y_i = \mathbf{x}^T \mathbf{y}$.

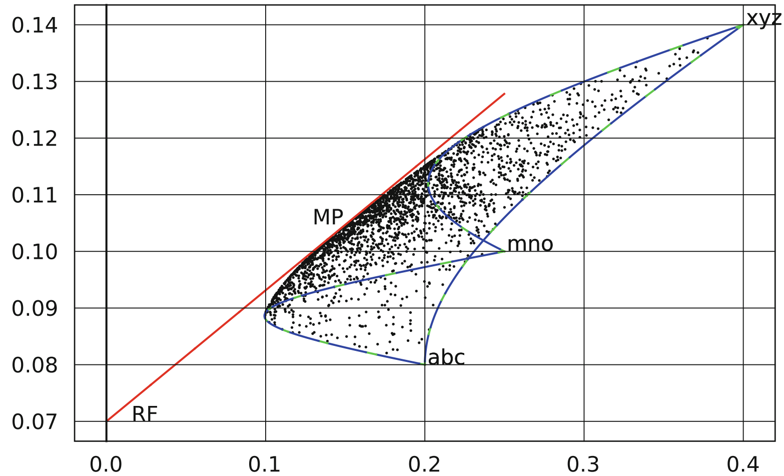


Fig. 2.8. Return vs. risk for three investments: $\mu_{abc} = 0.08$, $\sigma_{abc} = 0.2$, $\mu_{mno} = 0.10$, $\sigma_{mno} = 0.25$, $\mu_{xyz} = 0.14$, $\sigma_{xyz} = 0.4$, $\rho_{abc-mno} = -0.6$, $\rho_{abc-xyz} = 0.5$, $\rho_{mno-xyz} = -0.1$. Each *dot* is the (σ, μ) point for a mix of the three investments. The parabolas are the points for which one of the weights is zero

A portfolio is *efficient* if no other portfolio dominates it. A set of efficient portfolios among all attainable portfolios is called the *efficient frontier*. The efficient frontier of a set of individual investments is the upper left boundary of the risk-return region spanned by the set of all possible weights of those investments.

2.7 Capital Asset Pricing Model

Previously we assumed all investors are rational meaning they are risk averse and mean-variance optimizers. We now assume that all have the same information and therefore obtain the same estimates of returns, variances, and covariances. It then follows that all investors will have the same mix of stocks in the same proportion. This common portfolio is called the *market portfolio*. It is indicated as the point MP in Fig. 2.8.

The assumptions above are called the *equilibrium assumptions*. One of the implications of equilibrium is that the market portfolio consists of all the stocks in the market and in proportion to each stock's capitalization, that is, according to the total value of its shares relative to the total value of the entire market. This is because either all investors will own it or none will. If none own it, its price will be zero (or near zero). Then it will be undervalued, so now it is an attractive buy.

The next step in the CAPM development is to add the risk-free asset to the mix.

2.7.1 The Market Portfolio

Now allow portfolios to include a risk-free asset earning the risk-free return r_f. This asset appears in the risk-return plane at the point `RF` with coordinates $(0, r_f)$ in Fig. 2.8. Consider the line through $(0, r_f)$ and tangent to the efficient frontier of the risk-return region. This is called the *capital market line*. Let its point of tangency with the frontier be the point `MP` with coordinates (σ_M, μ_M). The mix of securities that gives this point on the efficient frontier is called the *market portfolio*. As mentioned, the market portfolio consists of all stocks in the market and in proportion to each stock's capitalization. In actual practice, various stock indexes such as the S&P-500 and the Russell 2000 try to approximate the market portfolio.

The slope of the line between `RF` and `MP` is

$$\frac{\mu_M - r_f}{\sigma_M} \tag{2.56}$$

and its intercept on the y-axis is r_f; therefore the capital market line has the equation

$$\mu = r_f + \frac{\mu_M - r_f}{\sigma_M}\sigma. \tag{2.57}$$

Example 2.20. Normally the capital market line must be determined numerically. But in the case of two investments simple calculus suffices.

Let A and B have mean returns μ_A and μ_B respectively, variances σ_A^2 and σ_B^2 and covariance covar. Let the risk-free rate be r_f and the market point be (σ_M, μ_M). The slope of the capital market line is given by (2.56). The plan is to equate this to the slope of the tangent to the risk-return curve. Let s play the role of w_B. The parameterization of the risk-return curve is, from (2.53)

$$\mu = (1-s)\mu_A + s\mu_B = \mu_A + (\mu_B - \mu_A)s, \tag{2.58}$$

hence

$$\frac{d\mu}{ds} = \mu_B - \mu_A.$$

And from (2.54)

$$\begin{aligned} \sigma^2 &= (1-s)^2\sigma_A^2 + 2s(1-s)\text{covar} + s^2\sigma_B^2 \\ &= \theta s^2 - 2\lambda s + \sigma_A^2 \end{aligned} \tag{2.59}$$

where $\theta = \sigma_A^2 - 2\text{covar} + \sigma_B^2$ and $\lambda = \sigma_A^2 - \text{covar}$. Differentiate both sides and substitute $\sigma = \sigma_M$ at the point of tangency

$$\begin{aligned} 2\sigma\frac{d\sigma}{ds} &= 2\theta s - 2\lambda \\ \frac{d\sigma}{ds} &= \sigma_M^{-1}(\theta s - \lambda). \end{aligned}$$

Therefore the slope of the tangent line is

$$\frac{d\mu}{d\sigma} = \frac{d\mu/ds}{d\sigma/ds} = \frac{(\mu_B - \mu_A)\sigma_M}{\theta s - \lambda}.$$

Equating slopes gives

$$\frac{\mu_M - r_f}{\sigma_M} = \frac{(\mu_B - \mu_A)\sigma_M}{\theta s - \lambda}. \tag{2.60}$$

Substituting for μ_M, (2.58), and σ_M, (2.59), gives

$$(\theta s - \lambda)\left((\mu_A - r_f) + (\mu_B - \mu_A)s\right) = (\mu_B - \mu_A)\left(\sigma_A^2 + (\theta s - \lambda)s - \lambda s\right)$$

The quadratic terms in s cancel; the resulting linear equation is solved for s to give

$$s = \frac{(\mu_B - \mu_A)\sigma_A^2 + (\mu_A - r_f)(\sigma_A^2 - \text{covar})}{(\mu_A - r_f)(\sigma_A^2 - 2\text{covar} + \sigma_B^2) + (\mu_B - \mu_A)(\sigma_A^2 - \text{covar})} \tag{2.61}$$

for the location of the market point.

As a numerical example, let the two-investment pair above be the equities mno and xyz in Fig. 2.8. If the portfolio consisted of these two only and the risk-free rate were 8 %, then, from the parameters given in the figure, the covariance is

$$(0.25)(0.4)(-0.1) = 0.01$$

and from (2.61)

$$\begin{aligned} s &= \frac{(0.14 - 0.1)0.25^2 + (0.1 - 0.08)(0.25^2 + 0.01)}{(0.1 - 0.08)(0.25^2 + 0.02 + 0.4^2) + (0.14 - 0.1)(0.25^2 + 0.01)} \\ &= 0.51. \end{aligned}$$

Therefore $\mu_M = 0.49(0.1) + 0.51(0.14) = 0.12$ and

$$\sigma_M = \sqrt{0.49^2 0.25^2 + 0.02(0.49)(0.51) + 0.51^2 0.4^2} = 0.23.$$

□

Since the capital market line is above and to the left of the risk-return region spanned by the risky securities, it becomes the new efficient frontier. It follows that rational investors will select a position along this line according to their personal level of risk tolerance. Hence all rational investors will have the same mix of risky securities, namely the market portfolio, differing only in their proportion as allocated between the market portfolio and a risk-free investment.

But investors may also go short. By doing so, their position on the capital market line need not be constrained between `RF` and `MP`. By going short on the risk-free asset and transferring the capital to the market portfolio, an investor's market weight will be greater than 1. The return for such a mix will exceed μ_M but at the same time, the risk will exceed σ_M.

The slope of the capital market line (2.56) is a very important investment parameter. It gives the rate at which one's level of return rises for taking on increments in the level of risk. It is called the *price of risk* or the *risk premium*.

2.7.2 Beta Factor of a Portfolio

A major result of CAPM, and perhaps a surprising one, is that the expected returns of any particular stock bears a simple relationship to that of the market portfolio.

Theorem (Capital Assets Pricing Theorem) *The expected rate of return of a portfolio, μ_B, (or an individual stock considered as a portfolio of one item) is given by*

$$\mu_B = r_f + \beta(\mu_M - r_f) \tag{2.62}$$

where μ_M is the market portfolio rate of return and β is given by

$$\beta = \frac{covar(B, M)}{\sigma_M^2}. \tag{2.63}$$

Here covar(B, M) is the covariance of the portfolios sequence of periodic returns versus those of the market portfolio.

Equation (2.62) is known as the *security market line*. We will have more say about it in the next section. Beta as given by (2.63) is the *beta factor* of the portfolio or of an individual stock; it is unique to each portfolio.

The difference $\mu_B - r_f$ in (2.62) is the *excess rate of return* of the portfolio above the risk-free rate. The theorem says that it is proportional to the excess rate of return of the market portfolio itself with proportionality factor equal to β. Moreover, since β is directly proportional to its covariance with the market portfolio, the theorem says the excess rate of return of a portfolio is proportional to its covariance with the market.

This last statement may sound surprising with respect to portfolios that are uncorrelated to the market and thus whose covariance is zero. But for a large portfolio of equities each uncorrelated with the market, and each other, their combined variance will be small and therefore their combined rate of return will be the risk-free rate.

The proof of (2.62) follows along the lines of the two investment example above, Example 2.20. Let asset A of that example be the market portfolio, M. From (2.58) and (2.59), the risk-return curve for pair M and B is

$$\begin{aligned}
\mu &= s\mu_B + (1-s)\mu_M \\
\sigma^2 &= (1-s)^2\sigma_M^2 + 2s(1-s)\text{covar} + s^2\sigma_B^2 \\
&= \theta s^2 - 2\lambda s + \sigma_M^2.
\end{aligned}$$

Note that for $s = 0$ the two investments degenerate into just the market portfolio. Further, it must be that the risk-return curve for M,B is tangent to the efficient frontier at that point since it cannot cross the efficient frontier. Therefore its slope at $s = 0$ equals $(\mu_M - r_f)/\sigma_M$. Hence we have, using (2.60),

$$\begin{aligned}
\frac{\mu_M - r_f}{\sigma_M} = \frac{d\mu}{d\sigma} &= \left.\frac{(\mu_B - \mu_M)\sigma_M}{\theta s - \lambda}\right|_{s=0} \\
&= \frac{(\mu_B - \mu_M)\sigma_M}{\text{covar} - \sigma_M^2}.
\end{aligned}$$

Solving for μ_B we get

$$\begin{aligned}\mu_B &= \mu_M + \frac{(\mu_M - r_f)(\text{covar} - \sigma_M^2)}{\sigma_M^2} \\ &= \mu_M + \frac{\text{covar}}{\sigma_M^2}(\mu_M - r_f) - (\mu_M - r_f) \\ &= r_f + \beta(\mu_M - r_f),\end{aligned}$$

where β is as in (2.63).

Beta and the Line of Best Fit

In Fig. 2.9 we plot the monthly returns of two securities versus the S&P-500 over a 1 year period. Let $x_1, x_2, \ldots, x_n$ be the sequence of S&P-500 returns and $y_1, y_2, \ldots, y_n$ be those for the security. Each graph plots the pairs (x_i, y_i) for $i = 1, 2, \ldots, n$; here $n = 12$.

We wish to calculate the straight line, $y = mx + b$, that best fits these data. This line is shown superimposed on each graph.

By the method of least-squares, the well-known equations for m and b are derived in appendix Section A.7. From (A.13) we find that

$$\begin{aligned}m &= \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} \\ &= \frac{\frac{1}{n}\sum x_i y_i - \left(\frac{1}{n}\sum x_i\right)\left(\frac{1}{n}\sum y_i\right)}{\frac{1}{n}\sum x_i^2 - \left(\frac{1}{n}\sum x_i\right)^2} \\ &= \frac{\text{covar}(x,y)}{\text{var}(x)}.\end{aligned} \tag{2.64}$$

Thus the slope of the best fit line is β

Example 2.21. The data points for MSP in Fig. 2.9 are the following

$$\begin{gathered}(-1.8, -1.8),\ (-.9, -.25),\ (-.7, -.45),\ (.09, .44),\ (.08, .5),\ (.2, .45), \\ (.3, .2),\ (.6, .35),\ (.68, .45),\ (.65, .7),\ (.9, .6),\ (1.35, 1.52).\end{gathered}$$

First calculate the means:

$$\begin{aligned}\bar{x} &= \frac{1}{12}(-1.8 + -0.9 + -0.7 + \ldots + 0.9 + 1.35) = 0.12 \\ \bar{y} &= \frac{1}{12}(-1.8 + -0.25 + -0.45 + \ldots + 0.6 + 1.52) = 0.23.\end{aligned}$$

Then the variance and covariance

$$\begin{aligned}\text{var}(x) &= \frac{1}{12}\left(-1.8^2 + -0.9^2 + \ldots + 1.35^2\right) - 0.12^2 = 0.6989 \\ \text{covar}(x,y) &= \frac{1}{12}\Big((-1.8)(-1.8) + (-.9)(-.25) + \\ &\qquad \ldots + (1.35)(1.52)\Big) - (0.12)(0.23) = 0.6038.\end{aligned}$$

Finally $\beta = 0.6038/0.6989 = 0.86$. □

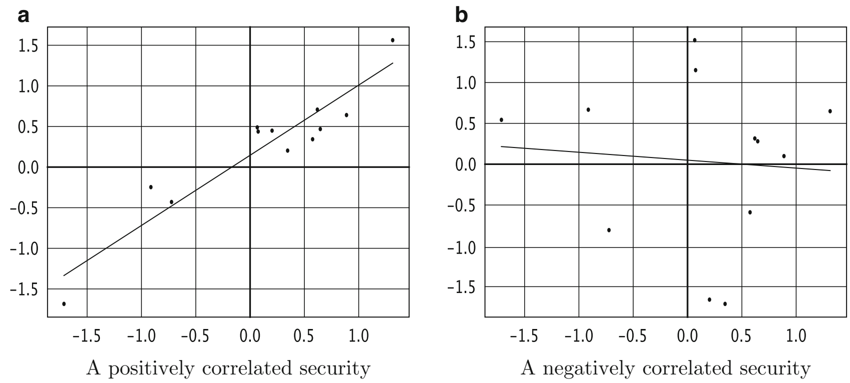


Fig. 2.9. A plot of monthly returns $\Delta S/S$ over 1 year for two securities versus those of the S&P-500 – a proxy for the entire market; MSP in (**a**) and GG in (**b**). The *straight lines* are the best least squares fit and the slope is beta in each case. Using the equations derived in the text, MSP has a beta of 0.86 and GG a beta of -0.10

2.7.3 The Security Market Line

Let $y_1, y_2, \ldots, y_n$ be a sequence of returns for an asset B and $x_1, x_2, \ldots, x_n$ those for the market portfolio. For each i let ϵ_i be the difference between the empirical return y_i and that predicted by the security market line, equation (2.62), so we can write

$$y_i = r_f + \beta(x_i - r_f) + \epsilon_i.$$

By the CAPM theorem, the expectation $\mathbb{E}(\epsilon_i) = 0$. So is the covariance,

$$\begin{aligned} \text{covar}(\epsilon_i, x_i) &= \text{covar}(y_i - r_f - \beta x_i + \beta r_f, x_i) \\ &= \text{covar}(y_i, x_i) - \beta\text{covar}(x_i, x_i) \\ &= \text{covar}(y_i, x_i) - \frac{\text{covar}(y_i, x_i)}{\text{var}(x_i)}\text{var}(x_i) = 0. \end{aligned}$$

It follows that the variance of the y_i is given by

$$\begin{aligned} \text{var}(y_i) &= \text{var}(r_f + \beta x_i - \beta r_f + \epsilon_i) \\ &= \beta^2\text{var}(x_i) + \text{var}(\epsilon_i). \end{aligned}$$

We obtain the important relationship

$$\sigma_B^2 = \beta^2\sigma_M^2 + \text{var}(\epsilon_i). \tag{2.65}$$

Equation (2.65) shows that the risk in a portfolio has two sources. The first, $\beta^2\sigma_M^2$, is unavoidable and is called the *systemic risk*. This risk is that of the market as a whole. This is the risk alluded to in the first chapter due to macro economic shocks arising from, for example, government policy, international economic forces, acts of nature. It cannot be diversified away.

The second source of risk is that specific to the portfolio itself. It is called *specific risk* or *diversifiable risk*. By adding more and more securities to the portfolio, this risk can be reduced to zero in the limit as the portfolio tends to the market portfolio.

Problems: Chapter 2

1. What should be the price of a 2 year \$100 zero coupon bond in order that the investment earns 6 % per year? (No compounding.)

2. An annuity starts with \$501,692 and pays out \$10,000 per month. If the remaining principle earns 4 % annual interest compounded continuously, for how many months will the annuity pay?
(Answer 55.)

3. Algorithm 6 assumes that the equity is purchased at the beginning of the dividend period. If the stock is held for an exact multiple of the dividend period, then the annualized return should be exactly the dividend yield. But what if the stock is bought or sold at mid-term intervals? For example shortly before ex-dividend day? Explore the annual return under various ownership periods with respect to the ex-dividend date.

4. Write a program to display a piecewise linear approximation of a price path as follows. Let S_0, S_1, S_2, $\ldots$, S_{365} be a 1 year sequence of prices. Select a subset of these, for example monthly S_0, S_{30}, $\ldots$, S_{364}, and generate the piecewise linear graph through these points, $(0, S_0)$, $(30, S_{30})$, $\ldots$, $(364, S_{364})$. Such an approximation could serve as an alternative to the moving average of the prices.

5. Use Algorithm 8 to construct several figures such as Fig. 2.4 and observe the extent to which correlated prices trend together (recall that it is the increments that are correlated, not the prices themselves). For each run, calculate the correlation between the prices.

6. Investigate the probability of losing money for the investment of Example 2.8 (pp. 44) if the stock is correlated with the market and, variously, $\rho = 0.8$, $\rho = 0$, $\rho = -0.5$. Do this for various market scenarios: a rising market, a falling market, a sideways market.

7. Run Algorithm 9 (pp. 54) with various correlations between the two stocks and the market. What is the risk of loss when: (a) $\rho_1 = 1$, $\rho_2 = 1$?, (b) $\rho_1 = 1$, $\rho_2 = -1$?, (c) $\rho_1 = 0$, $\rho_2 = 0$?, (d) $\rho_1 = -1$, $\rho_2 = -1$?, (e) $\rho_1 = 0.6$, $\rho_2 = -0.6$?

8. Calculate the VaR at the 99 % level over, variously, 1 month, 3 months, and 6 months, for a stock whose initial price is \$45, whose drift is 2 %, and whose volatility is 23 %.

9. Find the VaR at the 99 % level over 2 months by simulation for a portfolio of two stocks with parameters: for the first: $S_0 = 20$, $\mu = 3\,\%$, volatility= 26 %, for the second: $S_0 = 40$, $\mu = 1\,\%$, volatility= 33 %. Assume that the stocks are correlated, variously, $\rho = 0.9$, $\rho = 0.2$, $\rho = -0.8$.

10. Find the VaR for the stocks in Problem 9 by the historical method. For their price histories, use the GBM model to generate 2 months worth of prices for the equities. Only treat the $\rho = 0.2$ case.

11. Investigate how the probability of loss in Example 2.8 (pp. 44) varies as a function of volatility. Make a graph of loss vs. volatility.

12. An investor has a choice between two ventures A and B. As the investor sees it, the future holds three possibilities: (bull) A returns 12 %, B returns 3 %, (bear) A return −4 %, B returns 4 %, or (static) A returns 6 %, B returns 0 %. Assume the probabilities are: bull 0.2, bear 0.3, static 0.5. What are the expected returns and risks (standard deviations) for each venture? Same question for a 50–50 allocation of the investor's resources. What is the allocation giving least risk?
(Answer 0.1788 : 0.8212.)

13. If the risk-free rate is 3 % in Problem 12, what is the market point?
(Answer ($\mu = 1.9411, \sigma = 1.914$).)

14. Obtain recent price data for some security from among the list: AAPL, MON, KO, F, MCD, FDX. Along with data for the S&P-500, use it to calculate daily returns over the last month and to calculate beta for the stock.

15. Use the results of Problem 14 to calculate the risk premium for that stock.

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