Preface

This book aims both to synthesize much of undergraduate mathematics and to introduce research topics in geometric aspects of complex analysis in several variables. The topics all relate to orthogonality, real analysis, elementary complex variables, and linear algebra. I call the blend *Hermitian analysis.* The book developed from my teaching experiences over the years and specifically from Math 428, a capstone honors course taught in Spring 2013 at the University of Illinois. Many of the students in Math 428 had taken honors courses in analysis, linear algebra, and beginning abstract algebra. They knew differential forms and Stokes' theorem. Other students were strong in engineering, with less rigorous mathematical training, but with a strong sense of how these ideas get used in applications.

Rather than repeating and reorganizing various parts of mathematics, the course began with Fourier series, a new topic for many of the students. Developing some of this remarkable subject and related parts of analysis allows the synthesis of calculus, elementary real and complex analysis, and algebra. Proper mappings, unitary groups, complex vector fields, and differential forms eventually join this motley crew. Orthogonality and Hermitian analysis unify these topics. In the process, ideas arising on the unit circle in $\mathbf{C}$ evolve into more subtle ideas on the unit sphere in complex Euclidean space $\mathbf{C}^n$.

The book includes numerous examples and more than 270 exercises. These exercises sometimes appear, with a purpose, in the middle of a section. The reader should stop reading and start computing. Theorems, lemmas, propositions, etc. are numbered by chapter. Thus Lemma 2.4 means the fourth lemma in Chap. 2, and Fig. 1.8 means the eighth figure in Chap. 1.

Chapter 1 begins by considering the conditionally convergent series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$. We verify its convergence using summation by parts, which we discuss in some detail. We then review constant coefficient ordinary differential equations, the exponentiation of matrices, and the wave equation for a vibrating string. These topics motivate our development of Fourier series. We prove the Riesz–Fejer theorem characterizing nonnegative trig polynomials. We develop topics such as approximate identities and summability methods, enabling us to complete the discussion on the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$. The chapter closes with two proofs of Hilbert's inequality.

Chapter 2 discusses the basics of Hilbert space theory, motivated by orthonormal expansions, and includes the spectral theorem for compact Hermitian operators. We return to Fourier series after these Hilbert space techniques have become available. We also consider Sturm–Liouville theory in order to provide additional examples of orthonormal systems. The exercises include problems on Legendre polynomials, Hermite polynomials, and several other collections of special functions. The chapter ends with a section on spherical harmonics, whose purpose is to indicate one possible direction for Fourier analysis in higher dimensions. As a whole, this chapter links classical and modern analysis. It considerably expands the material on Hilbert spaces from my Carus monograph *Inequalities from Complex Analysis*. Various items here help the reader to think in a magical Hermitian way. Here are two specific examples:

- There exist linear transformations A, B on a real vector space satisfying the relationship $A^{-1} + B^{-1} = (A + B)^{-1}$ if and only if the vector space admits a complex structure.
- It is well known that a linear map on a complex space preserves inner products if and only it preserves norms. This fact epitomizes the polarization technique which regards a complex variable or vector z and its conjugate $\overline{z}$ as independent objects.

Chapter 3 considers the Fourier transform on the real line, partly to glimpse higher mountains and partly to give a precise meaning to distributions. We also briefly discuss Sobolev spaces and pseudo-differential operators. This chapter includes several standard inequalities (Young, Hölder, Minkowski) from real analysis and Heisenberg's inequality from physics. Extending these ideas to higher dimensions would be natural, but since many books treat this material well, we head in a different direction. This chapter is therefore shorter than the other chapters and it contains fewer interruptions.

Chapter 4, the heart of the book, considers geometric issues revolving around the unit sphere in complex Euclidean space. We begin with Hurwitz's proof (using Fourier series) of the isoperimetric inequality for smooth curves. We prove Wirtinger's inequality in two ways. We continue with an inequality on the areas of complex analytic images of the unit disk, which we also prove in two ways. One of these involves differential forms. This chapter therefore includes several sections on vector fields and differential forms, including the complex case. Other geometric considerations in higher dimensions include topics from my own research: finite unitary groups, group-invariant mappings, and proper mappings between balls. We use the notion of *orthogonal homogenization* to prove a sharp inequality on the volume of the images of the unit ball under certain polynomial mappings. This material naturally leads to the Cauchy–Riemann (CR) geometry of the unit sphere. The chapter closes with a brief discussion of positivity conditions for Hermitian polynomials, connecting the work on proper mappings to an analogue of the Riesz–Fejer theorem in higher dimensions. Considerations of orthogonality and Hermitian geometry weave all these topics into a coherent whole.

The prerequisites for reading the book include three semesters of calculus, linear algebra, and basic real analysis. The reader needs some acquaintance with complex numbers but does not require all of the material in the standard course. The appendix summarizes the prerequisites. We occasionally employ the notation of Lebesgue integration, but knowing measure theory is not a prerequisite for reading this book. The large number of exercises, many developed specifically for this book, should be regarded as crucial. They link the abstract and the concrete.

Books in the Cornerstones Series are aimed at aspiring young mathematicians ranging from advanced undergraduates to second-year graduate students. This audience will find the first three chapters accessible. The many examples, exercises, and motivational discussions make these chapters also accessible to students in physics and engineering. While Chap. 4 is more difficult, the mathematics there flows naturally from the earlier material. These topics require the synthesis of diverse parts of mathematics. The unity and beauty of the mathematics reward the reader while leading directly to current research. The author hopes someday to write a definitive account describing where in complex analysis and CR geometry these ideas lead.

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