Chapter 2
Invariants

2.1 Geodetic Closure and Convex Hull

For two vertices u and v of a graph G, a vertex $x \in V(G)$ is said to be *geodominated* by the pair $\{u,v\}$ if x lies on some $u-v$ geodesic in G. The *geodetic interval* $I_G[u,v]$ consists of u,v together with all vertices geodominated by the pair $\{u,v\}$. If S is a set of vertices of G, then the *geodetic closure* $I_G[S]$ is the union of all sets $I[u,v]$ for $u,v \in S$, i.e., it consists of S together with all vertices lying on some geodesic joining two vertices of S. When the graph G is clear from the context, $I_G[u,v]$ and $I_G[S]$ are usually replaced by $I[u,v]$ and $I[S]$, respectively.

A set S of vertices is called *geodesically convex*, *g-convex*, or simply *convex*, if $I[S] = S$, i.e., if for every pair $u,v \in S$, the interval $I[u,v] \subseteq S$. In any graph, the empty set, the whole vertex set, every singleton, and every two-path are convex. On the other hand, if $I[S] = V(G)$, then S is said to be a *geodetic set* (also known as *geodominating set*).

Let W be a set of vertices of a graph G. We define $I^k[W]$ recursively as follows: $I^0[W] = W$, $I^1[W] = I[W]$, and $I^k[W] = I[I^{k-l}[W]]$ for $k > 1$. Since the vertex u forms the only $u-u$ geodesic, we have $I[u,u] = u$, and hence $W \subseteq I[W]$. Observe that this implies that W is convex if and only if $I[W] = W$. The *geodetic iteration number* $\mathrm{gin}(W)$ of W is the smallest positive integer n such that $I^n[W] = I^{n+1}[W]$. Note that $I[W]$ is convex if and only if $\mathrm{gin}(W) = 1$. The *geodesic iteration number* of G, denoted by $\mathrm{gin}(G)$, is defined as $\mathrm{gin}(G) = \max\{\mathrm{gin}(W) : W \subseteq V(G)\}$.

The *convex hull* $[S]$ of a set S of vertices is the smallest convex set containing S. It is also called the *geodesic convex hull* or simply the *g-convex hull* of S and denoted by $[S]_g$. As an immediate consequence of this definition, the following properties hold.

Proposition 2.1. *Let S be a non-convex set of a graph G. Then, $S \subset I[S] \subseteq [S] \subseteq V(G)$. Moreover, the following statements are equivalent:*

- *$[S]$ is the smallest convex set containing S.*
- *$[S]$ is the intersection of all convex sets containing S.*

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- $[S] = I^k[S]$, *for every positive integer* $k \geq \mathrm{gin}(S)$.

If a set S satisfies $[S] = V(G)$, then it is called a *hull set* of G. Clearly, every geodetic set is a hull set, but the converse is not necessarily true. For example, if $U = \{x,y\}$ and $W = \{a,b,c\}$ are the partite sets of $K_{2,3}$, then the set $\{a,b\}$ is a hull set as $I^2[a,b] = V$, but it is not geodetic since $I[a,b] = \{a,b,x,y\}$.

2.2 Geodetic and Hull Numbers

The *geodetic number* of a graph G, denoted by $g(G)$, is the minimum cardinality of a geodetic set of $V(G)$ [113]. The *hull number* of a graph G, denoted by $h(G)$, is the minimum cardinality of a hull set of $V(G)$ [98].

If $G = \sum_{i=1}^{p} G_i$ is a non-connected graph with p components, then $g(G) = \sum_{i=1}^{p} g(G_i)$ and $h(G) = \sum_{i=1}^{p} h(G_i)$. Hence, the study of both parameters can be restricted to connected graphs.

Theorem 2.1 ([63]). *If G is a nontrivial graph of order n and diameter d, then* $2 \leq h(G) \leq g(G) \leq n-d+1$.

Proof. Every nontrivial graph satisfies $2 \leq h(G) \leq g(G)$, since every geodetic set is a hull set and the unique graph for which $h(G) = 1$ is $G \cong K_1$. Let u,v be vertices of G for which $d(u,v) = d$ and ρ a $u-v$ geodesic. If $V(\rho) = \{u, v_1, \ldots, v_{d-1}, v\}$ and $S = V(G) \setminus \{v_1, \ldots, v_{d-1}\}$, then $I[S] = V(G)$ and, consequently, $g(G) \leq |S| = n-d+1$. □

A corollary of this result is that the only connected graph of order and geodetic number n is the complete graph K_n. In Table 2.1, both the hull number and the geodetic number of some basic graphs are displayed.

Theorem 2.2 ([59,63]). *If n, d, and k are integers such that $2 \leq d < n$, $2 \leq k < n$, and $n-d-k+1 \geq 0$, then there exists a graph G of order n, diameter d, and $h(G) = g(G) = k$.*

Proof. Let P_{d+1} a path of order $d+1$ such that $V(P_{d+1}) = \{u_0, u_1, \ldots, u_d\}$. We first add $k-2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ to P_{d+1}, and join each to u_1, producing a tree T. Then, we add $n-d-k+1$ new vertices $w_1, w_2, \ldots, w_{n-d-k+1}$ and join each to both u_0 and u_2, thereby producing the graph G or order n and diameter d displayed

Table 2.1 Geodetic and hull numbers of some basic graph families

G[a]	P_n	C_{2l}	C_{2l+1}	T_n	K_n	$K_{p,q}$ [b]	W_n [c]	Q_n
$h(G)$	2	2	3	$\lvert \mathrm{Ext}(T_n) \rvert$	n	2	$\lfloor \frac{n}{2} \rfloor$	2
$g(G)$	2	2	3	$\lvert \mathrm{Ext}(T_n) \rvert$	n	$\min\{4,p\}$	$\lfloor \frac{n}{2} \rfloor$	2

[a] $G \ncong K_1$
[b] $2 \leq p \leq q$
[c] $n \geq 5$

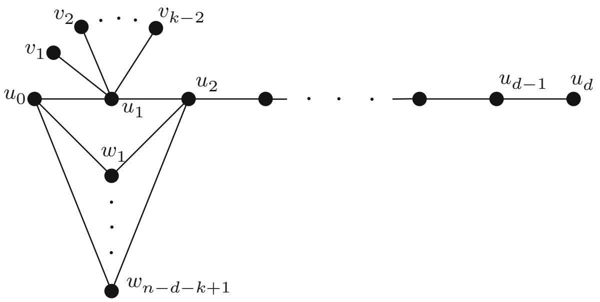

Fig. 2.1 $h(G) = g(G) = k$

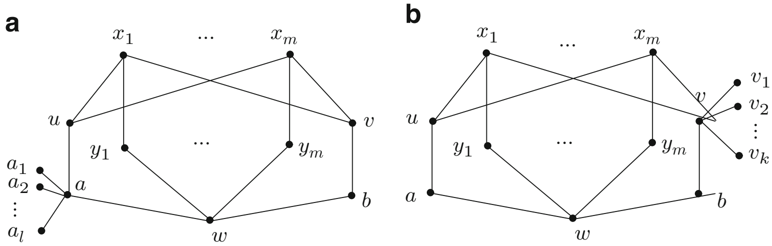

Fig. 2.2 (**a**) Graph $J_{7,m,l}$, (**b**) graph $J^k_{7,m}$

in Fig. 2.1. Finally, it is a routine exercise to check that $S = \{u_0, u_d, v_1, \ldots, v_{k-2}\}$ is both a minimum geodetic set and a minimum hull set of G. □

Theorem 2.3 ([59, 116]). *For every pair α, β of integers with $2 \leq \alpha \leq \beta$, there exists a graph G such that $h(G) = \alpha$ and $g(G) = \beta$.*

Proof. For $\alpha = \beta$, K_α has the desired properties. Suppose that $\alpha < \beta$ and consider the graph $J_{7,m,l}$ displayed in Fig. 2.2a. Observe that $\{v\} \cup \{a_j\}_{j=1}^{l}$ $\left(\text{resp. } \{v\} \cup \{a_j\}_{j=1}^{l} \cup \{y_i\}_{i=1}^{m}\right)$ is a minimum hull (resp. geodetic) set. Thus, taking $m = \beta - \alpha$ and $l = \alpha - 1$, we have a graph G satisfying $h(G) = \alpha$, $g(G) = \beta$. □

Proposition 2.2 ([98]). *A vertex x is an extreme vertex with respect to the geodesic convexity in a graph G if and only if it is simplicial.*

Proof. Let x be an extreme vertex. If $\deg_G(x) = 1$, then x is trivially simplicial. Suppose that $\deg_G(x) \geq 2$ and take a pair of vertices $u, v \in N(x)$. Since $V - x$ is convex, every vertex lying on some $u - v$ geodesic must belong to $V - x$, which is only true if u and v are adjacent.

Conversely, assume that x is a simplicial vertex of G and take a pair of vertices $u, v \in V(G) - x$. Let ρ be a $u - v$ geodesic containing vertex x and take the neighbors $\{a, b\}$ of x belonging to $V(\rho)$. Notice that $d_G(a, b) = 2$, a contradiction since $N(x)$ induces a clique. □

Every hull set of a graph $G = (V,E)$, and hence also every geodetic set, contains all vertices of $\mathrm{Ext}(G)$ since for every extreme vertex x and for every set S such that $S \subseteq V - x$, its convex hull $[S]$ is also contained in $V - x$ (see [98]). In other words, every graph G satisfies $|\mathrm{Ext}(G)| \le h(G) \le g(G)$.

An *extreme geodesic graph* is a graph G such that $|\mathrm{Ext}(G)| = g(G)$. In these graphs, the set $\mathrm{Ext}(G)$ of simplicial vertices is both its unique minimum geodetic and hull set. Complete graphs and trees provide two basic examples of extreme geodesic graphs.

Theorem 2.4 ([54]). *For every pair a,b of integers with $(a,b) \neq (0,1)$ and $0 \le a \le b$, there exists a graph G such that $|\mathrm{Ext}(G)| = a$ and $g(G) = b$.*

Proof. If $1 \le a = b$, then K_a satisfies $|\mathrm{Ext}(K_a)| = g(K_a) = a$. Thus we assume that $0 \le a < b$ and $2 \le b$. Take a copies $\{F_i\}_{i=1}^{a}$ of K_2 and $b-a$ copies $\{H_i\}_{i=1}^{b-a}$ of C_4. Let $V(F_i) = \{x_i, y_i\}$, $V(H_j) = \{u_j, v_j, w_j, z_j\}$, and $E(H_j) = \{u_jv_j, v_jw_j, w_jz_j, z_ju_j\}$ and consider the graph $G_{a,b}$ obtained from these two families by identifying the vertices $\{x_i\}_{i=1}^{a} \cup \{u_j\}_{j=1}^{b-a}$. Clearly, $\mathrm{Ext}(G_{a,b}) = \{y_i\}_{i=1}^{a}$. It is also easy to check that the set $\{y_i\}_{i=1}^{a} \cup \{w_j\}_{j=1}^{b-a}$ is a minimum geodetic set of $G_{a,b}$. Thus, $|\mathrm{Ext}(G_{a,b})| = a$ and $g(G_{a,b}) = b$. □

Another example of extreme geodesic graph is that obtained from a star $K_{1,k}$ by replacing each end-vertex by a complete graph and joining all new vertices to the cut-vertex v of the star. Observe that the extreme subgraph of any graph G of order n belonging to this family contains all vertices of G but v, which means that $h(G) = g(G) = n-1$. Moreover, as shown next, these are the only graphs verifying this double equality.

Theorem 2.5 ([22, 59, 98]). *Let G be a connected graph of order $n \ge 3$. Then, the following three statements are equivalent:*

1. $h(G) = n-1$.
2. $g(G) = n-1$.
3. $G \cong K_1 \vee (K_{n_1} + \cdots + K_{n_r})$ *where* $r \ge 2$ *and* $n_1 + \cdots + n_r = n-1$.

Proof. We shall show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.
$(1) \Rightarrow (2)$ The only connected graph such that $g(G) = n$ is the complete graph K_n. Therefore, $g(G) = n-1$, as $h(G) \le g(G)$ and $h(K_n) = n$.
$(2) \Rightarrow (3)$ Let S be a minimum geodetic set of G, where $V(G) - S = v$. We claim that v is adjacent to every vertex in S. For every geodesic ρ joining any two nonadjacent vertices u,v of G, the set $[V(G) - V(\rho)] \cup \{u,v\}$ is a geodetic set of G. This means, first, that $\mathrm{diam}(G) = 2$ and, second, that v is adjacent to every pair of nonadjacent vertices of S. Therefore, if u is a vertex of S nonadjacent to vertex v, then it must be adjacent to all other vertices of S. Since S is a geodetic set, v lies on some $s-t$ geodesic of length 2, where $s,t \in S$. Finally, since $us, ut \in E(G)$, it follows that $u \notin S$, a contradiction. Hence, as claimed, v is adjacent to every vertex in S. To show that $G \cong K_1 \vee (K_{n_1} + \cdots + K_{n_r})$, it suffices to observe that for any triple $x,y,z \in S$, if $xy, yz \in E(G)$, then $x,z \in E(G)$, since otherwise $d(x,z) = 2$, and hence $y = v$, a contradiction.

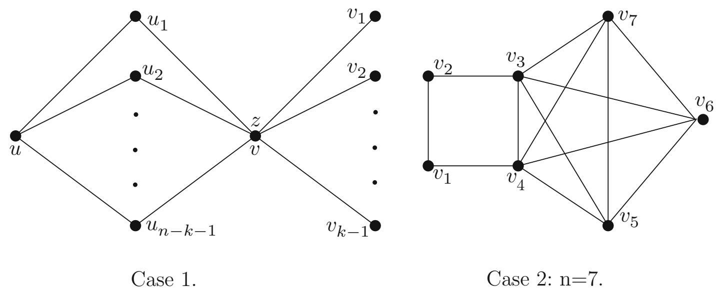

Fig. 2.3 Case 1: $2 \leq k \leq n-3$, Case 2: $2 \leq k = n-2$

(3) $\Rightarrow$ (1) All vertices of G but the cut-vertex of G are extreme vertices. Therefore, $h(G) = n-1$, since $|\mathrm{Ext}(G)| \leq h(G)$ and the only connected graph such that $h(G) = n$ is K_n. $\square$

Corollary 2.1 ([54]). *Every nontrivial graph G of order n with geodetic number $g(G) = n-1$ is an extreme geodesic graph.*

Theorem 2.6 ([54]). *For every pair k,n of integers with $2 \leq k \leq n-2$ there exists a graph G of order n and geodetic number $g(G) = k$ that is not an extreme geodesic graph.*

Proof. We consider two cases.

Case 1: $2 \leq k \leq n-3$. Take $G_1 = K_{2,n-k-1}$ and $G_2 = K_{1,k-1}$. Consider the graph G obtained from G_1 and G_2 by identifying v and z, where $V(G_1) = \{u,v\} \cup \{u_i\}_{i=1}^{n-k-1}$ and $V(G_2) = \{z\} \cup \{v_i\}_{i=1}^{k-1}$ (see Fig. 2.3). Clearly, $\mathrm{Ext}(G) = \{v_i\}_{i=1}^{k-1}$. It is also easy to check that the set $\{u\} \cup \{v_i\}_{i=1}^{k-1}$ is the unique minimum geodetic set of G. Thus, $|\mathrm{Ext}(G)| = k-1 < k = g(G)$.

Case 2: $2 \leq k = n-2$. Take $G_1 = K_2$ and $G_2 = K_{n-2}$. Consider the graph G obtained from G_1 and G_2 by adding the edges v_2v_3 and v_1v_4, where $V(G_1) = \{v_1, v_2\}$ and $V(G_2) = \{v_i\}_{i=3}^{n}$ (see Fig. 2.3). Clearly, $\mathrm{Ext}(G) = \{v_i\}_{i=4}^{n}$. It is also easy to check that the set $\{v_1, v_2\} \cup \{v_i\}_{i=4}^{n}$ is a minimum geodetic set of G. Thus, $|\mathrm{Ext}(G)| = n-4 < n-2 = g(G)$. $\square$

Theorem 2.7 ([54]). *If r, d, and k are integers such that $2 \leq k$ and $r \leq d \leq 2r$, then there exists an extreme geodesic graph G such that $\mathrm{rad}(G) = r$, $\mathrm{diam}(G) = d$, and $h(G) = g(G) = k$.*

Proof. When $r = 1$, we let $G = K_k$ or $G = K_{1,k}$ according to whether $d = 1$ or $d = 2$, respectively. For $r \geq 2$, we construct an extreme geodesic graph G with the desired properties. Take the cycle C_{2r} and the path P_{d-r+1}, where $V(C_{2r}) = \{v_1, \ldots, v_{2r}\}$ and $V(P_{d-r+1}) = \{u_0, u_1, \ldots, u_{d-r}\}$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying $v_1 \in V(C_{2r})$ and $u_0 \in V(P_{d-r+1})$ and adding the edge $v_r v_{r+2}$. Finally, the graph G is then obtained by adding $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ to H and joining all of them to the vertex u_{d-r+1} (see Fig. 2.4). Clearly, G is a graph of radius

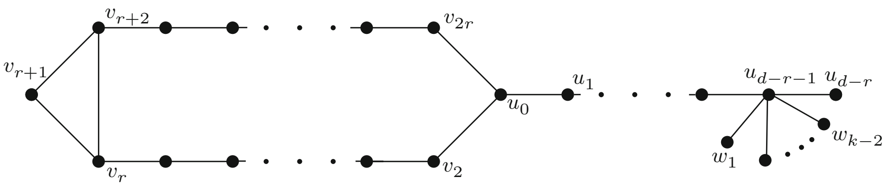

Fig. 2.4 Extreme geodesic graph G s.t. $\mathrm{rad}(G) = r$, $\mathrm{diam}(G) = d$, and $h(G) = g(G) = k$

r and diameter d. It is also easy to check that $\mathrm{Ext}(G) = \{v_{r+1}, w_1, \ldots, w_{k-2}, u_{d-r}\}$ is the sole geodetic set of G. Therefore, $g(G) = |\mathrm{Ext}(G)| = k$, as desired. $\square$

Theorem 2.8 ([54]). *If n, d, and k are integers such that $2 \le d < n$, $2 \le k < n$, and $n - d - k + 1 \ge 0$, then there exists an extreme geodesic graph G of order n, diameter d, and $g(G) = k$.*

Proof. Take the graph $H \cong \overline{K}_k \vee K_{n-d-k+2}$ where $V(\overline{K}_k) = \{v_1, \ldots, v_k\}$ and $V(K_{n-d-k+2}) = \{w_1, \ldots, w_{n-d-k+2}\}$. Consider the path P_{d-1}, where $V(P_{d-1}) = \{u_0, \ldots, u_{d-2}\}$. Let G be the graph obtained from H and P_{d-1} by identifying $v_1 \in V(H)$ and $u_0 \in V(P_{d-1})$. Certainly, G is a graph of order n and diameter d. It is also easy to check that the set $\mathrm{Ext}(G) = \{u_{d-2}, v_2, \ldots, v_k\}$ is the sole geodetic set of G. Therefore, $g(G) = |\mathrm{Ext}(G)| = k$, as desired. $\square$

Conjecture 2.1 ([54]). For any four integers a, b, d, n such that $a \le b \le n-2$, $2 \le b, d$, and $b + d - 1 \le n$, there exists a graph G of order n and diameter d such that $|\mathrm{Ext}(G)| = a$ and $g(G) = b$.

The *geodetic ratio* and *extreme ratio* of a graph G of order n are defined respectively as $r_g(G) = \frac{g(G)}{n}$ and $r_{\mathrm{ext}}(G) = \frac{|\mathrm{Ext}(G)|}{n}$.

Theorem 2.9 ([54]). *For every pair s, t of rational numbers with $0 \le s < t < \frac{1+s}{2} < 1$, there exists a connected graph G with $r_{\mathrm{ext}}(G) = s$ and $r_g(G) = t$.*

Proof. First, we assume that $s > 0$. Let $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$, where s_1, s_2, t_1, t_2 are positive integers. Since $0 < s < t < \frac{1+s}{2} < 1$, it follows that $s_2t_1 - s_1t_2 > 0$ and $s_2t_2 - 2s_2t_1 + s_1t_2 > 0$. Take an even integer $k > 1$ and consider the integers $a = ks_1t_2$, $k(s_2t_1 - s_1t_2)$, and $c = k(s_2t_2 - 2s_2t_1 + s_1t_2)$. Take $a-1$ copies $\{F_i\}_{i=1}^{a}$ of K_2, b copies $\{H_i\}_{i=1}^{b}$ of $K_{2,3}$ and the path P_{c+1}. Consider the graph G obtained from all of these graphs by identifying the vertices $\{y_i\}_{i=1}^{a-1} \cup \{w_{j1}\}_{j=1}^{b} \cup \{v_{c+1}\}$, where $V(F_i) = \{x_i, y_i\}$, $V(H_j) = \{u_{j1}, u_{j2}\} \cup \{w_{j1}, w_{j2}, w_{j3}\}$, and $V(P_{c+1}) = \{v_h\}_{h=1}^{c+1}$. Clearly, the order of G is $n = a + 4b + c = ks_2t_2$. It is also easy to check that $\mathrm{Ext}(G) = \{x_i\}_{i=1}^{a-1} \cup \{v_1\}$ and that the set $S = \{x_i\}_{i=1}^{a-1} \cup \{v_1\} \cup \{u_{j1}, u_{j2}\}_{j=1}^{b}$ is a minimum geodetic set of G. Hence, $|\mathrm{Ext}(G)| = a = ks_1t_2$ and $g(G) = a + 2b = ks_2t_1$, i.e., $r_{\mathrm{ext}}(G) = s$ and $r_g(G) = t$.

As for the case $s = 0$, it is similarly proved, by considering $b-1$ copies of $K_{2,3}$ and a single copy of $K_{2,c}$, where $b = t_1$ and $c = 2t_2 - 4t_1 + 2$. □

Conjecture 2.2 ([54]). Let G be a connected graph such that $r_{\text{ext}}(G) < r_g(G)$. Then, $2r_g(G) < r_{\text{ext}}(G) + 1$.

2.3 Monophonic and *m*-Hull Numbers

For two vertices u and v of a graph G, the *monophonic interval* $J[u,v]$ consists of u,v together with all vertices lying on some chordless $u-v$ path in G. If S is a set of vertices of G, then the *monophonic closure* $J[S]$ consists of S together with all vertices lying on some chordless path joining two vertices of S. If $J[S] = S$, then S is called *m-convex*, and if $J[S] = V(G)$, then S is said to be a *monophonic set*. Similarly to the geodetic case, the smallest m-convex set $[S]_m$ containing S is called the *m-convex hull* of S and an *m-hull set* is a set S such that $[S]_m = V(G)$.

Theorem 2.10 ([83]). *Let $G = (V,E)$ be a graph and $X \subseteq V$ a convex set. Then, X is m-convex if and only if for every pair of nonadjacent vertices $u,v \in X$ and every component of $G - X$ either $V(C) \cap N(u) = \emptyset$ or $V(C) \cap N(v) = \emptyset$.*

Proof. Assume that X is m-convex. The existence of a pair of nonadjacent vertices $u,v \in X$ and a component C of $G-X$ containing a pair u',v' such that $u' \in V(C) \cap N(u)$ and $v' \in V(C) \cap N(v)$ implies the existence of a sequence of vertices $\{w_i\}_{i=0}^{k+1}$ such that $k \geq 2$, $w_0 = u$, $w_1 = u'$, $w_k = v'$, $w_{k+1} = v$, and $\{w_i\}_{i=1}^{k}$ induces a chordless path in C. Hence, there exists a chordless path joining u and v containing at least one vertex outside X, a contradiction.

Conversely, suppose that X in not m-convex. Take a pair of vertices $u,v \in X$ and a chordless $u-v$ path ρ such that $V(\rho) \cap V(G-X) \neq \emptyset$. If $V(\rho) = \{w_i\}_{i=0}^{k+1}$, where $w_0 = u$ and $w_{k+1} = v$, then there exists a pair of indices $r,s \in \{w_i\}_{i=1}^{k}$ such that $r < s$, $w_r \in X$, $w_s \in X$, and $\{w_i\}_{i=r+1}^{s-1} \subseteq V(G-X)$. Hence, w_r, w_s is a pair of nonadjacent vertices in X and there is a component C of $G-X$ such that $V(C) \cap N(w_r) \neq \emptyset$ and $V(C) \cap N(w_s) \neq \emptyset$. □

The *monophonic number* of a graph G, denoted by $m(G)$, is the minimum cardinality of a monophonic set of $V(G)$ [116]. The *m-hull number* of a graph G, denoted by $h_m(G)$, is the minimum cardinality of an m-hull set of $V(G)$.

Certainly, $h_m(G) \leq m(G) \leq g(G)$ and $h_m(G) \leq h(G)$, since every monophonic set is an m-hull set, every geodetic set is monophonic, and every g-hull set is an m-hull set. Nevertheless, it is not true that every g-hull set be monophonic. For example, if $V_1 = \{a,b,c\}$ and $V_2 = \{e,f,g\}$ are the partite sets of the complete bipartite graph $K_{3,3}$, then it is easy to see that the set $W = \{a,b\}$ satisfies $[W]_g = V$ and $J[W] = V \setminus \{c\}$.

At this point, what remains to be done is to ask the following question: *Is there any other general relationship among the parameters $h_m(G)$, $m(G)$, $h(G)$, and $g(G)$,*

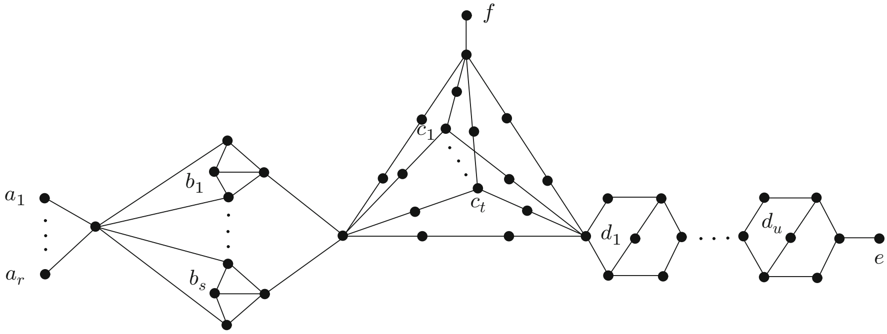

Fig. 2.5 $h_m(G) = r+2$, $m(G) = r+s+2$, $h(G) = r+t+2$, and $g(G) = r+t+u+2$

apart from the previous known inequalities? The following realization theorem shows that, unless we restrict ourselves to a specific class of graphs, the answer is negative.

Theorem 2.11 ([119]). *For any integers a, b, c, d such that* $3 \le a \le b \le c \le d$*, there exists a connected graph* $G = (V,E)$*, satisfying one of the following conditions:*

1. $a = h_m(G)$, $b = m(G)$, $c = h(G)$, *and* $d = g(G)$.
2. $a = h_m(G)$, $b = h(G)$, $c = m(G)$, *and* $d = g(G)$.

Proof. Let $G = (V,E)$ be the connected graph shown in Fig. 2.5. We consider the following subsets of vertices: $W_1 = \mathrm{Ext}(G) = \{a_1,\ldots,a_r,e,f\}$, $W_2 = W_1 \bigcup\{b_1,\ldots,b_s\}$, $W_3 = W_1 \bigcup\{c_1,\ldots,c_t\}$, and $W_4 = W_2 \bigcup\{c_1,\ldots,c_t\} \bigcup\{d_1,\ldots,d_u\}$. Next, we show that W_1 is a minimum m-hull set, W_2 is a minimum monophonic set, W_3 is a minimum g-hull set, and W_4 is a minimum geodetic set.

(i) W_1 is a minimum m-hull set. It is easy to see that every vertex $v \in V \setminus \{b_1,\ldots,b_s\}$ lies on some $a_i - e$ monophonic path. Hence, $J[W_1] = V \setminus \{b_1,\ldots,b_s\}$. Given a vertex b_i, we take $u_i, v_i \in N(b_i)$ such that $d(v_i,u_i) = 2$. Clearly, $b_i \in J[J[W_1]] = J^2[W_1]$ since $u_i b_i v_i$ is a monophonic path and $u_i, v_i \in J[W_1]$. As a consequence, we have proved that $[W_1]_m = J^2[W_1] = V$. This means that W_1 is an m-hull set of minimum cardinality since every m-hull set must contain all the simplicial vertices of the graph.

(ii) W_2 is a minimum monophonic set. From (i), we immediately conclude that W_2 is a monophonic set. In order to prove that W_2 is a minimum monophonic set, it is enough to remark the following fact. If W is a monophonic set of G, then for $i = 1,\ldots,s$ either $b_i \in W$ or $u_i, v_i \in W$ because, in any other case, every path containing b_i must also contain a chord.

(iii) W_3 is a minimum hull set. Notice that:

$$I[W_1] = V \setminus \{b_1,\ldots,b_s,c_1,\ldots,c_t,d_1,\ldots,d_u\} \cup N(c_1) \cup \ldots \cup N(c_t),$$

$$I[I[W_1]] = I^2[W_1] = V \setminus \{c_1,\ldots,c_t\} \cup N(c_1) \cup \ldots \cup N(c_t) = [\mathrm{Ext}(G)]_g.$$

Next, observe that every g-hull set W satisfies, first, $\mathrm{Ext}(G) \subseteq W$ and, second, $W \cap (c_i \cup N(c_i)) \neq \emptyset$, for every $i = 1, \ldots, t$. Hence, W_3 is a g-hull set of minimum cardinality.

(iv) W_4 is a minimum geodetic set. It is easy to see that every vertex $v \in V$, v lies on some $a_i - e$ geodesic joining two vertices of W_4. By the other hand, if we consider W as a geodetic set, then (a) either $b_i \in W$ or $u_i, v_i \in W$, $i = 1, \ldots, s$; (b) either $c_j \in W$ or $z_j \in W$ for some $z_j \in N(c_j)$, $j = 1, \ldots, t$; and (c) either $d_h \in W$ or $N(d_h) \subset W$, $h = 1, \ldots, u$. Hence, W_4 is a minimum geodetic set.

As a consequence, we have proved that $h_m(G) = r+2$, $m(G) = r+s+2$, $h(G) = r+t+2$, and $g(G) = r+s+t+u+2$, from which both statements of the theorem immediately follow. □

2.4 Convexity Number

The *convexity number*[1] of a connected graph G, denoted by $\mathrm{con}(G)$, is the maximum cardinality of a proper convex set of G [48,64]. The *clique number* $\omega(G)$ of a graph G is the maximum order of a clique in G. The *independence number* $\alpha(G)$ is the cardinality of a largest independent set, i.e., $\alpha(G) = \omega(\overline{G})$. Every non-complete graph G satisfies $2 \leq \omega(G) \leq \mathrm{con}(G) \leq n-1$, as the vertex set of every clique is convex. Moreover, $\mathrm{con}(G) = n-1$ if and only if G contains at least a simplicial vertex v since, as was shown in Proposition 2.2, $V(G) - v$ is a convex set if and only if $N(v)$ induces a clique.

Theorem 2.12 ([36]). *Let G be a graph of diameter* $\mathrm{diam}(G) = 2$. *If its complement $\overline{G}$ is not connected and at least two of the components of $\overline{G}$ are nontrivial, then* $\mathrm{con}(G) = \omega(G)$.

Proof. Let $\{H_i\}_{i=1}^k$ de components of $\overline{G}$. Certainly, $G = \vee_{i=1}^k \overline{H}_i$. Let C be a maximum proper convex set in G. Let $S_i = C \cap V(H_i)$, for every $i \in [k]$. To end the proof, it is enough to show that if $S_i \neq \emptyset$, then S_i induces a clique in $\overline{H}_i$. Suppose, to the contrary, that $S_1 \neq \emptyset$ and there exist two distinct vertices $u, v \in S_1$ such that $d_G(u,v) = 2$. Hence, $V(G) \setminus V(\overline{H}_1) \subseteq I_G[u,v] \subseteq C$. By hypothesis, there exists an index $j \neq 1$ such that H_j is nontrivial. Since H_i is connected, there exist two distinct vertices $a, b \in V(H_j)$ which are adjacent in H_j, i.e., a and b are not adjacent in $\overline{H}_j$. Hence, $d_G(a,b) = 2$ and both a and b are vertices in C adjacent to all vertices of $\overline{H}_1$, which means that $C = V(G)$, a contradiction. □

The converse of the preceding theorem is not always true. Take, for example, de graph $G \cong K_2 \vee \overline{K}_2$, and check that $\mathrm{diam}(G) = 2$, $\overline{G} = \overline{K}_2 + K_2$, and $\mathrm{con}(G) = \omega(G) = 3$.

[1] With respect to the geodesic convexity.

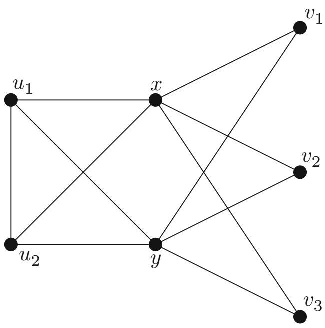

Fig. 2.6 $(K_2+\overline{K}_3)\vee\overline{K}_2$

It remains an *open problem* to characterize the family of graphs of diameter 2 for which the convexity number and the clique number are equal.

A graph G of order n such that $\omega(G)=l$ and $\mathrm{con}(G)=k$ is called an (l,k,n)-graph. The *clique ratio* and *convexity ratio* of an (l,k,n)-graph G are defined respectively as $r_\omega(G)=\frac{\omega(G)}{n}$ and $r_{\mathrm{con}}(G)=\frac{\mathrm{con}(G)}{n}$.

Theorem 2.13 ([64]). *For every triple l,k,n of integers with $2\le l\le k\le n-1$, there exists a non-complete (l,k,n)-graph.*

Proof. We consider two cases.

Case 1: $l=k$. For $k=2$ and $k=n-1$, the graphs $K_{2,n-2}$ and K_n-e, where $e\in E(K_n)$, have, respectively, the desired properties. So, we assume that $3\le k\le n-2$. Consider the graph $G=(K_{k-1}+\overline{K}_{n-k-1})\vee\overline{K}_2$ (in Fig. 2.6 the case $n=7$ and $k=3$ is shown). Clearly, G is a connected graph of order n and $\omega(G)=k\le\mathrm{con}(G)$. Let S be a maximum proper convex set of G. If $V(\overline{K}_2)=\{x,y\}$, then $|S\cap\{x,y\}|\le 1$, since $I[x,y]=V(G)$. We claim that $S\cap V(\overline{K}_{n-k-1})=\emptyset$. Assume, to the contrary, that this is not the case. First assume that S contains at least two vertices, say v_1,v_2, of $V(\overline{K}_{n-k-1})$. Then, $x,y\in I[v_1,v_2]$, and so $I[S]=V(G)$, a contradiction. Hence, S contains exactly one vertex of $V(\overline{K}_{n-k-1})$, say v_1. Since $k\ge 3$, it follows that S contains at least two distinct vertices $u,v\notin V(\overline{K}_{n-k-1})$. We may assume w.l.o.g. that $u\notin\{x,y\}$, as $|S\cap\{x,y\}|\le 1$. Since x and y lie on a $u-v_1$ geodesic, it follows that $x,y\in I[u,v_1]$, again a contradiction. Hence $S\cap V(\overline{K}_{n-k-1})=\emptyset$, as claimed. Because S contains at most one of x and y, $\mathrm{con}(G)=|S|\le k$ and so $\mathrm{con}(G)=k$.

Case 2: $l<k$. Take the graph $F=(K_{l-1}+\overline{K}_{k-l-1})\vee\overline{K}_2$, where $V(K_{l-1})=\{u_i\}_{i=1}^{l-1}$, $V(\overline{K}_{k-l-1})=\{v_i\}_{i=1}^{k-l-1}$, and $\overline{K}_2=\{x,y\}$. We consider two cases.

Subcase 2.1: $n-3\le k\le n-1$. If $k=n-1$, let F_1 be the graph obtained from F by adding a new vertex r and the pendant edge xr. If $k=n-2$, let F_2 be the graph obtained from F by adding two new vertices r,s and the edges xr,rs,sy. If $k=n-3$, let F_3 be the graph obtained from F by adding three new vertices r,s,t and the edges xr,rs,st,ty. It is a routine exercise to check that for $1\le i\le 3$, F_i is a graph of order n with $\omega(F_i)=\omega(F)=l$ and $\mathrm{con}(F_i)=k$.

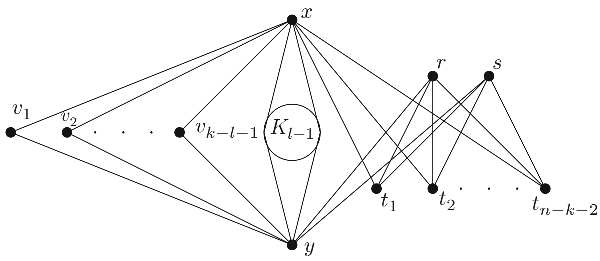

Fig. 2.7 Graph G of order $n \geq k+4$ with $\omega(G) = l$ and $\mathrm{con}(G) = k$

Subcase 2.2: $k \leq n-4$. Take the graphs F and $H = K_{2,n-k-2}$. Let G be the graph obtained from F and H by adding the edges $\{xt_i\}_{i=1}^{n-k-2} \cup \{yr, ys\}$ (see Fig. 2.7). Clearly, G is a graph of order n with $\omega(G) = l$. Observe that $V(F)$ is a convex set of G of order k. It remains thus only to show that $V(F)$ is a maximum proper convex of G. Assume, to the contrary, that S' is proper convex set of G with $|S'| \geq k+1$. Hence, $S' \cap V(H) \neq \emptyset$. Notice that $[r,s] = V(G)$ and, for every pair of distinct indexes $i, j \in \{1, \ldots, n-k-2\}$, $[t_i, t_j] = V(G)$. Thus, $1 \leq |S' \cap V(H)| \leq 2$, i.e., $k+1 \leq |S'| \leq k+2$ and $k-1 \leq |S' \cap V(F)| \leq k$. If we suppose that $x \in S'$ (resp. t_1), then $r \notin S'$ (resp. $s \notin S'$), as $[x,r] = V(G)$ (resp. $[y,t_1] = V(G)$). In other words, in all cases we arrive at a contradiction. $\square$

Corollary 2.2 ([61]). *For every pair s,t of rational numbers with $0 < s \leq t < 1$, there exists a graph with $r_\omega(G) = s$ and $r_{\mathrm{con}}(G) = t$.*

In the proof of Theorem 2.13, all the (l,k,n)-graphs constructed for $l \geq 3$ have the properties that there are exactly two maximum cliques, a unique maximum convex set S, and all vertices of both maximum cliques are in S. At this point, two natural questions arise: (1) Do (l,k,n)-graphs with arbitrarily many maximum convex sets exist? (2) Do (l,k,n)-graphs in which no maximum clique set is contained in any maximum convex set exist? In [61], both questions were partially answered.

Theorem 2.14 ([61]). *Let s,t be rational numbers with $0 < s < t < 1$ and let M be a positive integer. Then,*

1. *If $t \leq \frac{1}{2}$, then there exists a graph G with $r_\omega(G) = s$ and $r_{\mathrm{con}}(G) = t$, such that G contains at least M distinct maximum convex sets.*
2. *If $s+t < 1$ and $t > \frac{1}{2}$, then there exists a graph G with $r_\omega(G) = s$ and $r_{\mathrm{con}}(G) = t$, such that if Ω is the set of vertices of a maximum clique in G and S is a maximum convex set in G, then $d(\Omega, S) = \min\{d(u,v) : u \in \Omega, v \in S\} \geq M$.*

Sketch of proof.

1. Take $s = \frac{a}{b}$ and $t = \frac{c}{d}$ such that $0 < s+t < 1$ and $t \leq \frac{1}{2}$, where a,b,c,d are integers such that $0 < a < b$ and $0 < c < d$. Take $p = r(bc-ad)$ and $q = r(bd+$

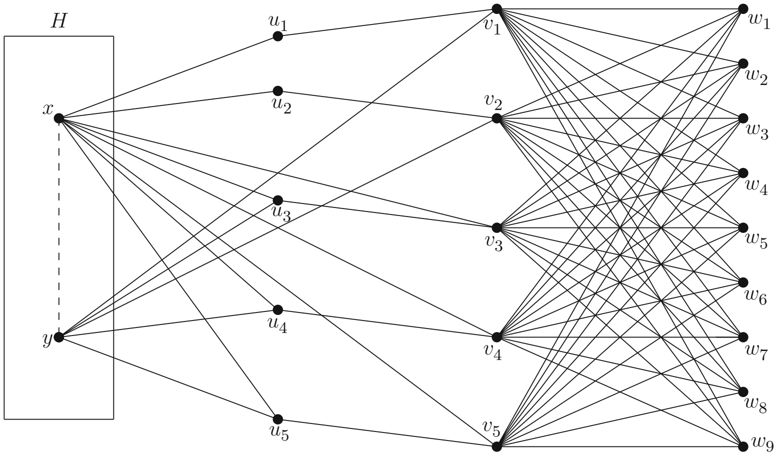

Fig. 2.8 A graph G with $r_\omega(G) = 1/3$ and $r_{\mathrm{con}}(G) = 1/2$. Note that $V(H) \cup \{u_3, u_4, u_5, v_3\}$, $V(H) \cup \{u_3, u_4, u_5, v_4\}$, and $V(H) \cup \{u_3, u_4, u_5, v_5\}$ are maximum convex sets

$ad - 2bc) - 1$, where $r = \lceil \max\{5, M+2\} \rceil$. Check that $p \geq \max\{5, M+2\}$ and $q \geq 4$. Consider the graph $H \cong K_{\mathrm{rad}+1} - e$ where $e = xy \in E(K_{\mathrm{rad}+1})$, the complete bipartite graph $F \cong K_{p,q}$ and the null graph $K \cong \overline{K}_p$.

Construct the graph $G = (V, E)$ such that $V(G) = V(H) \cup V(F) \cup V(K)$ and $E = E(H) \cup E(F) \cup E(K) \cup \{u_i v_i\}_{i=1}^{p} \cup \{xv_i, yu_i\}_{i=3}^{p} \cup \{yv_1, yv_2\}$, where $V(K) = \{u_i\}_{i=1}^{p}$ and $\{v_i\}_{i=1}^{p}$ is the stable set of F of order p. In Fig. 2.8, the graph G when $s = 1/3$, $r = 1/2$, and $M = 3$ is displayed.

Observe that G is a graph of order rbd and clique number $\omega(G) = \mathrm{rad}$. Check that, for $i = 3, \ldots, p$, the set $S_i = V(H) \cup \{u_3, \ldots, u_p, v_i\}$ is a convex set of G. Thus, $\mathrm{con}(G) \geq |S_i| = rbc$. Next, notice that every convex set of G cannot contain a pair of nonadjacent vertices of H. Finally, check that every set of cardinality at least $rbc + 1$ is not convex.

2. Take $s = \frac{a}{b}$ and $t = \frac{c}{d}$ such that $0 < s + t < 1$ and $t > \frac{1}{2}$, where a, b, c, d are integers such that $0 < a < b$ and $0 < c < d$. Choose an integer $r \geq 5M$ sufficiently large so that $t > \frac{1}{2} + \frac{1}{rbd}$. Consider the graph $H_0 \cong K_{\mathrm{rad}+1} - e$ where $e = x_0 y_0 \in E(K_{\mathrm{rad}+1})$. Let $h_1, h_2, \ldots, h_l$ be integers such that $3 \leq h_i \leq 5$ for every $i \in \{1, \ldots, l\}$ and $\sum_{i=1}^{l} h_i = r(bd - bc - ad) \geq r$. Since $r \geq 5M$, it follows that $l \geq r/5 \geq M$. For every $i \in \{1, \ldots, l\}$, consider the graph $H_i \cong K_{h_i} - e_i$, where $e_i = x_i y_i \in E(K_{h_i})$. Take the graph $F \cong K_{2,q}$, where $q = rbc - 3$.

   Construct the graph $G = (V, E)$ such that $V(G) = \cup_{i=0}^{l} V(H_i) \cup V(F)$ and $E = \cup_{i=0}^{l} E(H_i) \cup E(F) \cup \{x_i x_{i+1}, y_i x_{i+1}, x_i y_{i+1}, y_i y_{i+1}\}_{i=o}^{l-1} \cup \{x_l u_1, y_l u_1, x_l u_2, y_l u_2\}$, where $\{u_1, u_2\}$ is the stable set of F of order 2 (see Fig. 2.9).

   Observe that G is a graph of order rbd and clique number $\omega(G) = \mathrm{rad}$. Notice that G has two maximum cliques, namely, $\Omega_1 = V(H_0) \setminus \{x_0\}$ and $\Omega_2 = V(H_0) \setminus \{y_0\}$. Check that $S = V(F) \cup \{x_l\}$ is the unique maximum convex set of G. Finally, observe that $d(\Omega_1, S) = d(\Omega_2, S) = l \geq M$. □

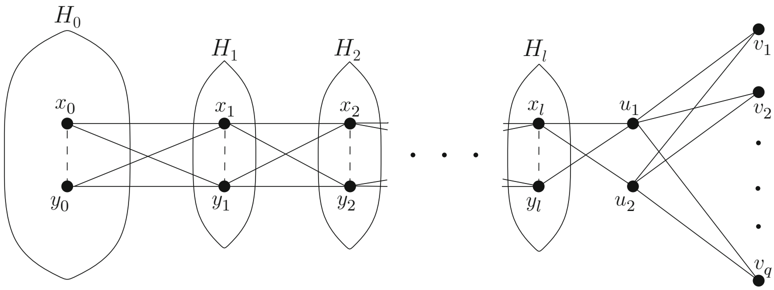

Fig. 2.9 A graph G such that $r_\omega(G) = s$ and $r_{\text{con}}(G) = t$, where $s+t < 1$ and $t > \frac{1}{2}$

Table 2.2 Clique and convexity numbers of some basic graph families

G[a]	P_n	C_{2l}	C_{2l+1}	T_n	K_n	$K_{p,q}$[b]	W_n[c]	Q_n
$\omega(G)$	2	2	2	2	n	2	3	2
$\text{con}(G)$	$n-1$	l	$l+1$	$n-1$	$n-1$	2	$n-2$	2^{n-1}

[a] $G \ncong K_1$
[b] $2 \le p \le q$
[c] $n \ge 5$

Question 2.1 ([61]). Are conditions $t \le \frac{1}{2}$ and $t > \frac{1}{2}$ of Theorem 2.14 needed?

A graph G is called *polyconvex* if for every integer i with $1 \le i \le \text{con}(G)$, there exists a convex set of cardinality i in G. All the graph families displayed in Table 2.2 are formed by polyconvex graphs, except the family of hypercubes [48].

Proposition 2.3 ([48]). *For $n \le 2$, a set S in the n-dimensional hypercube Q_n is convex if and only if S induces a k-dimensional hypercube Q_k in Q_n, for some integer k with $0 \le k \le n$. Hence, for every $n > 2$, Q_n is not polyconvex.*

Proof. If S is a set of vertices in Q_n that induces a Q_i for some $i \in [n]$, then S is convex in Q_n. So it remains only to prove the converse. We proceed by induction on n. The result is certainly true for $n = 2$. Assume that every convex set in Q_k ($k \ge 2$) induces a Q_i for some $i \in [k]$.

Let Q_{k+1} be a $(k+1)$-cube formed from two copies G_1 and G_2 of Q_k, i.e., $Q_{k+1} = Q_k \square K_2$, $V(G_1) = Q_k \times \{1\}$ and $V(G_2) = Q_k \times \{2\}$. Let S be a convex set in Q_{k+1}. If either $S = V(Q_{k+1})$ or $S \subseteq V(G_1)$ for $i = 1$ or $i = 2$, then the result holds. Thus we may assume that $S = S_1 \cup S_2$, where $S_1 = S \cap V(G_1)$ and $S_2 = S \cap V(G_2)$ are both nonempty. Certainly S_i is convex in G_i, for $i = 1,2$. Hence, by induction hypothesis, $G[S_1] = Q_r$ and $G[S_2] = Q_s$, where $0 \le r,s \le k$. Take $u_1 \in S_1$ and $v_2 \in S_2$. If u_2 is the neighbor of u_1 in G_2 and v_1 is the neighbor of v_2 in G_1, then $\{u_2, v_1\} \subseteq I[u_1, v_2]$. Hence, $r = s$ and $S = Q_{r+1}$. □

Theorem 2.15 ([48]). *Given any pair of integers n,k with $2 \le k \le n-1$, there exists a polyconvex connected graph of order n with* $\text{con}(G) = k$.

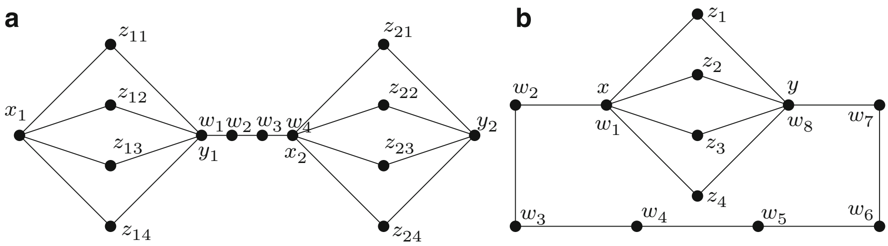

Fig. 2.10 (**a**) Graph G s.t. $n = 14$ and $\mathrm{con}(G) = k = 10$, (**b**) Graph G s.t. $n = 12$ and $\mathrm{con}(G) = k = 8$

Proof. For $k = n-1$, take $G = K_n$. So we may assume that $2 \le k \le n-2$. Consider the graph $G = (K_{k-1} + \overline{K}_{n-k-1}) \vee \overline{K}_2$ (in Fig. 2.6 the case $n = 7$ and $k = 3$ is shown). In the proof of Theorem 2.13 it was shown that G is a graph of order n with $\mathrm{con}(G) = k$. Finally, if $V(K_{k-1}) = \{u_i\}_{i=1}^{k-1}$ and $V(\overline{K}_2) = \{x, y\}$, notice first that $\{u_i\}_{i=1}^{k-1} \cup \{x\}$ induces a clique and hence it is a convex set of order k, and, second, that, for every $j \in \{1, \ldots, k-1\}$, $\{u_i\}_{i=1}^{j}$ induces a clique and thus it is a convex set of order j. $\square$

Theorem 2.16 ([124]). *Given any pair of integers n, k with $2 \le k \le n-1$, there exists a polyconvex connected K_3-free graph of order n with* $\mathrm{con}(G) = k$.

Proof. For $k = 2$, take $G = K_{2,n-2}$. So we may assume that $3 \le k \le n-1$. We consider three cases.

Case 1: $k+1 \le n \le \frac{3}{2}k - \frac{1}{2}$. If $3 \le k \le 4$, P_4 and P_5. Suppose thus that $k \ge 5$. Take two copies G_1 and G_2 of $K_{2,n-k}$ and a copy G_3 of P_{2k-n-2}. Let G be the graph obtained from G_1, G_2, and G_3 by identifying the vertices w_1 and w_{2k-n-2} with y_1 and x_2, respectively, where $V(G_i) = \{x_i, y_i\} \cup \{z_{ij}\}_{j=1}^{n-k}$ and $V(G_3) = \{w_i\}_{i=1}^{2k-n-2}$ (in Fig. 2.10a the case $n = 14$ and $k = 10$ is shown). Notice (1) G is a K_3-graph of order n, (2) $2k-n-2 \ge 0$, and (3) if $n = k+1$, then G is P_n. We form convex sets of orders 1 through $2k-n-2$ by taking a subpath of appropriate length from G_3. We form convex sets of orders $2k-n-1$ and $2k-n$ by taking all the vertices of G_3 and adding at most one vertex from each of the partite classes of order $n-k$ of G_1 and G_2. We form a convex set of orders $n-k+2$ to $k-1$ by taking all the vertices of G_1 together with an appropriate number of adjacent vertices of G_3. Finally, we form a convex set of order k by taking all the vertices of G_1 and G_3 and one more vertex from the partite class of order $n-k$ of G_2. Since $(n-k+2) - (2k-n) \le 1$, we have found convex sets of all orders between 1 and $k = \mathrm{con}(G)$.

Case 2: $\frac{3}{2}k \le n \le 2k-1$. Take a copy G_1 of $K_{2,2k-n}$ and a copy G_2 of P_{2n-2k}. Let G be the graph obtained from G_1 and G_2 by identifying the vertices w_1 and w_{2n-2k} with x and y, respectively, where $V(G_1) = \{x, y\} \cup \{z_i\}_{i=1}^{2k-n}$ and $V(G_2) = \{w_i\}_{i=1}^{2n-2k}$ (in Fig. 2.10b the case $n = 12$ and $k = 8$ is shown). Notice (1) G is a K_3-graph of order n, (2) $2k-n \ge 1$ and (3) if $n = 2k-1$ then G is C_n. We form convex sets of orders 1 through $n-k+1$ by taking a subpath of appropriate length from G_2. We form convex sets of orders $2+2k-n$ through k by taking all the vertices of G_1 and adding suitable numbers of adjacent vertices of G_2. Since $(2+2k-n)-(n-k+1) \leq 1$, we have found convex sets of all orders between 1 and $k = \mathrm{con}(G)$.

Case 3: $2k \leq n$. Let $m = \lfloor \frac{n-2k+4}{2} \rfloor$. Note that $m \geq 2$ since $n \geq 2k$. Start with a copy G_1 of $K_{m,m}$ or $K_{m,m+1}$, depending on whether $n-2k+4$ is even or odd. Let G be the graph obtained by replacing one edge of G_1 with a set Ω of $k-2$ paths of length 3. Let v be one of the vertices of Ω of degree greater than 2. Let S be the set of vertices consisting of v, all the neighbors of v in Ω and one more neighbor of v. Then S is a (maximum) convex set of order k. Convex sets of all smaller orders occur as subsets of S containing v. Notice that when $k = 3$ and $n = 7$, then this construction does not work. For this case, take the graph $K_{3,3}$ and replace one edge with a path of length 2. Then the three vertices of this path form a largest convex set. □

Finally, the following Nordhaus–Gaddum type result has been obtained.

Theorem 2.17 ([48, 104]). $2\log_4 n \leq \mathrm{con}(G) + \mathrm{con}(\overline{G}) \leq 2n-2$

Proof. The upper bound is obvious since for every graph G, $\mathrm{con}(G) \leq n-1$. To prove the lower bound we use the well-known result that, for every pair of positive integers h,k, the Ramsey number $R(h,k)$ is bounded above by $\binom{h+k-2}{h-1}$ (see Corollary 12.18 of [71]). In particular, $R(h,k) \leq 2^{h+k-2}$. Notice that if G is a graph of order n, then $R(\omega(G)+1, \omega(\overline{G})+1) > n$. Suppose that there is some graph G of order n such that $2\log_4 n > \mathrm{con}(G) + \mathrm{con}(\overline{G})$. Hence, $R(\omega(G)+1, \omega(\overline{G})+1) \leq 2^{\omega(G)+\omega(\overline{G})} \leq 2^{\mathrm{con}(G)+\mathrm{con}(\overline{G})} < 2^{2\log_4 n} = n$, a contradiction. □

Further results involving ideas, concepts, and invariants appearing in Sects. 2.1–2.4 can be found in [16, 19, 22, 48, 51, 54, 57, 59, 61, 63–65, 69, 72, 73, 75, 82, 98, 99, 101, 113, 119, 124, 154, 157, 160, 161, 163, 170].

2.5 Forcing Geodomination

Let S be a minimum geodetic set of a graph G. A subset T of S is called a *g-forcing* subset of S, if S is the unique minimum geodetic set that contains T. The minimum size of a forcing subset of a minimum geodetic set in G is called the *forcing geodetic number* $f_g(G)$ of G.

Let S be a minimum hull set of a graph G. A subset T of S is called an *h-forcing* subset of S, if S is the unique minimum hull set that contains T. The minimum size of an h-forcing subset of a minimum hull set in G is called the *forcing hull number* $f_h(G)$ of G.

As an immediate consequence of these definitions, the following properties hold.

Proposition 2.4. *Let G be a connected graph G. Then,*

- $f_g(G) \leq g(G)$ *and* $f_h(G) \leq h(G)$.

Table 2.3 Forcing and standard parameters of some basic graph families

G[a]	P_n	C_{2l}	C_{2l+1}	T_n^k	K_n	$K_{p,q}$[b]	Q_n
$f_h(G)$	0	1	0	0	0	2	1
$h(G)$	2	2	3	k	n	2	2
$f_g(G)$	0	1	0	0	0	4	1
$g(G)$	2	2	3	k	n	4	2
$f_c(G)$	1	2	2	$k-1$	$n-1$	2	2
$\mathrm{con}(G)$	$n-1$	l	$l+1$	$n-1$	$n-1$	2	2^{n-1}

[a] $G \ncong K_1$
[b] $5 \leq p \leq q$

- *$f_g(G) = 0$ (resp. $f_h(G) = 0$) if and only if G has a unique minimum geodetic set (resp. hull set).*
- *$f_g(G) = 1$ (resp. $f_h(G) = 1$) if and only if G has at least two minimum geodetic sets (resp. hull sets) and there exists a vertex belonging to exactly one minimum geodetic set (resp. hull set).*
- *$f_g(G) \geq 2$ (resp. $f_h(G) \geq 2$) if and only if G each vertex of each minimum geodetic set (resp. hull set) belongs to a least one more minimum geodetic set (resp. hull set).*
- *There are graphs G satisfying $f_g(G) < f_h(G)$. For example, if $2 < s$, then $f_h(K_{2,s}) = 1$ and $f_g(K_{2,s}) = 0$.*

In Table 2.3, both the forcing geodetic number (resp. forcing hull number) and the geodetic number (resp. hull number) of some basic graphs are displayed.

Theorem 2.18 ([49]). *Given any pair of integers a,b with $0 \leq a \leq b$, $2 \leq b$ and $(a,b) \neq (2,2)$, there exists a graph G such that $f_g(G) = a$ and $g(G) = b$.*

Proof. First check that if $g(G) = 2$, then $f_g(G) < 2$. If $a = 0$, take $G \cong K_b$. Next, suppose that $0 < a < b$. We distinguish three cases.

Case 1: $a = 1$. If $b = 2$, take any even cycle. Suppose that $b \geq 3$. Consider the graph G_1 in Fig. 2.11. Notice that $\{u_1,\dots,u_{b-2},v_3,x\}$ is a minimum geodetic set of G_1 and thus $g(G_1) = b$. To prove that $f_g(G_1) = 1$, notice first that $X = \{u_1,\dots,u_{b-2},v_3,x\}$ and $Y = \{u_1,\dots,u_{b-2},v_3,y\}$ are two distinct minimum geodetic sets and second that X is the unique minimum geodetic set containing $\{x\}$.

Case 2: $a > 1$ and $b = a+1$. Consider the graph G_2 in Fig. 2.11. Notice that $\{u_2,\dots,u_{a+1},v_1\}$ is a minimum geodetic set of G_2 and thus $g(G_2) = a+1$. Let W be a minimum geodetic set of G_2. Since $W \cap \{u_i\}_{i=1}^{a+1} \neq \emptyset$, $W \cap \{v_i\}_{i=1}^{a+1} \neq \emptyset$ and, for every $i \in \{1,\dots,a+1\}$, $W \cap \{u_i,v_i\} \neq \emptyset$, it follows that W is not the unique minimum geodetic set containing any of its subsets W' with $|W'| < a$. Thus $f_g(G_2) \geq a$. On the other hand, $W = \{u_2,\dots,u_{a+1},v_1\}$ is the unique minimum geodetic set containing $\{u_2,\dots,u_{a+1}\}$, which means that $f_g(G_2) = a$.

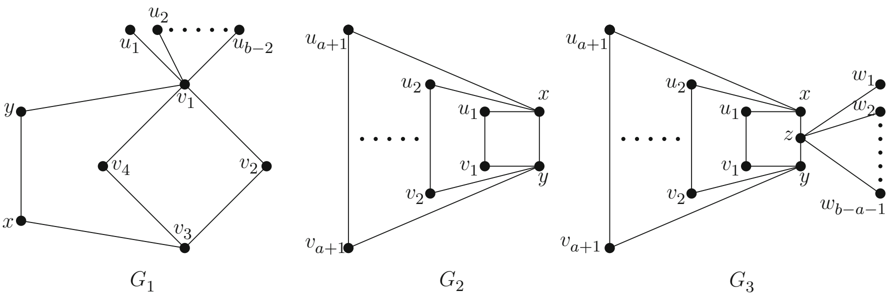

Fig. 2.11 Case 1: $f_g(G_1) = 1$, $g(G_2) = b$. Case 2: $f_g(G_2) = g(G_2) - 1 = a$. Case 3: $f_g(G_3) = a$, $g(G_3) = b$

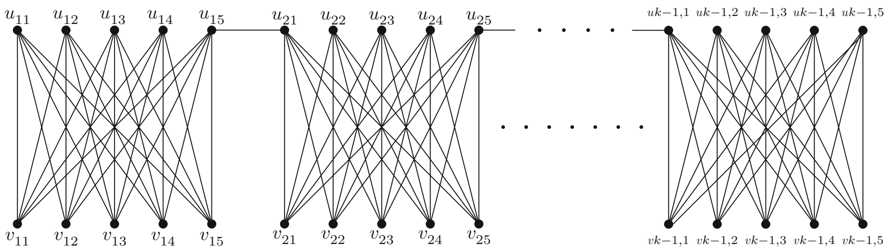

Fig. 2.12 Graph G_{2k} with $f_g(G_{2k} = g(G_{2k}) = a$

Case 3: $a > 1$ and $b \geq a+2$. Consider the graph G_3 in Fig. 2.11. Notice that $\{u_2, \ldots, u_{a+1}, v_1, w_1, \ldots, w_{b-a-1}\}$ is a minimum geodetic set of G_3 and thus $g(G_3) = b$. Next, notice that every minimum geodetic set of G_3 has the form $S = U \cup V \cup W$ where $U \subseteq \{u_i\}_{i=1}^{a+1}$, $V \subseteq \{v_i\}_{i=1}^{a+1}$, and $W = \{w_i\}_{i=1}^{b-a-1}$, with $U \neq \emptyset$ and $V \neq \emptyset$. This implies that if S is a minimum geodetic set of G_3, then it is not the unique minimum geodetic set containing any of its subsets W' with $|W'| < a$. Thus $f_g(G_3) \geq a$. On the other hand, $\{u_2, \ldots, u_{a+1}, v_1, w_1, \ldots, w_{b-a-1}\}$ is the unique minimum geodetic set containing $\{u_2, \ldots, u_{a+1}\}$, which means that $f_g(G_3) = a$.

Finally, assume that $3 \leq a = b$. We distinguish three cases.

Case 1: $a = 2k$ where $k \geq 2$. For $k = 2$, take $K_{5,5}$. Suppose that $k \geq 3$. Consider the graph G_{2k} obtained from $k-1$ copies $F_1, \ldots, F_{k-1}$ of $K_{5,5}$ by adding $k-2$ edges, namely, the set $\{u_{i5}u_{i+1,1}\}_{i=1}^{k-2}$, where, for every $i \in \{1, \ldots, k-1\}$, $V(F_i) = U_i \cup V_i = \{u_{ij}\}_{j=1}^{5} \cup \{v_{ij}\}_{j=1}^{5}$ (see Fig. 2.12). Notice that every minimum geodetic set of G_{2k} contains exactly one vertex from each of $U_1 \setminus \{u_{15}\}$ and $U_{k-1} \setminus \{u_{k-1,1}\}$ and exactly two vertices from each set V_i, $1 \leq i \leq k-1$. Therefore, $f_g(G_{2k}) = g(G_{2k}) = 2k = a$.

Case 2: $a = 2k+1$ where $k \geq 1$. For $k = 1$, take the graph H in Fig. 2.13. For $k = 2$, take the graph $C_5 \circ \overline{K}_5$. Suppose that $k \geq 3$. Let F_1' the graph obtained from

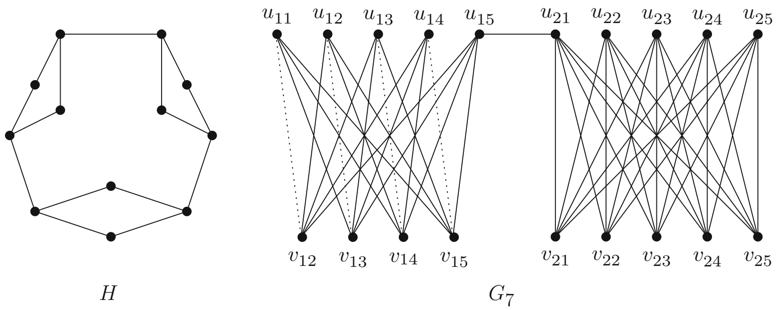

Fig. 2.13 $f_g(H) = g(H) = 3$ and $f_g(G_7) = g(G_7) = 7$

$K_{5,4}$ by deleting a maximum matching. Consider the graph G_{2k+1} constructed from the graph G_{2k+1} in Case 1 by replacing F_1 by F_1' (in Fig. 2.13 the graph G_7 is depicted). Notice that every minimum geodetic set of G_{2k} contains exactly two vertices from $U_1 \setminus \{u_{15}\}$, one vertex from $U_{k-1} \setminus \{u_{k-1,1}\}$ and exactly two vertices from each set V_i, $1 \leq i \leq k-1$. Therefore, $f_g(G_{2k}) = g(G_{2k}) = 2k+1 = a$. $\square$

In a quite similar way it is proved the corresponding realization theorem involving both hull parameters. Notice that, in this case, the equality $(a,b) = (2,2)$ is possible, as, for example, $f_h(K_{3,3}) = h(K_{3,3}) = 2$.

Theorem 2.19 ([53]). *Given any pair of integers a,b with $0 \leq a \leq b$ and $2 \leq b$, there exists a graph G such that $f_h(G) = a$ and $h(G) = b$.*

Lemma 2.1 ([171]). *Let m,t be positive integers with $m \geq 2$. Let $\{G_i\}_{i=1}^m$ be a family of m disjoint connected graphs of order at least $t+1$, such that each of them contains a t-set S_i such that $N_{G_i}[S_i]$ induces a clique. Let G be the graph obtained from $\{G_i\}_{i=1}^m$ by identifying $S_1, S_2, \ldots, S_m$. Then,*

(a) $h(G) = \sum_{i=1}^{m} h(G_i) - mt$.

(b) $g(G) = \sum_{i=1}^{m} g(G_i) - mt$.

(c) $f_h(G) = \sum_{i=1}^{m} f_h(G_i)$.

(d) $f_g(G) = \sum_{i=1}^{m} f_g(G_i)$.

Sketch of proof. To prove (a) and (b), check that a set T is a minimum hull (resp. geodetic) set of G if and only if $(T \cap V(G_i)) \cup S_i$ is a minimum hull (resp. geodetic) set of G_i for $i = 1, 2, \ldots, m$.

To prove (c) and (d) check that a set F is a forcing hull (resp. geodetic) set of G if and only if $F \cap V(G_i)$ is a forcing hull (resp. geodetic) set of G_i for $i = 1, 2, \ldots, m$. $\square$

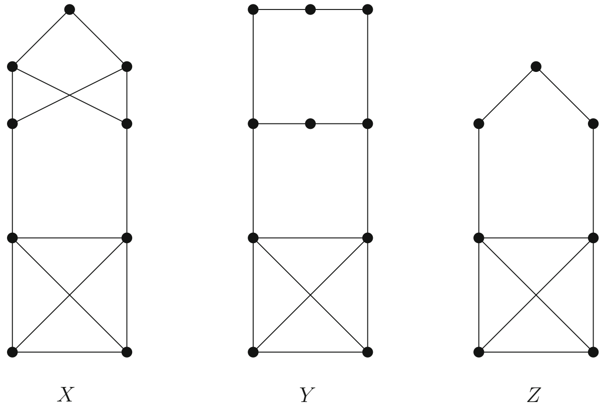

Fig. 2.14 $h(X) = h(Y) = h(Z) = 3$, $g(X) = g(Z) = 3$, $g(Y) = 4$

Theorem 2.20 ([171]). *Given any pair of nonnegative integers a, b with $a+b \geq 2$, there exists a two-connected graph G such that $f_h(G) = a$, $f_g(G) = b$, $h(G) = a + b + c$, and $g(G) = a + 2b + c$.*

Sketch of proof. Take a collection $\{X_i\}_{i=1}^{a}$ of disjoint graphs isomorphic to the graph X of Fig. 2.14. Take a collection $\{Y_i\}_{i=1}^{b}$ of disjoint graphs isomorphic to the graph Y of Fig. 2.14. Take a collection $\{X_Z\}_{i=1}^{c}$ of disjoint graphs isomorphic to the graph Z of Fig. 2.14. Observe that $\mathrm{Ext}(X) \cong \mathrm{Ext}(Y) \cong \mathrm{Ext}(Z) \cong K_2$. Let G be the graph obtained from these three families by identifying all the extreme sets. Hence, according to Lemma 2.1, $h(G) = 3a + 3b + 3c - 2(a+b+c) = a + b + c$, $h(G) = 3a + 4b + 3c - 2(a+b+c) = a + 2b + c$, $f_h(G) = a$, and $f_g(G) = b$, since

- $h(X) = 3$, $g(X) = 3$, $f_h(X) = 1$, $f_g(X) = 0$.
- $h(Y) = 3$, $g(Y) = 4$, $f_h(Y) = 0$, $f_g(Y) = 1$.
- $h(Z) = 3$, $g(Z) = 3$, $f_h(Z) = 0$, $f_g(Z) = 0$. □

Theorem 2.21 ([171]). *Given any pair of nonnegative integers a, b, there exists a two-connected graph G such that $f_h(G) = a$ and $f_g(G) = b$.*

Proof. If $a = b = 0$, take any extreme geodesic graph. If $a = 1$ and $b = 0$, take the graph X in Fig. 2.14. If $a = 0$ and $b = 1$, take the graph Y in Fig. 2.14. If $a + b \geq 1$, make use of Theorem 2.20. □

Conjecture 2.3 ([171]). For any integers a, b, c, d with $a \leq c \leq d$, $b \leq d$, and $c \geq 2$, there exists a connected graph G with $f_h(G) = a$, $f_g(G) = b$, $h(G) = c$, and $g(G) = d$.

Let S be a maximum proper convex set of a graph G. A subset T of S is called a *forcing* subset of S, if S is the unique maximum proper convex set that contains T. The minimum size of a forcing subset of a maximum proper convex set in G is called the *forcing convexity number* $f_c(G)$ of G.

As an immediate consequence of these definitions, the following properties hold.

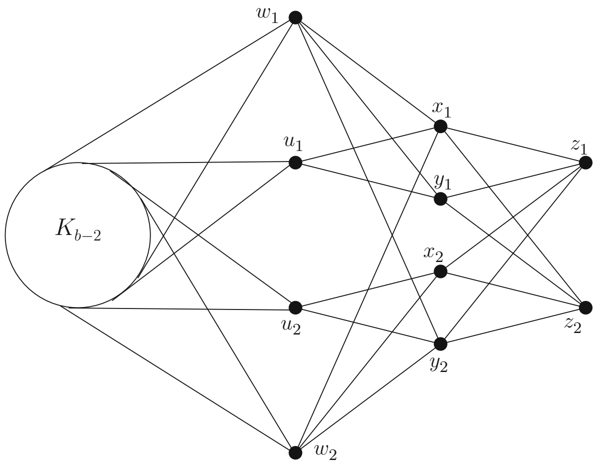

Fig. 2.15 $h(X) = h(Y) = h(Z) = 3$, $g(X) = g(Z) = 3$, $g(Y) = 4$

Proposition 2.5. *Let G be a connected graph G. Then,*

- $f_c(G) \leq g(G)$.
- $f_c(G) = 0$ *if and only if G has a unique maximum proper convex set.*
- $f_c(G) = 1$ *if and only if G has at least two maximum proper convex sets and there exists a vertex belonging to exactly one maximum proper convex set.*
- $f_c(G) \geq 2$ *if and only if G each vertex of each maximum proper convex set belongs to a least one more minimum geodetic set.*
- *There is no graph G satisfying* $(f_g(G), \mathrm{con}(G)) = (0, 2)$.

Proposition 2.6. *Let G be a connected graph G with* $\mathrm{con}(G) = 2$. *If* $G \ncong P_3$, *then* $f_c(G) = 2$.

Proof. Since $\mathrm{con}(G) = 2$, every pair of adjacent vertices forms a maximum convex set. If G contains a leaf, then either $G \cong P_3$ in which case $f_c(G) = 1$ or G has order at least 4, in which case $\mathrm{con}(G) = n - 1 \geq 3$. Hence, every vertex of G belongs to at least two maximum convex sets. □

Theorem 2.22 ([52]). *Given any pair of integers* a, b *with* $0 \leq a \leq b$ *and* $3 \leq b$, *there exists a graph G such that* $f_c(G) = a$ *and* $\mathrm{con}(G) = b$.

Sketch of proof. If $3 \leq a = b$, take K_{b+1}. If $1 \leq a < b$, take a tree T_{b+1}^{a+1}, with $b + 1$ vertices and $a + 1$ leaves. Suppose hence that $a = 0$ and $3 \leq b$. Let G be the graph depicted in Fig. 2.15. Observe that G is a graph of order $b + 8$, diameter 3, and radius 2. Check that $V(K_{b-2}) \cup \{u_1, u_2\}$ is the unique maximum proper set of G. □

Conjecture 2.4. For any integers a, n, d with $3 \leq a \leq n - 1$ and $2 \leq d$, there exists a graph G of order n and diameter d such that $f_c(G) = 0$ and $\mathrm{con}(G) = a$.

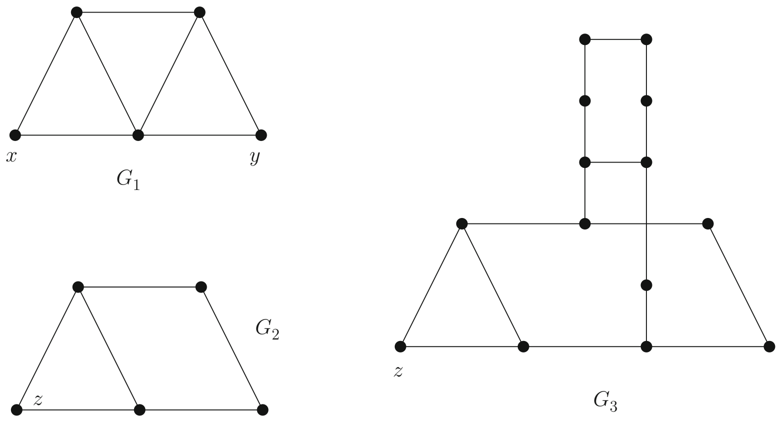

Fig. 2.16 G_1, G_2, and G_3 are $(1,1,3)$-, $(1,2,3)$-, and $(1,3,4)$-graphs, respectively

Let S be a minimum geodetic set of a graph G. A forcing subset T is called *critical* if no proper subset of T is a forcing subset of S. The maximum order of a critical forcing subset of a minimum geodetic set in G is called the *upper forcing geodetic number* $f_g^+(G)$ of G [181]. Clearly, $f_g(G) \leq f_g^+(G) \leq g(G)$ in any graph G.

Theorem 2.23 ([173]). *For any positive integers a,b,c with $1 \leq a \leq b \leq c-2$ or $4 \leq a+2 \leq b \leq c$, there exists a connected graph G with $f_g(G) = a$, $f_g^+(G) = b$, and $g(G) = c$.*

Sketch of proof. In what follows, a graph G such that $f_g(G) = a$, $f_g^+(G) = b$, and $g(G) = c$ is called an (a,b,c)-graph. Check that the graphs G_1 and G_2 of Fig. 2.16 are a $(1,1,3)$-graph and a $(1,2,3)$-graph, respectively. For $k \geq 3$, let $G_k = (V,E)$ be the graph defined by $V = \{z\} \cup \{u_i, v_i\}_{i=1}^{k} \cup [\cup_{i=1}^{k-1} \{a_{ij}\}_{j=1}^{7}]$ and $E = \{zv_1, zu_1, v_1u_1, v_ku_k\} \cup \{u_iu_{i+1}, v_iv_{i+1}\}_{i=1}^{k-1} \cup [\cup_{i=1}^{k-1} \{a_{ij}a_{i(j+1)}\}_{j=1}^{6}] \cup \{a_{i2}a_{i7}, a_{i1}v_i, a_{i7}u_i\}_{i=2}^{k-1}$ (in Fig. 2.16 the graph G_3 is shown). Check that, for every $k \geq 3$, the graph G_k is a $(1,k,k+1)$-graph. We distinguish cases.

Case 1: $1 \leq a \leq b \leq c-2$. If $a = b$, consider the graph H obtained from a copies $H_1,\ldots,H_a$ of G_1, by identifying y_{i-1} and x_i, where, for every $i \in \{1,\ldots,a\}$, $\text{Ext}(H_i) = \{x_i, y_i\}$. Check that H is $(a,a,a+2)$-graph. Take $d = c-a-1$ and consider d copies $B_1,\ldots,B_d$ of K_2, where, for every $i \in \{1,\ldots,d\}$, $V(B_i) = \{u_i, v_i\}$. Check that the graph G obtained from $H, B_1,\ldots,B_d$ by identifying $y_a, u_1,\ldots,u_d$, is a (a,a,c)-graph.

If $1 = a < b$, consider the graph G_b defined above. Take $d = c-b$ and check that the graph G obtained from $G_b, B_1,\ldots,B_d$ by identifying $z, u_1,\ldots,u_d$ is a $(1,b,c)$-graph.

If $2 = a < b$, take $d = c-b-1$ and check that the graph G obtained from $G_{b-1}, G_1, B_1,\ldots,B_d$ by identifying $z, x, u_1,\ldots,u_d$, is a $(2,b,c)$-graph.

Finally, if $3 \leq a < b$, take $k = b-a+1$, $d = c-b-1$ and check that the graph G obtained from $G_k, H, B_1, \ldots, B_d$ by identifying $z, x_1, u_1, \ldots, u_d$ is an (a,b,c)-graph.

Case 2: $4 \leq a+2 \leq b \leq c$. Consider the graph H_{a-2} obtained from G_2 and $a-2$ copies $F_1, \ldots, F_{a-2}$ of G_1, by identifying the pairs $z = x_1, y_1 = x_2, \ldots, y_{a-3} = x_{a-2}$, where, for every $i \in \{1, \ldots, a-2\}$, $\mathrm{Ext}(F_i) = \{x_i, y_i\}$ and $\mathrm{Ext}(G_2) = \{z\}$. Take $d = c-b$ and check that the graph G obtained from $G_{b-a}, H_{a-2}, B_1, \ldots, B_d$ by identifying $z, y_{a-2}, u_1, \ldots, u_d$ is an (a,b,c)-graph. □

2.6 Closed Geodomination

A geodetic set S of a graph G is a *closed geodetic set* if it can be written in canonical form as $S = \{v_1, v_2, \ldots, v_k\}$ such that $v_j \notin I_G[\{v_1, v_2, \ldots, v_{j-1}\}]$ for all $j \in \{2, \ldots, k\}$. The *closed geodetic number* of G, denoted by $cg(G)$, is the minimum cardinality of a closed geodetic set of G [21, 111].

Proposition 2.7 ([2]). *If G is a graph of order n and diameter d, then $g(G) \leq cg(G) \leq n-d+1$. Moreover, if either $p \in \{1,2,3,n-1,n\}$ or G is an extreme geodesic graph, then $g(G) = cg(G) = p$.*

Sketch of proof. Take a pair of antipodal vertices $u_1, u_2 \in V(G)$. Notice that $|I[u_1,u_2]| \geq d+1$. Complete the process to obtain a closed geodetic set $S = \{u_1, u_2, \ldots, u_k\}$. Hence, $n = |I[S]| \geq (d+1)+(k-2) = d+k-1$, i.e., $g(G) \leq cg(G) \leq k \leq n-d+1$.

If G is an extreme geodesic graph, then $\mathrm{Ext}(G)$ is a geodetic set, and it is also clearly a closed geodetic set. Thus, in this case, $g(G) = cg(G) = p$.

Notice that, $cg(G) = n$ if and only if $G \cong K_n$, since $g(G) \leq cg(G) \leq n-d+1$.

Next, suppose that $cg(G) = n-1$ and $g(G) \leq n-2$. If there exists a pair x, y of vertices such that either $d(x,y) \geq 3$ or $d(x,y) = 2$ and there are at least two $x-y$ geodesics, then $cg(G) \leq n-2$, a contradiction. Assume thus that $d = 2$ and that for every pair $x, y \in V(G)$ of nonadjacent vertices, $d(x,y) = 2$ and there is a unique $x-y$ geodesic. If $\mathrm{rad}(G) = 1$, then prove, first, that $|\mathrm{Cen}(G)| = 1$ and, second, that according to Theorem 2.5, $g(G) = n-1$, a contradiction. If $\mathrm{rad}(G) = 2$, prove that necessarily $cg(G) \leq n-2$, again a contradiction.

Finally, check that if S is a minimum geodetic set of cardinality at most 3, then it is also a closed geodetic set. □

In particular, all graphs considered in Table 2.1 satisfy $g(G) = cg(G)$, except the family of complete bipartite graphs $K_{p,q}$ with $4 \leq p \leq q$. In this case, as was proved in [2], $cg(K_{p,q}) = p$. This result allowed the authors of this paper to obtain the following realization theorem.

Theorem 2.24 ([2]). *Given any triple of integers n, h, k with $4 \leq h \leq k-1$ and $2k-h+3 \leq n$, there exists a graph G of order n such that $g(G) = h$ and $cg(G) = k$.*

Sketch of proof. Take $r = k - m + 3$ and $s = n - k$. Let $K_{r,s} = (U \cup V, E)$, where $U = \{u_i\}_{i=1}^{r}$ and $V = \{v_i\}_{i=1}^{s}$. Consider the graph G obtained from $K_{r,s}$ by adding a set $\{w_i\}_{i=1}^{m-3}$ of $m-3$ leaves, all of them joined to vertex v_1. Notice that $\{w_i\}_{i=1}^{m-3} \cup \{u_1, u_r, v_s\}$ is a minimum geodetic set. Check that $\{w_i\}_{i=1}^{m-3} \cup \{u_i\}_{i=1}^{r}$ is minimum closed geodetic set. □

2.7 Geodetic Domination

A vertex in a graph G *dominates* itself and its neighbors. A set of vertices S in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A dominating set S is called *geodetic dominating* if S is also a geodetic set of G. The *geodetic domination number* of G, denoted by $\gamma_g(G)$, is the minimum cardinality of a geodetic dominating set of G. Certainly, for every nontrivial connected graph G of order n, $2 \leq \max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq g(G) + \gamma(G)$, as the union of a geodetic set and a dominating set produces a geodetic dominating set. The *girth* $c(G)$ is the length of a shortest cycle contained in G.

Theorem 2.25 ([109]). *If G is a graph with $\delta(G) \geq 2$ and $c(G) \geq 6$, then $\gamma_g(G) = \gamma(G)$.*

Proof. Let D be a minimum dominating set of G. Suppose that $X = V(G) \setminus I[D] \neq \emptyset$ and take a vertex $x \in X$. Let $u \in N(x) \cap D$. Since $\delta(G) \geq 2$ and $c(G) \geq 4$, there is a vertex $v \in N(x) \setminus \{u\}$ and $uv \notin E(G)$. If $v \in D$, then $x \in I[u,v] \subseteq I[D]$, a contradiction. Thus $v \in V(G) \setminus D$, and there is a vertex $z \in N(v) \cap D$ as $c(G) \geq 6$, $d(u,z) \geq 3$. It follows that $x, v \in I[u,z] \subseteq I[D]$, a contradiction. Thus X is empty and D is a geodetic dominating set, as desired. □

Theorem 2.26 ([97]). *Let G be a graph of order $n \geq 2$ and diameter* $\mathrm{diam}(G) = d \geq 2$.

1. $\gamma_g(G) \leq n - \lfloor \frac{2d}{3} \rfloor$.
2. *If $c(G) = c \geq 6$, then $\gamma_g(G) \leq n - \lfloor \frac{2c}{3} \rfloor$.*

Moreover, both bounds are tight.

Proof.

1. Take nonnegative integers r, t such that $0 \leq r \leq 2$ and $d = 3t + r$. Take a pair u_0, u_d of antipodal vertices, i.e., s.t. $d(u_0, u_d) = d$. Let ρ be a $u_0 - u_d$ geodesic, where $V(\rho) = \{u_i\}_{i=0}^{d}$. Consider the set $A = \{u_0, u_3, \ldots, u_{3t}, u_d\}$. Check that the set $V(G) \setminus (V(\rho) \setminus A)$ is both geodetic and dominating. Observe that $t + 1 \leq |A| \leq t + 2$. Hence, $|V(G)| \setminus \gamma_g(G) \geq |(V(\rho) \setminus A| = \lfloor \frac{2d}{3} \rfloor = \lfloor \frac{2d}{3} \rfloor$.

   Finally, if $G \cong P_n$, then $\gamma_g(G) = \lceil \frac{n+2}{3} \rceil = n - \lfloor \frac{2(n-1)}{3} \rfloor = n - \lfloor \frac{2\mathrm{diam}(P_n)}{3} \rfloor$.

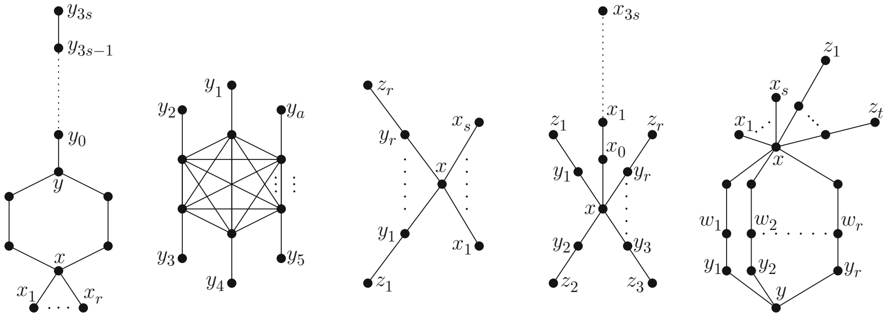

Fig. 2.17 $\gamma(G_1) = r+s+3$, $\gamma(G_2) = a$, $\gamma(G_3) = r+1$, $\gamma(G_4) = r+s+1$, $\gamma(G_1) = t+r+1$

2. Take nonnegative integers r,t such that $0 \le r \le 2$ and $c = c(G) = 3t + r$. Let $H \cong C_c$ be an induced cycle of length c, where $V(H) = \{w_i\}_{i=1}^{c}$. Take the set $A = \{u_1, u_4, \dots, u_{3t-2}\}$ if $r = 0$ and $A = \{u_1, u_4, \dots, u_{3t-2}, u_{3t+1}\}$ if $1 \le r \le 2$. Check that the set $V(G) \setminus (V(H) \setminus A)$ is both geodetic and dominating. Observe that $t \le |A| \le t+1$. Hence, $|V(G)| \setminus \gamma_g(G) \ge |(V(H) \setminus A| = \lfloor \frac{6t+2r}{3} \rfloor = \lfloor \frac{2c}{3} \rfloor$.

   Finally, if $G \cong C_n$, then $\gamma_g(G) = \lceil \frac{n}{3} \rceil = n - \lfloor \frac{2n}{3} \rfloor = n - \lfloor \frac{2c(C_n)}{3} \rfloor$. □

Theorem 2.27 ([97]). *Let a,c,b be integers such that $a,b \ge 2$ and $\max\{a,b\} \le c \le a+b$. Then there is a connected graph G such that $\gamma(G) = a$, $g(G) = b$, and $\gamma_g(G) = c$.*

Sketch of proof. Take integers $a,b \ge 2$. We distinguish cases.

Case 1: $c = a+b$. Consider the graph G_1 shown in Fig. 2.17. Check that $\{x, y, y_2, \dots, y_{3s-1}\}$ and $\{y_{3s}, x_1, \dots, x_r\}$ are a minimum dominating set and a minimum geodetic set of G, respectively. Notice also that the union of these two sets produces a minimum geodetic dominating set. Hence, taking $r = b-1$ and $s = a-2$, we obtain the desired values.

Case 2: $a = b = c$. Consider the graph $G_2 \cong K_a \odot K_1$ shown in Fig. 2.17. Check that $\{y_i\}^a i = 1$ is a minimum geodetic dominating set.

Case 3: $a < b = c$. Consider the graph G_3 shown in Fig. 2.17. Check that $\{x, z_1, \dots, z_r\}$ is a minimum dominating set and that $\{z_1, \dots, z_r, x_1, \dots, x_s\}$ is both a minimum geodetic set and a minimum geodetic dominating set. Hence, taking $r = a-1$ and $s = b-a+1$, we obtain the desired values.

Case 4: $b < a = c$. Consider the graph G_4 shown in Fig. 2.17. Check that $\{z_1, \dots, z_r, x_0, x_3, \dots, x_{3s}\}$ is both a minimum dominating set and a minimum geodetic dominating set and that $\{z_1, \dots, z_r, x_{3s}\}$ is a minimum geodetic set. Hence, taking $r = b-1$ and $s = a-b$, we obtain the desired values.

Case 5: $\max\{a,b\} < c < a+b$. Consider the graph G_5 shown in Fig. 2.17. Check that $\{x, z_1, \dots, z_t, y_1, \dots, y_r\}$ is a minimum dominating set, that $\{y, x_1, \dots, x_s, z_1, \dots, z_t\}$ is a minimum geodetic set, and that $\{y, x_1, \dots, x_s, z_1, \dots, z_t, w_1, \dots, w_r\}$ is a minimum geodetic dominating set. Hence, taking $r = c-b$, $s = c-a$, and $t = a+b-c-1$, we obtain the desired values. □

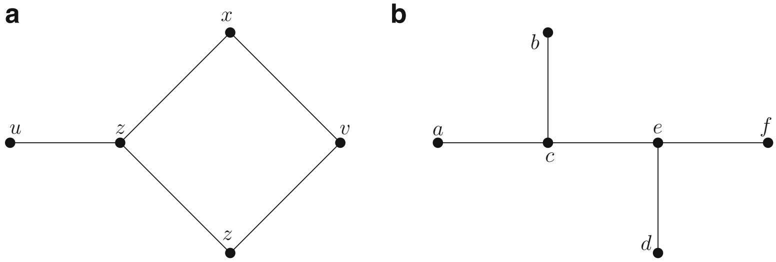

Fig. 2.18 (**a**) $g(G) = g_3(G) = |\{u,v\}|$, $g_2(G) = |\{u,x,y\}| = 3$. (**b**) $g(T) = g_2(G) = g_3(G) = 4$

2.8 *k*-Geodomination

Given a graph G of diameter d and an integer $2 \leq k \leq d$, a vertex v is said to be *k-geodominated* by a pair of vertices x and y if $d(x,y) = k$ and v lies on a shortest path between vertices x and y. A subset S of vertices in a graph $G = (V,E)$ is a *k-geodetic set* if each vertex in $V \setminus S$ is k-geodominated by some pair of vertices in S and the minimum cardinality of a k-geodetic set is the *k-geodetic number* of a graph $g_k(G)$ [147].

It is clear that, for every $k \in \{2,\ldots,d\}$, $2 \leq g(G) \leq g_k(G) \leq n-k+1$, where $n \geq 2$ denotes the order of G.

Theorem 2.28 ([147]). *If G is a graph of diameter* 2*, then* $g_2(G) = g(G)$.

Proof. Let S be a minimum geodetic set of G. Since $\mathrm{diam}(G) = 2$, it follows that $S \subsetneq V(G)$. Take an arbitrary vertex $v \notin S$. As S is a geodetic set, it follows that v is geodominated by a pair of vertices $x,y \in S$. Because $\mathrm{diam}(G)$, it follows that $d(x,y)$. Thus S is a also two-geodetic set of G. □

Certainly, the condition that the diameter of the graph is 2 is not necessary as the graph in Fig. 2.18a shows. Also, observe that the converse of this result is not true. For example, the tree T in Fig. 2.18b has diameter 3, and the set $\mathrm{Ext}(T)$ of leaves of T is as minimum geodetic set as well as a minimum two-geodetic set.

Theorem 2.29 ([147]). *If $G = (V,E)$ is a graph of diameter $d \geq 3$ and $g(G) = 2$, then $g_k(G) = 2$ if and only if $k = d$.*

Sketch of proof. Let $S = \{x,y\}$ a minimum geodetic set of G. Then, x and y are antipodal, that is, $d(x,y) = d$. Thus S is a d-geodetic set and so $g_d(G) = 2$.

Conversely, if $k = 1$ or $k > d$, then V is the unique geodetic set of G, and so $g_k(G) = n > 2 = g(G)$, a contradiction. Assume next that $2 \leq k \leq d-1$. Since $g(G) = 2$, every minimum geodetic set S contains antipodal vertices and so S is d-geodetic. As $k < d$, it follows that S is not k-geodetic. Thus no two subset of V is a k-geodetic set. Therefore, $g_k(G) > 2 = g(G)$, again a contradiction. □

Theorem 2.30 ([147]). *Let $k \geq 2$ be an integer. For each pair a, b of integers with $2 \leq a \leq b$, there exists a tree T such that $g(T) = a$ and $g_k(T) = b$.*

Sketch of proof. Suppose first that $a = b$. For $k = 2$, take the star $K_{1,a}$ and check that $g(K_{1,a}) = g_2(K_{1,a}) = a$. For $k \geq 3$, take the tree T obtained from P_k by joining $a-1$ vertices to one leaf of P_k. Check that $g(T) = g_k(T) = a$.

Assume now that $b = a+1$. For $1 \leq i \leq a$, let P^i_{k+1} be a copy of the path P_{k+1}. Consider the tree T obtained from these a paths by identifying the vertices $\{x_{i1}\}_{i=1}^{a}$, where $V(P^i_{k+1}) = \{x_{ij}\}_{j=1}^{k+1}$. Check that $g(T) = a$ and $g_k(T) = a+1$.

Finally, suppose that $b = a + j$, with $j \geq 2$. Consider the path P_{jk}. It is routine to check that $g_k(P_{jk}) = j+2$. Hence, if $a = 2$, P_{jk} fulfills the desired property. So, we may assume that $a \geq 3$. Let T be the tree obtained from P_{jk} by joining $a-2$ vertices to one support vertex of this path. Check that $g(T) = a$ and $g_k(T) = j + a = b$. □

Conjecture 2.5. For any integers a, b, n, k, d with $2 \leq k \leq d$ and $2 \leq a \leq b \leq n-k+1$, there exists a graph G of order n and diameter d such that $g(G) = a$ and $g_k(G) = b$.

Conjecture 2.6. For any integers $2 \leq a, b, h, k$ there exists a graph G such that $g_h(G) = a$ and $g_k(G) = b$.

Theorem 2.31 ([13]). *If $G = (V, E)$ is a graph of order n, diameter d, and maximum degree Δ, then, for every $k \in \{2, \ldots, d\}$, $g_k(G) \geq \lceil \frac{2n}{\Delta(\Delta-1)^{k-1}(k-1)+2} \rceil$.*

Proof. Let S be a k-geodetic set of G. We know that every vertex not in S lies on a geodesic of length k joining to vertices of S. For every vertex u, the number of length k beginning in u is bounded above by $\Delta(\Delta-1)^{k-1}$. If we consider all vertices in S, we have that the number of paths of length k beginning and ending in S is at most $\frac{|S|\Delta(\Delta-1)^{k-1}}{2}$ and the number of vertices of $V(G) \setminus S$ which can lie on those paths is bounded above by $\frac{|S|\Delta(\Delta-1)^{k-1}(k-1)}{2}$. Therefore, $n - |S| \leq \frac{|S|\Delta(\Delta-1)^{k-1}(k-1)}{2}$. □

2.9 Edge Geodomination

A set of vertices U of a graph G is said to be *edge geodetic* if the union of all edges belonging to all geodesics joining pairs of vertices of U is the whole edge set $E(G)$ of G [8]. The *edge geodetic number* $g_e(G)$ of G is defined as the minimum cardinality of an edge geodetic set of G.

Clearly, if G is a nontrivial graph of order n and diameter d, then $2 \leq g(G) \leq g_e(G) \leq \min\{g(G) + n - 2, n - d + 1\}$. The equality $g(G) = g_e(G)$ holds, among others, for trees, cycles, and complete graphs. The equality $g_e(G) = g(G) + n - 2$ holds, for example, for the graph $K_n - e$, obtained from K_n by deleting one edge e. Finally, the equality $g_e(G) = n - d + 1$ holds, for example, for paths, stars, and complete graphs.

A collection $\mathscr{S}$ of subsets of $[p]$ is said to *separate pairs in* $[p]$ if, for any two distinct $i, j \in [p]$, there exists a pair of disjoint sets $A, A' \in \mathscr{S}$ such that $i \in A$ and $j \in A'$.

Lemma 2.2 ([8]). *The minimum cardinality of a collection $\mathscr{S}$ of subsets $[p]$ that separates pairs in $[p]$ is $\lceil 3\log_3 p \rceil$.*

Theorem 2.32 ([7,8]). *For any connected graph G, $\lceil 3\log_3 \omega(G) \rceil \leq g_e(G)$. Moreover, this bound is tight.*

Sketch of proof. Take a maximum clique $\Omega_p \cong K_p$ of G, where $p = \omega(G)$ and $V(K_p) = [p]$. Take a minimum edge geodetic set U of G. Given a vertex $u \in U$ and a geodesic γ starting at u such that $E(\gamma) \cap E(\Omega_p) \neq \emptyset$, let u_γ denote the vertex in $V(\gamma) \cap V(\Omega_p)$ closer to u.

Consider the set $\mathscr{S} = \{A_u\}_{u \in U}$, where $A_u = \{u_\gamma : \gamma$ is a geodesic with end-vertex $u\}$. Check that $\mathscr{S}$ separates pairs in $[p]$. Hence, $g_e(G) = |U| = |\mathscr{S}| \geq \lceil 3\log_3 p \rceil$.

Finally, to prove the sharpness, given a positive integer p, take the complete graph K_p and a set $S = \{u_i\}_{i=1}^{q}$ of $q = \lceil 3\log_3 p \rceil$ new vertices. Take a collection $\mathscr{S} = \{S_i\}_{i=1}^{q}$ of $q = \lceil 3\log_3 p \rceil$ subsets of $[p] = V(K_p)$ that separates pairs in $[p]$. Consider that graph G obtained from K_p and S by joining, for every $i \in \{1, \ldots, q\}$, vertex u_i to all vertices of S_i. Check that S is a minimum edge geodetic set of G. □

Theorem 2.33 ([166]). *For each pair a, b of integers with $2 \leq a \leq b$, there exists a graph G such that $g(G) = a$ and $g_e(G) = b$.*

Sketch of proof. If $a = b$, take the star $K_{1,a}$. If $2 = a < b$, take the graph $K_1 \vee K_{2,b-2}$. Finally, if $2 < a < b$, take the graph G obtained from $H \cong K_1 \vee K_{2,b-2}$ by adding $a - 2$ leaves, all of them joined to the universal vertex of H. □

Theorem 2.34 ([118]). *The boundary $\partial(G)$ of every graph is an edge geodetic set. Moreover, if G is bipartite, then for every $u \in V(G)$, $\{u\} \cup \partial(u)$ is an edge geodetic set.*

Proof. Let $e = xy \in E(G)$. Take a maximal geodesic ρ containing e. This means that if a, b are the endpoints vertices of ρ, then $b \in \partial(a)$ and $a \in \partial(b)$. Thus e is in a geodesic with endpoints in $\partial(G)$.

Let $u \in V(G)$ and $e = (x, y) \in E(G)$. Because G is a bipartite graph, $d(x, u) \neq d(y, u)$. Suppose that $d(x, u) < d(y, u)$. Take a $u - x$ geodesic ρ. Then, the path ρ' obtained from ρ by joining vertex y with the edge e is a $u - y$ geodesic that can be extended to a maximal $u - z$ geodesic. Then $z \in \partial(u)$ and hence e is in a geodesic with endpoints in $\{u\} \cup \partial(u)$. □

Further results involving geodominating invariants related to geodesic convexity in graphs can be found in [2, 7, 8, 13, 49, 52, 53, 58, 60, 77, 79, 97, 109, 111, 125, 126, 137, 146, 147, 166, 171, 173, 176, 177, 181].

2.10 Classical Parameters

A subset $A \subset V$ of a graph $G = (V,E)$ is called *Helly independent*[2] if $\bigcap_{a\in A}[A-a] = \emptyset$. The *Helly number* $H(G)$ of a graph G is the maximum cardinality of a Helly independent set. In other words, $H(G)$ is the smallest integer such that every family of convex sets with an empty intersection contains a subfamily of at most $H(G)$ members with an empty intersection.

Given a subset $A \subset V$ of a graph $G = (V,E)$, a partition $\{A_1,A_2\}$ of A is called a *Radon partition* if $[A_1] \cap [A_2] \neq \emptyset$. A subset $A \subset V$ of a graph $G = (V,E)$ is called *Radon dependent* if it has a Radon partition, and otherwise it is said to be *Radon independent*. The *Radon number*[3] $R(G)$ of a graph $G = (V,E)$ is the maximum cardinality of a Radon independent set.

A subset $A \subset V$ of a graph $G = (V,E)$ is called *redundant* when $[A] = \bigcup_{a\in A}[A-a]$, and otherwise it is said to be *irredundant*. The *Carathéodory number* $C(G)$ of a graph G is the maximum cardinality of an irredundant set.

Theorem 2.35 ([138]). *Let $G = (V,E)$ be a connected graph of order $n \geq 2$.*

1. *Every subset of a Helly independent set is Helly independent.*
2. *Every subset of a Radon independent set is Radon independent.*
3. $2 \leq \omega(G) \leq H(G)$.
4. $H(G) \leq R(G)$.

Proof.

1. Take $B \subset A \subseteq V$. Suppose that B is Helly dependent. Observe that $\cap_{b\in B}[B-b] \subseteq \cap_{b\in B}[A-b]$. Note also that, for every $a \in A-B$, $\cap_{b\in B}[B-b] \subseteq [B] \subseteq [A-a]$. Hence, $\emptyset \neq \cap_{b\in B}[B-b] \subseteq (\cap_{b\in B}[A-b]) \cap (\cap_{a\in A-B}[A-a]) = \cap_{a\in A}[A-a]$.

Table 2.4 Classical parameters of some basic graph families

G[a]	P_n[b]	C_n[c]	T_n[d]	K_n	$K_{p,q}$[e]	Q_n[f]
$\omega(G)$	2	2	2	n	2	2
$H(G)$	2	3	2	n	2	2
$R(G)$	2	3	3	n	2	$\lceil \log_2(n+1) \rceil + 1$
$C(G)$	2	2	2	1	2	n

[a] $G \ncong K_1$
[b] $2 < n$
[c] $4 < n$
[d] $\Delta(T_n) \geq 3$
[e] $2 \leq p \leq q$
[f] $2 < n$

[2] As in the remaining sections and chapters, unless otherwise stated, all terms, invariants and results are referred to the geodesic convexity.

[3] Some authors [90, 132] define the Radon number to be one unit larger, i.e., as the smaller value r such that each set with at least r points admits a Radon partition.

2. Take $B \subset A \subseteq V$. Suppose that B is Radon dependent. Let $\{B_1, B_2\}$ be a Radon partition of B. Consider the partition of A, $\{B_1, A_2\}$, where $A_2 = B_2 \cup (A - B)$. Then, as $\emptyset \neq [B_1] \cap [B_2] \subseteq [B_1] \cap [A_2]$, it follows that $\{B_1, A_2\}$ is a Radon partition of A.
3. Let $A \subseteq V$ such that $G[A] \cong K_h$. Then, for every $a \in A$, $G[A - a] \cong K_{h-1}$, which means that $\cap_{a \in A}[A - a] = \emptyset$.
4. Let $A \subseteq V$ be Radon dependent. Consider a Radon partition $\{A_1, A_2\}$ of A and a vertex $p \in [A_1] \cap [A_2]$. For each $a \in A$, we have either $A_1 \subseteq A - a$ or $A_2 \subseteq A - a$. Hence, $p \in [A - a]$ for every $a \in A$, which means that A is Helly dependent. □

Remark 2.1. Consider the Euclidean convexity, derived from the Euclidean distance, in R^n. Then, the Helly number, the Radon number, and the Carathéodory number are all equal to $n + 1$ [175].

Let $W \subset V(G)$ be a set of vertices of a connected graph G such that $G[W]$ is connected. Take the subgraph $G - W$ and a new vertex x_W, and, finally, join this vertex to every vertex in $y \in (V(G) \setminus W) \cap N_G(W)$. This operation is called a contraction of a connected subgraph of G. A *contraction* of G is every graph that can be obtained by a sequence of contractions of connected subgraphs. The *Hadwiger number* $\eta(G)$ of G is the number of vertices in the largest clique that can be formed as a contraction of G.

Theorem 2.36 ([93]). *For every connected graph G, $H(G) \leq \eta(G)$, $R(G) \leq 2\eta(G)$. Moreover, the first bound is tight.*

According to Duchet's research [90, 91], the geodesic convexity is in the following sense universal with respect to Carathéodory, Helly, and Radon parameters: Given any finite convexity space $(X, \mathscr{C})$, there exists a finite graph G such that the Carathéodory, Helly, and Radon numbers of the geodesic convexity in G coincide with those of $\mathscr{C}$.

Theorem 2.37 ([89, 129]). *Let G be a non-complete connected graph. Let $H_m(G)$, $R_m(G)$, and $C_m(G)$ denote the Helly, Radon, and Carathéodory numbers of G with respect to the monophonic convexity. Then,*

1. $H_m(G) = \omega(G)$.
2. $R_m(G) = \omega(G)$ *if* $\omega(G) \geq 3$ *and* $2 \leq R_m(G) \leq 3$ *if* $\omega(G) = 2$.
3. $C_m(G) = 2$.

A *distance-hereditary graph* is a graph in which every chordless path is a geodesic [120].

Corollary 2.3. *If G is a non-complete distance-hereditary graph, then $C(G) = 2$, $H(G) = \omega(G)$, and $R(G) = \omega(G)$ if $\omega(G) \geq 3$ ($2 \leq R(G) \leq 3$ if $\omega(G) = 2$).*

In the case of the geodesic convexity of a graph G, the Helly number $H(G)$ may well exceed the clique number $\omega(G)$, the five-cycle C_5 being the smallest example.

A graph is *weakly modular* if, for any triple $\{x,y,z\}$ of vertices such that $I[x,y] \cap I[x,z] = \{x\}$, $I[y,x] \cap I[y,z] = \{y\}$, and $I[z,x] \cap I[z,y] = \{z\}$, all vertices on every $y-z$ geodesic have the same distance to x.

A graph is called *constructible* if there exists a well-ordering $\leq$ of its vertices such that, for every vertex x which is not the smallest element, there is a vertex $y < x$ which is adjacent to x and to every neighbor z of x with $z < x$.

Note that there exists no relation between constructible graphs and weakly modular graphs, as is shown by the following two examples. Let C_4 be a cycle of length four. Then C_4 is clearly weakly modular but not constructible. Now let x and y be two new vertices, and let G be the graph obtained by joining x to all vertices of C_4 and y to two adjacent vertices of C_4. Then G is constructible but not weakly modular.

Theorem 2.38 ([9, 162]). *Every weakly modular (resp. constructible) graph G satisfies $\omega(G) = H(G)$.*

Further results on classical convexity invariants directly or indirectly related to geodesic convexity in graphs can be found in [9, 12, 40, 43, 87, 89–93, 128, 129, 132, 137, 149, 159, 162, 167, 168, 175].

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