Chapter 2
Surface Topology and Geometry

This chapter briefly reviews the fundamental concepts and theorems in algebraic topology [1], surface differential geometry [6], and surface Ricci flow [4, 7]. Detailed discussion on Ricci flow on general Riemannian manifolds can be found in [5]. Advanced topics on differential geometry related to Yamabe equations can be found in [9].

2.1 Surface Topology

Topology studies the invariants under homeomorphism transformation group. Algebraic topology studies the topologies of spaces and the mappings among spaces by algebraic means. Generally, different groups are associated with different spaces, such as fundamental group, homology group, and cohomology group. The structures of these groups convey the topological information about the spaces. The homomorphisms among these groups reflect the properties of the mappings among the spaces. In reality, most surfaces are the boundaries of some finite volumes; therefore, they are compact and orientable. In the following, we focus on the fundamental groups and covering spaces of compact orientable surfaces.

Definition 2.1 (Connected Sum). The connected sum $S_1 \# S_2$ is formed by deleting the interior of disks $D_i \subset S_i$ and attaching the resulting punctured surfaces $S_i - D_i$ to each other by an orientation reversing homeomorphism $h : \partial D_1 \to \partial D_2$, where ∂D_i represents the boundary of D_i. Let $p \in \partial D_1$ and $q \in \partial D_2$. p is equivalent to q, $p \sim q$, if $q = h(p)$. So $S_1 \# S_2 := \{(S_1 - D_1) \cup (S_2 - D_2)\}/\sim$. See Fig. 2.1.

Theorem 2.1 (Classification for Compact Orientable Surfaces). *Any closed connected orientable surface is homeomorphic to either a sphere or a finite connected sum of tori,*

$$S = \mathbb{S}^2 \# T_1 \# T_2 \cdots \# T_g,$$

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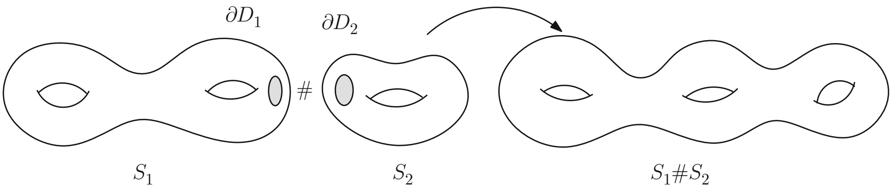


Fig. 2.1 Connected sum

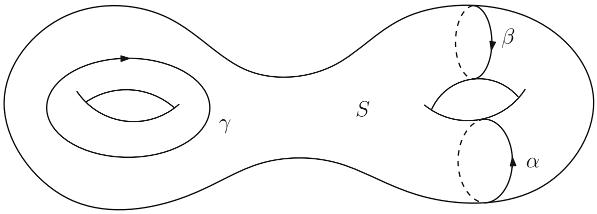


Fig. 2.2 α is homotopic to β, not homotopic to γ

where $\mathbb{S}^2$ *is the unit sphere,* T_i *is a torus,* $i = 1, 2, \cdots, g$. g *is called the genus of the surface, and each* T_i *is a handle.*

In general, the genus g and the number of boundary components b are the total topological invariants. Surface topology is usually represented by its fundamental group.

2.1.1 Fundamental Group

Definition 2.2 (Homotopy). Two continuous maps $f_0, f_1 : M \to N$ are *homotopic* if there is a continuous map $F : M \times [0,1] \to N$ such that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$. The map F is called a *homotopy* between f_0 and f_1, denoted as $f_0 \cong f_1$ or $F : f_0 \cong f_1$. For each $t \in [0,1]$, we denote $F(\cdot, t)$ by $f_t : M \to N$, where f_t is a continuous map.

A map $f : [0,1] \to M$ from the unit interval to a topological space M is called a *path* in M. If f and g are two paths in M with $f(1) = g(0)$, then the *product* of f and g is a path $f \cdot g$, which is defined as

$$f \cdot g(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}. \tag{2.1}$$

Fix a base point $q \in M$, a *loop* with base point q is a path such that $f(0) = f(1) = q$. Two loops on a surface are homotopic to each other, if they can deform to each other without leaving the surface, as shown in Fig. 2.2.

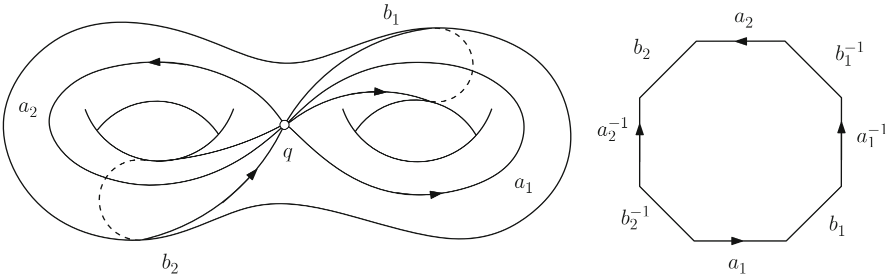


Fig. 2.3 A set of canonical basis of the fundamental group $\pi_1(M, q)$

Definition 2.3 (Fundamental Group). All the homotopy classes of loops with base point q under the product (2.1) form a group, the so-called *fundamental group* of the surface, denoted as $\pi_1(M, q)$.

The fundamental group is finitely generated. Intuitively, each handle T_i is a torus, which is the direct product of two circles, $T_i = \mathbb{S}^1 \times \mathbb{S}^1$. We denote the first circle as a_i, and the second circle b_i, then all such $\{(a_i, b_i)\}$'s are the generators of $\pi_1(M, q)$.

Definition 2.4 (Canonical Fundamental Group Basis). A fundamental group basis $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$ is canonical, if

(1) a_i and b_i intersect at the same point q.
(2) a_i and a_j, b_i and b_j, a_i and b_j only touch at q, $i \neq j$.

As shown in Fig. 2.3, if we slice the surface along the canonical fundamental group generators, then we will get a $4g$-gon. The boundary is $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, which can shrink to a point. For general compact orientable closed surfaces, the following theorem holds.

Theorem 2.2 (Fundamental Groups of General Surfaces). *The fundamental group of the surface $M = \mathbb{S}^2 \# g\mathbb{T}^2$ (connected sum of g tori with $\mathbb{S}^2$) is the group with generators $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$ and one relation $\prod_{k=1}^{g} [a_k, b_k] = e$, where $[a, b] = aba^{-1}b^{-1}$.*

2.1.2 Covering Space

Definition 2.5 (Covering Space). Let $p : \tilde{M} \to M$ be a continuous map and p is onto. Suppose for all $q \in M$, there is an open neighborhood U of q such that

$$p^{-1}(U) = \cup_{j \in J} \tilde{U}_j,$$

for some collection $\{\tilde{U}_j, j \in J\}$ of subsets of $\tilde{M}$, satisfying $\tilde{U}_j \cap \tilde{U}_k = \emptyset$ if $j \neq k$, and with $p|_{\tilde{U}_j} : \tilde{U}_j \to U$ a homeomorphism for each $j \in J$. Then $p : \tilde{M} \to M$ is a *covering*.

The automorphisms of the covering space which are commutative with the projection are called *Deck transformations*.

Definition 2.6 (Deck Transformation). Suppose $p : \tilde{M} \to M$ is a covering. An automorphism $\tau : \tilde{M} \to \tilde{M}$ is called a *Deck transformation* if $p \circ \tau = p$.

All the deck transformations form a group $Deck(\tilde{M})$, called the *deck transformation group*. M is homeomorphic to the quotient space

$$\tilde{M}/Deck(\tilde{M}) \cong M.$$

Definition 2.7 (Fundamental Domain). A closed subset $D \in \tilde{M}$ is called a *fundamental domain* of the $Deck(\tilde{M})$, if $\tilde{M}$ is the union of conjugates of D,

$$\tilde{M} = \bigcup_{\tau \in Deck} \tau D,$$

and the intersection of any two conjugates has no interior.

Among all covering spaces for a given surface, the one with the simplest topology is the so-called *universal covering*.

Definition 2.8 (Universal Covering). Suppose $p : \tilde{M} \to M$ is a covering. If $\tilde{M}$ is simply connected ($\pi(\tilde{M}, \tilde{q}) = \langle e \rangle$), then the covering is the *universal covering*.

Theorem 2.3 (Universal Covering Space for Surfaces). *The universal covering spaces of orientable closed surfaces are sphere $\mathbb{S}^2$ (genus zero), plane $\mathbb{E}^2$ (genus one), and hyperbolic disk $\mathbb{H}^2$ (high genus).*

Figure 1.4 shows the universal covering spaces of all orientable closed surfaces.

2.2 Surface Differential Geometry

Movable frame and exterior differentiation are the power methods for studying surface differential geometry. We will briefly review the fundamental concepts and theorems for surface differential geometry using the movable frame method, due to its simplicity. A thorough introduction to exterior calculus and movable frame can be found in [3]. We use bold letters to represent vectors, such as $\mathbf{r}$ and $\mathbf{e}$, and Greek letters for differential forms, such as ω and τ.

2.2.1 Movable Frame Method

We apply movable frame method to study surfaces in $\mathbb{E}^3$. Assume the equation for a surface S is $\mathbf{r} = \mathbf{r}(u, v) : D \to \mathbb{E}^3$. Select a smooth orthonormal frame field locally, and at each point $\mathbf{r}(u, v)$ define an orthonormal frame

$$\{\mathbf{r}(u, v); \mathbf{e}_1(u, v), \mathbf{e}_2(u, v), \mathbf{e}_3(u, v)\},$$

such that $\mathbf{e}_3$ is the normal field, $\mathbf{e}_3(u, v) = \mathbf{n}(u, v)$. Taking the exterior derivative of the movable frame $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we get the surface structure equation

$$d\mathbf{r} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2,$$

$$d\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} = \begin{pmatrix}0 & \omega_{12} & \omega_{13}\\ -\omega_{12} & 0 & \omega_{23}\\ -\omega_{13} & -\omega_{23} & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}.$$

From $d^2\mathbf{r} = 0$, we obtain

$$d\omega_1 = \omega_{12} \wedge \omega_2, \tag{2.2}$$

$$d\omega_2 = \omega_{21} \wedge \omega_1, \tag{2.3}$$

and

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0.$$

From (2.2) and (2.3), we directly obtain

$$\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2. \tag{2.4}$$

From $d^2\mathbf{e}_1 = 0$, we get the *Gauss equation*

$$d\omega_{12} = \omega_{13} \wedge \omega_{32} \tag{2.5}$$

and *Codazzi equation*

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}. \tag{2.6}$$

Similarly, from $d^2\mathbf{e}_2 = 0$, we obtain another Codazzi equation

$$d\omega_{23} = \omega_{21} \wedge \omega_{13}. \tag{2.7}$$

2.2.2 First and Second Fundamental Forms

The first fundamental form of the surface is given by

$$I = \langle d\mathbf{r}, d\mathbf{r}\rangle = \langle \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2, \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2\rangle = \omega_1^2 + \omega_2^2. \tag{2.8}$$

The second fundamental form is given by

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3\rangle = -\langle \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2, \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2\rangle = \omega_1\omega_{13} + \omega_2\omega_{23}. \tag{2.9}$$

The first and second fundamental forms are not independent, but related by the Gauss and Codazzi equations. Further, Gauss–Codazzi equations are also sufficient conditions for the existence and the uniqueness of a surface.

Let Ω be a matrix. Each entry is a differential form,

$$\Omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}.$$

Equations (2.2) and (2.3) can be summarized as

$$d\begin{pmatrix}\omega_1 & \omega_2 & 0\end{pmatrix} = \begin{pmatrix}\omega_1 & \omega_2 & 0\end{pmatrix} \wedge \Omega,$$

where the wedge product can be interpreted as the matrix product, and the product of two entries is replaced by the wedge product of two differential forms. Similarly, the Gauss–Codazzi equations (2.5), (2.6) and (2.7) can be summarized as

$$d\Omega = \Omega \wedge \Omega.$$

The fundamental theorem for surface differential geometry is as follows.

Theorem 2.4. *Suppose $D \subset \mathbb{R}^2$ is a domain on the (u, v) plane. Given 5 differential 1-forms, $\omega_1, \omega_2, \omega_{12}, \omega_{13}, \omega_{23}$, satisfying*

$$d\begin{pmatrix}\omega_1 & \omega_2 & 0\end{pmatrix} = \begin{pmatrix}\omega_1 & \omega_2 & 0\end{pmatrix} \wedge \Omega,$$
$$d\Omega = \Omega \wedge \Omega,$$

then for any given initial condition at $(u_0, v_0) \in D$, the position $\mathbf{r}(u_0, v_0)$ and an orthonormal frame $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}(u_0, v_0)$, there is a unique surface patch $\mathbf{r}(u, v)$ with an orthonormal frame field $\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}(u, v)$ in a neighborhood of (u_0, v_0), such that

$$d\mathbf{r} = \begin{pmatrix}\omega_1 & \omega_2 & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3\end{pmatrix}$$

and

$$d\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} = \Omega\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}.$$

The proof can be found in classical differential geometry textbook, such as [6].

Because $d\mathbf{r} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$, the area element of the surface is $\omega_1 \wedge \omega_2$. Similarly, because $d\mathbf{e}_3 = \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2$, the area element of the unit sphere is $\omega_{31} \wedge \omega_{32}$. Then the determinant of the Jacobian matrix of the Weingarten map $d\mathbf{r} \to d\mathbf{e}_3$ is given by

$$K = \frac{\omega_{31} \wedge \omega_{32}}{\omega_1 \wedge \omega_2}.$$

From Gauss equation (2.5), we get the important equation for Gauss curvature,

$$d\omega_{12} = -\omega_{31} \wedge \omega_{32} = -K\omega_1 \wedge \omega_2. \tag{2.10}$$

We say a geometric quantity is intrinsic, if it is solely determined by the first fundamental form, namely, Riemannian metric. From (2.4) and (2.10), we see that Gauss curvature K is solely determined by ω_1 and ω_2; therefore, we obtain Gauss' *Theorema Egregium*.

Theorem 2.5. *The Gaussian curvature solely depends on the first fundamental form, namely, is intrinsic.*

Therefore, the Gaussian curvature of a surface is invariant under local isometry.

2.2.3 Curves on Surfaces

Consider a curve C on a surface S with a local representation $C : (u(s), v(s))$, where s is the arc length parameter. Let $\boldsymbol{\alpha}$ be the tangent direction of C, and $\theta(s)$ be the angle from $\mathbf{e}_1$ to $\boldsymbol{\alpha}$. By direct computation, the *geodesic curvature* is given by

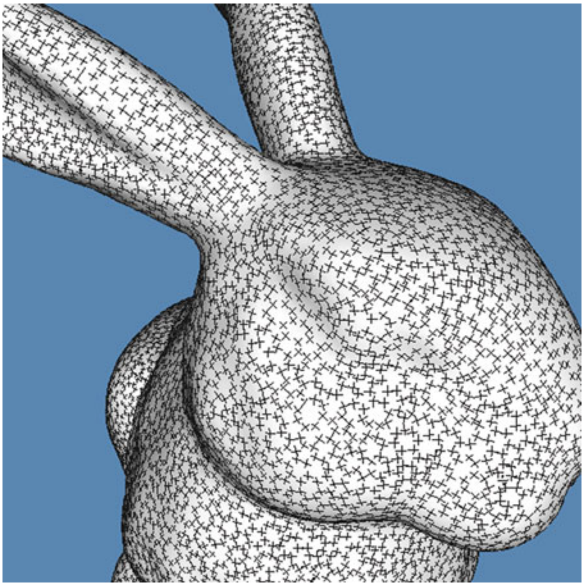
$$k_g = \frac{d\theta + \omega_{12}}{ds}. \tag{2.11}$$

If $k_g \equiv 0$, then the curve is called a *geodesic*. The *normal curvature* is

$$k_n = \frac{\omega_1\omega_{13} + \omega_2\omega_{23}}{ds^2} = \frac{II}{I}.$$

At each point, there are two orthogonal tangent directions, along which the normal curvature reaches the minimum k_1 and maximum k_2. k_1 and k_2 are called the *principle curvatures,* and the two directions are called the *principle directions*, as shown in Fig. 2.4. The Gauss curvature is the product of principle curvatures, $K = k_1k_2$; the *mean curvature* is the mean value of the principle curvatures, $H = (k_1 + k_2)/2$.

Fig. 2.4 Principle directions on the Stanford bunny surface



From (2.10) and (2.11), we can prove the Gauss–Bonnet theorem, which claims that although the Gauss curvature is determined by the Riemannian metric, the total curvature is solely determined by the surface topology.

Theorem 2.6 (Gauss–Bonnet). *Suppose S is a compact two-dimensional Riemannian manifold with piecewise-smooth boundary ∂S. Let K be the Gauss curvature, k_g the geodesic curvature of ∂S, and $\theta_k, k = 1, 2 \cdots, n$ be the exterior angles of ∂S. Then*

$$\int_S K dA + \int_{\partial S} k_g ds + \sum_{k=1}^{n} \theta_k = 2\pi \chi(S),$$

where $\chi(S)$ is the Euler characteristics of the surface.

2.3 Conformal Metric Deformation

2.3.1 Isothermal Coordinates

Given a metric surface, one can choose *isothermal coordinates* to facilitate geometric computations, as shown in Fig. 2.5. Most differential operators, such as gradient and Laplace–Beltrami operators, have the simplest form under isothermal coordinates.

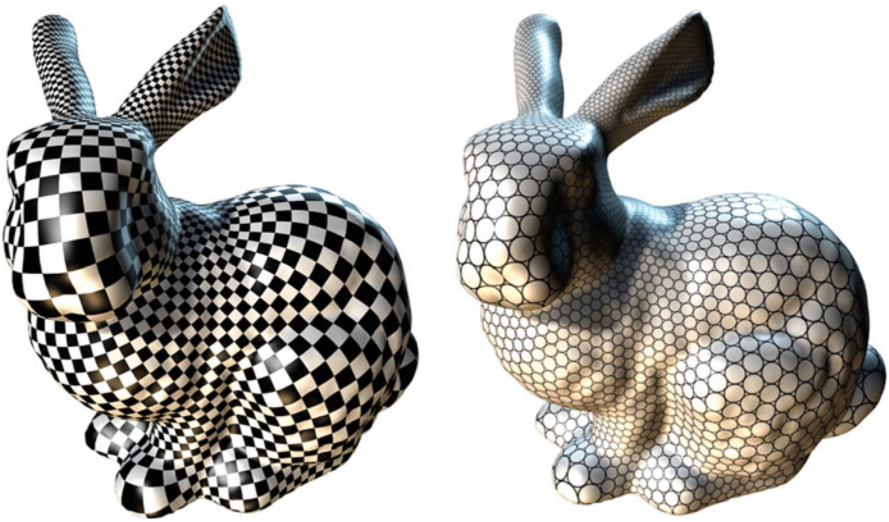


Fig. 2.5 Isothermal coordinate system on the Stanford bunny surface. The mapping from the surface to the parameter plane is conformal, which preserves angles and infinitesimal circles

Definition 2.9 (Isothermal Coordinates). On a surface S with a Riemannian metric $\mathbf{g}$, a local coordinates system (u, v) is an isothermal coordinate system, if

$$\mathbf{g}(u, v) = e^{2\lambda(u,v)}(du^2 + dv^2), \tag{2.12}$$

where $\lambda : S \to \mathbb{R}$ is a function defined on the surface, and called *conformal factor*.

Isothermal coordinates on metric surfaces always exist, which can be proved either using surface Ricci flow or quasi-conformal mapping. In the later part, we give a proof by solving a Beltrami equation. An elementary proof can be found in Chern's work [2].

Theorem 2.7 (Existence of Isothermal Coordinates). *Let* $(S, \mathbf{g})$ *be a compact orientable surface, then every point of* S *has a neighborhood whose local coordinates are isothermal parameters.*

2.3.2 *Gauss Curvature Under Conformal Deformation*

We use movable frame method to deduce Gauss curvature under isothermal coordinates. Let $(S, \mathbf{g})$ be a surface embedded in $\mathbb{E}^3$, with position vector function $\mathbf{r}(u, v)$ and isothermal coordinates (u, v). We denote $\partial_u \mathbf{r}$ as $\mathbf{r}_u$, and $\partial_v \mathbf{r}$ as $\mathbf{r}_v$, then

$$\langle \mathbf{r}_u, \mathbf{r}_u \rangle = e^{2\lambda}, \ \langle \mathbf{r}_v, \mathbf{r}_v \rangle = e^{2\lambda}, \ \langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0.$$

Choose an orthonormal frame

$$\mathbf{e}_1 = e^{-\lambda}\mathbf{r}_u,\ \mathbf{e}_2 = e^{-\lambda}\mathbf{r}_v,\ \mathbf{e}_3 = \mathbf{n}.$$

Then

$$d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv = \mathbf{e}_1 e^{\lambda} du + \mathbf{e}_2 e^{\lambda} dv = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

where $\omega_1 = e^{\lambda} du$ and $\omega_2 = e^{\lambda} dv$. From (2.4), we get $\omega_{12} = -\lambda_v du + \lambda_u dv$. Therefore

$$d\omega_{12} = (\lambda_{vv} + \lambda_{uu}) du \wedge dv = -K\omega_1 \wedge \omega_2 = -Ke^{2\lambda} du \wedge dv,$$

then we obtain

$$K(u,v) = -e^{-2\lambda(u,v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\lambda = -\Delta_{\mathbf{g}}\lambda, \tag{2.13}$$

where the Laplace–Beltrami operator is

$$\Delta_{\mathbf{g}} = e^{-2\lambda(u,v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right).$$

Let $\bar{\mathbf{g}}$ be another Riemannian metric, conformal to the original metric,

$$\bar{\mathbf{g}} = e^{2\tau}\mathbf{g},$$

where τ is a scalar function. We choose isothermal coordinates for both $\mathbf{g}$ and $\bar{\mathbf{g}}$, then

$$\mathbf{g} = e^{2\lambda}(du^2 + dv^2),\ \bar{\mathbf{g}} = e^{2(\lambda+\tau)}(du^2 + dv^2),$$

The Gauss curvature $\bar{K}$ induced by $\bar{\mathbf{g}}$ becomes

$$\bar{K} = -e^{-2(\lambda+\tau)}\Delta(\lambda+\tau) = e^{-2\tau}(-e^{-2\lambda}\Delta\lambda - e^{-2\lambda}\Delta\tau) = e^{-2\tau}(K - \Delta_{\mathbf{g}}\tau).$$

So we obtain the Yamabe equation

$$\bar{K} = e^{-2\tau}(K - \Delta_{\mathbf{g}}\tau). \tag{2.14}$$

2.3.3 Geodesic Curvature Under Conformal Deformation

Suppose C is a curve on the surface and the tangent direction $\boldsymbol{\alpha}$ of the curve has the angle θ to the $\mathbf{e}_1$ direction. Then the geodesic curvature of C is

$$k_g = \frac{d\theta + \omega_{12}}{ds}.$$

Choose the isothermal coordinates (u, v),

$$\begin{aligned}
\omega_{12} &= \lambda_v du - \lambda_u dv, \\
ds &= e^{\lambda}\sqrt{du^2 + dv^2}, \\
\frac{du}{ds} &= e^{-\lambda}\cos\theta, \ \frac{dv}{ds} = e^{-\lambda}\sin\theta, \\
k_g &= \frac{d\theta}{ds} + \frac{\lambda_v du - \lambda_u dv}{ds} = \frac{d\theta}{ds} - e^{-\lambda}(\lambda_u \sin\theta - \lambda_v \cos\theta) \\
&= \frac{d\theta}{ds} - \langle \nabla_{\mathbf{g}}\lambda, \mathbf{n}\rangle,
\end{aligned}$$

where $\nabla_{\mathbf{g}} = e^{-\lambda}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$, and $\mathbf{n} = (\sin\theta, -\cos\theta)$ is the outward normal of the curve on the tangent plane. The geodesic curvature can also be written as

$$k_g = \frac{d\theta}{ds} - \partial_{\mathbf{n},\mathbf{g}}\lambda.$$

Assume $\bar{\mathbf{g}}$ is another Riemannian metric conformal to $\mathbf{g}$, $\bar{\mathbf{g}} = e^{2\tau}\mathbf{g}$. Choose the isothermal coordinates (u, v),

$$\bar{\mathbf{g}} = e^{2(\tau+\lambda)}(du^2 + dv^2), \mathbf{g} = e^{2\lambda}(du^2 + dv^2),$$

then

$$\bar{k}_g = \frac{d\theta}{d\bar{s}} - \partial_{\mathbf{n},\bar{\mathbf{g}}}(\lambda + \tau).$$

Because $d\bar{s} = e^{\tau}ds$,

$$\frac{d\theta}{d\bar{s}} = e^{-\tau}\frac{d\theta}{ds}.$$

Because $\nabla_{\bar{\mathbf{g}}} = e^{-\tau-\lambda}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = e^{-\tau}\nabla_{\mathbf{g}}$,

$$\partial_{\mathbf{n},\bar{\mathbf{g}}} = \langle \nabla_{\bar{\mathbf{g}}}, \mathbf{n}\rangle = e^{-\tau}\langle \nabla_{\mathbf{g}}, \mathbf{n}\rangle = e^{-\tau}\partial_{\mathbf{n},\mathbf{g}}.$$

Therefore

$$\bar{k}_g = e^{-\tau}\frac{d\theta}{ds} - e^{-\tau}\partial_{\mathbf{n},\mathbf{g}}(\lambda + \tau) = e^{-\tau}\left(\frac{d\theta}{ds} - \partial_{\mathbf{n},\mathbf{g}}\lambda - \partial_{\mathbf{n},\mathbf{g}}\tau\right) = e^{-\tau}(k_g - \partial_{\mathbf{n},\mathbf{g}}\tau).$$

Theorem 2.8 (Surface Yamabe Problem). *Suppose S is a surface with a Riemannian metric $\mathbf{g}$, which induces Gauss curvature K and geodesic curvature k_g on the boundary. Let*

$$\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$$

be another metric conformal to the original one, which induces Gauss curvature $\bar{K}$ and geodesic curvature $\bar{k}_g$. Then the changes of Gauss curvature and geodesic curvature associated with the conformal metric change respectively are

$$\begin{aligned}\bar{K} &= e^{-2\lambda}(K - \Delta_{\mathbf{g}}\lambda),\\ \bar{k}_g &= e^{-\lambda}(k_g - \partial_{\mathbf{n},\mathbf{g}}\lambda).\end{aligned}$$

Surface Yamabe problem is to find the conformal factor λ from the prescribed curvatures $\bar{K}$ and $\bar{k}_g$, which can be solved using surface Ricci flow.

2.4 Surface Ricci Flow

Given an n dimensional Riemannian manifold M with metric tensor $\mathbf{g} = (g_{ij})$, the normalized Ricci flow is defined by the geometric evolution equation

$$\partial_t \mathbf{g}(t) = -2Ric(\mathbf{g}(t)) + \rho \mathbf{g}(t).$$

where Ric is the Ricci curvature tensor and ρ is the mean value of the scalar curvature,

$$\rho = \frac{2}{n}\frac{\int_M R_{\mathbf{g}} d\mu_{\mathbf{g}}}{\int_M d\mu_{\mathbf{g}}},$$

where $R_{\mathbf{g}}$ and $\mu_{\mathbf{g}}$ are the scalar curvature and the volume element with respect to the evolving metric $\mathbf{g}(t)$, respectively. Recall that a one-parameter family of metrics $\{\mathbf{g}(t)\}$, where $t \in [0, T)$ for some $0 < T \leq \infty$, is called a solution to the normalized Ricci flow if it satisfies the above equation at all $p \in M$ and $t \in [0, T)$.

In two dimensions, the Ricci curvature for a metric $\mathbf{g}$ is equal to $\frac{1}{2}R\mathbf{g}$, where R is the scalar curvature (or twice the Gauss curvature). Therefore, the normalized Ricci flow equation for surfaces takes the form

$$\partial_t \mathbf{g}(t) = (\rho - R(t))\mathbf{g}(t), \tag{2.15}$$

where ρ is the mean value of the scalar curvature,

$$\rho = \frac{4\pi\chi(M)}{A(0)},$$

where $\chi(M)$ is the Euler characteristic number of M, and $A(0)$ is the total area of the surface M at time $t = 0$.

Let $(g^{ij}) = (g_{ij})^{-1}$ be the inverse of the matrix (g_{ij}). Set the area element with respect to metric $\mathbf{g}$ to be

$$\mu_{\mathbf{g}} = \sqrt{det g_{ij}}.$$

Then along the Ricci flow, we compute

$$\partial_t \mu_{\mathbf{g}} = \frac{1}{2} g^{ij} \partial_t g_{ij} \mu_{\mathbf{g}} = (\rho - R)\mu_{\mathbf{g}}.$$

For the total area $A = \int_M d\mu$, we have

$$\partial_t A(t) = \int_M (\rho - R) d\mu_{\mathbf{g}} = 0.$$

Therefore, the normalized Ricci flow preserves the total area, $A(t) = A(0), \forall t > 0$. During the Ricci flow (2.15), the metric deforms conformally, $\mathbf{g}(t) = e^{2\lambda(t)}\mathbf{g}(0)$,

$$\partial_t \lambda = \frac{1}{2}(\rho - R),\ \lambda(0) = 0, \tag{2.16}$$

and from Yamabe equation (2.14),

$$\Delta_0 \lambda - \frac{1}{2} R_0 + \frac{1}{2} R e^{2\lambda} = \Delta_0 \lambda - K_0 + K e^{2\lambda} = 0.$$

We obtain the curvature evolution equation

$$\partial_t R = \Delta_{g(t)} R + R(R - \rho). \tag{2.17}$$

Let $u = e^{2\lambda}$, then we get the evolution equation for u,

$$\partial_t u = (\rho - R)u.$$

Plug in $R = u^{-1}(R_0 - \Delta_0 \log u)$, we get an evolution flow for u,

$$\partial_t u = \Delta_0 \log u + \rho u - R_0,\ u(0) = u_0. \tag{2.18}$$

For most evolution equations, one proved that the solutions exist for all $t \geq 0$ by combining a short-time existence (and uniqueness) result with prior bounds which show that the solutions cannot develop singularities in finite time. Equation (2.18) is a parabolic equation. It can be set up as a fixed point problem for a contraction mapping. The mapping is obtained by applying the fundamental solution of the linearization at any given $u_0 > 0$ to (2.18); it is a contraction on any sufficiently short-time interval. This gives the proof for the short-time existence of the solution. The long-time existence can be obtained by estimating both the lower and upper bounds of $R(t)$ and $u(t)$, which requires a generalization of Li-Yau's Harnack inequality [8]. The proofs require advanced background knowledge and sophisticated geometric skills, which is beyond the scope of the current book. Details can be found in Hamilton's [7] and Chow's [4] works.

Theorem 2.9 (Hamilton [7]). *Let (M^2, g_0) be compact. If $\rho \leq 0$, or if $R(0) \geq 0$ on all of M^2, then the solution to (2.15) exists for all $t \geq 0$ and converges to a metric of constant curvature.*

Theorem 2.10 (Chow [4]). *If g_0 is any metric on $\mathbb{S}^2$, then its evolution under (2.15) develops positive scalar curvature in finite time, and hence by Theorem 2.9 converges to the round metric as t goes to ∞.*

In Chap. 4, we will give a discrete version of surface Ricci flow theory and prove its convergence (Theorem 4.6). Discrete surface Ricci flow shares the same theoretical framework, the same fundamental principles, and even the same formulae (comparing (2.16) and (4.10)), but only requires elementary geometric knowledge to prove. Furthermore, discrete surface Ricci flow theory leads to the practical computational algorithms directly.

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