An Alternative to Cronbach's Alpha: An L-Moment-Based Measure of Internal-Consistency Reliability

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1 Introduction

Coefficient alpha (Cronbach 1951; Guttman 1945) is a commonly used index for measuring internal-consistency reliability. Consider alpha (α) in terms of a model that decomposes an observed score into the sum of two independent components: a true unobservable score t_i and a random error component e_{ij}. The model can be summarized as

$$X_{ij} = t_i + e_{ij} \tag{1}$$

where X_{ij} is the observed score associated with the i-th examinee on the j-th test item, and where $i = 1, \ldots, n$; $j = 1, \ldots, k$; and the error terms (e_{ij}) are independent with a mean of zero. Inspection of (1) indicates that this particular model restricts the true score t_i to be the same across all k test items. The reliability measure associated with the test items in (1) is a function of the true score variance and cannot be computed directly. Thus, estimates of reliability such as coefficient α have been derived and will be defined herein as (e.g., Christman and Van Aelst 2006)

$$\alpha = \frac{k}{k-1}\left(1 - \frac{\sum_j \sigma_j^2}{\sum_j \sigma_j^2 + \sum\sum_{j \neq j'} \sigma_{jj'}}\right). \tag{2}$$

A conventional estimate of α can be obtained by substituting the usual OLS sample estimates associated with σ_j^2 and $\sigma_{jj'}$ into (2) as

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$$\hat{\alpha}_C = \frac{k}{k-1}\left(1 - \frac{\sum_j s_j^2}{\sum_j s_j^2 + \sum\sum_{j \neq j'} s_{jj'}}\right) \tag{3}$$

where s_j^2 and $s_{jj'}$ are the diagonal and off-diagonal elements from the variance–covariance matrix, respectively.

Although coefficient α is often used as an index for reliability, it is also well known that its use is limited when data are non-normal, in particular leptokurtic, or when sample sizes are small (e.g., Bay 1973; Christman and Van Aelst 2006; Sheng and Sheng 2012; Wilcox 1992). These limitations are of concern because data sets in the social and behavioral sciences can often possess heavy tails or consist of small sample sizes (e.g., Micceri 1989; Yuan et al. 2004). Specifically, it has been demonstrated that $\hat{\alpha}_C$ can substantially underestimate α when heavy-tailed distributions are encountered. For example, Sheng and Sheng (2012, Table 1) sampled from a symmetric leptokurtic distribution and found the empirical estimate of α to be approximately $\hat{\alpha}_C = 0.70$ when the true population parameter was $\alpha = 0.80$. Further, it is not uncommon that data sets consist of small sample sizes, e.g., $n = 10$ or 20. More specifically, small sample sizes are commonly encountered in the contexts of rehabilitation (e.g., alcohol treatment programs, group therapy, etc.) and special education as student–teacher ratios are often small. Furthermore, Monte Carlo evidence has demonstrated that $\hat{\alpha}_C$ can underestimate α—even when small samples are drawn from a normal distribution (see Sheng and Sheng 2012, Table 1).

L-moment estimators (e.g., Hosking 1990; Hosking and Wallis 1997) have demonstrated to be superior to the conventional product-moment estimators in terms of bias, efficiency, and their resistance to outliers (e.g., Headrick 2011; Hodis et al. 2012; Hosking 1992; Vogel and Fennessy 1993). Further, L-comoment estimators (Serfling and Xiao 2007) such as the L-correlation have demonstrated to be an attractive alternative to the conventional Pearson correlation in terms of relative bias when heavy-tailed distributions are of concern (Headrick and Pant 2012a,b,c,d,e).

In view of the above, the present aim here is to propose an L-comoment-based coefficient L-α, and its estimator denoted as $\hat{\alpha}_L$, as an alternative to conventional alpha $\hat{\alpha}_C$ in (3). Empirical results associated with the simulation study herein indicate that $\hat{\alpha}_L$ can be substantially superior to $\hat{\alpha}_C$ in terms of relative bias and relative standard error (RSE) when distributions are heavy-tailed and sample sizes are small.

The rest of the paper is organized as follows. In Sect. 2, summaries of univariate L-moments and L-comoments are first provided. Coefficient L-α ($\hat{\alpha}_L$) is then introduced and numerical examples are provided to illustrate the computation and sampling distribution associated with $\hat{\alpha}_L$. In Sect. 3, a Monte Carlo study is carried out to evaluate the performance of $\hat{\alpha}_C$ and $\hat{\alpha}_L$. The results of the study are discussed in Sect. 4.

2 L-Moments, L-Comoments, and Coefficient L-α

The system of univariate L-moments (Hosking 1990, 1992; Hosking and Wallis 1997) can be considered in terms of the expectations of linear combinations of order statistics associated with a random variable Y. Specifically, the first four L-moments are expressed as

$$\begin{aligned}
\lambda_1 &= E[Y_{1:1}] \\
\lambda_2 &= \frac{1}{2}E[Y_{2:2} - Y_{1:2}] \\
\lambda_3 &= \frac{1}{3}E[Y_{3:3} - 2Y_{2:3} + Y_{1:3}] \\
\lambda_4 &= \frac{1}{4}E[Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}]
\end{aligned}$$

where $Y_{\ell:m}$ denotes the ℓth smallest observation from a sample of size m. As such, $Y_{1:m} \leq Y_{2:m} \leq \ldots \leq Y_{m:m}$ are referred to as order statistics drawn from the random variable Y. The values of λ_1 and λ_2 are measures of location and scale and are the arithmetic mean and one-half of the coefficient of mean difference (or Gini's index of spread), respectively. Higher order L-moments are transformed to dimensionless quantities referred to as L-moment ratios defined as $\tau_r = \lambda_r/\lambda_2$ for $r \geq 3$, where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L-moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L-skew (τ_3) where a symmetric distribution implies that all L-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L-kurtosis (τ_4) has the boundary condition for continuous distributions of $(5\tau_3^2 - 1)/4 < \tau_4 < 1$.

L-comoments (Olkin and Yitzhuki 1992; Serfling and Xiao 2007) are introduced by considering two random variables Y_j and Y_k with distribution functions $F(Y_j)$ and $F(Y_k)$. The second L-moments associated with Y_j and Y_k can alternatively be expressed as

$$\begin{aligned}
\lambda_2(Y_j) &= 2\mathrm{Cov}(Y_j, F(Y_j)) \\
\lambda_2(Y_k) &= 2\mathrm{Cov}(Y_k, F(Y_k)).
\end{aligned} \tag{4}$$

The second L-comoments of Y_j toward Y_k and Y_k toward Y_j are

$$\begin{aligned}
\lambda_2(Y_j, Y_k) &= 2\mathrm{Cov}(Y_j, F(Y_k)) \\
\lambda_2(Y_k, Y_j) &= 2\mathrm{Cov}(Y_k, F(Y_j)).
\end{aligned} \tag{5}$$

The ratio $\eta_{jk} = \lambda_2(Y_j, Y_k)/\lambda_2(Y_j)$ is defined as the L-correlation of Y_j with respect to Y_k, which measures the monotonic relationship (not just linear) between two variables (Headrick and Pant 2012c). Note that in general, $\eta_{jk} \neq \eta_{kj}$. The estimators of (4) and (5) are U-statistics (Serfling 1980; Serfling and Xiao 2007) and their sampling distributions converge to a normal distribution when the sample size is sufficiently large.

In terms of coefficient L-α, an approach that can be taken to equate the conventional and L-moment (comoment) definitions of α is to express (2) as

Table 1 Data (Items) for computing the second L-moment–comoment matrix in Table 2

X_{i1}	X_{i2}	X_{i3}	$\hat{F}(X_{i1})$	$\hat{F}(X_{i2})$	$\hat{F}(X_{i3})$
2	4	3	0.15	0.45	0.15
5	7	7	0.75	0.95	1.00
3	5	5	0.35	0.65	0.40
6	6	6	0.90	0.80	0.75
7	7	6	1.00	0.95	0.75
5	2	6	0.75	0.10	0.75
2	3	3	0.15	0.25	0.15
4	3	6	0.55	0.25	0.75
3	5	5	0.35	0.65	0.40
4	4	5	0.55	0.45	0.40

The data are part of the "Satisfaction With Life Data" from McDonald (1999, p. 47)

Table 2 Second L-moment–comoment matrix for coefficient $\hat{\alpha}_L$ in Eq. (9)

Item	1	2	3
1	$\ell_{2(1)} = 0.989$	$\ell_{2(12)} = 0.500$	$\ell_{2(13)} = 0.789$
2	$\ell_{2(21)} = 0.500$	$\ell_{2(2)} = 1.022$	$\ell_{2(23)} = 0.411$
3	$\ell_{2(31)} = 0.667$	$\ell_{2(32)} = 0.333$	$\ell_{2(3)} = 0.733$

$$\alpha = \frac{1}{1+(R-1)/k} = \frac{k}{k-1}\left(1 - \frac{\sum_j \sigma_j^2}{\sum_j \sigma_j^2 + \sum\sum_{j\neq j'} \sigma_{jj'}}\right) \tag{6}$$

where $R > 1$ is the common ratio between the main and off-diagonal elements of the variance–covariance matrix, i.e. $R = \sigma_j^2/\sigma_{jj'}$. (See the appendix for the derivation of Eq. (6)). As such, given a fixed value of R in (6) will allow for α to be defined in terms of the second L-moments and second L-comoments as

$$\alpha = \frac{1}{1+(R-1)/k} = \frac{k}{k-1}\left(1 - \frac{\sum_j \lambda_{2(j)}}{\sum_j \lambda_{2(j)} + \sum\sum_{j\neq j'} \lambda_{2(jj')}}\right) \tag{7}$$

where $R = \lambda_{2(j)}/\lambda_{2(jj')}$. Thus, the estimator of L-α is expressed as

$$\hat{\alpha}_L = \frac{k}{k-1}\left(1 - \frac{\sum_j \ell_{2(j)}}{\sum_j \ell_{2(j)} + \sum\sum_{j\neq j'} \ell_{2(jj')}}\right) \tag{8}$$

where $\ell_{2(j)}$ ($\ell_{2(jj')}$) denotes the sample estimate of the second L-moments (second L-comoment) in (4) and (5). An example demonstrating the computation of $\hat{\alpha}_L$ is provided below in Eq. (9). The computed estimate of $\hat{\alpha}_L = 0.807$ in (9) is based on the data in Table 1 and the second L-moment–comoment matrix in Table 2. The corresponding conventional estimate for the data in Table 1 is $\hat{\alpha}_C = 0.798$.

$$\begin{aligned}\hat{\alpha}_L = 0.807 = {}& (3/2)(1-(\ell_{2(1)}+\ell_{2(2)}+\ell_{2(3)})/(\ell_{2(1)}+\ell_{2(2)}+\ell_{2(3)} \\ &+\ell_{2(21)}+\ell_{2(31)}+\ell_{2(32)}+\ell_{2(12)}+\ell_{2(13)}+\ell_{2(23)})).\end{aligned} \tag{9}$$

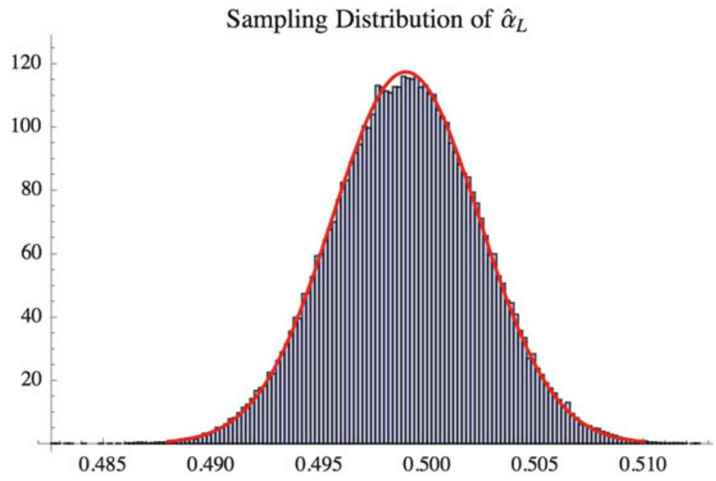

Fig. 1 Approximate normal sampling distribution of $\hat{\alpha}_L$ with $\alpha = 0.50$. The distribution consists of 25,000 statistics based on samples of size $n = 100{,}000$ and the heavy-tailed distribution (kurtosis of 25) in Fig. 2

The estimator $\hat{\alpha}_L$ in (8) and (9) is a ratio of the sums of U-statistics and thus a consistent estimator of α in (7) with a sampling distribution that converges, for large samples, to the normal distribution (e.g., Olkin and Yitzhuki 1992; Schechtman and Yitzhaki 1987; Serfling and Xiao 2007). For convenience to the reader, provided in Fig. 1 is the sampling distribution of $\hat{\alpha}_L$ that is approximately normal and based on $\alpha = 0.50$, $n = 100{,}000$, and a symmetric heavy-tailed distribution (kurtosis of 25, see Fig. 2) that would be associated with t_i in (1).

3 Monte Carlo Simulation

An algorithm was written in MATLAB (Mathworks 2010) to generate 25,000 independent sample estimates of conventional and *L*-comoment α. The estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$ were based on the parameters (α, k, R) given in Tables 3 and 4 and the distributions in Figs. 2–4. The parameters of α were selected because they represent commonly used references of various degrees of reliability, i.e. 0.50 (poor); 5/7 = 0.714 (acceptable); 0.80 (good); and 0.90 (excellent). Further, for each set of parameters in Tables 3 and 4, the empirical estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$ were generated based on sample sizes of $n = 10, 20, 1{,}000$. For all cases in the simulation, the error term e_{ij} in (1) was normally distributed with zero mean and with the variance parameters (σ_e^2) listed in Tables 3 and 4.

The three distributions depicted in Figs. 2–4 are associated with the true scores t_i in Eq. (1). These distributions are referred to as: Distribution 1 is symmetric and leptokurtic (skew = 0, kurtosis = 25; *L*-skew = 0, *L*-kurtosis = 0.4225);

Table 3 Parameters for the Conventional covariance (*L*-comoment) matrix and distributions in Figs. 2–4

Distribution-matrix	Diagonal	Off-diagonal	σ_e^2
1-C	3.420	1.710	1.710
1-L	0.848	0.424	1.000
2-C	3.224	1.612	1.612
2-L	0.842	0.421	1.000
3-C	2.000	1.000	1.000
3-L	0.798	0.399	1.000

Reliability is α =0.80, 0.90; number of items are k =4, 9
Ratio of diagonal to off-diagonal is $R = 2$

Table 4 Parameters for the Conventional covariance (*L*-comoment) matrix and distributions in Figs. 2–4

Distribution-matrix	Diagonal	Off-diagonal	σ_e^2
1-C	8.550	1.710	6.840
1-L	1.470	0.294	5.313
2-C	8.060	1.612	6.448
2-L	1.443	0.2886	5.135
3-C	5.000	1.000	4.000
3-L	1.262	0.2524	4.000

Reliability is α =0.50, 0.714; number of items are k =4, 10
Ratio of diagonal to off-diagonal is $R = 5$

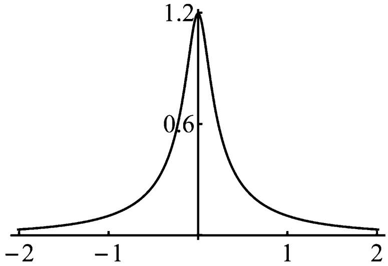

Fig. 2 Distribution 1 with skew (*L*-skew) of 0 (0) and kurtosis (*L*-kurtosis) of 25 (0.4225)

Distribution 2 is asymmetric and leptokurtic (skew = 3, kurtosis = 21; *L*-skew = 0.3130, *L*-kurtosis = 0.3335); and Distribution 3 is standard normal (skew = 0, kurtosis = 0; *L*-skew = 0, *L*-kurtosis = 0.1226). We would note that Distributions 1 and 2 have been used in several studies in the social and behavioral sciences (e.g., Berkovits et al. 2000; Enders 2001; Harwell and Serlin 1988; Headrick and Sawilowsky 1999, 2000; Olsson et al. 2003).

The pseudo-random deviates associated with the distributions in Figs. 2–4 were generated for this study using the *L*-moment-based power method transformation derived by Headrick (2011). Specifically, the true scores t_i in (1) were generated using the following (Fleishman 1978) type polynomial

$$t_i = c_1 + c_2 Z_i + c_3 Z_i^2 + c_4 Z_i^3 \tag{10}$$

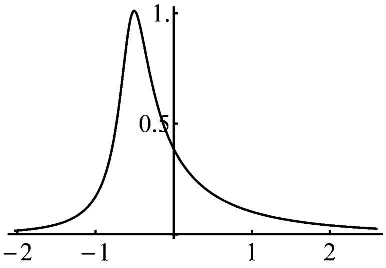

Fig. 3 Distribution 2 with skew (L-skew) of 3 (0.3130) and kurtosis (L-kurtosis) of 21 (0.3335)

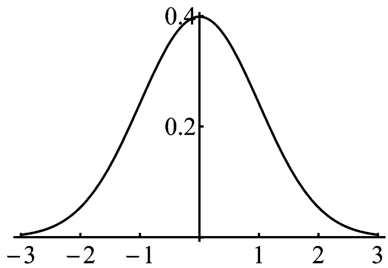

Fig. 4 Distribution 3 is standard normal with skew (L-skew) of 0 (0) and kurtosis (L-kurtosis) of 0 (0.1226)

where $Z_i \sim$ iid $N(0,1)$. The shape of the distribution of the true scores t_i in (10) is contingent on the values of the coefficients, which are computed based on Headrick's equations (2.14)–(2.17) in Headrick (2011) as

$$
\begin{aligned}
c_1 &= -c_3 = -\tau_3\sqrt{\frac{\pi}{3}} \\
c_2 &= \frac{-16\delta_2 + \sqrt{2}(3+2\tau_4)\pi}{8(5\delta_1 - 2\delta_2)} \\
c_4 &= \frac{40\delta_1 - \sqrt{2}(3+2\tau_4)\pi}{20(5\delta_1 - 2\delta_2)}.
\end{aligned}
\tag{11}
$$

The three sets of coefficients for the distributions in Figs. 2–4 are (respectively): (1) $c_1 = 0.0$, $c_2 = 0.3338$, $c_3 = 0.0$, $c_4 = 0.2665$; (2) $c_1 = -0.3203$, $c_2 = 0.5315$, $c_3 = 0.3203$, $c_4 = 0.1874$; and (3) $c_1 = 0.0$, $c_2 = 1.0$, $c_3 = 0.0$, $c_4 = 0.0$. The values of the three sets of coefficients are based on the values of L-skew and L-kurtosis given in Figs. 2–4 and where $\delta_1 = 0.36045147$ and $\delta_2 = 1.15112868$ in (11) (see Headrick 2011, Eqs. A.1, A.2). The solutions to the coefficients in (11) ensure that $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$, which are associated with the unit normal distribution.

The estimator $\hat{\alpha}_C$ was computed using Eq. (3). The estimator $\hat{\alpha}_L$ was computed using Eqs. (4), (5), and (8) as was demonstrated in Tables 1 and 2. The estimators were both transformed to the form of an intraclass correlation as

$\bar{\rho}_{C,L} = \hat{\alpha}_{C,L}/(1-(k-1)\hat{\alpha}_{C,L})$ (e.g., Headrick 2010, p. 104) and were subsequently Fisher z' transformed, i.e. $z'_{\bar{\rho}_{C,L}}$. Bias-corrected accelerated bootstrapped average (mean) estimates, confidence intervals (C.I.s), and standard errors were subsequently obtained for $z'_{\bar{\rho}_{C,L}}$ using 10,000 resamples. The bootstrap results associated with the means and C.I.s were then transformed back to their original metrics (i.e., the estimators $\hat{\alpha}_C$ and $\hat{\alpha}_L$). Further, percentages of relative bias (RBias) and RSE were computed for $\hat{\alpha}_{C,L}$ as: RBias $= ((\hat{\alpha}_{C,L} - \alpha)/\alpha) \times 100$ and RSE $= (\text{standarderror}/\hat{\alpha}_{C,L}) \times 100$. The results of the simulation are reported in Tables 5–7 and are discussed in the next section.

4 Discussion and Conclusion

One of the advantages that L-moment ratios have over conventional product-moment estimators is that they can be far less biased when sampling is from distributions with more severe departures from normality (Hosking and Wallis 1997; Serfling and Xiao 2007). And, inspection of the simulation results in Tables 5

Table 5 Simulation results for α based on the Conventional (C) and L-moment (L) procedures (Proc) based on samples of size $n = 10$

Parameters	Dist–Proc	Estimate (α)	95 % C.I.	RSE (%)	RBias (%)
$\alpha = 0.50, k = 4$	1-C	0.4416	0.4367, 0.4465	0.5661	−11.68
$\alpha = 0.50, k = 4$	1-L	0.4847	0.4801, 0.4891	0.4725	−3.06
$\alpha = 0.50, k = 4$	2-C	0.4448	0.4400, 0.4495	0.3237	−11.04
$\alpha = 0.50, k = 4$	2-L	0.4839	0.4796, 0.4883	0.2583	−3.22
$\alpha = 0.50, k = 4$	3-C	0.4888	0.4852, 0.4922	0.3621	−2.24
$\alpha = 0.50, k = 4$	3-L	0.5003	0.4968, 0.5040	0.3698	0.06
$\alpha = 0.714, k = 10$	1-C	0.6617	0.6581, 0.6652	0.2720	−7.36
$\alpha = 0.714, k = 10$	1-L	0.6960	0.6931, 0.6989	0.2155	−2.56
$\alpha = 0.714, k = 10$	2-C	0.6662	0.6628, 0.6697	0.2612	−6.73
$\alpha = 0.714, k = 10$	2-L	0.6975	0.6946, 0.7003	0.2079	−2.35
$\alpha = 0.714, k = 10$	3-C	0.7069	0.7051, 0.7086	0.1273	−1.03
$\alpha = 0.714, k = 10$	3-L	0.7131	0.7113, 0.7149	0.1290	−0.17
$\alpha = 0.80, k = 4$	1-C	0.7306	0.7275, 0.7336	0.2053	−8.67
$\alpha = 0.80, k = 4$	1-L	0.7887	0.7866, 0.7908	0.1357	−1.41
$\alpha = 0.80, k = 4$	2-C	0.7398	0.7371, 0.7426	0.1906	−7.52
$\alpha = 0.80, k = 4$	2-L	0.7924	0.7904, 0.7944	0.1287	−0.95
$\alpha = 0.80, k = 4$	3-C	0.7908	0.7893, 0.7922	0.0923	−1.15
$\alpha = 0.80, k = 4$	3-L	0.8030	0.8016, 0.8044	0.0909	0.37
$\alpha = 0.90, k = 9$	1-C	0.8591	0.8575, 0.8609	0.0989	−4.54
$\alpha = 0.90, k = 9$	1-L	0.8924	0.8914, 0.8936	0.0628	−0.84
$\alpha = 0.90, k = 9$	2-C	0.8636	0.8620, 0.8651	0.0926	−4.04
$\alpha = 0.90, k = 9$	2-L	0.8933	0.8922, 0.8944	0.0605	−0.74
$\alpha = 0.90, k = 9$	3-C	0.8934	0.8927, 0.8941	0.0381	−0.73
$\alpha = 0.90, k = 9$	3-L	0.8991	0.8985, 0.8998	0.0378	−0.10

See Tables 3 and 4 for the parameters and Figs. 2–4 for the distributions (Dist)

Table 6 Simulation results for α based on the Conventional (C) and L-moment (L) procedures (Proc) based on samples of size $n = 20$

Parameters	Dist–Proc	Estimate (α)	95 % C.I.	RSE (%)	RBias (%)
$\alpha = 0.50, k = 4$	1-C	0.4643	0.4606, 0.4679	0.3977	−7.15
$\alpha = 0.50, k = 4$	1-L	0.4903	0.4870, 0.4933	0.3263	−1.94
$\alpha = 0.50, k = 4$	2-C	0.4697	0.4663, 0.4732	0.3732	−6.05
$\alpha = 0.50, k = 4$	2-L	0.4938	0.4909, 0.4967	0.306	−1.24
$\alpha = 0.50, k = 4$	3-C	0.4945	0.4921, 0.4968	0.2389	−1.11
$\alpha = 0.50, k = 4$	3-L	0.4995	0.4971, 0.5019	0.2456	−0.11
$\alpha = 0.714, k = 10$	1-C	0.6852	0.6826, 0.6878	0.1926	−4.07
$\alpha = 0.714, k = 10$	1-L	0.7056	0.7036, 0.7077	0.1485	−1.22
$\alpha = 0.714, k = 10$	2-C	0.6858	0.6834, 0.6882	0.1831	−3.98
$\alpha = 0.714, k = 10$	2-L	0.7047	0.7028, 0.7066	0.1414	−1.34
$\alpha = 0.714, k = 10$	3-C	0.7098	0.7086, 0.7111	0.0881	−0.62
$\alpha = 0.714, k = 10$	3-L	0.7130	0.7117, 0.7142	0.0882	−0.19
$\alpha = 0.80, k = 4$	1-C	0.7569	0.7549, 0.7591	0.1404	−5.39
$\alpha = 0.80, k = 4$	1-L	0.7937	0.7923, 0.7952	0.0917	−0.78
$\alpha = 0.80, k = 4$	2-C	0.7612	0.7592, 0.7631	0.1330	−4.85
$\alpha = 0.80, k = 4$	2-L	0.7940	0.7926, 0.7954	0.0893	−0.75
$\alpha = 0.80, k = 4$	3-C	0.7944	0.7935, 0.7954	0.0627	−0.7
$\alpha = 0.80, k = 4$	3-L	0.8000	0.7990, 0.8010	0.0613	−0.002
$\alpha = 0.90, k = 9$	1-C	0.8750	0.8737, 0.8761	0.0690	−2.79
$\alpha = 0.90, k = 9$	1-L	0.8958	0.8950, 0.8966	0.0431	−0.47
$\alpha = 0.90, k = 9$	2-C	0.8784	0.8773, 0.8795	0.0644	−2.4
$\alpha = 0.90, k = 9$	2-L	0.8965	0.8958, 0.8972	0.0411	−0.39
$\alpha = 0.90, k = 9$	3-C	0.8969	0.8965, 0.8974	0.0247	−0.34
$\alpha = 0.90, k = 9$	3-L	0.8998	0.8994, 0.9002	0.0250	−0.02

See Tables 3 and 4 for the parameters and Figs. 2–4 for the distributions (Dist)

and 6 clearly indicates that this is the case. That is, the superiority that the L-comoment-based estimator $\hat{\alpha}_L$ has over its corresponding conventional counterpart $\hat{\alpha}_C$ is obvious in the contexts of Distributions 1 and 2. For example, inspection of the first entry in Table 5 ($\alpha = 0.50$, $k = 4$, $n = 10$) indicates that the estimator $\hat{\alpha}_C$ associated with Distribution 1 was, on average, 88.32 % of its associated population parameter whereas the estimator $\hat{\alpha}_L$ was 96.94 % of its parameter. Further, and in the context of Distribution 1, it is also evident that $\hat{\alpha}_L$ is a more efficient estimator as its RSE is smaller than its corresponding conventional estimator (see Table 5, $\alpha = 0.50$, $k = 4$, $n = 10$). This demonstrates that $\hat{\alpha}_L$ has more precision because it has less variance around its estimate.

In summary, the L-comoment-based $\hat{\alpha}_L$ is an attractive alternative to the traditional Cronbach alpha $\hat{\alpha}_C$ when distributions with heavy tails and small samples sizes are encountered. It is also worthy to point out that $\hat{\alpha}_L$ had a slight advantage over $\hat{\alpha}_C$ when sampling was from normal populations (see Table 5; $\alpha = 0.50$, $k = 4$, $n = 10$, 3-C, 3-L). When sample sizes was large the performance of the two estimators $\hat{\alpha}_{C,L}$ were similar (see Table 7; $n = 1{,}000$).

Table 7 Simulation results for α based on the Conventional (C) and L-moment (L) procedures (Proc) based on samples of size $n = 1,000$

Parameters	Dist–Proc	Estimate (α)	95 % C.I.	RSE (%)	RBias (%)
$\alpha = 0.50, k = 4$	1-C	0.4988	0.4982, 0.4994	0.05814	−0.24
$\alpha = 0.50, k = 4$	1-L	0.4988	0.4984, 0.4992	0.04210	−0.24
$\alpha = 0.50, k = 4$	2-C	0.4993	0.4987, 0.4998	0.05613	−0.14
$\alpha = 0.50, k = 4$	2-L	0.5001	0.4997, 0.5005	0.04200	0.02
$\alpha = 0.50, k = 4$	3-C	0.5000	0.4997, 0.5003	0.03200	0.00
$\alpha = 0.50, k = 4$	3-L	0.5000	0.4997, 0.5004	0.03400	0.00
$\alpha = 0.714, k = 10$	1-C	0.7134	0.7129, 0.7138	0.03084	−0.12
$\alpha = 0.714, k = 10$	1-L	0.7132	0.7129, 0.7135	0.02103	−0.15
$\alpha = 0.714, k = 10$	2-C	0.7133	0.7129, 0.7137	0.02804	−0.14
$\alpha = 0.714, k = 10$	2-L	0.7140	0.7137, 0.7143	0.01961	−0.04
$\alpha = 0.714, k = 10$	3-C	0.7141	0.7140, 0.7143	0.01120	−0.03
$\alpha = 0.714, k = 10$	3-L	0.7142	0.7140, 0.7144	0.01260	−0.01
$\alpha = 0.80, k = 4$	1-C	0.7991	0.7987, 0.7994	0.02127	−0.11
$\alpha = 0.80, k = 4$	1-L	0.8017	0.8015, 0.8019	0.01247	0.21
$\alpha = 0.80, k = 4$	2-C	0.7990	0.7987, 0.7993	0.02003	−0.12
$\alpha = 0.80, k = 4$	2-L	0.8011	0.8009, 0.8013	0.01248	0.14
$\alpha = 0.80, k = 4$	3-C	0.7999	0.7998, 0.8000	0.00875	−0.01
$\alpha = 0.80, k = 4$	3-L	0.8000	0.7998, 0.8001	0.00875	0.00
$\alpha = 0.90, k = 9$	1-C	0.8992	0.8990, 0.8994	0.01001	−0.09
$\alpha = 0.90, k = 9$	1-L	0.9008	0.9007, 0.9009	0.00555	0.09
$\alpha = 0.90, k = 9$	2-C	0.8994	0.8992, 0.8995	0.01000	−0.07
$\alpha = 0.90, k = 9$	2-L	0.9005	0.9004, 0.9006	0.00556	0.06
$\alpha = 0.90, k = 9$	3-C	0.8999	0.8999, 0.9000	0.00333	−0.01
$\alpha = 0.90, k = 9$	3-L	0.9000	0.8999, 0.9000	0.00333	0.00

See Tables 3 and 4 for the parameters and Figs. 2–4 for the distributions (Dist)

Appendix

Under the assumption of parallel measures, the error term e_{ij} in Eq. (1) has constant variance σ_e^2, the variance–covariance matrix assumes compound-symmetry, and thus the main and off-diagonal elements are $\sigma_j^2 = \sigma_X^2$ and $\sigma_{jj'} = \sigma_t^2$, respectively. Hence, Eq. (2) can be expressed using the true score and observed score variances as

$$\alpha = \frac{k}{k-1}\left(1 - \frac{k\sigma_X^2}{k\sigma_X^2 + k(k-1)\sigma_t^2}\right),$$

which can be simplified to

$$\alpha = \frac{k}{k-1}\left(1 - \frac{\sigma_X^2}{\sigma_X^2 + (k-1)\sigma_t^2}\right)$$

$$= \frac{k}{k-1}\left(\frac{(k-1)\sigma_t^2}{\sigma_X^2+(k-1)\sigma_t^2}\right)$$
$$= \frac{k\sigma_t^2}{\sigma_X^2+(k-1)\sigma_t^2}.$$

If we let $R = \sigma_j^2 / \sigma_{jj'} = \sigma_X^2 / \sigma_t^2$, then it follows that

$$\alpha = \frac{k}{R+k-1} = \frac{1}{1+(R-1)/k},$$

which is given in Eq. (6).

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